Matrix Formulas That Is Useful In Least Square Analysis

(You could always swap indexes properly to get similar results)

1 Difference of Quadratic Forms, Inverse Matrices and Solution of Linear Systems

For invertable $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ matrix R (no need symmetric), define Q

$$Q = \begin{bmatrix} Q^{(xx)} & Q^{(xy)} & Q^{(xz)} \\ Q^{(yx)} & Q^{(yy)} & Q^{(yz)} \\ Q^{(zx)} & Q^{(zy)} & Q^{(zz)} \end{bmatrix} = R^{-1} = \begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix}^{-1}. \tag{1}$$

For any $\begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}$ $(n_4 \times (n_1 + n_2 + n_3)$ matrix),
define $\mathrm{SS}_{R[x,z]}$, $\mathrm{SS}_{R[x,y,z]}$, $\begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{bmatrix}$ and $\begin{bmatrix} \vec{a}_T^{(11)} & \vec{a}_T^{(12)} & \vec{a}_T^{(13)} \end{bmatrix}$ as
(replace transpose $\cdot^T$ to conjugate transpose $\cdot^H$ for matrix with complex entries)

$$\mathrm{SS}_{R[x,z]} = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|z)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xz)} \\ R^{(zx)} & R^{(zz)} \end{bmatrix}^{-1} \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|z)} \end{bmatrix}^T,$$

$$\mathrm{SS}_{R[x,y,z]} = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix}^{-1} \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}^T,$$

$$\begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix} = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix},$$

$$\begin{bmatrix} \vec{a}_T^{(11)} & \vec{a}_T^{(12)} & \vec{a}_T^{(13)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix}^T = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}.$$

Then the difference of the quadratic form is
(Assume the inverse matries in the expressions are all valid, pseudo-inverse might also work)

$$\mathrm{SS}_{R[x,y,z]} - \mathrm{SS}_{R[x,z]} = \vec{a}^{(12)} \left(Q^{(yy)} \right)^{-1} \left(\vec{a}_T^{(12)} \right)^T. \tag{2}$$

The $\left(Q^{(yy)} \right)^{-1}$ might be expressed in R blocks as
(Swap index x and z will give another expression)

$$\left(Q^{(yy)} \right)^{-1} = R^{(yy)} - R^{(yx)} \left(R^{(xx)} \right)^{-1} R^{(xy)} \tag{3}$$

$$- \left(R^{(yz)} - R^{(yx)} \left(R^{(xx)} \right)^{-1} R^{(xz)} \right) \left(R^{(zz)} - R^{(zx)} \left(R^{(xx)} \right)^{-1} R^{(xz)} \right)^{-1} \left(R^{(zy)} - R^{(zx)} \left(R^{(xx)} \right)^{-1} R^{(xy)} \right)$$

Define P

$$\left[\begin{array}{cc} P^{(xx)} & P^{(xy)} \\ P^{(yx)} & P^{(yy)} \end{array}\right] \triangleq \left[\begin{array}{cc} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{array}\right]^{-1} = \left[\begin{array}{cc} Q^{(xx)} & Q^{(xy)} \\ Q^{(yx)} & Q^{(yy)} \end{array}\right] - \left[\begin{array}{c} Q^{(xz)} \\ Q^{(yz)} \end{array}\right] \left(Q^{(zz)}\right)^{-1} \left[\begin{array}{cc} Q^{(zx)} & Q^{(zy)} \end{array}\right] \tag{4}$$

$$\left(Q^{(zz)}\right)^{-1} = R^{(zz)} - \left[\begin{array}{cc} R^{(zx)} & R^{(zy)} \end{array}\right] \left[\begin{array}{cc} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{array}\right]^{-1} \left[\begin{array}{c} R^{(xz)} \\ R^{(yz)} \end{array}\right]$$

If we solve

$$\left[\begin{array}{cc} \vec{b}^{(11)} & \vec{b}^{(12)} \end{array}\right] \left[\begin{array}{cc} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{array}\right] = \left[\begin{array}{cc} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} \end{array}\right]$$

Then there is relation
(There are 6 of these relations)

$$b^{(12)} = a^{(12)} - a^{(13)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} \tag{5}$$

$$a^{(12)} \left(I - \left(Q^{(yy)}\right)^{-1} Q^{(yz)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)}\right) = b^{(12)} + b^{(13)}_{[xz]} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} \tag{6}$$

