Approximation Theory Notes

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1 Approximation

See Ref. [3, 5]. A field active in 1910s Bernstein et al.

1.1 Chebyshev Approximations

Theorems for absolutely continuous functioins.

Lemma 1 (Theorem 7.1 in [5]). *Assume $f, \ldots, f^{(\nu-1)}$ be absolutely continuous on $[-1,1]$ and $f^{(\nu)}$ is of bounded variation V. Then for $\forall k \geq \nu+1$, the Chebyshev coefficients of f satisfy*

$$|a_k| \leq \frac{2V}{\pi k(k-1)\cdots(k-\nu)} \leq \frac{2V}{\pi(k-\nu)^{\nu+1}}.$$

Lemma 2 (Theorem 7.2 in [5]). *Same condition as Lemma 1, then for $\forall n > \nu$, its Chebyshev projections satisfy*

$$\|f - f_n\|_\infty \leq \frac{2V}{\pi\nu(n-\nu)^\nu}$$

and its Chebyshev interpolants satisfy

$$\|f - p_n\|_\infty \leq \frac{4V}{\pi\nu(n-\nu)^\nu}.$$

Theorems for analytical functions.

Lemma 3 (Theorem 8.1 in [5]). *Assume f analytic in $[-1,1]$ be analytically continuable to the open Bernstein ellipse E_ρ (note [1]), where it satisfies $|f(x)| \leq M$ for some M. Then its Chebyshev coefficients satisfy $|a_0| \leq M$ and*

$$|a_k| \leq 2M\rho^{-k}, \quad k \geq 1.$$

Lemma 4 (Theorem 8.2 in [5]). *Same condition as Lemma 3, then for $\forall n \geq 0$ its Chebyshev projections satisfy*

$$\|f - f_n\|_\infty \leq \frac{2M\rho^{-n}}{\rho - 1}$$

and its Chebyshev interpolants satisfy

$$\|f - p_n\|_\infty \leq \frac{4M\rho^{-n}}{\rho - 1}$$

[1] Bernstein ellipse E_ρ: foci 1 and -1 with semimajor and semiminor axis lengths summing to ρ in complex plane.

1.2 General Theorms

General theorms about polynomial approximations.

Lemma 5 (Theorem 16.1 in [5]). *Let $f \in C([-1,1])$ have degree n Chebyshev projection f_n, Chebyshev interpolant p_n, and best approximant p_n^*, $n \geq 1$. Then*

$$\|f - f_n\|_\infty \leq \left(4 + \frac{4}{\pi^2}\log(n+1)\right)\|f - p_n^*\|_\infty$$

$$\|f - p_n\|_\infty \leq \left(2 + \frac{2}{\pi}\log(n+1)\right)\|f - p_n^*\|_\infty$$

Two theorems about near-best approximation.

Lemma 6 (Theorem 12.1 in [5]). *Given $f \in C([-1,1])$, let ρ $(1 \leq \rho \leq \infty)$ be the parameter of the largest Bernstein ellipse E_ρ to which f can be analytically continued, and let $\{p_n\}$ be the interpolants to f in any sequence of grids $\{x_k\}$ in $[-1,1]$ such that*

$$\lim_{n\to\infty} M_n^{1/n} = \frac{1}{2}, \quad M_n = \sup_{x\in[-1,1]} |l(x)|, \quad l(x) = \prod_{k=0}^{n}(x - x_k),$$

then the errors satisfy

$$\lim_{n\to\infty} \|f - p_n\|_\infty^{1/n} = \rho^{-1}.$$

Lemma 7 (Theorem 15.1 in [5]). *Let Λ be the Lebesgue constant for a linear projection L of $C([-1,1])$ onto $\mathcal{P}_n$. Let f be a function in $C([-1,1])$, $p = Lf$ the corresponding polynomial approximant to f, and p^* the best approximation. Then*

$$\|f - p\|_\infty \leq (\Lambda + 1)\|f - p^*\|_\infty$$

where the Lebesgue constant is defined as

$$\Lambda \triangleq \sup_f \frac{\|p\|_\infty}{\|f\|_\infty} = \sup_{x\in[-1,1]} \sum_{j=0}^{n} |l_j(x)|, \quad l_j(x) = \frac{\prod_{k\neq j}(x - x_k)}{\prod_{k\neq j}(x_j - x_k)}.$$

Lemma 6 and Lemma 7 imply that $\lim_{n\to\infty} \Lambda_n^{1/n} = 1$ concludes near best convergent for $f \in C([-1,1])$.

Lemma 8 (Theorem 2.2.1 (b) and (a stronger version)Theorem 3.1.4 in [4]. See below for the symbols.). *Assume $c_\mu > 0$ and $\mathrm{Int}(\mathrm{Pc}(S(\mu))) = \emptyset$. If*

$$\lim_{n\to\infty} \gamma_n(\mu)^{1/n} = \frac{1}{\mathrm{cap}(S(\mu))} \tag{1}$$

for some subsequence $N \in \mathbb{N}$, then we have

$$\frac{1}{n}\nu_{p_n(\mu;\cdot)} \to \omega_{S(\mu)} \quad \text{as } n \to \infty,\ n \in N \tag{2}$$

in the weak sense.*

Definition 9 (See [4]: Definition 1.1.1, Appendix A.I, A.II, A.III, A.IV etc.). μ: a finite Borel measure on $\mathbb{C}$ with compact support $S(\mu)$ and infinitely many points. $\mathrm{Pc}(S(\mu))$ th polynomial convex hull of $S(\mu)$. Int: interior of K.

