Chapter 3: Splines

Contents

1 Introduction

Disdvantages of Polynomial Interpolation $P_n(x)$:

- n-time differentiable. We do not need such high smoothness.
- big error in certain intervals (esp near the end points)
- There is no convergence results
- Heavy to compute for large n

Suggestion: use piecewise polynomial interpolation

Usage:

- Visualization of discrete data
- graphic design

Requirement:

- Interpolation
- certain degree of smoothness

2 Problem Setting

Given a set of data

x	t_0	t_1	...	t_n
y	y_0	y_1	...	y_n

Find a function S(x) which interpolates points (t_i, y_i) for all i.

The set $t_0 < t_1 < ... < t_n$ are called knots. Note they need to be ordered.

S(x) consists of piecewise polynomials.

S(x) is called a spline of degree k, if

$$\mathcal{S}(x) \doteq \begin{cases} \mathcal{S}_0(x), & t_0 \le x \le t_1 \\ \mathcal{S}_1(x), & t_1 \le x \le t_2 \\ \vdots & \\ \mathcal{S}_{n-1}(x), & t_{n-1} \le x \le t_n \end{cases}$$

Figure 1:

$$S(x) = \begin{cases} x & x \in [-1, 0] \\ 1 - x & x \in (0, 1) \\ 2x - 2 & x \in [1, 2] \end{cases}$$

Figure 2:

- $S_i(x)$ is a polynomial of degree k
- S(x) is (k-1) times continuous differentiable i.e. for i = 1,2,. . . ,k-1 we have:

$$S_{i-1}(t_i) = S_i(t_i)$$
$$S'_{i-1}(t_i) = S'_i(t_i)$$

. . .

$$S_{i-1}^{(k-1)}(t_i) = S_i^{(k-1)}(t_i)$$

Commonly used splines:

- n = 1: linear spline (simplest)
- n = 2: quadratic spline (less popular)
- n = 3: cubic spline (most used)

3 Examples

Determine whether this function is a first-degree spline function.

Answer Check the properties of a linear spline.

- Linear polynomial for each piece - OK (degree 1 or less)

$$S(x) = \begin{cases} x^2 & x \in [-10, 0] \\ -x^2 & x \in (0, 1) \\ 1 - 2x & x \geq 1 \end{cases}$$

Figure 3:

- S(x) is continuous at inner knots
- At x= 0, S(x) is discontinuous, because from the left we get 0 and from the right we get 1.

Therefore this is NOT linear spline.

Determine whether the following function is a quadratic spline.

Answer Let's label each piece:

$$Q_0(x) = x^2; Q_1(x) = -x^2; Q_2(x) = 1 - 2x$$

We now check all conditions. Let's check the continuity at the inner knots of Q and Q'.

$$Q_0(0) = 0 = Q_1(0)$$
$$Q_1(1) = -1 = Q_2(1)$$
$$Q_0'(0) = 0 = Q_1'(0)$$
$$Q_1'(1) = -2 = Q_2'(1)$$

Therefore, since all conditions pass, this IS a quadratic spline.

4 Linear Spline

n = 1: piecewise linear interpolation, i.e., straight line between 2 neighboring points.

Requirements:

$$S_0(t_0) = y_0$$
$$S_{i-1}(t_i) = S_i(t_i) = y_i; i = 1, 2, ..., n-1$$
$$S_{n-1}(t_n) = y_n$$

Easy to find: write the equation for a line through two points (t_i, y_i) and (t_{i+1}, y_{i+1})

$$S_i(x) = y_i + \frac{y_{i+1} - y_i}{t_{i+1} - t_i}(x - t_i); i = 0, 1, 2, ..., n-1$$

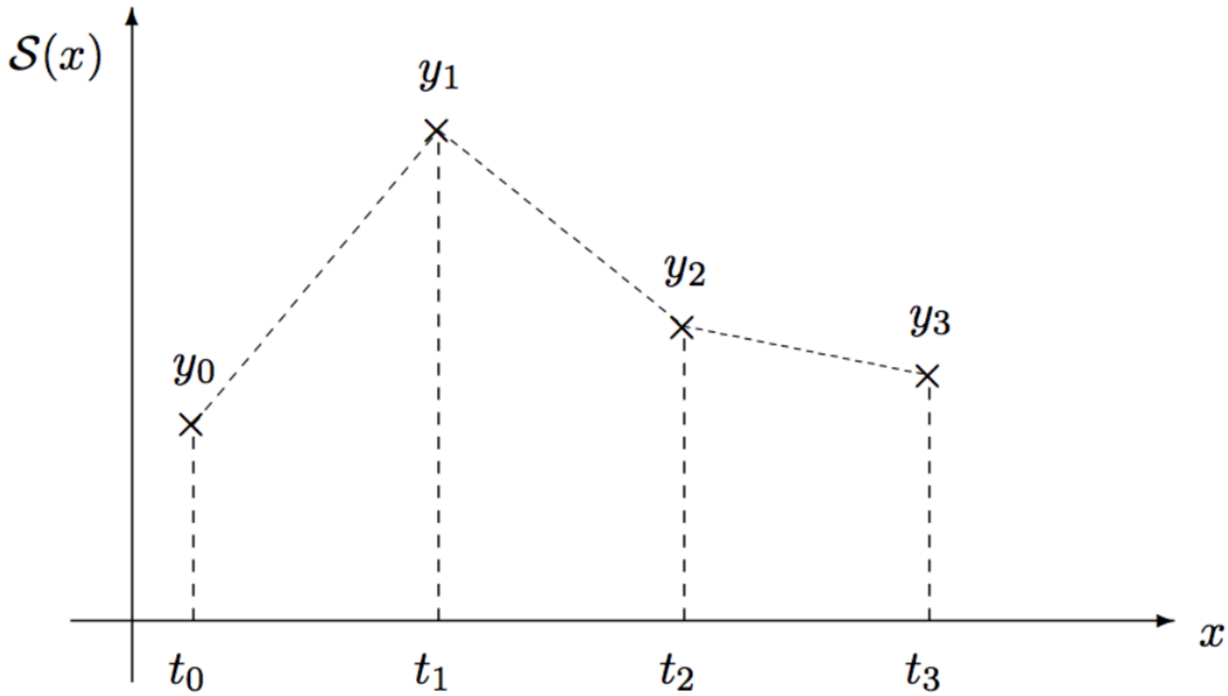

Figure 4:

4.1 Accuracy Theorem for linear spline

Assume $t_0 < t_1 < ... < t_n$ and let $h_i = t_{i+1} - t_i, h = max_i h_i$

f(x) is the given function and S(x) is the linear spline that interpolates the function such that

$$S(t_i) = f(t_i); i = 0, 1, ..., n$$

Then we have the following for $x \in [t_0, t_n]$

- If f" exists and is continuous, then $|f(x) - S(x)| \leq \frac{1}{8} h^2 max_x |f''(x)|$