Note

$$\begin{aligned} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} &= -\left(R^{(zy)} - R^{(zx)} \left(R^{(xx)}\right)^{-1} R^{(xy)}\right) \left(R^{(yy)} - R^{(yx)} \left(R^{(xx)}\right)^{-1} R^{(xy)}\right)^{-1} \\ &= -\left[\begin{array}{cc} R^{(zx)} & R^{(zy)} \end{array}\right] \left[\begin{array}{c} P^{(xy)} \\ P^{(yy)} \end{array}\right] \end{aligned}$$

$$I - \left(Q^{(yy)}\right)^{-1} Q^{(yz)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} = \left(Q^{(yy)}\right)^{-1} P^{(yy)}$$

Also

$$\vec{b}^{(11)} = \vec{a}^{(11)} - a^{(13)} \left(Q^{(zz)}\right)^{-1} Q^{(zx)}$$

In matrix form

$$\left[\begin{array}{cc} \vec{b}^{(11)} & \vec{b}^{(12)} \\ \vec{b}^{(21)} & \vec{b}^{(22)} \end{array}\right] = \left[\begin{array}{cc} \vec{a}^{(11)} & \vec{a}^{(12)} \\ \vec{a}^{(21)} & \vec{a}^{(22)} \end{array}\right] - \left[\begin{array}{c} a^{(13)} \\ a^{(23)} \end{array}\right] \left(Q^{(zz)}\right)^{-1} \left[\begin{array}{cc} Q^{(zx)} & Q^{(zy)} \end{array}\right]$$

$$\left[\begin{array}{cc} \vec{b}^{(11)} & \vec{b}^{(12)} \\ \vec{b}^{(21)} & \vec{b}^{(22)} \end{array}\right] = \left[\begin{array}{cc} \vec{a}^{(11)} & \vec{a}^{(12)} \\ \vec{a}^{(21)} & \vec{a}^{(22)} \end{array}\right] + \left[\begin{array}{c} a^{(13)} \\ a^{(23)} \end{array}\right] \left[\begin{array}{cc} R^{(zx)} & R^{(zy)} \end{array}\right] \left[\begin{array}{cc} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{array}\right]^{-1}$$

$$\left[\begin{array}{c} a^{(13)} \\ a^{(23)} \end{array}\right] \left[\begin{array}{cc} R^{(zx)} & R^{(zy)} \end{array}\right] = \left[\begin{array}{cc} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} \\ \vec{r}^{(y|x)} & \vec{r}^{(y|y)} \end{array}\right] - \left[\begin{array}{cc} \vec{a}^{(11)} & \vec{a}^{(12)} \\ \vec{a}^{(21)} & \vec{a}^{(22)} \end{array}\right] \left[\begin{array}{cc} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{array}\right]$$

$$\left[\begin{array}{c} a^{(13)} \\ a^{(23)} \end{array}\right] \left[\begin{array}{cc} R^{(zx)} & R^{(zy)} \end{array}\right] \left[\begin{array}{cc} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{array}\right]^{-1} = \left[\begin{array}{cc} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} \\ \vec{r}^{(y|x)} & \vec{r}^{(y|y)} \end{array}\right] \left[\begin{array}{cc} R^{(xx)} & R^{(xy)} \\ R^{(yx)} & R^{(yy)} \end{array}\right]^{-1} - \left[\begin{array}{cc} \vec{a}^{(11)} & \vec{a}^{(12)} \\ \vec{a}^{(21)} & \vec{a}^{(22)} \end{array}\right]$$

Q: partial coef and coef??–should correct.

1.1 Partial correlation coefficient

$$\begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} & O \\ R^{(yx)} & R^{(yy)} & O \\ O & O & R^{(zz)} \end{bmatrix} = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & 0 \end{bmatrix}$$

$$\left(Q^{(yy)} \right)^{-1} = R^{(yy)} - R^{(yx)} \left(R^{(xx)} \right)^{-1} R^{(xy)}$$

$$x'_t = x_t - \alpha Y$$

What if z is totally whitened? i.e.

$$R^{(zz)} = I$$

not verified

$$c^{(33)} = a^{(33)} - \begin{bmatrix} a^{(31)} & a^{(32)} \end{bmatrix} \begin{bmatrix} Q^{(xx)} & Q^{(xy)} \\ Q^{(yx)} & Q^{(yy)} \end{bmatrix}^{-1} \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix}$$

