$$\mathrm{var}(\epsilon_t^{(1)}) = \mathrm{var}(x_t^{(1)}) - \left[\begin{array}{ccc} 0 & \vec{v}^{(x|y)} & \vec{v}^{(x|z)} \end{array}\right] \left[\begin{array}{ccc} v^{(x)}I & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & v^{(y)}I & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & v^{(z)}I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{v}^{(x|y)T} \\ \vec{v}^{(x|z)T} \end{array}\right]. \tag{1}$$

$$\mathrm{var}(\epsilon_t^{(1)}) = \mathrm{var}(x_t^{(1)}) \left(I - \left[\begin{array}{ccc} 0 & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{array}\right] \left[\begin{array}{ccc} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{array}\right] \right). \tag{2}$$

$$\mathrm{var}(\epsilon_t^{A(1)}) = \mathrm{var}(x_t^{(1)}) - \left[\begin{array}{cc} 0 & \vec{v}^{(x|z)} \end{array}\right] \left[\begin{array}{cc} v^{(x)}I & R^{(xz)} \\ R^{(zx)} & v^{(z)}I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{v}^{(x|z)T} \end{array}\right]. \tag{3}$$

$$\mathrm{var}(\epsilon_t^{A(1)}) = \mathrm{var}(x_t^{(1)}) \left(I - \left[\begin{array}{cc} 0 & \vec{s}^{(x|z)} \end{array}\right] \left[\begin{array}{cc} I & S^{(xz)} \\ S^{(zx)} & I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{s}^{(x|z)T} \end{array}\right] \right) \tag{4}$$

$$F_{y\to x|z} = \ln\left(\frac{1 - \frac{1}{\mathrm{var}(x_t^{(1)})} \left[\begin{array}{cc} 0 & \vec{v}^{(x|z)} \end{array}\right] \left[\begin{array}{cc} v^{(x)}I & R^{(xz)} \\ R^{(zx)} & v^{(z)}I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{v}^{(x|z)T} \end{array}\right]}{1 - \frac{1}{\mathrm{var}(x_t^{(1)})} \left[\begin{array}{ccc} 0 & \vec{v}^{(x|y)} & \vec{v}^{(x|z)} \end{array}\right] \left[\begin{array}{ccc} v^{(x)}I & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & v^{(y)}I & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & v^{(z)}I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{v}^{(x|y)T} \\ \vec{v}^{(x|z)T} \end{array}\right]} \right). \tag{5}$$

$$F_{y\to x|z} = \ln\left(\frac{1 - \left[\begin{array}{cc} 0 & \vec{s}^{(x|z)} \end{array}\right] \left[\begin{array}{cc} I & S^{(xz)} \\ S^{(zx)} & I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{s}^{(x|z)T} \end{array}\right]}{1 - \left[\begin{array}{ccc} 0 & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{array}\right] \left[\begin{array}{ccc} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{array}\right]} \right). \tag{6}$$

---

note that

$$\left[\begin{array}{cc} v^{(x)}I & R^{(xz)} \\ R^{(zx)} & v^{(z)}I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{v}^{(x|z)T} \end{array}\right] = \left[\begin{array}{c} \vec{d}^{(11)} \\ \vec{d}^{(12)} \end{array}\right]$$

$$\left[\begin{array}{ccc} v^{(x)}I & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & v^{(y)}I & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & v^{(z)}I \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \vec{v}^{(x|y)T} \\ \vec{v}^{(x|z)T} \end{array}\right] = \left[\begin{array}{c} \vec{a}^{(11)} \\ \vec{a}^{(12)} \\ \vec{a}^{(13)} \end{array}\right]$$

$$F_{y\to x|z} = \ln\left(\frac{1 - \frac{1}{\mathrm{var}(x_t^{(1)})} \left[\begin{array}{cc} \vec{d}^{(11)} & \vec{d}^{(12)} \end{array}\right] \left[\begin{array}{cc} v^{(x)}I & R^{(xz)} \\ R^{(zx)} & v^{(z)}I \end{array}\right] \left[\begin{array}{c} \vec{d}^{(11)} \\ \vec{d}^{(12)} \end{array}\right]}{1 - \frac{1}{\mathrm{var}(x_t^{(1)})} \left[\begin{array}{ccc} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{array}\right] \left[\begin{array}{ccc} v^{(x)}I & R^{(xy)} & R^{(xz)} \\ R^{(yx)} & v^{(y)}I & R^{(yz)} \\ R^{(zx)} & R^{(zy)} & v^{(z)}I \end{array}\right] \left[\begin{array}{c} \vec{a}^{(11)} \\ \vec{a}^{(12)} \\ \vec{a}^{(13)} \end{array}\right]} \right). \tag{7}$$

