Chapter 3: Splines

Contents

1 Introduction

Disdvantages of Polynomial Interpolation $P_n(x)$:

- n-time differentiable. We do not need such high smoothness.
- big error in certain intervals (esp near the end points)
- There is no convergence results
- Heavy to compute for large n

Suggestion: use piecewise polynomial interpolation

Usage:

- Visualization of discrete data
- graphic design

Requirement:

- Interpolation
- certain degree of smoothness

2 Problem Setting

Given a set of data

x	t_0	t_1	...	t_n
y	y_0	y_1	...	y_n

Find a function S(x) which interpolates points (t_i, y_i) for all i.

$$\mathcal{S}(x) \doteq \begin{cases} \mathcal{S}_0(x), & t_0 \leq x \leq t_1 \\ \mathcal{S}_1(x), & t_1 \leq x \leq t_2 \\ \vdots & \\ \mathcal{S}_{n-1}(x), & t_{n-1} \leq x \leq t_n \end{cases}$$

Figure 1:

The set $t_0 < t_1 < ... < t_n$ are called knots. Note they need to be ordered.

S(x) consists of piecewise polynomials.

S(x) is called a spline of degree k, if

- $S_i(x)$ is a polynomial of degree k
- S(x) is (k-1) times continuous differentiable i.e. for i = 1,2,. . . ,k-1 we have:

$$S_{i-1}(t_i) = S_i(t_i)$$
$$S'_{i-1}(t_i) = S'_i(t_i)$$

. . .

$$S_{i-1}^{(k-1)}(t_i) = S_i^{(k-1)}(t_i)$$

Commonly used splines:

- n = 1: linear spline (simplest)
- n = 2: quadratic spline (less popular)
- n = 3: cubic spline (most used)

3 Examples

Determine whether this function is a first-degree spline function.

Answer Check the properties of a linear spline.

- Linear polynomial for each piece - OK (degree 1 or less)
- S(x) is continuous at inner knots
- At x= 0, S(x) is discontinuous, because from the left we get 0 and from the right we get 1.

Therefore this is NOT linear spline.

Determine whether the following function is a quadratic spline.

Answer Let's label each piece:

$$S(x) = \begin{cases} x & x \in [-1, 0] \\ 1 - x & x \in (0, 1) \\ 2x - 2 & x \in [1, 2] \end{cases}$$

Figure 2:

$$S(x) = \begin{cases} x^2 & x \in [-10, 0] \\ -x^2 & x \in (0, 1) \\ 1 - 2x & x \geq 1 \end{cases}$$

Figure 3:

$$Q_0(x) = x^2; Q_1(x) = -x^2; Q_2(x) = 1 - 2x$$

We now check all conditions. Let's check the continuity at the inner knots of Q and Q'.

$$Q_0(0) = 0 = Q_1(0)$$
$$Q_1(1) = -1 = Q_2(1)$$
$$Q_0'(0) = 0 = Q_1'(0)$$
$$Q_1'(1) = -2 = Q_2'(1)$$

Therefore, since all conditions pass, this IS a quadratic spline.

4 Linear Spline

n = 1: piecewise linear interpolation, i.e., straight line between 2 neighboring points.

Requirements:

$$S_0(t_0) = y_0$$
$$S_{i-1}(t_i) = S_i(t_i) = y_i; i = 1, 2, ..., n-1$$
$$S_{n-1}(t_n) = y_n$$

Easy to find: write the equation for a line through two points (t_i, y_i) and (t_{i+1}, y_{i+1})