1.2 Spectrum domain approximation

From Eq. (5) get analog formula for frequency domain:

$$B_{xy} \approx A_{xy} - \left[A_{xz} Q_{zz}^{-1} Q_{zy} \right]_+$$

where $[\cdot]_+$ stand for extract the positive time part. That is

$$[f(w)]_+ = \mathscr{F}_{\mathrm{DFT}} \left[\mathscr{F}_{\mathrm{DFT}}^{-1} \left[f(w), t \right] (t) \cdot u(t), w \right]$$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Under assumption (blocked diagonal matrix)

$$\Sigma = \mathrm{diag} \left(\begin{bmatrix} \Sigma_{xx} & \Sigma_{yy} & \Sigma_{zz} \end{bmatrix} \right)$$

We have

$$Q_{zz} = A_{xz}^{*T} \Sigma_{xx}^{-1} A_{xz} + A_{yz}^{*T} \Sigma_{yy}^{-1} A_{yz} + A_{zz}^{*T} \Sigma_{zz}^{-1} A_{zz}$$

$$Q_{zy} = A_{xz}^{*T} \Sigma_{xx}^{-1} A_{xy} + A_{yz}^{*T} \Sigma_{yy}^{-1} A_{yy} + A_{zz}^{*T} \Sigma_{zz}^{-1} A_{zy}$$

Hence

$$A_{xz}/Q_{zz} * Q_{zy} \approx A_{xz} \left(A_{zz}^{-1} \Sigma_{zz} \left(A_{zz}^{*T} \right)^{-1} \right) \left(A_{yz}^{*T} \Sigma_{yy}^{-1} A_{yy} + A_{zz}^{*T} \Sigma_{zz}^{-1} A_{zy} \right) \approx A_{xz} \left(A_{zy} + A_{yz}^* \right)$$

$$B_{xy} \approx A_{xy} - \left[A_{xz} \left(A_{zy} + A_{yz}^* \right) \right]_+$$

$$Q_{zz}^{-1} = S_{zz} - S_{z,xy} S_{xy,xy}^{-1} S_{xy,z}$$

—————-

$$\begin{bmatrix} O & A_{xy} & A_{xz} \end{bmatrix} \left(\begin{bmatrix} A_{xy} \\ O \\ A_{zy} \end{bmatrix} + \begin{bmatrix} A_{yz} & O & A_{yz} \end{bmatrix}^{*T} \right) = A(A + A^*)$$

as

2 Relation to Linear Regression For Time Series

Assume $E(x_t) = E(y_t) = E(y_t) = 0$, as usual, let's do a 3-var AR of order m:

$$\begin{cases} x_t = \sum_{j=1}^{m} a_j^{(11)} x_{t-j} + \sum_{j=1}^{m} a_j^{(12)} y_{t-j} + \sum_{j=1}^{m} a_j^{(13)} z_{t-j} + \epsilon_t^{(1)} \quad \text{(a)} \\[2mm] y_t = \sum_{j=1}^{m} a_j^{(21)} x_{t-j} + \sum_{j=1}^{m} a_j^{(22)} y_{t-j} + \sum_{j=1}^{m} a_j^{(23)} z_{t-j} + \epsilon_t^{(2)} \quad \text{(b)} \\[2mm] z_t = \sum_{j=1}^{m} a_j^{(31)} x_{t-j} + \sum_{j=1}^{m} a_j^{(32)} y_{t-j} + \sum_{j=1}^{m} a_j^{(33)} z_{t-j} + \epsilon_t^{(3)} \quad \text{(c)} \end{cases} \tag{7}$$

The least square solution for $\vec{a}^{(11)} \quad \vec{a}^{(12)} \quad \vec{a}^{(13)}$ in min var $\epsilon_t^{(1)}$ ($E(\epsilon_t^{(u)} x_{t-k}^{(v)T}) = 0, \ \forall k = 1 \ldots m$) satisfies

$$\begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix} = \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}, \tag{8}$$

where (here $\mathbb{E}$ can be expectation operator or $\frac{1}{L}\sum_{t=1}^{L} \cdot$)

$$\vec{a}^{(uv)} = \left(a_j^{(uv)} \right)_{j=1\ldots m}^{\text{row}}, \ (u, v \in \{1, 2, 3\}),$$

$$R^{(uv)} = \left(\mathbb{E}(x_{t-j}^{(u)} x_{t-k}^{(v)T}) \right)_{j,k=1\ldots m}, \ (u, v \in \{x, y, z\}),$$

$$\vec{r}^{(u|v)} = \left(\mathbb{E}(x_t^{(u)} x_{t-k}^{(v)T}) \right)_{k=1\ldots m}^{\text{row}}, \ (u, v \in \{x, y, z\}).$$

And get $\text{var}(\epsilon_t^{(1)})$ from

$$\text{var}(x_t) = \vec{a}^{(11)} \vec{r}^{(x|x)T} + \vec{a}^{(12)} \vec{r}^{(x|y)T} + \vec{a}^{(13)} \vec{r}^{(x|z)T} + \text{var}(\epsilon_t^{(1)}).$$

or

$$\text{var}(\epsilon_t^{(1)}) = \text{var}(x_t) - \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & R^{(yy)} & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & R^{(zz)} \end{bmatrix}^{-1} \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|y)} & \vec{r}^{(x|z)} \end{bmatrix}^T. \tag{9}$$

Similarly for least square min var $\epsilon_t^{(1|3)}$ in

$$x_t = \sum_{j=1}^{m} c_j^{(11)} x_{t-j} + \sum_{j=1}^{m} c_j^{(13)} z_{t-j} + \epsilon_t^{(1|3)}, \tag{10}$$

that is

$$\text{var}(\epsilon_t^{(1|3)}) = \text{var}(x_t) - \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|z)} \end{bmatrix} \begin{bmatrix} R^{(xx)} & R^{(xz)} \\ R^{(zx)} & R^{(zz)} \end{bmatrix}^{-1} \begin{bmatrix} \vec{r}^{(x|x)} & \vec{r}^{(x|z)} \end{bmatrix}^T. \tag{11}$$

Note Granger causality $F_{y \to x | z}$ is defined as

$$F_{y \to x | z} = \ln \frac{\text{var}(\epsilon_t^{(1|3)})}{\text{var}(\epsilon_t^{(1)})} = \ln \left(1 + \frac{\text{var}(\epsilon_t^{(1|3)}) - \text{var}(\epsilon_t^{(1)})}{\text{var}(\epsilon_t^{(1)})} \right). \tag{12}$$

With help of Eq.(2), we get

$$F_{y \to x | z} = \ln \left(1 + \frac{\vec{a}^{(12)} \left(Q^{(yy)} \right)^{-1} \left(\vec{a}^{(12)} \right)^T}{\text{var}(\epsilon_t^{(1)})} \right)$$

3 Approximation of Toeplitz Matrix Operations

Mul of band toeplitz matrix

$$
\begin{bmatrix} a_0 & a_1 & \\ a_{-1} & a_0 & a_1 \\ & a_{-1} & a_0 \end{bmatrix}
\begin{bmatrix} b_0 & b_1 & \\ b_{-1} & b_0 & b_1 \\ & b_{-1} & b_0 \end{bmatrix}
=
\begin{bmatrix} c_0 - a_{-1}b_1 & c_1 & c_2 \\ c_{-1} & c_0 & c_1 \\ c_{-2} & c_{-1} & c_0 - a_1 b_{-1} \end{bmatrix}
\tag{13}
$$

where $c_0 = a_{-1}b_1 + a_0 b_0 + a_1 b_{-1}$, $c_{-1} = a_{-1}b_0 + a_0 b_{-1}$, $c_1 = a_0 b_1 + a_1 b_0$.

Inverse of band toeplitz ??????????

$$R(Q + G) = I$$

$$GR = I - QR$$

$$GRQ = Q - QRQ$$

since

$$RQ \approx I$$

so

$$G \approx Q - QRQ$$

$$R^{-1} \approx Q + Q - QRQ$$

In summary, that is

$$K = I - QR$$

$$R^{-1} = (I - K)^{-1}Q$$

Bad......since it possible that $\|K\| > 1$. Indeed, there is a 10 neuron case that $\|K\| = 8.34$ (for non-whitenned covz). But seems work for whitened signal and $p < 40$.

