Chapter 9: Numerical Methods for ODE

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Contents

1 Problem 1 Scalar ODE

$$x' = 2x^2 + x - 1, x(1) = 1$$

1.1 a.

Write out Euler's method for this ODE. Compute the value of x(1.2) usign Euler's method, with h = 0.1. Recall Euler's method itself:

$$x_1 = x_0 + hx'(t_0) = x_0 + h * f(t_0, x_0)$$

Let's compute to x(1.2):

$$x(1.1) = x_0 + hf(t_0, x_0) = 1 + 0.1 * 2 = 1.2$$
$$x(1.2) = x_1 + hf(t_1, x_1) = 1.2 + 0.1 * 3.08 = 1.508$$

1.2 b.

Write out Heun's method for this ODE. Compute the value x(1.2) by Heun's method with h = 0.1. Recall Heun's method:

$$x_{k+1} = x_k + \frac{1}{2}(K_1 + K_2)$$
$$K_1 = h * f(t_k, x_k)$$

$$K_2 = h * f(t_k + h, x_k + K_1)$$

Let's compute to x(1.2):

At $x_0 = 1, t_0 = 1$:

$$K_1 = 0.1 * 2 = 0.2$$
$$K_2 = 0.1 * 3.08 = 0.308$$
$$x(1.1) = 1 + \frac{1}{2}(0.2 + 0.308) = 1.254$$

At $x_1 = 1.254, t_0 = 1.1$:

$$K_1 = 0.1 * 3.399 = 0.3399$$
$$K_2 = 0.1 * 5.6749 = 0.5675$$
$$x(1.2) = 1.254 + \frac{1}{2}(0.3399 + 0.5675) = 1.7077$$

1.3 c

Write out the classic 4th order RK method for this ODE. Compute the value x(1.2) by RK4 method with h = 0.1.

Recall RK4 method:

$$x_{k+1} = x_k + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$K_1 = h * f(t_k, x_k)$$
$$K_2 = h * f(t_k + \frac{1}{2}h, x_k + \frac{1}{2}K_1)$$
$$K_3 = h * f(t_k + \frac{1}{2}h, x_k + \frac{1}{2}K_2)$$
$$K_4 = h * f(t_k + h, x_k + K_3)$$

At $x_0 = 1, t_0 = 1$:

$$K_1 = 0.1 * 2 = 0.2$$
$$K_2 = 0.1 * 2.52 = 0.252$$
$$K_3 = 0.1 * 2.6618 = 0.2662$$
$$K_4 = 0.1 * 3.2333 = 0.3233$$
$$x(1.1) = 1 + \frac{1}{6}(0.2 + 2 * 0.252 + 2 * 0.2262 + 0.3233) = 1.247$$

At $x_1 = 1.247, t_0 = 1.1$

$$K_1 = 0.1 * 1.802 = 0.1802$$
$$K_2 = 0.1 * 2.467 = 0.2467$$
$$K_3 = 0.1 * 2.647 = 0.2647$$
$$K_4 = 0.1 * 3.464 = 0.3464$$
$$x(1.2) = 1.247 + \frac{1}{6}(0.1802 + 2 * 0.2467 + 2 * 0.2647 + 0.3464) = 1.505$$

1.4 d

Write out 2nd order ABM method for this ODE. Note that this is a multi-step method. it needs 2 initial values to initiate the iterations. For the second initial value you can use the result obtained in part b with Heun's method. Compute the values x(1.2) and x(1.3) using the ABM method.

Here we add the intial value $x_1 = 1.1, t_1 = 1.254$. Recall the ABM method:

$$x_{n+1} = x_n + \frac{h}{2}(3f(t_n, x_n) - f(t_{n-1}, x_{n-1}))$$

To compute $x(1.2) = x_2$:

$$x(1.2) = 1.254 + 0.1/2 * (3 * 3.399 - 2) = 1.664$$

To compute $x(1.3) = x_3$:

$$x(1.3) = 1.664 + 0.1/2 * (3 * 6.2018 - 3.399) = 2.424$$

2 Problem 2

Given the following high order ordinary differential equation

$$x'''(t) = -2x'' + xt, x(0) = 1, x'(0) = 2, x''(0) = 3$$

2.1 a

Rewrite the equation into a system of first order equations. Make sure to include the intial conditions.

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -2x_3 + x_1 t \end{cases}$$

Initial values at $t = 0$:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

2.2 b

Set up the forward Euler step for the system in (a) with h = 0.1

Recall the forward Euler method:

$$x_{k+1} = x_k + hx'(t_k) = x_k + h * f(t_k, x_k)$$

Here we have the values:

$$x_k = (x_1^0, x_2^0, x_3^0)^T$$

This first vector is our initial condition.

$$x_k' = (x_1', x_2', x_3')^T = (x_2^0, x_3^0, -2x_3^0 + x_1^0 t_0)^T$$

This gives us the setup:

$$x_{k+1} = x_k + 0.1 * x_k' = (x_1^0, x_2^0, x_3^0)^T + 0.1 * (x_2^0, x_3^0, -2x_3^0 + x_1^0 t_0)^T$$

2.3 c

Write a Matlab script that computes 10 Euler steps, to obtain the value of $x_1 0 \approx x(1)$

My scipt:

```
% Setup intial conditions
% t starts at zero
t = 0;
h = 0.1;
xPrev = [1; 2; 3;];
xPrevPrime = [xPrev(2); xPrev(3); myFunc(xPrev, t)];

for i = 1:10
   xPrev =  xPrev + h .* xPrevPrime;
   xPrevPrime = [xPrev(2); xPrev(3); myFunc(xPrev, t)];
   % increment t
   t = t + h;
end

disp(xPrev)
disp(t)

% differential equations
function out=myFunc(x, t)
    out = -2 * x(3) + x(1) * t;
end
```

My output (value of x(1) and value of t):

Problem2Script

- 3.8514
- 3.4875
- 1.0241
- 1.0000