Eig Tracking

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In this note, we study the connection between matrix A and B in the point of view of eigenvalues and potentially eigenvectors.

1 Characteristic polynomial of linear matrix function

For the $n \times n$ matrix

$$C(t) = (1-t)A + tB \quad t \in [0,1] \tag{1}$$

the characteristic polynomial is

$$p_t(\lambda) = \det\left(C(t) - \lambda I\right) = \det\left((1-t)A + tB - \lambda I\right). \tag{2}$$

It easy to show that $p_t(\lambda)$ has the form

$$p_t(\lambda) = -\lambda^n + q_1(t)\lambda^{n-1} + \cdots + q_{n-1}(t)\lambda + q_n(t), \tag{3}$$

where $q_k(t) \in \mathbb{C}[t]$ and $\deg(q_k) \leq k$. Intuitively, due to $p_t(t)$ is of degree n at most, rigorously due to $q_k(t)$ is the sum of all k-rowed diagonal minors of the matrix $C(t)$.

1.1 Taylor expansion of characteristic polynomial around repeated eigenvalues

Due to Eq. (3), for characteristic polynomial $p_t(\lambda)$ at $t = t_0$, the second-order Taylor expansion around eigenvalue λ_0 is

$$p_t(\lambda) = a_2\left(\lambda - \lambda_0\right)^2 + a_1\left(\lambda - \lambda_0\right) + b_2\left(t - t_0\right)^2 + b_1\left(t - t_0\right) + c_2\left(t - t_0\right)\left(\lambda - \lambda_0\right) + h.o.t. \tag{4}$$

In principle, a_1, a_2, b_1, b_2, c_2 can be expressed in terms of q_n, q_{n-1}, q_{n-2}, i.e. by

$$p_t(\lambda) = q_{n-2}(t_0)\lambda^2 + \left(q'_{n-1}(t_0)t + q_{n-1}(t_0)\right)\lambda + \left(q''_n(t_0)t^2/2 + q'_n(t_0)t + q_n(t_0)\right) + h.o.t. \tag{5}$$

Since λ_0 is a repeated root of $p_{t_0}(\lambda) = 0$, it must has form

$$p_{t_0}(\lambda) = a_2\left(\lambda - \lambda_0\right)^2 + h.o.t. \tag{6}$$

i.e. $a_1 = 0$.

Assume $\lambda_1(t)$ and $\lambda_2(t)$ are two continued trajectories that crossed somewhere at $t = t_0 \in [t_1, t_2]$. We have

$$\begin{cases} p_{t_1}(\lambda_1(t_1)) = 0 \\ p_{t_1}(\lambda_2(t_1)) = 0 \\ p_{t_2}(\lambda_1(t_2)) = 0 \\ p_{t_2}(\lambda_2(t_2)) = 0. \end{cases} \tag{7}$$

Eq. (4) has six unknowns (including λ_0 and t_0), we still need to simplify it:

- When $b_1 \neq 0$, then $b_2\left(t - t_0\right)^2$ is a higher-order-term, which can be dropped

$$p_t(\lambda)|_{b_1 \neq 0} = a_2\left(\lambda - \lambda_0\right)^2 + b_1\left(t - t_0\right) + c_2\left(t - t_0\right)\left(\lambda - \lambda_0\right) + h.o.t. \tag{8}$$

this is the parabolic-like case. Note

$$p_t(\lambda)|_{b_1\neq 0} = \left(\sqrt{a_2}\,(\lambda-\lambda_0) + \frac{1}{2}\frac{c_2}{\sqrt{a_2}}\,(t-t_0)\right)^2 - \left(\frac{1}{2}\frac{c_2}{\sqrt{a_2}}\,(t-t_0)\right)^2 + b_1\,(t-t_0) + h.o.t. \tag{9}$$

$$= \left(\sqrt{a_2}\,(\lambda-\lambda_0) + \frac{1}{2}\frac{c_2}{\sqrt{a_2}}\,(t-t_0)\right)^2 + b_1\,(t-t_0) + h.o.t. \tag{10}$$

when evaluating λ near λ_0, the following might be good enough

$$p_t(\lambda)|_{b_1\neq 0, c_2=0} = a_2\,(\lambda-\lambda_0)^2 + b_1\,(t-t_0) + h.o.t. \tag{11}$$

since if $\lambda-\lambda_0 \sim h$ then $t-t_0 \sim h^2$, $(t-t_0)\,(\lambda-\lambda_0) \sim h^3$ a h.o.t. What's more, Eq. (7) tells only the root, thus only the ratio between a_2 and b_1 is important, rather than their absolute value. Now we can solve λ_0 and t_0.

- When $b_1 = 0$

$$p_t(\lambda)|_{b_1=0} = a_2\,(\lambda-\lambda_0)^2 + b_2\,(t-t_0)^2 + c_2\,(t-t_0)\,(\lambda-\lambda_0) + h.o.t. \tag{12}$$

  this is the crossing line case, which can be seen after a rotation of eliminating c_2. Again, with Eq. (7), only the ratio between a_2 and b_2 and c_2 are important. Note that in geometry, solving of line crossing need only four points.

- It is possible that both $b_1 = 0$ and $b_2 = 0$, which is a higher order case. It is might related to triple-repeated root. It is too rarem, no further discussion.

1.2 Numerical computation of crossing eigenvalues

Assume

$$(\lambda_j - \lambda_0)^2 + b_1\,(t_k - t_0) = 0$$

How to solve λ_0, t_0, b_1? It is a nonlinear equation:

$$\lambda_j^2 - 2\lambda_j\lambda_0 + \lambda_0^2 + b_1 t_k - b_1 t_0 = 0$$

Method 1: solve coefficients in expanded form

$$\lambda_j^2 + d_1\lambda_j + d_2 t_k + d_3 = 0$$

so

$$\begin{cases} d_1 = -2\lambda_0 \\ d_2 = b_1 \\ d_3 = \lambda_0^2 - b_1 t_0 \end{cases}$$

where d_1, d_2 and d_3 can be solved by SVD.