For standard whitened data

$$F_{y\to x|z} = \ln\left(\frac{1 - \left[\begin{array}{cc} \vec{d}^{(11)} & \vec{d}^{(12)} \end{array}\right] \left[\begin{array}{cc} I & S^{(xz)} \\ S^{(zx)} & I \end{array}\right] \left[\begin{array}{c} \vec{d}^{(11)} \\ \vec{d}^{(12)} \end{array}\right]}{1 - \left[\begin{array}{ccc} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{array}\right] \left[\begin{array}{ccc} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{array}\right] \left[\begin{array}{c} \vec{a}^{(11)} \\ \vec{a}^{(12)} \\ \vec{a}^{(13)} \end{array}\right]} \right).$$

$$(D-A)^{-1} = D^{-1}(I - AD^{-1})^{-1}$$

==

$$c = \begin{bmatrix} 0 & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} I & S^{(xz)} \\ S^{(zx)} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vec{s}^{(x|z)T} \end{bmatrix}$$

$$c = \begin{bmatrix} 0 & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} I & 0 & S^{(xz)} \\ 0 & I & 0 \\ S^{(zx)} & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{bmatrix} - \vec{s}^{(x|y)}\vec{s}^{(x|y)T}$$

$$d = \begin{bmatrix} 0 & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{bmatrix}$$

$$d - c = \vec{s}^{(x|y)}\vec{s}^{(x|y)T} + \begin{bmatrix} 0 & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix} \left(\begin{bmatrix} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{bmatrix}^{-1} - \begin{bmatrix} I & 0 & S^{(xz)} \\ 0 & I & 0 \\ S^{(zx)} & 0 & I \end{bmatrix}^{-1} \right) \begin{bmatrix} 0 \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{bmatrix}$$

$$E = \begin{bmatrix} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{bmatrix}, \; F = \begin{bmatrix} I & 0 & S^{(xz)} \\ 0 & I & 0 \\ S^{(zx)} & 0 & I \end{bmatrix}, \; G = \begin{bmatrix} 0 & S^{(xy)} & 0 \\ S^{(yx)} & 0 & S^{(yz)} \\ 0 & S^{(zy)} & 0 \end{bmatrix}$$

$$(F+G)^{-1} - F^{-1} = (F(I + F^{-1}G))^{-1} - F^{-1} = ((I + F^{-1}G)^{-1} - I)F^{-1}, \text{not good}$$

$$E^{-1} - F^{-1} = (I - F^{-1}E)E^{-1}$$

$$E^{-1} - F^{-1} = E^{-1}FF^{-1} - E^{-1}EF^{-1} = E^{-1}(F-E)F^{-1}$$

———-

For standard whitened data

$$F_{y\to x|z} \approx \begin{bmatrix} \vec{a}^{(11)} & \vec{a}^{(12)} & \vec{a}^{(13)} \end{bmatrix} \begin{bmatrix} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{bmatrix} \begin{bmatrix} \vec{a}^{(11)} \\ \vec{a}^{(12)} \\ \vec{a}^{(13)} \end{bmatrix} - \begin{bmatrix} \vec{d}^{(11)} & \vec{d}^{(12)} \end{bmatrix} \begin{bmatrix} I & S^{(xz)} \\ S^{(zx)} & I \end{bmatrix} \begin{bmatrix} \vec{d}^{(11)} \\ \vec{d}^{(12)} \end{bmatrix}.$$

......

Way 2:

to verify

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -C' & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & C \\ 0 & I & 0 \\ C' & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & -C \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I - C'C \end{bmatrix}$$

$\left\| S^{(zx)} S^{(xz)} \right\|$ can be small.

......