The uniquely existing orthonormal polynomials

$$p_n(\mu; z) = \gamma_n(\mu) z^n + \cdots, \quad \gamma_n(\mu) > 0,\ n \in \mathbb{N},$$

are defined by orthonoality relations

$$\int p_n(\mu; z)\overline{p_m(\mu; z)} d\mu(z) = \delta_{n,m}.$$

Counting measure of zeros of polynomial P: ν_P, i.e. places mass 1 to every zeros of P (counting multiplicity). Then $\|\nu_P\| = \deg(P)$. $\nu_P/\deg(P)$ the normalized counting measure.

Carriers of measure μ

$$\Gamma(\mu) := \{C \subseteq \mathbb{C} | C \text{ a Borel set and } \mu(\mathbb{C}\backslash\mathbb{C}) = 0\}.$$

Minimal-carrier capacity of the measure μ:

$$c_\mu := \inf \{\mathrm{cap}(C) | C \in \Gamma(\mu),\ C \text{ bounded}\}$$

Logarithmic energy:

$$I(\mu) = \int\int \log \frac{1}{|z-t|} d\mu(t) d\mu(z).$$

The capacity of K is defined as

$$\mathrm{cap}(K) := e^{-V_K},$$

where

$$V_K = \inf \{I(\mu) \mid \mathrm{supp}(\mu) \subseteq K,\ \mu \geq 0,\ \|\mu\| = 1\}.$$

The logarithmic potential of measure μ is defined by

$$p(\mu; z) = \int \log \frac{1}{|z-t|} d\mu(t).$$

Weak* convergence $\mu_n \to \mu$ means

$$\int f\, d\mu_n \to \int f\, d\mu \quad \text{as } n \to \infty$$

for every f continuous on $\overline{\mathbb{C}}$.

Equilibrium Measure ω_K of compact positive capacity set $K \subseteq \overline{\mathbb{C}}$:

$$p(\omega_K; z) = \int \log \frac{1}{|z-t|} d\omega_K(t)$$

with

$$p(\omega_K; z) \leq \log \frac{1}{\mathrm{cap}(K)} \quad \forall z \in \mathbb{C}$$

where "=" for quasi every $z \in K$. quasi everywhere: holds outside a set of zero capacity.

Remarks:

- Lemma 8 says that the asymptotics of the leading coefficient tells the limiting distribution of all zeros.
- Once the equilibrium measure $\omega_{S(\mu)}$ is approached, due to Lemma 6 (see also the book [5]), the polynomials $\{p_n(\mu; z)\}$ is near optimal.

- In the language of Book [4], this property is called "regular" of the measure μ (Eq. (1)) and "regular" asymptotic zero distribution (Eq. (2)).
- Now the remaining problem is the Criteria of Regularity.

Lemma 10 (By P. Erdős and P. Turán)**.** *For $S(\mu) = [-1, 1]$, and $\mu'(x) > 0$ a.e., then μ is regular.*

Note: $\mu'(x)$ is the Radon-Nikodym derivative of μ with respect to Lebesgue measure. In this case, it is the usual derivative of $\mu([-1, x])$ a.e..

Lemma 11 (Ullman's criterion. See Theorem 4.1.4 in [4])**.** *$c_\mu = \mathrm{cap}(S(\mu))$ implies the regularity of μ.*

Lemma 12 (Criterion Λ, Theorem 4.2.1 in [4])**.** *If*

$$\mathrm{cap}\left(\left\{z \,\middle|\, \limsup_{r\to 0+} \frac{\log 1/\mu(\Delta_r(z))}{\log 1/r} < \infty\right\}\right) = \mathrm{cap}(S(\mu))$$

where $\Delta_r(z) = \{z' \,|\, |z' - z| \le r\}$, then μ is regular.

Lemma 13 (Theorem 9.2 in [2])**.** *Same condition as 10, then the distribution of zeros in interval $[\cos\theta_1, \cos\theta_2]$ follows*

$$\lim_{n\to\infty} \frac{1}{n} \nu_{p_n(\mu;z)}([\cos\theta_2, \cos\theta_1]) = \frac{\theta_2 - \theta_1}{\pi}.$$

2 Root Finding

Theorem 14 (Theorem 18.1 in [5])**.** *The roots of the polynomial*

$$p(x) = \sum_{k=0}^{n} a_k T_k(x), \quad a_n \neq 0$$

are the eigenvalues of the matrix (Called colleague matrix)

$$C = \begin{bmatrix} 0 & 1 & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & \ddots & \ddots & \\ & & & \frac{1}{2} & 0 \end{bmatrix} - \frac{1}{2a_n} \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}$$

Proof. Let

$$\vec{v}(x) = \begin{bmatrix} T_0(x) & T_1(x) & \cdots & T_{n-1}(x) \end{bmatrix}^T$$

then due to the 3-term recurrence relation

$$C\vec{v}(x) = x\vec{v}(x) - \frac{1}{2a_n} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p(x) \end{bmatrix}$$

$\square$

Remarks:

- Easily extend to other orthogonal polynomials. Notably other near-best polynomial basis.
- The eigenvalues of the matrix C can be solved in $O(n^2)$ time instead of $O(n^3)$.
- Not limited to polynomial root finding. It can be used for analytical function root finding, along real line.
- The extra roots (for non-polynomial) are clustered along the maximum Bernstein ellipse.
- It is a global root finding method.
- When apply to non-analytical function, the extra roots are distributed near the real line. The partial sums of Taylor series also has roots clustering along the circle of convergence ([1]).
- By transform the last row of C to first column (or row?), may reduce rounding errors.

References

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[5] L. N. Trefethen, *Approximation Theory and Approximation Practice*, SIAM, Jan. 2013.