Matrix Formulas That Is Usefull In Least Square Analysis

(You could always swap indexes properly to get similar results)

1 Difference of Quadratic Forms, Inverse Matrices and Solution of Linear Systems

For invertable $(n_1+n_2+n_3)\times(n_1+n_2+n_3)$ matrix R (no need symmetric)

$$R=\begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix}^{-1}=\begin{bmatrix} Q^{(xx)} & Q^{(xy)} & Q^{(xz)} \\ Q^{(yx)} & Q^{(yy)} & Q^{(yz)} \\ Q^{(zx)} & Q^{(zy)} & Q^{(zz)} \end{bmatrix} \tag{1}$$

For any $\begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}$ ($n_4\times(n_1+n_2+n_3)$ matrix)
Define
(replace transpose $\cdot^T$ to conjugate transpose $\cdot^H$ for matrix with complex entries)

$$\mathrm{SS}_{R[x,z]}=\begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|z)} \end{bmatrix}\begin{bmatrix} R^{(xx)} & R^{(xz)} \\ R^{(zx)} & R^{(zz)} \end{bmatrix}^{-1}\begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|z)} \end{bmatrix}^T$$

$$\mathrm{SS}_{R[x,y,z]}=\begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}\begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix}^{-1}\begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}^T$$

$$\begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{bmatrix}\begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix}=\begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_T^{(11)} & \vec{a}_T^{(12)} & \vec{a}_T^{(13)} \end{bmatrix}\begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix}^T=\begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}$$

Then the difference of the quadratic form is
(Assume the inverse matries in the expressions are all valid, pseudo-inverse could also work)

$$\mathrm{SS}_{R[x,y,z]}-\mathrm{SS}_{R[x,z]}=\vec{a}^{(12)}\left(Q^{(yy)}\right)^{-1}\left(\vec{a}_T^{(12)}\right)^T \tag{2}$$

The $\left(Q^{(yy)}\right)^{-1}$ might be expressed as
(Swap index x and z will give another expression)

$$\begin{aligned}\left(Q^{(yy)}\right)^{-1} &= R^{(yy)}-R^{(yx)}\left(R^{(xx)}\right)^{-1}R^{(xy)} \\ &\quad -\left(R^{(yz)}-R^{(yx)}\left(R^{(xx)}\right)^{-1}R^{(xz)}\right)\left(R^{(zz)}-R^{(zx)}\left(R^{(xx)}\right)^{-1}R^{(xz)}\right)^{-1}\left(R^{(zy)}-R^{(zx)}\left(R^{(xx)}\right)^{-1}R^{(xy)}\right)\end{aligned} \tag{3}$$

The inverse matrices have relation

$$\begin{bmatrix} P^{(xx)} & P^{(xy)} \\ P^{(yx)} & P^{(yy)} \end{bmatrix} \triangleq \begin{bmatrix} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{bmatrix}^{-1} = \begin{bmatrix} Q^{(xx)} & Q^{(xy)} \\ Q^{(yx)} & Q^{(yy)} \end{bmatrix} - \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix} \left(Q^{(zz)}\right)^{-1} \begin{bmatrix} Q^{(zx)} & Q^{(zy)} \end{bmatrix} \tag{4}$$

$$\left(Q^{(zz)}\right)^{-1} = R^{(zz)} - \begin{bmatrix} R^{(zx)} & R^{(zy)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{bmatrix}^{-1} \begin{bmatrix} R^{(xz)} \\ R^{(yz)} \end{bmatrix}$$

If we solve

$$\begin{bmatrix} \vec{b}^{(11)} & \vec{b}^{(12)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{bmatrix} = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} \end{bmatrix}$$

Then there is relation
(There are 6 of these relations)

$$b^{(12)} = a^{(12)} - a^{(13)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} \tag{5}$$

$$a^{(12)} \left(I - \left(Q^{(yy)}\right)^{-1} Q^{(yz)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)}\right) = b^{(12)} + b^{(13)}_{[xz]} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} \tag{6}$$

