1 Freq domain formula

$$\mathscr{F}\left[\begin{array}{c} x_t \\ y_t \end{array}\right] = \left[\begin{array}{cc} \hat{a}(w) & \hat{b}(w) \\ \hat{c}(w) & \hat{d}(w) \end{array}\right] \mathscr{F}\left[\begin{array}{c} \epsilon_t \\ \eta_t \end{array}\right]$$

$$\operatorname{cov}(\epsilon_t, \eta_t) = 0$$

$$\begin{aligned} S_{yy}(w) &= \left(\hat{c}(w)\mathscr{F}[\epsilon_t] + \hat{d}(w)\mathscr{F}[\eta_t]\right)\left(\hat{c}^*(w)\mathscr{F}^*[\epsilon_t] + \hat{d}^*(w)\mathscr{F}^*[\eta_t]\right) \\ &= \hat{c}(w)\hat{c}^*(w)\operatorname{var}(\epsilon_t) + \hat{d}(w)\hat{d}^*(w)\operatorname{var}(\eta_t) \end{aligned}$$

$$\frac{\hat{c}(w)\hat{c}^*(w)\operatorname{var}(\epsilon_t) + \hat{d}(w)\hat{d}^*(w)\operatorname{var}(\eta_t)}{\hat{d}(w)\hat{d}^*(w)\operatorname{var}(\eta_t)}$$

3-var, $\operatorname{var}(\epsilon^*_{1t}) = S_{\epsilon^*_1}(\omega)$

$$f_{Y\to X|Z-}(\omega) = \ln \frac{|\operatorname{var}(\epsilon^*_{1t})|}{\left|\hat{F}_{11}(\omega)\operatorname{var}(\epsilon_{1t})\hat{F}_{11}(\omega)'\right|}$$

Assume $\operatorname{var}(\epsilon^*_{1t})$ is scalar

$$= \ln \frac{\hat{F}_{11}(\omega)\operatorname{var}(\epsilon_{1t})\hat{F}_{11}(\omega)' + \hat{F}_{12}(\omega)\operatorname{var}(\epsilon_{2t})\hat{F}_{12}(\omega)' + \hat{F}_{13}(\omega)\operatorname{var}(\epsilon_{3t})\hat{F}_{13}(\omega)'}{\hat{F}_{11}(\omega)\operatorname{var}(\epsilon_{1t})\hat{F}_{11}(\omega)'}$$

$$= \ln\left(1 + \frac{\hat{F}_{12}(\omega)\operatorname{var}(\epsilon_{2t})\hat{F}_{12}(\omega)' + \hat{F}_{13}(\omega)\operatorname{var}(\epsilon_{3t})\hat{F}_{13}(\omega)'}{\hat{F}_{11}(\omega)\operatorname{var}(\epsilon_{1t})\hat{F}_{11}(\omega)'}\right)$$

$$\hat{F}_{11}(\omega) = \hat{D}_{11}(\omega)\hat{A}_{11}(\omega) + \hat{D}_{12}(\omega)\hat{A}_{31}(\omega)$$

$$\begin{aligned} \hat{F}_{12}(\omega) &= \hat{D}_{11}(\omega)\hat{A}_{12}(\omega) + \hat{D}_{12}(\omega)\hat{A}_{32}(\omega) \\ \hat{F}_{13}(\omega) &= \hat{D}_{11}(\omega)\hat{A}_{13}(\omega) + \hat{D}_{12}(\omega)\hat{A}_{33}(\omega) \end{aligned}$$

$$\begin{aligned} f_{Y\to X|Z-}(\omega) &\approx \frac{\hat{F}_{12}(\omega)\operatorname{var}(\epsilon_{2t})\hat{F}_{12}(\omega)'}{\hat{F}_{11}(\omega)\operatorname{var}(\epsilon_{1t})\hat{F}_{11}(\omega)'} + \frac{\hat{F}_{13}(\omega)\operatorname{var}(\epsilon_{3t})\hat{F}_{13}(\omega)'}{\hat{F}_{11}(\omega)\operatorname{var}(\epsilon_{1t})\hat{F}_{11}(\omega)'} \\ &\approx f_{Y\to X}(\omega)? + \frac{\hat{F}_{13}(\omega)\operatorname{var}(\epsilon_{3t})\hat{F}_{13}(\omega)'}{\hat{F}_{11}(\omega)\operatorname{var}(\epsilon_{1t})\hat{F}_{11}(\omega)'} \end{aligned}$$

