A Case to Show the Power of Non-uniform Fourier Transform in Spectrum Based Granger Causality Calculation

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1 Aim

Calculate the Granger causality between a set of given time series (generated by Integrate-and-Fire (I&F) neuron model) through its spectrum which is estimated by non-uniform discrete Fourier transform. Non-uniform here means samples are unequally spaced.

I&F model parameters: $\mu = 1\,\mathrm{kHz}$, $f = 0.012$, $S = 0.02$, 1->2. $T = 10^7\,\mathrm{ms}$.

2 Methods

2.1 The Non-uniform Fourier Transform

In this article, we use the following definition of Non-Uniform Fourier Transform (NUFT)

$$F(k) = \sum_{j=0}^{J-1} x_j e^{-2\pi \mathrm{i} k t_j} \tag{1}$$

where x_j is the sample value at time $t_j \in [0, 1]$, $F(k)$ is the frequency component at k-Hz. Here $k \in \mathbb{Z}$. Alghouth it is possible to generalize to $\mathbb{R}$, it's sufficient use only $k \in \mathbb{Z}$ in a finite closed interval $[0, 1]$.

The fast calculation algorithm used here follows this article: "Accelerating the Nonuniform Fast Fourier Transform" by Leslie Greengard and June-Yub Lee.

2.2 Calculate Spectrum from NUFT

Obtain Samples The data generated by our I&F simulation code is equally spaced. In order to obtain unequally spaced data, we use a high output sample rate (8 kHz) in I&F code, then do a random (uniformly distributed) sample in a certain time window.

Calculate Spectrum We use Welch's method to estimate the spectrum, which split input date into (possible) overlapping segments, then calculate periodogram of each segments, the average of periodogram is the desired spectrum estimation. The periodogram is obtain by NUFT. (Note: no window function applied here)

More precisely:

The j-th sample data of n-th segment is

$$X_j^{(n)} = X(\lfloor t_j M + n \cdot (M - D) \rfloor) \tag{2}$$

where $t_j \sim U([0,1])$ (standard uniform distribution), M is the time length of each segment, D is the overlaped length of two segments. For data $X(t)$, $t \in [0,T]$, we may obtain $N = \lfloor (T-M)/(M-D) \rfloor + 1$ segments. $\lfloor o \rfloor$ is floor function.

Spectrum $\hat{S}(k)$ is obtained by (Here assume $\mathrm{E}X(t) = \vec{0}$)

$$F^{(n)}(k) = \sum_{j=0}^{J-1} X_j^{(n)} e^{-2\pi \mathrm{i} k t_j / M}$$

$$\hat{S}(k) = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{J} F^{(n)}(k) \left(F^{(n)}(k)\right)^{T*} \tag{3}$$

where A^{T*} means conjugate transpose of matrix A. Here $X(t)$ is p-dim column vector, hence $\hat{S}(k)$ is $p \times p$ matrix. p denote the number of variables (neurons) in input data.

In our case, $M = 1000\,\mathrm{ms}$, choose D so that all cases use the same number of input data points.

Remove Bias The $\hat{S}(k)$ contains systemic bias, we have to subtract it.

$$S(k) = \frac{J}{J-1} \Delta t \left[\hat{S}(k) - \mathrm{E}(X(t)\, X^T(t))\right]$$

where $\Delta t = M/J$ is the average sample interval. $\frac{J}{J-1}\Delta t$ is a normalization factor that make the scale of $S(k)$ invariant to Δt.

2.3 Approximate Spectral GC Formula

For simplicity, let $p = 2$ here, and we only write down the formula for $F_{x \to y}$.

We know

$$F_{x \to y} \approx \frac{\sum_{k=1}^{m} r_k^2}{\mathrm{var}(\epsilon_1^A)\,\mathrm{var}(\epsilon_2^A)} \tag{4}$$

where $\epsilon_1^A(t)$, $\epsilon_2^A(t)$ is the whitened signal of $x_1(t)$, $x_2(t)$, $X(t) = [x_1(t)\ x_2(t)]^T$, $r_k = \mathrm{cov}(\epsilon_1^A(t),\, \epsilon_2^A(t+k))$. Originally, m means fitting order, here we just choose a high enough value ($m = 30$ in code).

Through the whiten spectrum Q, we can obtain r_k.

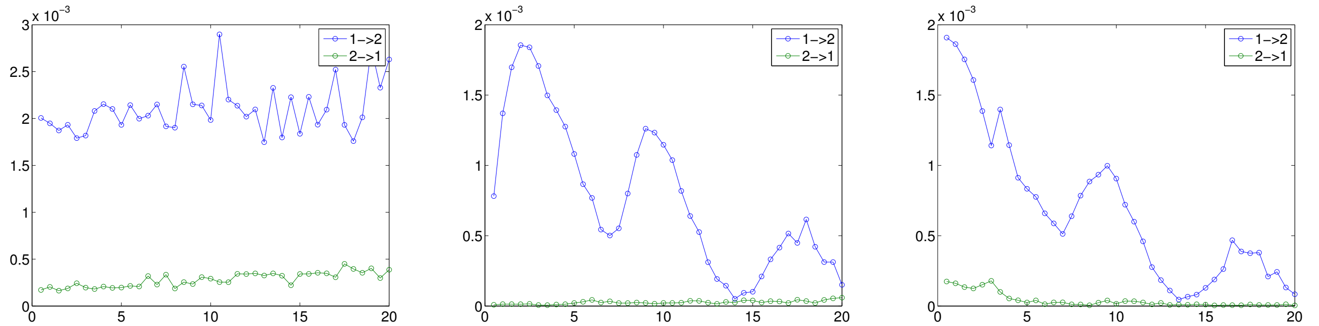
$$u_l(k) = e^{\frac{1}{2}\ln S_{ll}(k) + \mathrm{i}\,\mathcal{H}(\frac{1}{2}\ln S_{ll}(k))} \tag{5}$$