Note

$$\begin{aligned} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} &= -\left(R^{(zy)} - R^{(zx)} \left(R^{(xx)}\right)^{-1} R^{(xy)}\right) \left(R^{(yy)} - R^{(yx)} \left(R^{(xx)}\right)^{-1} R^{(xy)}\right)^{-1} \\ &= -\begin{bmatrix} R^{(zx)} & R^{(zy)} \end{bmatrix} \begin{bmatrix} P^{(xy)} \\ P^{(yy)} \end{bmatrix} \end{aligned}$$

$$I - \left(Q^{(yy)}\right)^{-1} Q^{(yz)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} = \left(Q^{(yy)}\right)^{-1} P^{(yy)}$$

Also

$$\vec{b}^{(11)} = \vec{a}^{(11)} - a^{(13)} \left(Q^{(zz)}\right)^{-1} Q^{(zx)}$$

In matrix form

$$\begin{bmatrix} \vec{b}^{(11)} & \vec{b}^{(12)} \end{bmatrix} = \begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} \end{bmatrix} - a^{(13)} \left(Q^{(zz)}\right)^{-1} \begin{bmatrix} Q^{(zx)} & Q^{(zy)} \end{bmatrix}$$

$$\begin{bmatrix} \vec{b}^{(11)} & \vec{b}^{(12)} \end{bmatrix} = \begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} \end{bmatrix} + a^{(13)} \begin{bmatrix} R^{(zx)} & R^{(zy)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{bmatrix}^{-1}$$

$$b^{(21)} = a^{(21)} - a^{(23)} \left(Q^{(zz)}\right)^{-1} Q^{(zx)}$$

$$\vec{b}^{(22)} = \vec{a}^{(22)} - a^{(23)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)}$$

$$\begin{bmatrix} \vec{b}^{(21)} & \vec{b}^{(22)} \end{bmatrix} = \begin{bmatrix} \vec{a}^{(21)} & \vec{a}^{(22)} \end{bmatrix} - a^{(23)} \left(Q^{(zz)}\right)^{-1} \begin{bmatrix} Q^{(zx)} & Q^{(zy)} \end{bmatrix}$$

$$\begin{bmatrix} \vec{b}^{(11)} & \vec{b}^{(12)} \\ \vec{b}^{(21)} & \vec{b}^{(22)} \end{bmatrix} = \begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} \\ \vec{a}^{(21)} & \vec{a}^{(22)} \end{bmatrix} - \begin{bmatrix} a^{(13)} \\ a^{(23)} \end{bmatrix} \left(Q^{(zz)}\right)^{-1} \begin{bmatrix} Q^{(zx)} & Q^{(zy)} \end{bmatrix}$$

$$\begin{bmatrix} \vec{b}^{(11)} & \vec{b}^{(12)} \\ \vec{b}^{(21)} & \vec{b}^{(22)} \end{bmatrix} = \begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} \\ \vec{a}^{(21)} & \vec{a}^{(22)} \end{bmatrix} + \begin{bmatrix} a^{(13)} \\ a^{(23)} \end{bmatrix} \begin{bmatrix} R^{(zx)} & R^{(zy)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} a^{(13)} \\ a^{(23)} \end{bmatrix} \begin{bmatrix} R^{(zx)} & R^{(zy)} \end{bmatrix} = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} \\ \vec{r}^{(y|x)} & \vec{r}^{(y|y)} \end{bmatrix} - \begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} \\ \vec{a}^{(21)} & \vec{a}^{(22)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{bmatrix}$$

$$\begin{bmatrix} a^{(13)} \\ a^{(23)} \end{bmatrix} \begin{bmatrix} R^{(zx)} & R^{(zy)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{bmatrix}^{-1} = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} \\ \vec{r}^{(y|x)} & \vec{r}^{(y|y)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{bmatrix}^{-1} - \begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} \\ \vec{a}^{(21)} & \vec{a}^{(22)} \end{bmatrix}$$

Q: partial coef and coef??–should correct.