crude approximate (assume x, y, z have been whitened)

$$\hat{F}_{13}(\omega) \approx \hat{A}_{13}(\omega) + \hat{D}_{12}(\omega)$$

$$f_{Y\to X|Z-}(\omega) \approx f_{Y\to X}(\omega) + \frac{\hat{F}_{13}(\omega)\operatorname{var}(\epsilon_{3t})\hat{F}_{13}(\omega)'}{\operatorname{var}(\epsilon_{1t})}$$

And $\operatorname{var}(\epsilon_{3t})$ is diagonal matrix.

1.1 Only 3-varable case

$$\hat{A} = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} \\ \hat{B}_{31} & \hat{B}_{32} & \hat{B}_{33} \end{bmatrix}^{-1} = \left| \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} \\ \hat{B}_{31} & \hat{B}_{32} & \hat{B}_{33} \end{bmatrix} \right|^{-1} \begin{bmatrix} \begin{vmatrix} \hat{B}_{22} & \hat{B}_{23} \\ \hat{B}_{32} & \hat{B}_{33} \end{vmatrix} & -\begin{vmatrix} \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{32} & \hat{B}_{33} \end{vmatrix} & \begin{vmatrix} \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{22} & \hat{B}_{23} \end{vmatrix} \\ -\begin{vmatrix} \hat{B}_{21} & \hat{B}_{23} \\ \hat{B}_{31} & \hat{B}_{33} \end{vmatrix} & \begin{vmatrix} \hat{B}_{11} & \hat{B}_{13} \\ \hat{B}_{31} & \hat{B}_{33} \end{vmatrix} & -\begin{vmatrix} \hat{B}_{11} & \hat{B}_{13} \\ \hat{B}_{21} & \hat{B}_{23} \end{vmatrix} \\ \begin{vmatrix} \hat{B}_{21} & \hat{B}_{22} \\ \hat{B}_{31} & \hat{B}_{32} \end{vmatrix} & -\begin{vmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{31} & \hat{B}_{32} \end{vmatrix} & \begin{vmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{vmatrix} \end{bmatrix}$$

$$\hat{F}_{12}(\omega) = \left| \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} \\ \hat{B}_{31} & \hat{B}_{32} & \hat{B}_{33} \end{bmatrix} \right|^{-1} (\hat{D}_{11}(\omega)(-\begin{vmatrix} \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{32} & \hat{B}_{33} \end{vmatrix}) + \hat{D}_{12}(\omega)(-\begin{vmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{31} & \hat{B}_{32} \end{vmatrix}))$$

$$\hat{F}_{13}(\omega) = \hat{D}_{11}(\omega)\hat{A}_{13}(\omega) + \hat{D}_{12}(\omega)\hat{A}_{33}(\omega)$$

2 inverse of $(1+1+(n-2)) \times (1+1+(n-2))$ maxtrix

$$\begin{bmatrix} 1 & 0 & O \\ -a_{21}/a_{11} & 1 & O \\ -a_{31}/a_{11} & O & I \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & a_{23} - a_{21}a_{13}/a_{11} \\ O & a_{32} - a_{31}a_{12}/a_{11} & a_{33} - a_{31}a_{13}/a_{11} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & a_{23} - a_{21}a_{13}/a_{11} \\ O & a_{32} - a_{31}a_{12}/a_{11} & a_{33} - a_{31}a_{13}/a_{11} \end{bmatrix} \begin{bmatrix} 1 & -a_{12}/a_{11} & -a_{13}/a_{11} \\ 0 & 1 & O \\ O & O & I \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & 0 & O \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & a_{23} - a_{21}a_{13}/a_{11} \\ O & a_{32} - a_{31}a_{12}/a_{11} & a_{33} - a_{31}a_{13}/a_{11} \end{bmatrix}, \text{ verified}$$