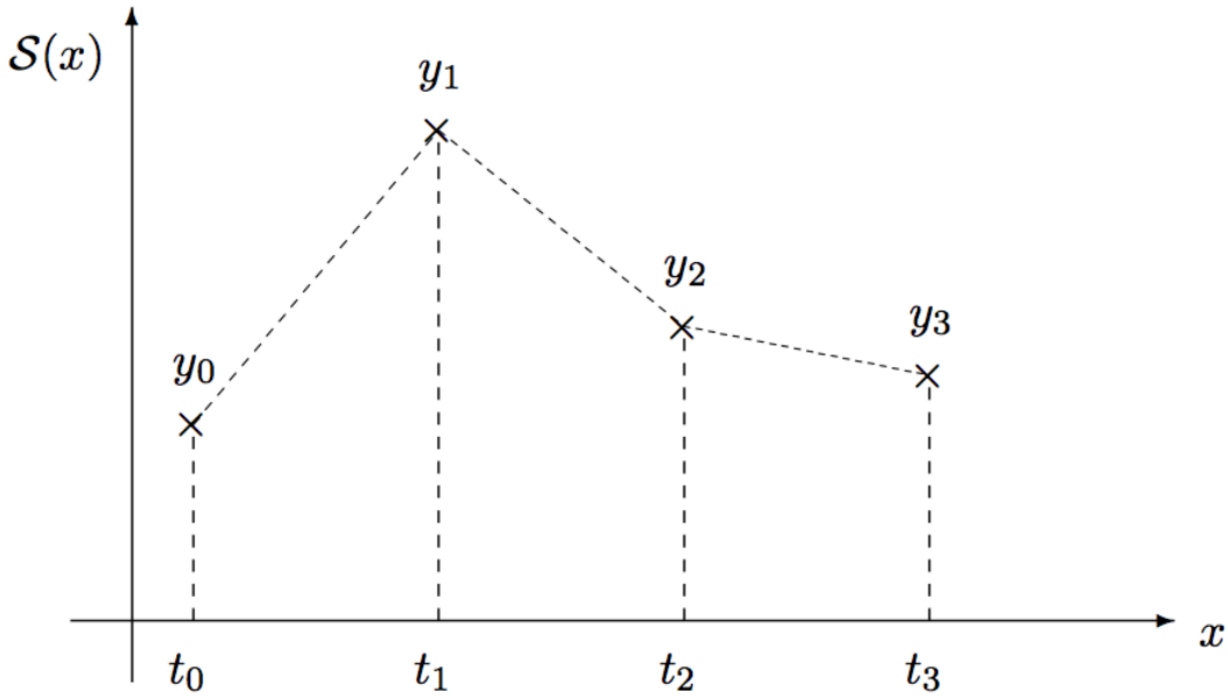

Figure 4:

$$S_i(x) = y_i + \frac{y_{i+1} - y_i}{t_{i+1} - t_i}(x - t_i); i = 0, 1, 2, ..., n - 1$$

4.1 Accuracy Theorem for linear spline

Assume $t_0 < t_1 < ... < t_n$ and let $h_i = t_{i+1} - t_i, h = max_i h_i$

f(x) is the given function and S(x) is the linear spline that interpolates the function such that

$$S(t_i) = f(t_i); i = 0, 1, ..., n$$

Then we have the following for $x \in [t_0, t_n]$

- If f" exists and is continuous, then $|f(x) - S(x)| \leq \frac{1}{8}h^2 max_x|f''(x)|$
- If f' exists and is continous, then $|f(x) - S(x)| \leq \frac{1}{2}h max_x|f'(x)|$

To minimize error, it is obvious that one should add more knots where the function has large first or second derivative.

Since a quadratic spline is less used, please reference it within the book.

5 Natural Cubic Spline

Given $t_0 < t_1 < ... < t_n$, we define the cubic spline with

$$S(x) = S_i(x); t_i \leq x \leq t_{i+1}$$

Write

$$S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i, i = 0, 1, ..., n-1$$

Total number of unknowns is 4 * n. Requirements are that S, S', S" are all continuous.

