Inverse Covariance Matrix Age

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Inspired by:

- [2], gives form of inverse covariance from AR coefficients, univariate.
  - Method based on spectral polynomial and matrix form of lag operator.
  - The method can be extended to multivariate case.
  - It involves calculation of semi-infinite matrix, and the calculation detail is not given.
  - No discussion about the effect of finite truncation or starting point.
- [1], gives form of inverse covariance from AR coefficients, univariate.
  - Metod based on spectral polynomial and algebra operations to construct related linear equations.
  - With modification, can be extended to multivariate case to some extend (no close form solution).

1 Solving Covariance Matrix And Its Inverse Using Method in [1]

The spectral polynomial of an (stable) AR process satisfies ($\cdot^H$ means conjugate transpose)

$$\sum_{k=-\infty}^{\infty} R_k z^k = \left(\sum_{k=0}^{m} A_k z^k\right)^{-1} \Sigma \left(\sum_{k=0}^{m} A_k^H z^{-k}\right)^{-1}.$$

$$R_k = R_{-k}^H, \quad A_0 = I$$

Match the coefficients of z in both side of

$$\left(\sum_{k=-\infty}^{\infty} R_k z^k\right)\left(\sum_{k=0}^{m} A_k^H z^{-k}\right) = \left(\sum_{k=0}^{m} A_k z^k\right)^{-1} \Sigma,$$

get

$$\sum_{k=0}^{m} R_{k+j} A_k^H = \begin{cases} B_j \Sigma & j \geq 0 \\ 0 & j < 0 \end{cases}, \tag{1}$$

$$\sum_{k=0}^{\infty} B_k z^k = \left(\sum_{k=0}^{m} A_k z^k\right)^{-1}. \tag{2}$$

Define (V is the covariance matrix up to order m)

$$V = \begin{bmatrix} R_0 & R_1 & \cdots & R_{m-1} \\ R_{-1} & R_0 & & \vdots \\ \vdots & & \ddots & \\ R_{1-m} & \cdots & & R_0 \end{bmatrix}, \quad V_\star = \begin{bmatrix} R_{-m} & \cdots & R_{-2} & R_{-1} \\ R_{-m-1} & R_{-m} & & R_{-2} \\ \vdots & & \ddots & \vdots \\ R_{1-2m} & \cdots & & R_{-m} \end{bmatrix}$$

$$A^H = \begin{bmatrix} A_0^H & 0 & \cdots & 0 \\ A_1^H & A_0^H & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{m-1}^H & \cdots & A_1^H & A_0^H \end{bmatrix}, \quad A_\star^H = \begin{bmatrix} A_m^H & \cdots & A_2^H & A_1^H \\ 0 & A_m^H & & A_2^H \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_m^H \end{bmatrix}$$

$$B = \begin{bmatrix} B_0 & B_1 & \cdots & B_{m-1} \\ 0 & B_0 & \ddots & \vdots \\ \vdots & & \ddots & B_1 \\ 0 & \cdots & 0 & B_0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & \Sigma & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Sigma \end{bmatrix}$$

Eq.(2) in matrix form is

$$B = A^{-1}$$

Write Eq,(1) in matrix form:

$$\begin{bmatrix} V & V_\star^H \end{bmatrix} \begin{bmatrix} A^H \\ A_\star^H \end{bmatrix} = B\Gamma \tag{3}$$

$$\begin{bmatrix} V_\star & V \end{bmatrix} \begin{bmatrix} A^H \\ A_\star^H \end{bmatrix} = O \tag{4}$$

From Eq.(3)(4) we get

$$\begin{cases} VA^H + V_\star^H A_\star^H = A^{-1}\Gamma \\ V_\star A^H + VA_\star^H = O \end{cases}$$

$$AVA^H - A_\star V A_\star^H = \Gamma \tag{5}$$

If define

$$G = A^{-1}A_\star$$

$$U = A^{-1}\Gamma A^{-1H}$$

Then Eq.(5) become

$$GVG^H - V + U = 0, \tag{6}$$

which is the discrete Lyapunov equation. The solution is

$$V = \sum_{j=0}^{\infty} G^j U G^{jH}.$$

To solve V^{-1}, start from Eq.(6), and use Binomial inverse theorem

$$\begin{aligned} V^{-1} &= \left(U + GVG^H\right)^{-1} \\ &= U^{-1} - U^{-1}G(V^{-1} + G^H U^{-1} G)^{-1} G^H U^{-1} \\ &= A^H\Gamma^{-1}A - A^H\Gamma^{-1}A_\star(V^{-1} + A_\star^H\Gamma^{-1}A_\star)^{-1}A_\star^H\Gamma^{-1}A \end{aligned} \tag{7}$$

Note that by sufficient zero padding to A_k (increase the size of V or V^{-1} but keep the real fitting order), non-zero terms of $A_\star$ will locate at lower left corner, non-zero terms of A will locate around (upper) diagonal. Therefore the non-zero terms of $A^H\Gamma^{-1}A_\star$ will locate at lower left corner, and hence (no matter what V^{-1} is) the non-zeros of $A^H\Gamma^{-1}A_\star(V^{-1} + A_\star^H\Gamma^{-1}A_\star)^{-1}A_\star^H\Gamma^{-1}A$ will locate at lower right, the width is (the real) m.