$$\begin{bmatrix} 1 & 0 & O \\ 0 & 1 & O \\ O & -\frac{a_{32} - a_{31}a_{12}/a_{11}}{a_{22} - a_{21}a_{12}/a_{11}} & I \end{bmatrix} \begin{bmatrix} a_{11} & 0 & O \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & a_{23} - a_{21}a_{13}/a_{11} \\ O & a_{32} - a_{31}a_{12}/a_{11} & a_{33} - a_{31}a_{13}/a_{11} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & 0 & O \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & a_{23} - a_{21}a_{13}/a_{11} \\ O & O & a_{33} - a_{31}a_{13}/a_{11} - \frac{a_{32} - a_{31}a_{12}/a_{11}}{a_{22} - a_{21}a_{12}/a_{11}}(a_{23} - a_{21}a_{13}/a_{11}) \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & O \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & a_{23} - a_{21}a_{13}/a_{11} \\ O & O & a_{33} - a_{31}a_{13}/a_{11} - \frac{a_{32} - a_{31}a_{12}/a_{11}}{a_{22} - a_{21}a_{12}/a_{11}}(a_{23} - a_{21}a_{13}/a_{11}) \end{bmatrix} \begin{bmatrix} 1 & 0 & O \\ 0 & 1 & -\frac{a_{23} - a_{21}a_{13}/a_{11}}{a_{22} - a_{21}a_{12}/a_{11}} \\ O & O & I \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & 0 & O \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & O \\ O & O & a_{33} - a_{31}a_{13}/a_{11} - \frac{(a_{32} - a_{31}a_{12}/a_{11})(a_{23} - a_{21}a_{13}/a_{11})}{a_{22} - a_{21}a_{12}/a_{11}} \end{bmatrix}, \text{ verified}$$

$$A^{-1} = Q_1Q_2B^{-1}P_2P_1$$

$$Q_1Q_2 = \begin{bmatrix} 1 & -a_{12}/a_{11} & -a_{13}/a_{11} \\ 0 & 1 & O \\ O & O & I \end{bmatrix} \begin{bmatrix} 1 & 0 & O \\ 0 & 1 & -\frac{a_{23} - a_{21}a_{13}/a_{11}}{a_{22} - a_{21}a_{12}/a_{11}} \\ O & O & I \end{bmatrix}$$

$$= \begin{bmatrix} I & -a_{12}/a_{11} & \frac{(a_{23}-a_{21}a_{13}/a_{11})a_{12}/a_{11}}{a_{22}-a_{21}a_{12}/a_{11}} - a_{13}/a_{11} \\ O & I & -\frac{a_{23}-a_{21}a_{13}/a_{11}}{a_{22}-a_{21}a_{12}/a_{11}} \\ O & O & I \end{bmatrix}$$

$$P_2P_1 = \begin{bmatrix} 1 & 0 & O \\ 0 & 1 & O \\ O & \frac{a_{32}-a_{31}a_{12}/a_{11}}{a_{22}-a_{21}a_{12}/a_{11}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 & O \\ a_{21}/a_{11} & 1 & O \\ a_{31}/a_{11} & O & I \end{bmatrix} =$$

$$= \begin{bmatrix} I & O & O \\ -a_{21}/a_{11} & I & O \\ \frac{(a_{32}-a_{31}a_{12}/a_{11})a_{21}/a_{11}}{a_{22}-a_{21}a_{12}/a_{11}} - a_{31}/a_{11} & -\frac{a_{32}-a_{31}a_{12}/a_{11}}{a_{22}-a_{21}a_{12}/a_{11}} & I \end{bmatrix}$$

3 Inverse of $(n_1+n_2+n_3)\times(n_1+n_2+n_3)$ maxtrix

$$\begin{bmatrix} I & O & O \\ -a_{21}a_{11}^{-1} & I & O \\ -a_{31}a_{11}^{-1} & O & I \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ O & a_{22}-a_{21}a_{11}^{-1}a_{12} & a_{23}-a_{21}a_{11}^{-1}a_{13} \\ O & a_{32}-a_{31}a_{11}^{-1}a_{12} & a_{33}-a_{31}a_{11}^{-1}a_{13} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ O & a_{22}-a_{21}a_{11}^{-1}a_{12} & a_{23}-a_{21}a_{11}^{-1}a_{13} \\ O & a_{32}-a_{31}a_{11}^{-1}a_{12} & a_{33}-a_{31}a_{11}^{-1}a_{13} \end{bmatrix} \begin{bmatrix} I & -a_{11}^{-1}a_{12} & -a_{11}^{-1}a_{13} \\ O & I & O \\ O & O & I \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & O & O \\ O & a_{22}-a_{21}a_{11}^{-1}a_{12} & a_{23}-a_{21}a_{11}^{-1}a_{13} \\ O & a_{32}-a_{31}a_{11}^{-1}a_{12} & a_{33}-a_{31}a_{11}^{-1}a_{13} \end{bmatrix}, \text{ verified}$$

$$\begin{bmatrix} I & O & O \\ O & I & O \\ O & -\left(a_{32}-a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1} & I \end{bmatrix} \begin{bmatrix} a_{11} & O & O \\ O & a_{22}-a_{21}a_{11}^{-1}a_{12} & a_{23}-a_{21}a_{11}^{-1}a_{13} \\ O & a_{32}-a_{31}a_{11}^{-1}a_{12} & a_{33}-a_{31}a_{11}^{-1}a_{13} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & O & O \\ O & a_{22}-a_{21}a_{11}^{-1}a_{12} & a_{23}-a_{21}a_{11}^{-1}a_{13} \\ O & O & a_{33}-a_{31}a_{11}^{-1}a_{13}-\left(a_{32}-a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{11}^{-1}a_{13}\right) \end{bmatrix}$$

$$xx \begin{bmatrix} I & O & O \\ O & I & -\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{11}^{-1}a_{13}\right) \\ O & O & I \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & O & O \\ O & a_{22}-a_{21}a_{11}^{-1}a_{12} & O \\ O & O & a_{33}-a_{31}a_{11}^{-1}a_{13}-\left(a_{32}-a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{11}^{-1}a_{13}\right) \end{bmatrix} = B,$$

$$A^{-1} = Q_1Q_2B^{-1}P_2P_1$$

$$Q_1Q_2 = \begin{bmatrix} I & -a_{11}^{-1}a_{12} & -a_{11}^{-1}a_{13} \\ O & I & O \\ O & O & I \end{bmatrix} \begin{bmatrix} I & O & O \\ O & I & -\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{11}^{-1}a_{13}\right) \\ O & O & I \end{bmatrix}$$

$$= \begin{bmatrix} I & -a_{11}^{-1}a_{12} & a_{11}^{-1}a_{12}\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{11}^{-1}a_{13}\right)-a_{11}^{-1}a_{13} \\ O & I & -\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{11}^{-1}a_{13}\right) \\ O & O & I \end{bmatrix}$$

$$P_2P_1 = \begin{bmatrix} I & O & O \\ O & I & O \\ O & -\left(a_{32}-a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1} & I \end{bmatrix} \begin{bmatrix} I & O & O \\ -a_{21}a_{11}^{-1} & I & O \\ -a_{31}a_{11}^{-1} & O & I \end{bmatrix}$$

$$= \begin{bmatrix} I & O & O \\ -a_{21}a_{11}^{-1} & I & O \\ \left(a_{32}-a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1}-a_{31}a_{11}^{-1} & -\left(a_{32}-a_{31}a_{11}^{-1}a_{12}\right)\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1} & I \end{bmatrix}$$

verified.

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ c_{21} & 1 & 0 \\ c_{31} & c_{32} & 1 \end{bmatrix} = \begin{bmatrix} b_1+a_{12}b_2c_{21}+a_{13}b_3c_{31} & a_{12}b_2+a_{13}b_3c_{32} & a_{13}b_3 \\ b_2c_{21}+a_{23}b_3c_{31} & b_2+a_{23}b_3c_{32} & a_{23}b_3 \\ b_3c_{31} & b_3c_{32} & b_3 \end{bmatrix}$$

3.1 If a[i,i]==I

$$B = \begin{bmatrix} I & O & O \\ O & I-a_{21}a_{12} & O \\ O & O & I-a_{31}a_{13}-(a_{32}-a_{31}a_{12})(I-a_{21}a_{12})^{-1}(a_{23}-a_{21}a_{13}) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} I & -a_{12} & a_{12}(I-a_{21}a_{12})^{-1}(a_{23}-a_{21}a_{13})-a_{13} \\ O & I & -(I-a_{21}a_{12})^{-1}(a_{23}-a_{21}a_{13}) \\ O & O & I \end{bmatrix} B^{-1} * x \tag{1}$$

$$x = \begin{bmatrix} I & O & O \\ -a_{21} & I & O \\ (a_{32}-a_{31}a_{12})(I-a_{21}a_{12})^{-1}a_{21}-a_{31} & -(a_{32}-a_{31}a_{12})(I-a_{21}a_{12})^{-1} & I \end{bmatrix}$$

4 Inverse of $n_1 \times n_2$ matrix

Assume a_{11}, a_{22}, $a_{11}-a_{12}a_{22}^{-1}a_{21}$, $a_{22}-a_{21}a_{11}^{-1}a_{12}$ are invertable.

Use row elimination, get

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \left(a_{11}-a_{12}a_{22}^{-1}a_{21}\right)^{-1} & -\left(a_{11}-a_{12}a_{22}^{-1}a_{21}\right)^{-1}a_{12}a_{22}^{-1} \\ -\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1} & \left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^{-1}+a_{11}^{-1}a_{12}\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1}a_{21}a_{11}^{-1} & -a_{11}^{-1}a_{12}\left(a_{22}-a_{21}a_{11}^{-1}a_{12}\right)^{-1} \\ -a_{22}^{-1}a_{21}\left(a_{11}-a_{12}a_{22}^{-1}a_{21}\right)^{-1} & a_{22}^{-1}+a_{22}^{-1}a_{21}\left(a_{11}-a_{12}a_{22}^{-1}a_{21}\right)^{-1}a_{12}a_{22}^{-1} \end{bmatrix}$$

(verified, PDC/test_2varinv.m)

4.1 Know inverse of $(n_1+n_2)\times(n_1+n_2)$ matrix, get inverse of $n_2 \times n_2$ matrix

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$$\Rightarrow \quad b_{21}a_{12}+b_{22}a_{22} = I$$

$$\Rightarrow \quad (I-b_{21}a_{12})^{-1}b_{22}a_{22} = I$$

so

$$a_{22}^{-1} = (I-b_{21}a_{12})^{-1}b_{22}$$

(verified)

Note 1 (when $n_1 = 1$):

http://math.stackexchange.com/questions/55165/eigenvalues-for-the-rank-one-matrix-uvt

The eigen values of $b_{21}a_{12}$ are $n-1$ 0s and $a_{12}b_{21}$.

?? if $n_1 = 2$?

Note 2 (when $n_1 = 1$):

Due to Sherman-Morrison formula,

$$(I-b_{21}a_{12})^{-1} = I + \frac{b_{21}a_{12}}{1-a_{12}b_{21}}$$

as long as $a_{12}b_{21} \neq 1$.

More generally, by Sherman–Morrison–Woodbury formula,

$$(I_{n_2} - b_{21}a_{12})^{-1} = I + b_{21}\left(I_{n_1} - a_{12}b_{21}\right)^{-1} a_{12}$$

https://en.wikipedia.org/wiki/Woodbury_matrix_identity

$$a_{11}^{-1} = \left(I + b_{12}\left(I_{n_2} - a_{21}b_{12}\right)^{-1} a_{21}\right) b_{11}$$

5 Cal v_{12} from B

$$\hat{B}\hat{X} = \hat{E}, \quad \hat{X} = H\hat{E}, \quad H = \hat{B}^{-1}$$

$$S = H\Omega H^{*T}$$

$$S_{11} = H_{11}\Omega_1 H_{11} + H_{12}\Omega_2 H_{12}^* + H_{13}\Omega_3 H_{13}^*$$

$$S_{12} = H_{11}\Omega_1 H_{21}^* + H_{12}\Omega_2 H_{22}^* + H_{13}\Omega_3 H_{23}^*$$

$$F_{y\to x} \approx \sum_{j=1} \left(\mathscr{F}^{-1}\left[S_{12}, \omega\right](j)\right)^2$$

$$= \sum_{j=1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{12}(\omega)e^{\mathrm{i}\omega j}\,\mathrm{d}\omega\right)^2$$

$$= \frac{1}{2\pi}\times\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{j=1} S_{12}(\omega)S_{12}^*(w)e^{\mathrm{i}\omega j}e^{-\mathrm{i}wj}\,\mathrm{d}\omega\,\mathrm{d}w$$

$$= \frac{1}{2\pi}\times\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} S_{12}(\omega)S_{12}^*(w)\sum_{j=1} e^{\mathrm{i}(\omega-w)j}\,\mathrm{d}\omega\,\mathrm{d}w$$

$$= \frac{1}{2\pi}\times\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} S_{12}(u+w)S_{12}^*(w)\sum_{j=1} e^{\mathrm{i}uj}\,\mathrm{d}u\,\mathrm{d}w$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{12}(u+w)S_{12}^*(w)\,\mathrm{d}w\ \sum_{j=1} e^{\mathrm{i}uj}\,\mathrm{d}u$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(u)\left(\frac{\mathrm{i}}{2}\cot\frac{u}{2} + \pi\delta(u) - \frac{1}{2}\right)\mathrm{d}u$$

$$= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} U(u)\frac{\mathrm{i}}{2}\cot\frac{u}{2}\,\mathrm{d}u + \frac{1}{2}U(0) - \frac{1}{4\pi}\int_{-\pi}^{\pi} U(u)\,\mathrm{d}u\right)$$

$$= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} U(u)\frac{\mathrm{i}}{2}\cot\frac{u}{2}\,\mathrm{d}u + \frac{1}{2}U(0) - \frac{1}{2}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{12}(w)\,\mathrm{d}w\right|^2\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(u)\frac{\mathrm{i}}{2}\cot\frac{u}{2}\,\mathrm{d}u + \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{12}(w)S_{12}^*(w)\,\mathrm{d}w - \frac{1}{2}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{12}(w)\,\mathrm{d}w\right|^2$$

$$?? \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} U(u)\frac{\mathrm{i}}{2}\cot\frac{u}{2}\,\mathrm{d}u + \frac{F_{y\to x} + F_{x\to y}}{2} - \frac{1}{2}\left|v_{00}^{(xy)}\right|^2$$

$$U(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{12}(u+w)S_{12}^*(w)\,\mathrm{d}w$$

$$U(-u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{12}^{*}(u-w) S_{12}^{*}(w)\,\mathrm{d}w$$
$$= \sum_{k}\left(v_k^{(12)}\right)^2 e^{-\mathrm{i}ku}$$

Note

$$\sum_{j=1} e^{\mathrm{i}uj} \to \frac{\mathrm{i}}{2}\cot\frac{u}{2} + \pi\delta(u) - \frac{1}{2},\ \text{(week)}$$

– approx

$$b_1 = \hat{B}_{11}, \quad b_2 \approx \hat{B}_{22} - \hat{B}_{21}\hat{B}_{12},$$
$$b_3 \approx \hat{B}_{33} - \hat{B}_{31}\hat{B}_{13} - \left(\hat{B}_{32} - \hat{B}_{31}\hat{B}_{12}\right)\left(\hat{B}_{23} - \hat{B}_{21}\hat{B}_{13}\right)$$