Equations we have:

- $S_i(t_i) = y_i, i = 0, 1, ..., n-1$
- $S_i(t_{i+1}) = y_{i+1}, i = 0, 1, ..., n-1$
- $S_i'(t_{i+1}) = S_{i+1}'(t_{i+1}), i = 0, 1, ..., n-2$
- $S_i''(t_{i+1}) = S_{i+1}''(t_{i+1}), i = 0, 1, ..., n-2$

This gives us 4n-2 equations total, so we still need two more conditions.

- $S_0''(t_0) = 0, S_{n-1}''(t_n) = 0$

These last two equations choice makes our spline a "natural" spline.

How to compute $S_i(x)$? We know:

- S_i is polynomial degree 3
- S_i' is degree 2
- S_i" is degree 1

Procedure

1. Start with S_i"(x), they are all inear, one can use Lagrange form
2. Integrate S_i"(x) twice to get S_i(x), you will get 2 integration constant
3. Determine these constants. Various tricks on the way…

Define: $z_i = S''(t_i), i = 1, 2, ..., n-1, z_0 = z_n = 0$

These z_i's are our unknowns.

Let $h_i = t_{i+1} - t_i$. Lagrange form for S_i".

$$S_i''(x) = \frac{z_{i+1}}{h_i}(x - t_i) - \frac{z_i}{h_i}(x - t_{i+1})$$

Then

$$S_i'(x) = \frac{z_{i+1}}{2h_i}(x - t_i)^2 - \frac{z_i}{2h_i}(x - t_{i+1})^2 + C_i - D_i$$
$$S_i(x) = \frac{z_{i+1}}{6h_i}(x - t_i)^3 - \frac{z_{i+1}}{6h_i}(x - t_{i+1})^3 + c_i(x - t_i) - D_i(x - t_{i+1})$$

You can check yourself that these two equations are correct.

Interpolating Properties: $S_i(t_i) = y_i$

$$\begin{cases} h_{i-1}z_{i-1} + 2(h_{i-1} + h_i)z_i + h_i z_{i+1} = 6(b_i - b_{i-1}), & i = 1, 2, \cdots, n-1 \\ z_0 = z_n = 0. \end{cases}$$

Figure 5:

$$y_i = -\frac{z_i}{6h_i}(-h_i)^3 - D_i(-h_i) = \frac{1}{6}z_i h_i^2 + D_i h_i$$

$$D_i = \frac{y_i}{h_i} - \frac{h_i}{6}z_i$$

$S_i(t_{i+1}) = y_{i+1}$ gives

$$y_{i+1} = \frac{z_{i+1}}{6h_i}h_i^3 + C_i h_i$$

$$C_i = \frac{y_{i+1}}{h_i} - \frac{h_i}{6}z_{i+1}$$

We see that, once z_i's are known, then (C_i, D_i)'s are known, and so are S_i, S_i' are known.

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x - t_i)^3 - \frac{z_i}{6h_i}(x - t_{i+1})^3 + \left(\frac{y_{i+1}}{h_i} - \frac{h_i}{6}z_{i+1}\right)(x - t_i) - \left(\frac{y_i}{h_i} - \frac{h_i}{6}z_i\right)(x - t_{i+1})$$

$$S_i'(x) = \frac{z_{i+1}}{2h_i}(x - t_i)^2 - \frac{z_i}{2h_i}(x - t_{i+1})^2 + \frac{y_{i+1} - y_i}{h_i} - \frac{z_{i+1} - z_i}{6}h_i$$

Continuity of S'(x) requires

$$S_{i-1}'(t_i) = S_i'(t_i), i = 1, 2, ..., n-1$$

$$S_i'(t_i) = -\frac{1}{6}h_i z_{i+1} - \frac{1}{3}h_i z_i + b_i$$

$$S_{i-1}'(t_i) = \frac{1}{6}z_{i-1}h_{i-1} + \frac{1}{3}z_i h_{i-1} + b_{i-1}$$

Set them equal to each other and we get:

Summarizing the algorithm:

- Set up the matrix-vector equation and solve for z_i
- Compute S_i(x) using these z_i's

See Matlab codes.