We can then filter out fake solutions by:

- Smallest singular value should be zero or small
- The imaginary part of t_0 should be zero or small (seems not very useful in practice, it is always not small, order of 0.01~0.1).
- The roots of $p_{t_0}(\lambda) = 0$ should be repeated and around λ_1 and λ_2.

If further $C(t)$ is a real matrix, then b_1 should be real too. Might be used to further simplify the code to find crossing point, i.e. we could solve the crossing point by geometry of parabolic.

1.2.1 Refinement step

From Eq. (8)

$$
\begin{aligned}
p_t(\lambda)|_{b_1\neq 0} &= a_2\lambda^2 + \left(-2a_2\lambda_0 - c_2t_0\right)\lambda + c_2t\lambda + \left(b_1 - c_2\lambda_0\right)t + \left(-b_1t_0 + a_2\lambda_0^2 + c_2t_0\lambda_0\right) + h.o.t. \\
&= d_0\lambda^2 + d_1\lambda + d_2t\lambda + d_3t + d_4 + h.o.t.
\end{aligned}
\tag{13}
$$

thus

$$
\begin{cases}
a_2 = d_0 \\
c_2 = d_2
\end{cases}
$$

$$
-b_1t_0 + c_2t_0\lambda_0 = -t_0d_3
$$

$$
\begin{aligned}
a_2\lambda_0^2 - d_3t_0 &= d_4 \\
-2a_2\lambda_0 - c_2t_0 &= d_1
\end{aligned}
$$

i.e. solve λ_0 in

$$
\begin{aligned}
a_2\lambda_0^2 + d_3/c_2\left(2a_2\lambda_0 + d_1\right) &= d_4 \\
a_2\lambda_0^2 + 2a_2d_3/c_2\lambda_0 + d_1d_3/c_2 - d_4 &= 0 \\
d_0\lambda_0^2 + 2d_0d_3/d_2\lambda_0 + d_1d_3/d_2 - d_4 &= 0 \\
d_0d_2\lambda_0^2 + 2d_0d_3\lambda_0 + d_1d_3 - d_2d_4 &= 0
\end{aligned}
$$

then find a real t_0 in between t_1 and t_2:

$$
t_0 = -\left(2d_0\lambda_0 + d_1\right)/d_2,
$$

and

$$
b_1 = d_3 + d_2\lambda_0.
$$

If d_2 is too small such that the above procedure is unstable, we could solve t_0 first instead of λ_0.

1.3 Case of no repeated root

Let's start from Eq. (5), it is now

$$
p_t(\lambda) = q_{n-1}(t_0)\lambda + q_n'(t_0)t + q_n(t_0) + h.o.t. \tag{14}
$$

We know that

$$
q_n(t_0) = -(-1)^n \det\left((1-t_0)A + t_0B\right)
$$

For q_{n-1}, we need expression of sub determinates.

If we have both the left and right eigenvalue-eigenvector set, and they are complete

$$
C = P\Lambda P^{-1} = Q^{-1}\Lambda Q
$$

so

$$
C = P\Lambda Q
$$

1.4 How eigenvector change

Model case, see the 2×2 case in LyX doc "The eigensystem problem".

More complex cases might be reduced to this 2×2 cases.

1.5 How to consider the eigen-space of two crossing eigenvalues as a whole

Purpose: avoid singularity of eigen-space when t change.

$$
C\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \tag{15}
$$

...

1.6 Coarse grained asymptotic behaviour of eigenvalues

Often we see eigenvalues nearly across eachother, where their tendency is very similar to the crossing eigenvalues. Is it possible to develop a asymptotic that skip the "crossing" point, but describe the asymptotics away from the point?????!!!!

1.7 Why the eigenvalues will cross or will not cross??

Consider the two paths: $A_1 \to A_2 \to A_3$ v.s. $A_1 \to A_3$, do they have the same number of crossing in general? why and why not?

In principle, topology should help, since it is a continuous change. Also if they have the same number, we must able to construct a super complex case that make the number of crossing huge. So how are they diminished??

What is the maximun number of crossing???

2 Appendix

2.1 Projection of a conic section is still a conic section

The conic section is the curve $\{(x, y)\}$ that satisfy quadratic equation:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

After rotation P and project back to x-y plane:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P[1:2, 1:2] \begin{bmatrix} x \\ y \end{bmatrix}$$

It is obvious that (x', y') still satisfy a quadratic equation (or degenerated). Also the type of the conic section is conserved, as can seen by geometry of the trends of (x, y).

2.2 Imaging of roots of complex coefficient quadratic equation is hyperbolic in complex plane

i.e. the image $\{\lambda(t) | t \in \mathbb{R}\}$ to the solution of $\lambda(t)$ where

$$a\lambda^2(t) + b\lambda(t) + c = t, \quad a, b, c \in \mathbb{C} \tag{16}$$

is hyperbolic curves (or degenerated into cross lines) in complex plain $(\mathrm{Im}(\lambda), \mathrm{Re}(\lambda))$.

Proof: Consider the model quadratic equation

$$\lambda^2 = c + t. \tag{17}$$

We have

$$\mathrm{Im}(c + t) = \mathrm{Im}(c) = \mathrm{Im}(\lambda^2) = 2\mathrm{Im}(\lambda)\mathrm{Re}(\lambda).$$

The full case Eq. (16) is just rotation and zooming of Eq. (17), as can be seen by complete-the-square technique.