6 Solve $F_{y\to x|z}$

Use Eq.(1),

$$B \approx \begin{bmatrix} I & O & O \\ O & I - a_{21}a_{12} & O \\ O & O & I - a_{31}a_{13} - a_{32}a_{23} \end{bmatrix}$$

$$A^{-1} \approx \begin{bmatrix} I & a_{12} & a_{13} \\ O & I & (I + a_{21}a_{12})(a_{23} - a_{21}a_{13}) \\ O & O & I \end{bmatrix} B^{-1} \begin{bmatrix} I & O & O \\ a_{21} & I & O \\ a_{31} & (a_{32} - a_{31}a_{12})(I + a_{21}a_{12}) & I \end{bmatrix}$$

$$(I + a_{21}a_{12})(a_{23} - a_{21}a_{13}) = a_{23} - a_{21}a_{13} + a_{21}a_{12}a_{23} - a_{21}a_{12}a_{21}a_{13}$$
$$(a_{32} - a_{31}a_{12})(I + a_{21}a_{12}) = a_{32} - a_{31}a_{12} + a_{32}a_{21}a_{12} - a_{31}a_{12}a_{21}a_{12}$$

$$A^{-1} \approx \begin{bmatrix} I & a_{12} & a_{13} \\ O & I & a_{23} - a_{21}a_{13} \\ O & O & I \end{bmatrix} \begin{bmatrix} I & O & O \\ O & I + a_{21}a_{12} & O \\ O & O & I + a_{31}a_{13} + a_{32}a_{23} \end{bmatrix} \begin{bmatrix} I & O & O \\ a_{21} & I & O \\ a_{31} & a_{32} - a_{31}a_{12} & I \end{bmatrix}$$
$$= \begin{bmatrix} I & a_{12} + a_{12}a_{21}a_{12} & a_{13} + a_{13}a_{31}a_{13} + a_{13}a_{32}a_{23} \\ O & I + a_{21}a_{12} & a_{23} + a_{23}a_{31}a_{13} + a_{23}a_{32}a_{23} - a_{21}a_{13} - a_{21}a_{13}a_{31}a_{13} - a_{21}a_{13}a_{32}a_{23} \\ O & O & I + a_{31}a_{13} + a_{32}a_{23} \end{bmatrix} * x$$

$$\approx \begin{bmatrix} I & a_{12} + a_{12}a_{21}a_{12} & a_{13} + a_{13}a_{31}a_{13} + a_{13}a_{32}a_{23} \\ O & I + a_{21}a_{12} & a_{23} - a_{21}a_{13} \\ O & O & I + a_{31}a_{13} + a_{32}a_{23} \end{bmatrix} \begin{bmatrix} I & O & O \\ a_{21} & I & O \\ a_{31} & a_{32} - a_{31}a_{12} & I \end{bmatrix}$$

$$\approx \begin{bmatrix} I + a_{12}a_{21} + a_{13}a_{31} & a_{12} + a_{13}a_{32} & a_{13} \\ a_{21} + a_{23}a_{31} & I + a_{21}a_{12} + a_{23}a_{32} & a_{23} - a_{21}a_{13} \\ a_{31} & a_{32} - a_{31}a_{12} & I + a_{31}a_{13} + a_{32}a_{23} \end{bmatrix} ???$$

$$A = \begin{bmatrix} & a_{12} & a_{13} \\ a_{21} & & a_{23} \\ a_{31} & a_{32} & \end{bmatrix} \begin{bmatrix} & a_{12} & a_{13} \\ a_{21} & & a_{23} \\ a_{31} & a_{32} & \end{bmatrix}, \quad A^2 = \begin{bmatrix} a_{12}a_{21} + a_{13}a_{31} & a_{13}a_{32} & a_{12}a_{23} \\ a_{23}a_{31} & a_{21}a_{12} + a_{23}a_{32} & a_{21}a_{13} \\ a_{32}a_{21} & a_{31}a_{12} & a_{31}a_{13} + a_{32}a_{23} \end{bmatrix}$$

.
.
.
.
.
.