Way 3:

$$\begin{bmatrix} I & & \\ & & I \\ & I & \end{bmatrix} \begin{bmatrix} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{bmatrix} \begin{bmatrix} I & & \\ & & I \\ & I & \end{bmatrix} = \begin{bmatrix} I & S^{(xz)} & S^{(xy)} \\ S^{(zx)} & I & S^{(zy)} \\ S^{(yx)} & S^{(yz)} & I \end{bmatrix}$$

$$\begin{bmatrix} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{bmatrix}^{-1} = \begin{bmatrix} Q^{(xx)} & Q^{(xy)} & Q^{(xz)} \\ Q^{(yx)} & Q^{(yy)} & Q^{(yz)} \\ Q^{(zx)} & Q^{(zy)} & Q^{(zz)} \end{bmatrix}$$

$$\begin{bmatrix} I & S^{(xz)} \\ S^{(zx)} & I \end{bmatrix}^{-1} = \left(I + \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix} (I - \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix})^{-1} \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \right) \begin{bmatrix} Q^{(xx)} & Q^{(xz)} \\ Q^{(zx)} & Q^{(zz)} \end{bmatrix}$$

verified.

$$F_{y \to x|z} \approx d - c = \vec{s}^{(x|y)} \vec{s}^{(x|y)T} + \begin{bmatrix} 0 & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix} \left(\begin{bmatrix} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{bmatrix}^{-1} - \begin{bmatrix} I & 0 & S^{(xz)} \\ 0 & I & 0 \\ S^{(zx)} & 0 & I \end{bmatrix}^{-1} \right) \begin{bmatrix} 0 \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{bmatrix}$$

$$\begin{bmatrix} I & 0 & S^{(xz)} \\ 0 & I & 0 \\ S^{(zx)} & 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} P^{(xx)} & 0 & P^{(xz)} \\ 0 & I & 0 \\ P^{(zx)} & 0 & P^{(zz)} \end{bmatrix}$$

$$\left(I + \begin{bmatrix} Q^{(xy)} \\ O \\ Q^{(zy)} \end{bmatrix} (I - \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix})^{-1} \begin{bmatrix} S^{(yx)} & O & S^{(yz)} \end{bmatrix} \right) \begin{bmatrix} Q^{(xx)} & & Q^{(xz)} \\ & I & \\ Q^{(zx)} & & Q^{(zz)} \end{bmatrix}$$

verified

$$D_s = \begin{bmatrix} I & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & I & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & I \end{bmatrix}^{-1} - \begin{bmatrix} I & 0 & S^{(xz)} \\ 0 & I & 0 \\ S^{(zx)} & 0 & I \end{bmatrix}^{-1}$$

$$D_s = - \begin{bmatrix} Q^{(xy)} \\ O \\ Q^{(zy)} \end{bmatrix} (I - \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix})^{-1} \begin{bmatrix} S^{(yx)} & O & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xx)} & & Q^{(xz)} \\ & I & \\ Q^{(zx)} & & Q^{(zz)} \end{bmatrix} + \begin{bmatrix} & Q^{(xy)} & \\ Q^{(yx)} & Q^{(yy)} - I & Q^{(y} \\ & Q^{(zy)} & \end{bmatrix}$$

$$d - c - \vec{s}^{(x|y)} \vec{s}^{(x|y)T} = - \vec{s}^{(x|z)} Q^{(zy)} (I - \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix})^{-1} \left(\begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xz)} \\ Q^{(zz)} \end{bmatrix} \right) \vec{s}^{(x|z)T} + ..$$

$$d - c - \vec{s}^{(x|y)} \vec{s}^{(x|y)T} = \vec{s}^{(x|z)} Q^{(zy)} (Q^{(yy)})^{-1} Q^{(yz)} \vec{s}^{(x|z)T} + 2 \vec{s}^{(x|y)} Q^{(yz)} \vec{s}^{(x|z)T} + \vec{s}^{(x|y)} (Q^{(yy)} - I) \vec{s}^{(x|y)T}$$

$$d - c = \vec{s}^{(x|z)} Q^{(zy)} (Q^{(yy)})^{-1} Q^{(yz)} \vec{s}^{(x|z)T} + 2 \vec{s}^{(x|y)} Q^{(yz)} \vec{s}^{(x|z)T} + \vec{s}^{(x|y)} Q^{(yy)} \vec{s}^{(x|y)T}$$

$$d - c = \begin{bmatrix} \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} Q^{(yy)} & Q^{(yz)} \\ Q^{(zy)} & Q^{(zy)} (Q^{(yy)})^{-1} Q^{(yz)} \end{bmatrix} \begin{bmatrix} \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{bmatrix}$$