6 Smoothness Theorem for natural splines

$$\begin{cases} h_{i-1}z_{i-1} + 2(h_{i-1}+h_i)z_i + h_i z_{i+1} = 6(b_i - b_{i-1}), \qquad i = 1, 2, \cdots, n-1 \\ z_0 = z_n = 0. \end{cases}$$

In matrix-vector form: $\mathbf{H} \cdot \vec{z} = \vec{b}$

$$\mathbf{H} = \begin{pmatrix} 2(h_0+h_1) & h_1 & & & & \\ h_1 & 2(h_1+h_2) & h_2 & & & \\ & h_2 & 2(h_2+h_3) & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix}$$

$\mathbf{H}$: tri-diagonal, symmetric, and diagonal dominant

$$2\,|h_{i-1}+h_i| > |h_i| + |h_{i-1}|$$

which implies unique solution.

Figure 6:

$$\begin{cases} h_{i-1}z_{i-1} + 2(h_{i-1}+h_i)z_i + h_i z_{i+1} = 6(b_i - b_{i-1}), \qquad i = 1, 2, \cdots, n-1 \\ z_0 = z_n = 0. \end{cases}$$

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 6(b_1-b_0) \\ 6(b_2-b_1) \\ 6(b_3-b_2) \\ \vdots \\ 6(b_{n-2}-b_{n-3}) \\ 6(b_{n-1}-b_{n-2}) \end{pmatrix}.$$

Figure 7:

Theorem. *If $\mathcal{S}$ is the natural cubic spline function that interpolates a twice-continuously differentiable function f at knots*

$$a = t_0 < t_1 < \cdots < t_n = b$$

then

$$\int_a^b \left[\mathcal{S}''(x)\right]^2 \, dx \le \int_a^b \left[f''(x)\right]^2 \, dx.$$

Note that $\int (f'')^2$ is related to the curvature of f.
Cubic spline gives the least curvature, $\Rightarrow$ most smooth, so best choice.

Figure 8:

Proof. Let

$$g(x) = f(x) - \mathcal{S}(x)$$

Then

$$g(t_i) = 0, \qquad i = 0, 1, \cdots, n$$

and

$$f'' = \mathcal{S}'' + g'', \qquad (f'')^2 = (\mathcal{S}'')^2 + (g'')^2 + 2\mathcal{S}''g''$$

$$\Rightarrow \quad \int_a^b (f'')^2 \, dx = \int_a^b (\mathcal{S}'')^2 \, dx + \int_a^b (g'')^2 \, dx + \int_a^b 2\mathcal{S}''g'' \, dx$$

We claim that

$$\int_a^b \mathcal{S}''g'' \, dx = 0$$

then this would imply

$$\int_a^b (f'')^2 \, dx \ge \int_a^b (\mathcal{S}'')^2 \, dx$$

and we are done.

Figure 9:

Proof of the claim:

$$\int_a^b \mathcal{S}'' g''\, dx = 0$$

Using integration-by-parts,

$$\int_a^b \mathcal{S}'' g''\, dx = \mathcal{S}'' g' \Big|_a^b - \int_a^b \mathcal{S}''' g'\, dx$$

Since $S''(a) = S''(b) = 0$, the first term is 0.
For the second term, since $\mathcal{S}'''$ is piecewise constant. Call

$$c_i = \mathcal{S}'''(x), \quad \text{for} \quad x \in [t_i, t_{i+1}].$$

Then

$$\int_a^b S''' g'\, dx = \sum_{i=0}^{n-1} c_i \int_{t_i}^{t_{i+1}} g'(x)\, dx = \sum_{i=0}^{n-1} c_i \left[g(t_{i+1}) - g(t_i) \right] = 0,$$

(b/c $g(t_i) = 0$).

Figure 10: