—
to get

$$A^{-1}=\begin{bmatrix} I & a_{12} & O \\ a_{21} & I & a_{23} \\ O & a_{32} & I \end{bmatrix}^{-1}$$

$$Q_1Q_2=\begin{bmatrix} I & -a_{12} & a_{12}\left(I-a_{21}a_{12}\right)^{-1}a_{23} \\ O & I & -\left(I-a_{21}a_{12}\right)^{-1}a_{23} \\ O & O & I \end{bmatrix}$$

$$P_1P_2=\begin{bmatrix} I & O & O \\ -a_{21} & I & O \\ a_{32}\left(I-a_{21}a_{12}\right)^{-1}a_{21} & -a_{32}\left(I-a_{21}a_{12}\right)^{-1} & I \end{bmatrix}$$

$$B=\begin{bmatrix} I & O & O \\ O & I-a_{21}a_{12} & O \\ O & O & I-a_{32}\left(I-a_{21}a_{12}\right)^{-1}a_{23} \end{bmatrix}$$

$$A^{-1}=Q_1Q_2B^{-1}P_2P_1$$

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ c_{21} & 1 & 0 \\ c_{31} & c_{32} & 1 \end{bmatrix}=\begin{bmatrix} b_1+a_{12}b_2c_{21}+a_{13}b_3c_{31} & a_{12}b_2+a_{13}b_3c_{32} & a_{13}b_3 \\ b_2c_{21}+a_{23}b_3c_{31} & b_2+a_{23}b_3c_{32} & a_{23}b_3 \\ b_3c_{31} & b_3c_{32} & b_3 \end{bmatrix}$$

$$Q^{(yy)}=b_2+a_{23}b_3c_{32}=\left(I-a_{21}a_{12}\right)^{-1}+\left(I-a_{21}a_{12}\right)^{-1}a_{23}\left(I-a_{32}\left(I-a_{21}a_{12}\right)^{-1}a_{23}\right)^{-1}a_{32}\left(I-a_{21}a_{12}\right)^{-1}$$

verified

$$\left(Q^{(yy)}\right)^{-1}=I-a_{21}a_{12}-a_{23}a_{32}$$

verified
—
to get

$$A^{-1}=\begin{bmatrix} I & a_{12} & a_{13} \\ a_{21} & I & a_{23} \\ a_{31} & a_{32} & I \end{bmatrix}^{-1}$$

$$\begin{bmatrix} I & O & O \\ O & I-a_{21}a_{12} & O \\ O & O & I-a_{31}a_{13}-\left(a_{32}-a_{31}a_{12}\right)\left(I-a_{21}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{13}\right) \end{bmatrix}=B,$$

$$Q_1Q_2=\begin{bmatrix} I & -a_{12} & a_{12}\left(I-a_{21}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{13}\right)-a_{13} \\ O & I & -\left(I-a_{21}a_{12}\right)^{-1}\left(a_{23}-a_{21}a_{13}\right) \\ O & O & I \end{bmatrix}$$

$$P_2P_1=\begin{bmatrix} I & O & O \\ -a_{21} & I & O \\ \left(a_{32}-a_{31}a_{12}\right)\left(I-a_{21}a_{12}\right)^{-1}a_{21}-a_{31} & -\left(a_{32}-a_{31}a_{12}\right)\left(I-a_{21}a_{12}\right)^{-1} & I \end{bmatrix}$$

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ c_{21} & 1 & 0 \\ c_{31} & c_{32} & 1 \end{bmatrix}=\begin{bmatrix} b_1+a_{12}b_2c_{21}+a_{13}b_3c_{31} & a_{12}b_2+a_{13}b_3c_{32} & a_{13}b_3 \\ b_2c_{21}+a_{23}b_3c_{31} & b_2+a_{23}b_3c_{32} & a_{23}b_3 \\ b_3c_{31} & b_3c_{32} & b_3 \end{bmatrix}$$

$$Q^{(yy)} = b_2 + a_{23} b_3 c_{32}$$

$$Q^{(yy)} = \left(I - a_{21}a_{12}\right)^{-1} + \left(I - a_{21}a_{12}\right)^{-1}\left(a_{23} - a_{21}a_{13}\right)\Big(I - a_{31}a_{13} - \left(a_{32} - a_{31}a_{12}\right)\left(I - a_{21}a_{12}\right)^{-1}\left(a_{23} - a_{21}a_{13}\right)\Big)\left(a_{32} - a\right.$$

$$Q^{(yy)} = \Big(I - a_{21}a_{12} - \left(a_{23} - a_{21}a_{13}\right)\left(I - a_{31}a_{13}\right)^{-1}\left(a_{32} - a_{31}a_{12}\right)\Big)^{-1}$$

$$\left(Q^{(yy)}\right)^{-1} = I - a_{21}a_{12} - \left(a_{23} - a_{21}a_{13}\right)\left(I - a_{31}a_{13}\right)^{-1}\left(a_{32} - a_{31}a_{12}\right)$$

$$\left(Q^{(yy)}\right)^{-1} = I - a_{23}a_{32} - \left(a_{21} - a_{23}a_{31}\right)\left(I - a_{13}a_{31}\right)^{-1}\left(a_{12} - a_{13}a_{32}\right)$$

verified.

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$$c = \begin{bmatrix} 0 & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} I & S^{(xz)} \\ S^{(zx)} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vec{s}^{(x|z)T} \end{bmatrix}$$

$$c = \vec{s}^{(x|z)} \left(I - S^{(zx)} S^{(xz)}\right)^{-1} \vec{s}^{(x|z)T}$$

$$c = \vec{s}^{(x|z)} \left(I + S^{(zx)} \left(I - S^{(xz)} S^{(zx)}\right)^{-1} S^{(xz)}\right) \vec{s}^{(x|z)T} = \vec{s}^{(x|z)} \vec{s}^{(x|z)T} + \vec{s}^{(x|z)} S^{(zx)} \left(I - S^{(xz)} S^{(zx)}\right)^{-1} S^{(xz)} \vec{s}^{(x|z)T}$$

—-

$$d = c + (d - c) = \vec{s}^{(x|z)} \left(I - S^{(zx)} S^{(xz)}\right)^{-1} \vec{s}^{(x|z)T} + \vec{a}^{(12)} \left(Q^{(yy)}\right)^{-1} \vec{a}^{(12)T}$$

—-

$$F_{y \to x|z} = \ln \frac{1 - c}{1 - d} = -\ln \frac{1 - d}{1 - c} = -\ln \left(1 - \frac{d - c}{1 - c}\right) = -\ln \left(1 - \frac{\vec{a}^{(12)} \left(Q^{(yy)}\right)^{-1} \vec{a}^{(12)T}}{1 - \vec{s}^{(x|z)} \left(I - S^{(zx)} S^{(xz)}\right)^{-1} \vec{s}^{(x|z)T}}\right)$$

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Use $\vec{a}^{(12)} \left(Q^{(yy)}\right)^{-1} \vec{a}^{(12)T}$ to calculate approximate GC could get $O(m^3 p^2)$ time cost.

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frequency domain: relation of pairwise GC and GC

$$A(w) = \begin{bmatrix} A_{xx}(w) & A_{xy}(w) & A_{xz}(w) \\ A_{yx}(w) & A_{yy}(w) & A_{yz}(w) \\ A_{zx}(w) & A_{zy}(w) & A_{zz}(w) \end{bmatrix}$$

$$S(w) = \begin{bmatrix} S_{xx}(w) & S_{xy}(w) & S_{xz}(w) \\ S_{yx}(w) & S_{yy}(w) & S_{yz}(w) \\ S_{zx}(w) & S_{zy}(w) & S_{zz}(w) \end{bmatrix} = A^{-1}(w) \Sigma A^{-1H}(w)$$

$$Q(w) = S^{-1}(w)$$

$$B(w) = \begin{bmatrix} B_{xx}(w) & B_{xz}(w) \\ B_{yx}(w) & B_{yz}(w) \end{bmatrix}$$

$$S_{[x,y]}(w) = \begin{bmatrix} S_{xx}(w) & S_{xy}(w) \\ S_{yx}(w) & S_{yy}(w) \end{bmatrix} = B^{-1}(w) \Sigma_{[xy]} B^{-1H}(w)$$

$$[f(w)]_+ = \mathscr{F}_{\mathrm{DFT}} \left[\mathscr{F}_{\mathrm{DFT}}^{-1} \left[f(w), t\right](t) \cdot u(t), w\right]$$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then

$$B_{xy} \approx \left[A_{xy} - A_{xz} Q_{zz}^{-1} Q_{zy}\right]_+$$

Under assumption

$$\Sigma = \mathrm{diag}\left(\left[\begin{array}{ccc} \Sigma_{xx} & \Sigma_{yy} & \Sigma_{zz} \end{array}\right]\right)$$

We have

$$Q_{zz} = A_{xz}^{*T} \Sigma_{xx}^{-1} A_{xz} + A_{yz}^{*T} \Sigma_{yy}^{-1} A_{yz} + A_{zz}^{*T} \Sigma_{zz}^{-1} A_{zz}$$

$$Q_{zy} = A_{xz}^{*T} \Sigma_{xx}^{-1} A_{xy} + A_{yz}^{*T} \Sigma_{yy}^{-1} A_{yy} + A_{zz}^{*T} \Sigma_{zz}^{-1} A_{zy}$$

Hence

$$A_{xz}/Q_{zz} * Q_{zy} \approx A_{xz} \left(A_{zz}^{-1} \Sigma_{zz} \left(A_{zz}^{*T}\right)^{-1}\right) \left(A_{yz}^{*T} \Sigma_{yy}^{-1} A_{yy} + A_{zz}^{*T} \Sigma_{zz}^{-1} A_{zy}\right) \approx A_{xz} \left(A_{zy} + A_{yz}^{*}\right)$$

$$B_{xy} \approx \left[A_{xy} - A_{xz}\left(A_{zy} + A_{yz}^{*}\right)\right]_+$$

...

$$gc = A_{xz} A_{xz}^{*} \left(A_{zy} A_{zy}^{*} + A_{zy} A_{yz} + A_{yz}^{*} A_{zy}^{*} + A_{yz}^{*} A_{yz}\right)$$

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