$$d - c = \left(\vec{s}^{(x|z)} Q^{(zy)} (Q^{(yy)})^{-1} + \vec{s}^{(x|y)} \right) Q^{(yz)} \vec{s}^{(x|z)T} + \vec{s}^{(x|y)} \left(Q^{(yz)} \vec{s}^{(x|z)T} + Q^{(yy)} \vec{s}^{(x|y)T} \right)$$

$$d - c = \left(\vec{s}^{(x|z)} Q^{(zy)} + \vec{s}^{(x|y)} Q^{(yy)} \right) (Q^{(yy)})^{-1} \left(Q^{(yy)} \vec{s}^{(x|y)T} + Q^{(yz)} \vec{s}^{(x|z)T} \right)$$

$$d - c = \vec{a}^{(12)} (Q^{(yy)})^{-1} \vec{a}^{(12)T}$$

For any invertable matrix S, also assume $\begin{bmatrix} S^{(xx)} & S^{(xz)} \\ S^{(zx)} & S^{(zz)} \end{bmatrix}$ is invertable, then

$$S = \begin{bmatrix} S^{(xx)} & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & S^{(yy)} & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & S^{(zz)} \end{bmatrix}^{-1} = \begin{bmatrix} Q^{(xx)} & Q^{(xy)} & Q^{(xz)} \\ Q^{(yx)} & Q^{(yy)} & Q^{(yz)} \\ Q^{(zx)} & Q^{(zy)} & Q^{(zz)} \end{bmatrix}$$

$$\begin{bmatrix} S^{(xx)} & S^{(xz)} \\ S^{(zx)} & S^{(zz)} \end{bmatrix}^{-1} = \left(I + \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix} (I - \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix})^{-1} \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix}\right) \begin{bmatrix} Q^{(xx)} & Q^{(xz)} \\ Q^{(zx)} & Q^{(zz)} \end{bmatrix}$$

Define

$$D_s = \begin{bmatrix} S^{(xx)} & S^{(xy)} & S^{(xz)} \\ S^{(yx)} & S^{(yy)} & S^{(yz)} \\ S^{(zx)} & S^{(zy)} & S^{(zz)} \end{bmatrix}^{-1} - \begin{bmatrix} S^{(xx)} & 0 & S^{(xz)} \\ 0 & I & 0 \\ S^{(zx)} & 0 & S^{(zz)} \end{bmatrix}^{-1}$$

$$D_s = -\begin{bmatrix} Q^{(xy)} \\ O \\ Q^{(zy)} \end{bmatrix} (I - \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix})^{-1} \begin{bmatrix} S^{(yx)} & O & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xx)} & & Q^{(xz)} \\ & I & \\ Q^{(zx)} & & Q^{(zz)} \end{bmatrix} + \begin{bmatrix} & Q^{(xy)} & \\ Q^{(yx)} & Q^{(yy)} - I & Q^{(y} \\ & Q^{(zy)} & \end{bmatrix}$$

$$D_s = -\begin{bmatrix} Q^{(xy)} \\ O \\ Q^{(zy)} \end{bmatrix} \left(Q^{(yy)}\right)^{-1} \begin{bmatrix} Q^{(yx)} & O & Q^{(yz)} \end{bmatrix} + \begin{bmatrix} & Q^{(xy)} & \\ Q^{(yx)} & Q^{(yy)} - I & Q^{(yz)} \\ & Q^{(zy)} & \end{bmatrix}$$

verified.

For any row vector $\begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix}$

$$\begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix} D_s \begin{bmatrix} \vec{s}^{(x|x)T} \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{bmatrix}$$

$$= -\begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix} (I - \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix})^{-1} \begin{bmatrix} S^{(yx)} & S^{(yz)} \end{bmatrix} \begin{bmatrix} Q^{(xx)} & Q^{(xz)} \\ Q^{(zx)} & Q^{(zz)} \end{bmatrix} \begin{bmatrix} \vec{s}^{(x|x)T} \\ \vec{s}^{(x|z)T} \end{bmatrix} + ext.$$

$$= \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix} \left(S^{(yy)} Q^{(yy)}\right)^{-1} S^{(yy)} \begin{bmatrix} Q^{(yx)} & Q^{(yz)} \end{bmatrix} \begin{bmatrix} \vec{s}^{(x|x)T} \\ \vec{s}^{(x|z)T} \end{bmatrix} + ext.$$

$$ext. = \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} & Q^{(xy)} & \\ Q^{(yx)} & Q^{(yy)} - I & Q^{(yz)} \\ & Q^{(zy)} & \end{bmatrix} \begin{bmatrix} \vec{s}^{(x|x)T} \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{bmatrix}$$