1.1 Partial correlation coefficient

$$\left[\begin{array}{ccc} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{array}\right] \left[\begin{array}{ccc} R^{(xx)} & R^{(xy)} & O \\ R^{(yx)} & R^{(yy)} & O \\ O & O & R^{(zz)} \end{array}\right] = \left[\begin{array}{ccc} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & 0 \end{array}\right]$$

$$\left(Q^{(yy)}\right)^{-1} = R^{(yy)} - R^{(yx)} \left(R^{(xx)}\right)^{-1} R^{(xy)}$$

$$x_t' = x_t - \alpha Y$$

2 Appendix: Formulas Used to Derive Above Results

2.1 Binomial inverse theorem

$$(A + UBV)^{-1} = A^{-1} - A^{-1}UB\left(B + BVA^{-1}UB\right)^{-1}BVA^{-1}$$

Specially

$$\left(A - BD^{-1}C\right)^{-1} = A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1}CA^{-1}$$

2.2 Inverse of $(n_1 + n_2) \times (n_1 + n_2)$ matrix

(Assume the inverse matries in the expressions are all valid)

$$\left(R^{(zy)} - R^{(zx)}\left(R^{(xx)}\right)^{-1}R^{(xy)}\right)\left(R^{(yy)} - R^{(yx)}\left(R^{(xx)}\right)^{-1}R^{(xy)}\right)^{-1}$$

Use row elimination, get

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \left(a_{11} - a_{12}a_{22}^{-1}a_{21}\right)^{-1} & -\left(a_{11} - a_{12}a_{22}^{-1}a_{21}\right)^{-1}a_{12}a_{22}^{-1} \\ -\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1} & \left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^{-1} + a_{11}^{-1}a_{12}\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1} & -a_{11}^{-1}a_{12}\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1} \\ -a_{22}^{-1}a_{21}\left(a_{11} - a_{12}a_{22}^{-1}a_{21}\right)^{-1} & a_{22}^{-1} + a_{22}^{-1}a_{21}\left(a_{11} - a_{12}a_{22}^{-1}a_{21}\right)^{-1}a_{12}a_{22}^{-1} \end{bmatrix}$$

2.2.1 Known inverse matrix, get inverse of sub matrix

Known

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then a_{22}^{-1} can get from

$$a_{22}^{-1} = b_{22} - b_{21}\left(b_{11}\right)^{-1}b_{12}$$

2.3 Inverse of $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ maxtrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = UD^{-1}L$$

$$U = \begin{bmatrix} I & -a_{11}^{-1}a_{12} & a_{11}^{-1}a_{12}\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23} - a_{21}a_{11}^{-1}a_{13}\right) - a_{11}^{-1}a_{13} \\ O & I & -\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23} - a_{21}a_{11}^{-1}a_{13}\right) \\ O & O & I \end{bmatrix}$$

$$L = \begin{bmatrix} I & O & O \\ -a_{21}a_{11}^{-1} & I & O \\ \left(a_{32} - a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1} - a_{31}a_{11}^{-1} & -\left(a_{32} - a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1} & I \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & O & O \\ O & a_{22} - a_{21}a_{11}^{-1}a_{12} & O \\ O & O & a_{33} - a_{31}a_{11}^{-1}a_{13} - \left(a_{32} - a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23} - a_{21}a_{11}^{-1}a_{13}\right) \end{bmatrix}$$

Note:

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ c_{21} & 1 & 0 \\ c_{31} & c_{32} & 1 \end{bmatrix} = \begin{bmatrix} b_1 + a_{12}b_2c_{21} + a_{13}b_3c_{31} & a_{12}b_2 + a_{13}b_3c_{32} & a_{13}b_3 \\ b_2c_{21} + a_{23}b_3c_{31} & b_2 + a_{23}b_3c_{32} & a_{23}b_3 \\ b_3c_{31} & b_3c_{32} & b_3 \end{bmatrix}$$