$$Q(k) = \begin{bmatrix} u_1^{-1} & \\ & u_2^{-1} \end{bmatrix} S(k) \begin{bmatrix} u_1^{-1} & \\ & u_2^{-1} \end{bmatrix}^{T*}$$

$$\frac{r_k}{\sqrt{\mathrm{var}(\epsilon_1^A)\,\mathrm{var}(\epsilon_2^A)}} = \mathcal{F}_{DFT}^{-1}\left[Q_{21}\right] \tag{6}$$

where $\mathcal{H}(s(k))$ denote discrete Hilbert transform, $\mathcal{H}(s(k)) = \mathcal{F}_{DFT}^{-1}\left[-\mathrm{i} \cdot \mathrm{sgn}(w) \cdot \mathcal{F}_{DFT}\left[s(k)\right]\right]$.

3 Result

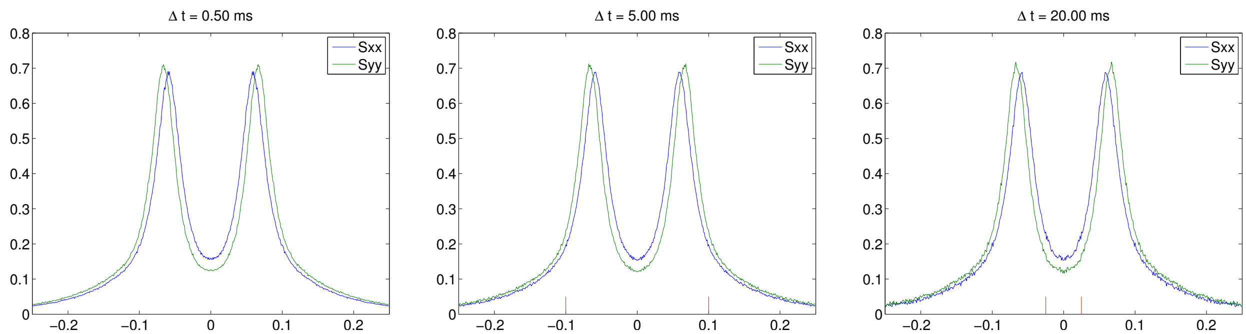


(a) Use NUFT and uniformly distributed sample time.

(b) Use equally spaced data and time domain algorithm.

(c) Use NUFT and near equally spaced data.

Figure 1: GC v.s. sample interval Δt

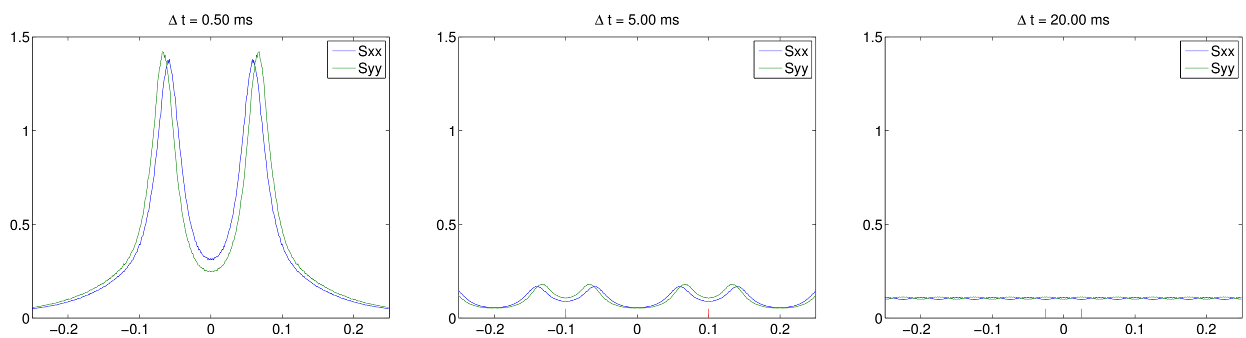


(a) $\Delta t = 0.5\,\text{ms}$

(b) $\Delta t = 5.0\,\text{ms}$

(c) $\Delta t = 20.0\,\text{ms}$

Figure 2: Spectrum when use NUFT and uniformly distributed sample. The short red line denote the classical Nyquist frequency corresponding to Δt.



(a) $\Delta t = 0.5\,\text{ms}$

(b) $\Delta t = 5.0\,\text{ms}$

(c) $\Delta t = 20.0\,\text{ms}$

Figure 3: Spectrum when use NUFT and near equally spaced sample. The short red line denote the classical Nyquist frequency corresponding to Δt.

Conclusion: No visible aliasing (folding) phenomena when use NUFT with uniformly distributed data. Large Δt will introduce large noise.