For

$$d - c = \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} & \vec{s}^{(x|z)} \end{bmatrix} D_s \begin{bmatrix} \vec{s}^{(x|x)T} \\ \vec{s}^{(x|y)T} \\ \vec{s}^{(x|z)T} \end{bmatrix} + \vec{s}^{(x|y)} \vec{s}^{(x|y)T}$$

+assume $S = S^T$

$$= \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|z)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(zy)} \end{bmatrix} \left(Q^{(yy)}\right)^{-1} \begin{bmatrix} Q^{(yx)} & Q^{(yz)} \end{bmatrix} \begin{bmatrix} \vec{s}^{(x|x)T} \\ \vec{s}^{(x|z)T} \end{bmatrix} + 2\vec{s}^{(x|x)} Q^{(xy)} \vec{s}^{(x|y)T} + 2\vec{s}^{(x|y)} Q^{(yz)} \vec{s}^{(x|z)T} + \vec{s}^{(x|y)} Q^{(yy)} \vec{s}^{(}$$

verified

$$= \left(\vec{s}^{(x|x)} Q^{(xy)} + \vec{s}^{(x|y)} Q^{(yy)} + \vec{s}^{(x|z)} Q^{(zy)}\right) \left(Q^{(yy)}\right)^{-1} \left(Q^{(yx)} \vec{s}^{(x|x)T} + Q^{(yy)} \vec{s}^{(x|y)T} + Q^{(yz)} \vec{s}^{(x|z)T}\right)$$

verified
Hence ()

$$d - c = \vec{a}^{(12)} \left(Q^{(yy)}\right)^{-1} \left(\vec{a}^{(12)}\right)^T$$

verified
??

$$\left(Q^{(yy)}\right)^{-1} = a_{22} - a_{21} a_{11}^{-1} a_{12} - \left(a_{23} - a_{21} a_{11}^{-1} a_{13}\right) \left(a_{33} - a_{31} a_{11}^{-1} a_{13}\right)^{-1} \left(a_{32} - a_{31} a_{11}^{-1} a_{12}\right)$$

$$\left(Q^{(yy)}\right)^{-1} = S^{(yy)} - S^{(yx)} \left(S^{(xx)}\right)^{-1} S^{(xy)} - \left(S^{(yz)} - S^{(yx)} \left(S^{(xx)}\right)^{-1} S^{(xz)}\right) \left(S^{(zz)} - S^{(zx)} \left(S^{(xx)}\right)^{-1} S^{(xz)}\right)^{-1} \Big(S^{(zy)} - S^{(z\cdots}$$

verified (also swap x and z)
For $S^{(xx)} = I$

$$\left(Q^{(yy)}\right)^{-1} = S^{(yy)} - S^{(yx)} S^{(xy)} - \left(S^{(yz)} - S^{(yx)} S^{(xz)}\right) \left(S^{(zz)} - S^{(zx)} S^{(xz)}\right)^{-1} \left(S^{(zy)} - S^{(zx)} S^{(xy)}\right)$$

—-
Note that:

$$F_{y \to x|z} = -\ln \left(1 - \frac{\frac{1}{\mathrm{var}(x_t)} (d - c)}{1 - \frac{1}{\mathrm{var}(x_t)} c}\right)$$

hence

$$F_{y \to x|z} \approx \frac{1}{\mathrm{var}(x_t)} \vec{a}^{(12)} \left(Q^{(yy)}\right)^{-1} \left(\vec{a}^{(12)}\right)^T$$

Relation to two-var GC

$$\begin{bmatrix} S^{(xx)} & S^{(xy)} \\ S^{(yx)} & S^{(yy)} \end{bmatrix}^{-1} = \left(I + \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix} (I - \begin{bmatrix} S^{(zx)} & S^{(zy)} \end{bmatrix} \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix})^{-1} \begin{bmatrix} S^{(zx)} & S^{(zy)} \end{bmatrix}\right) \begin{bmatrix} Q^{(xx)} & Q^{(xy)} \\ Q^{(yx)} & Q^{(yy)} \end{bmatrix}$$

$$= \left(I + \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix} \left(S^{(zz)}Q^{(zz)}\right)^{-1} \begin{bmatrix} S^{(zx)} & S^{(zy)} \end{bmatrix}\right) \begin{bmatrix} Q^{(xx)} & Q^{(xy)} \\ Q^{(yx)} & Q^{(yy)} \end{bmatrix}$$