So as a consequence, except the $m \times m$ block of the V^{-1}, it can be calculated as $A^H\Gamma^{-1}A$

$$V^{-1} \approx A^H\Gamma^{-1}A \tag{8}$$

It happens that only the upper left of V^{-1} is in our interesting.

Following [2] will get the same result as Eq.(8).

2 Exact Solution by Zero Padding

Suppose we expend the fitting order to $3m$ by zero padding. (to be verified)

That is:

$$V_{\{3m\}} = \begin{bmatrix} R_0 & R_1 & \cdots & R_{3m-1} \\ R_{-1} & R_0 & & \vdots \\ \vdots & & \ddots & \\ R_{1-3m} & \cdots & & R_0 \end{bmatrix}$$

$$A^H_{\{3m\}} = \begin{bmatrix} A^H & 0 & 0 \\ A^H_\star & A^H & 0 \\ 0 & A^H_\star & A^H \end{bmatrix}, \quad A^H_{\star\{3m\}} = \begin{bmatrix} 0 & 0 & A^H_\star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the solution is:

$$V^{-1}_{\{3m\}} = A^H_{\{3m\}}\Gamma^{-1}_{\{3m\}}A_{\{3m\}} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A^H\Gamma^{-1}A_\star & 0 & 0 \end{bmatrix} (V^{-1}_{\{3m\}} + \begin{bmatrix} A^H_\star\Gamma^{-1}A_\star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) \begin{bmatrix} 0 & 0 & A^H_\star\Gamma^{-1}A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V^{-1}_{\{3m\}} = A^H_{\{3m\}}\Gamma^{-1}A_{\{3m\}} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A^H\Gamma^{-1}A_\star(A^H\Gamma^{-1}A + A^H_\star\Gamma^{-1}A_\star)A^H_\star\Gamma^{-1}A \end{bmatrix}$$

$$V^{-1}_{\{3m\}} = \begin{bmatrix} A^H\Gamma^{-1}A & A^H\Gamma^{-1}A_\star & 0 \\ A^H_\star\Gamma^{-1}A & A^H_\star\Gamma^{-1}A_\star + A^H\Gamma^{-1}A & A^H\Gamma^{-1}A_\star \\ 0 & A^H_\star\Gamma^{-1}A & A^H_\star\Gamma^{-1}A_\star + A^H\Gamma^{-1}A - A^H\Gamma^{-1}A_\star(A^H\Gamma^{-1}A + A^H_\star\Gamma^{-1}A_\star)A^H_\star\Gamma^{-1}A \end{bmatrix}$$

3 Comparison to Spectral Density Inversion Approximation

The inverse if spectral density of AR process can be easily calculated as

$$S^{-1}(w) = A^H(w)\Sigma^{-1}A(w). \tag{9}$$

Then by inverse Fourier transform, we can construct an (another) approximation of V^{-1}

$$Q = \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{m-1} \\ Q_{-1} & Q_0 & & \vdots \\ \vdots & & \ddots & \\ Q_{1-m} & \cdots & & Q_0 \end{bmatrix}$$

$$Q_j = \mathscr{F}^{-1}\left[S^{-1}(w)\right](j)$$

where the j is time.

The difference of this Q approximation to $A^H\Gamma^{-1}A$ is

$$Q - A^H\Gamma^{-1}A = A^H_{\#}\Gamma^{-1}A_{\#} \tag{10}$$

$$A_{\#} = \begin{bmatrix} 0 & & \cdots & 0 \\ A_{m-1} & 0 & & \vdots \\ \vdots & \ddots & \ddots & \\ A_1 & \cdots & A_{m-1} & 0 \end{bmatrix}.$$

Which can be seen as the triangular partition of circulant matrix.

Note that for Eq.(10) to be correct, a much longer FFT length should be choosen (e.g. fftlen $> 2m$).

4 Expression For Certain Block

Define

$$A^{(:,u)} = \begin{bmatrix} A_{0,[:,u]} & A_{1,[:,u]} & \cdots & A_{m-1,[:,u]} \\ 0 & A_{0,[:,u]} & & \vdots \\ \vdots & & \ddots & A_{1,[:,u]} \\ 0 & \cdots & 0 & A_{0,[:,u]} \end{bmatrix}$$

$$A_{k,[:,u]} = \begin{bmatrix} A_{k,1,u} \\ \vdots \\ A_{k,p,u} \end{bmatrix}$$

$$Q^{(zy)} \approx A^{(:,z)H}\Gamma^{-1}A^{(:,y)}$$

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References

[1] E. J. Godolphin and J. M. Unwin. Evaluation of the Covariance Matrix for the Maximum Likelihood Estimator of a Gaussian Autoregressive-Moving Average Process. *Biometrika*, 70(1):279–284, 1983.

[2] J. Wise. The Autocorrelation Function and the Spectral Density Function. *Biometrika*, 42(1/2):151–159, 1955.