4 Appendix: Formulas Used to Derive Above Results

4.1 Binomial inverse theorem

$$(A + UBV)^{-1} = A^{-1} - A^{-1}UB\left(B + BVA^{-1}UB\right)^{-1} BVA^{-1}$$

Specially

$$\left(A - BD^{-1}C\right)^{-1} = A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1} CA^{-1}$$

4.2 Inverse of $(n_1 + n_2) \times (n_1 + n_2)$ matrix

(Assume the inverse matries in the expressions are all valid)

$$\left(R^{(zy)} - R^{(zx)}\left(R^{(xx)}\right)^{-1} R^{(xy)}\right)\left(R^{(yy)} - R^{(yx)}\left(R^{(xx)}\right)^{-1} R^{(xy)}\right)^{-1}$$

Use row elimination, get

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \left(a_{11} - a_{12}a_{22}^{-1}a_{21}\right)^{-1} & -\left(a_{11} - a_{12}a_{22}^{-1}a_{21}\right)^{-1}a_{12}a_{22}^{-1} \\ -\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1} & \left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^{-1} + a_{11}^{-1}a_{12}\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1} & -a_{11}^{-1}a_{12}\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1} \\ -a_{22}^{-1}a_{21}\left(a_{11} - a_{12}a_{22}^{-1}a_{21}\right)^{-1} & a_{22}^{-1} + a_{22}^{-1}a_{21}\left(a_{11} - a_{12}a_{22}^{-1}a_{21}\right)^{-1}a_{12}a_{22}^{-1} \end{bmatrix}$$

4.2.1 Known inverse matrix, get inverse of sub matrix

Known

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then a_{22}^{-1} can get from

$$a_{22}^{-1} = b_{22} - b_{21}\left(b_{11}\right)^{-1} b_{12}$$

4.3 Inverse of $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ maxtrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = UD^{-1}L$$

$$U = \begin{bmatrix} I & -a_{11}^{-1}a_{12} & a_{11}^{-1}a_{12}\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23} - a_{21}a_{11}^{-1}a_{13}\right) - a_{11}^{-1}a_{13} \\ O & I & -\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23} - a_{21}a_{11}^{-1}a_{13}\right) \\ O & O & I \end{bmatrix}$$

$$L = \begin{bmatrix} I & O & O \\ -a_{21}a_{11}^{-1} & I & O \\ \left(a_{32} - a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1} - a_{31}a_{11}^{-1} & -\left(a_{32} - a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1} & I \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & O & O \\ O & a_{22} - a_{21}a_{11}^{-1}a_{12} & O \\ O & O & a_{33} - a_{31}a_{11}^{-1}a_{13} - \left(a_{32} - a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22} - a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23} - a_{21}a_{11}^{-1}a_{13}\right) \end{bmatrix}$$

Note:

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ c_{21} & 1 & 0 \\ c_{31} & c_{32} & 1 \end{bmatrix} = \begin{bmatrix} b_1 + a_{12}b_2c_{21} + a_{13}b_3c_{31} & a_{12}b_2 + a_{13}b_3c_{32} & a_{13}b_3 \\ b_2c_{21} + a_{23}b_3c_{31} & b_2 + a_{23}b_3c_{32} & a_{23}b_3 \\ b_3c_{31} & b_3c_{32} & b_3 \end{bmatrix}$$

5 Useful Theorems

5.1 Spectral Representation of Multivariate Stationary Time Series [1]

$$R(h) = \int_{-\pi}^{\pi} e^{ihw} \mathrm{d}F(w), \quad h = 0, \pm 1, \dots$$

$F(-\pi) = 0$, F is right-continuous, $F(\mu) - F(\lambda)$ non-negative define for all $\lambda < \mu$.

- $E\left(Z(\lambda)Z^*(\lambda)\right)$ finite

- $E\left(Z(\lambda)\right) = 0$, $-\pi < \lambda < \pi$

- $E\left(Z(\lambda_4) - Z(\lambda_3)\right)\left(Z(\lambda_2) - Z(\lambda_1)\right)^* = 0$ $(\lambda_1, \lambda_2] \cap (\lambda_3, \lambda_4] = \phi$

- $E\left(Z(\lambda + \delta) - Z(\lambda)\right)\left(Z(\lambda + \delta) - Z(\lambda)\right)^* \to 0$, as $\delta \downarrow 0$.

unique matrix distribution F, $F(\mu) - F(\lambda) = E\left((Z(\mu) - Z(\lambda))(Z(\mu) - Z(\lambda))^*\right)$, $\lambda \le \mu$. o

$$E\left(\mathrm{d}Z(\lambda)\mathrm{d}Z^*(\lambda)\right) = \begin{cases} \mathrm{d}F(\lambda) & \mu = \lambda \\ 0 & \text{otherwise} \end{cases}$$

$$X_t = \int_{-\pi}^{\pi} e^{itv} \mathrm{d}Z(v), \quad t = 0, \pm 1, \dots \quad \text{with probability 1}$$

References

[1] Peter J. Brockwell and Richard A. Davis. *Time Series: Theory and Methods.* Springer Science & Business Media, November 2013.