$$= \begin{bmatrix} Q^{(xx)} & Q^{(xy)} \\ Q^{(yx)} & Q^{(yy)} \end{bmatrix} - \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix} \left(Q^{(zz)}\right)^{-1} \begin{bmatrix} Q^{(zx)} & Q^{(zy)} \end{bmatrix}$$

$$\begin{bmatrix} b^{(11)} & b^{(12)} \end{bmatrix} = \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} \end{bmatrix} \begin{bmatrix} S^{(xx)} & S^{(xy)} \\ S^{(yx)} & S^{(yy)} \end{bmatrix}^{-1}$$

$$b^{(12)} = \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} \end{bmatrix} \left(I + \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix} (I - \begin{bmatrix} S^{(zx)} & S^{(zy)} \end{bmatrix} \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix})^{-1} \begin{bmatrix} S^{(zx)} & S^{(zy)} \end{bmatrix}\right) \begin{bmatrix} Q^{(xy)} \\ Q^{(yy)} \end{bmatrix}$$

$$b^{(12)} = \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} \end{bmatrix} \left(I + \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix} \left(S^{(zz)}Q^{(zz)}\right)^{-1} \begin{bmatrix} S^{(zx)} & S^{(zy)} \end{bmatrix}\right) \begin{bmatrix} Q^{(xy)} \\ Q^{(yy)} \end{bmatrix}$$

$$b^{(12)} = \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} \end{bmatrix} \begin{bmatrix} Q^{(xy)} \\ Q^{(yy)} \end{bmatrix} - \begin{bmatrix} \vec{s}^{(x|x)} & \vec{s}^{(x|y)} \end{bmatrix} \begin{bmatrix} Q^{(xz)} \\ Q^{(yz)} \end{bmatrix} \left(Q^{(zz)}\right)^{-1} Q^{(zy)}$$

$$b^{(12)} = a^{(12)} - a^{(13)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} \tag{8}$$

(inference: $a^{(12)} = 0,\ a^{(13)} = 0 \ \Rightarrow\ b^{(12)} = 0$, also $b^{(13)} = 0$)

$$d_{xy} - c_{xy} = b^{(12)} \left(Q^{(yy)} - Q^{(yz)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)}\right)^{-1} \left(b^{(12)}\right)^T$$

verified

—

rude approximation (very small corralation)

$$F_{y\to x} \approx b^{(12)} \left(b^{(12)}\right)^T \approx \left(a^{(12)} - a^{(13)}Q^{(zy)}\right)\left(a^{(12)T} - Q^{(yz)}a^{(13)T}\right) \approx F_{y\to x|z} + a^{(13)}Q^{(zy)}Q^{(yz)}a^{(13)T} - 2a^{(12)}Q^{(yz)}a^{(13)T}$$

??? get a formula that contain $a^{(23)}$ which like Eq.(8).

$$b^{(12)} = a^{(12)} - a^{(13)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)}$$

$$b^{(13)} = a^{(13)} - a^{(12)} \left(Q^{(yy)}\right)^{-1} Q^{(yz)}$$

$$b^{(23)} = a^{(23)} - a^{(21)} \left(Q^{(xx)}\right)^{-1} Q^{(xz)}$$

$$b^{(32)} = a^{(32)} - a^{(31)} \left(Q^{(xx)}\right)^{-1} Q^{(xy)}$$

->

$$b^{(12)} = a^{(12)} - \left(b^{(13)} + a^{(12)} \left(Q^{(yy)}\right)^{-1} Q^{(yz)}\right) \left(Q^{(zz)}\right)^{-1} Q^{(zy)} = a^{(12)} \left(I - \left(Q^{(yy)}\right)^{-1} Q^{(yz)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)}\right) - b^{(13)} \left(Q^{(zz)}\right.$$

$$b^{(12)} + b^{(13)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)} = a^{(12)} \left(I - \left(Q^{(yy)}\right)^{-1} Q^{(yz)} \left(Q^{(zz)}\right)^{-1} Q^{(zy)}\right)$$

(inference: $b^{(12)} = 0,\ b^{(13)} = 0 \ \Rightarrow\ a^{(12)} = 0$, also $a^{(13)} = 0$)

??? get starting point formula

Possible fast algorithm for GC

.

.

.

.

.