Chapter 2: Polynomial Interpolation

Contents

1 Problem 1

Consider the polynomial interpolation for the following data points:

x	1	2	3
y	3	4	5

1.1   a. Write down the linear system in matrix form for solving the coefficients a_i of the polynomial $p_n(x)$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

1.2   b. Use the lagrange interpolation process to obtain a polynomial to approximate these data points.

First let's define some cardinal functions:

$$I_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 2)(x - 3)}{(1 - 2)(1 - 3)} = \frac{(x - 2)(x - 3)}{2} = \frac{1}{2}(x - 2)(x - 3)$$

$$I_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 1)(x - 3)}{(2 - 1)(2 - 3)} = \frac{(x - 1)(x - 3)}{-1} = -(x - 1)(x - 3)$$

$$I_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 1)(x - 2)}{(3 - 1)(3 - 2)} = \frac{(x - 1)(x - 2)}{2} = \frac{1}{2}(x - 1)(x - 2)$$

Then plug our cardinal functions into the equation:

$$P_2(x) = I_0(x)y_0 + I_1(x)y_1 + I_2(x)y_2 = 3*(\frac{1}{2}(x-2)(x-3)) + 4*(-(x-1)(x-3)) + 5*(\frac{1}{2}(x-1)(x-2))$$

$$P_2(x) = \frac{3}{2}(x^2 - 5x + 6) - 4(x^2 - 4x + 3) + \frac{5}{2}(x^2 - 3x + 2)$$

$$P_2(x) = \frac{3}{2}x^2 - 4x^2 + \frac{5}{2}x^2 - 15x + 16x + 9 - 12 + 5$$

$$P_2(x) = x + 2$$

Which is a polynomial that does in fact contain all of our given data points!

2 Problem 2

2.1 a. Use the Lagrange interpolation process to obtain a polynomial of least degree that assumes these values

x	0	2	3	4
y	7	11	28	63

First let's define some cardinal functions.

$$I_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-2)(x-3)(x-4)}{(0-2)(0-3)(0-4)} = \frac{(x-2)(x-3)(x-4)}{-24}$$

$$I_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-3)(x-4)}{(2-0)(2-3)(2-4)} = \frac{x(x-3)(x-4)}{4}$$

$$I_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-2)(x-4)}{(3-0)(3-2)(3-4)} = \frac{x(x-2)(x-4)}{-3}$$

$$I_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-2)(x-3)}{(4-0)(4-2)(4-3)} = \frac{x(x-2)(x-3)}{8}$$

Then plug our cardinal functions into the equation:

$$P_3(x) = I_0(x)y_0 + I_1(x)y_1 + I_2(x)y_2 + I_3(x)y_3$$

$$P_3(x) = 7*(\frac{(x-2)(x-3)(x-4)}{-24}) + 11*(\frac{x(x-3)(x-4)}{4}) + 28*(\frac{x(x-2)(x-4)}{-3}) + 63*(\frac{x(x-2)(x-3)}{8})$$

$$P_3(x) = \frac{-7}{24}(x^3 - 9x^2 + 26x - 24) + \frac{11}{4}(x^3 - 7x^2 + 12x) + \frac{-28}{3}(x^3 - 6x^2 + 8x) + \frac{63}{8}(x^3 - 5x^2 + 6x)$$

$$P_3(x) = \frac{-7}{24}x^3 + \frac{11}{4}x^3 - \frac{28}{3}x^3 + \frac{63}{8}x^3 + \frac{63}{24}x^2 - \frac{77}{4}x^2 + \frac{168}{3}x^2 - \frac{315}{8}x^2 - \frac{182}{24}x + \frac{132}{4}x - \frac{224}{3}x + \frac{378}{8}x + 7$$

$$P_3(x) = x^3 - 2x + 7$$

And this polynomial includes all of our sample points!

2.2 b. For the points in the previous table, find the Newton's form of the interpolating polynomial. Show that the two polynomials obtained are identical, although their forms may differ.

The general equation for n = 3:

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

Then let's calculate the coefficients using our triangular matrix:

$$
\begin{array}{c|c|c|c|c}
0 & 7 & & & \\
2 & 11 & 2 & & \\
3 & 28 & 17 & 5 & \\
4 & 63 & 35 & 9 & 1
\end{array}
$$

Which results in the polynomial:

$$P_3(x) = 7 + 2x + 5x(x - 2) + x(x - 2)(x - 3)$$
$$P_3(x) = x^3 - 2x + 7$$

This is identical to the simplified polynomial in step a!

2.3 c. The polynomial $p(x) = x^4 - x^3 + x^2 - x + 1$ has the values shown

x	-2	-1	0	1	2	3
p(x)	31	5	1	1	11	61

Find a polynomial q(x) that takes the same values (you don't need to expand it):

If we proceed with Lagrange form, then we can start with cardinal functions.

$$I_0(x) = \frac{(x+1)x(x-1)(x-2)(x-3)}{(-2+1)(-2)(-2-1)(-2-2)(-2-3)} = \frac{-1}{120}x(x+1)(x-1)(x-2)(x-3)$$

$$I_1(x) = \frac{(x+2)x(x-1)(x-2)(x-3)}{(-1+2)(-1)(-1-1)(-1-2)(-1-3)} = \frac{1}{24}x(x+2)(x-1)(x-2)(x-3)$$

$$I_2(x) = \frac{(x+2)(x+1)(x-1)(x-2)(x-3)}{(0+2)(0+1)(0-1)(0-2)(0-3)} = \frac{-1}{12}(x+2)(x+1)(x-1)(x-2)(x-3)$$

$$I_3(x) = \frac{x(x+2)(x+1)(x-2)(x-3)}{(1+2)(1+1)(1)(1-2)(1-3)} = \frac{1}{12}x(x+2)(x+1)(x-2)(x-3)$$

$$I_4(x) = \frac{x(x+2)(x+1)(x-1)(x-3)}{(2+2)(2+1)(2)(2-1)(2-3)} = \frac{-1}{24}x(x+2)(x+1)(x-1)(x-3)$$

$$I_5(x) = \frac{x(x+2)(x+1)(x-1)(x-2)}{(3+2)(3+1)(3)(3-1)(3-2)} = \frac{1}{120}x(x+2)(x+1)(x-1)(x-2)$$

Then plug them into our polynomial:

$$P_5(x) = 31 * (\frac{-1}{120}x(x+1)(x-1)(x-2)(x-3)) + 5 * (\frac{1}{24}x(x+2)(x-1)(x-2)(x-3))...$$

$$+\frac{-1}{12}(x+2)(x+1)(x-1)(x-2)(x-3) + \frac{1}{12}x(x+2)(x+1)(x-2)(x-3) + 11*(\frac{-1}{24}x(x+2)(x+1)(x-1)(x-3))...$$

$$+61 * (\frac{1}{120}x(x+2)(x+1)(x-1)(x-2))$$