

Exploring mathematics behind the Cody Dock bridge

Extended essay

Research Question: How can catenary curves be used in construction of non-circular, “rolling” bridges?

Word count: 2172

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1 Introduction

Bridges are the “structures that span horizontally between supports, whose function is to carry vertical loads”(Billington et al., 2024). They allow us people to travel more easily and by this, they connect us. They have already existed for thousands of years, yet there is still so little of real innovation in their design.

That is what the Cody Dock bridge is for me: a change. For once, the simplicity of a drawbridge was overruled by (quite literally) a simple revolution. The artists and architects who designed it wanted for it to be different from the thousands or even millions of other bridges. To achieve this, they used really interesting mathematics, which I will be investigating in this essay. Those innovators created one of the most unique drawbridges (if it even can be called a “draw”bridge) in the world.



Figure 1: Bridge photo¹

However, the sheer innovation of this “structure” was not the only reason I decided to look more into it and broaden my mathematical knowledge by investigating it. I am also very interested in urban planning and architecture, which can be combined in this work with my passion for Mathematics.

I also believe, that this bridge is a sort of mathematical phenomena existing in the real world. It is using the generally forgotten field of catenary curves to create a functional art piece, perhaps to inspire others to explore the mathematics behind it.

The Cody Dock bridge was constructed in 2023 in East London, replacing a much older and simpler drawbridge. It was designed by architects Thomas Randall-Page and Tim Lucas. The “revolution” in this bridge is how it rotates (or “rolls”) to let boats pass

¹Photo source: <https://newatlas.com/architecture/cody-dock-rolling-bridge/>

under it. The whole bridge is like a wheel rolling smoothly on a perfectly built for it road. What is also interesting, that because of how little work is required to move the structure (and how rarely it needs to be turned), it is operated by a hand crank, requiring approximately 20 minutes of continuous cranking to completely roll over.(Williams, 2023) On Figure 2 below, a man in the back can be seen operating the hand crank.



Figure 2: Rolling the bridge using the hand crank²

The Cody Dock bridge was constructed in 2023 in East London, replacing a much older and simpler drawbridge. It was designed by architects Thomas Randall-Page and Tim Lucas. The “revolution” in this bridge is how it rotates (or “rolls”) to let boats pass under it. The whole bridge is like a wheel rolling smoothly on a perfectly built for it road. What is also interesting, that because of how little work is required to move the structure (and how rarely it needs to be turned), it is operated by a hand crank, requiring approximately 20 minutes of continuous cranking to completely roll over.(Williams, 2023) On Figure 2 below, a man in the back can be seen operating the hand crank.

2 Defining “smooth” rolling

In order to even start the discussion, “smooth rolling” has to be defined. If we e.g. take a circle shape and roll it on a straight surface, then it will roll “smoothly” - its center of mass will move only horizontally and not vertically. However, when a square is rolled on the same surface, its center of mass will move vertically as well and hence its movement will not be “smooth”.

²Photo source: <https://newatlas.com/architecture/cody-dock-rolling-bridge/>

Even so, why is it important to roll “smoothly”? The answer is connected to physics, more specifically to the work needed to be done to roll the shape. If the figure is rolled “smoothly”, then there is no need to do work in moving the center of mass up and down. Therefore, in the ideal world, the shape will roll without any work needed to be done (in reality, there is always some friction, so some work will always be needed to be done).

Another important property for this “rolling” is that when any point of the shape is in contact with the road surface, its velocity relative to it has to be zero. Otherwise, if the point was moving while touching the road, it would be sliding and not rolling.

However, there must exist a surface (other than straight road) on which a square can roll “smoothly”. It will be described parametrically as follows:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad (1)$$

Similarly, the square shape will also be parametrized. However, it will be parametrized in polar coordinates, as this representation will be easier to work with. The parametrization is as follows:

$$\begin{cases} r = r(t) \\ \theta = \theta(t) \end{cases} \quad (2)$$

To simplify the calculations, the line of movement of the center of mass of the square will be chosen to be the x-axis. Hence, the road will have to be under it, so $y(t) < 0$. Likewise, the center of mass of the square point (axle) will be chosen to be at the origin of its local coordinate system. The parameter t can be thought of as the time, at which there is a point on the square and on the road, which are touching. Therefore, the point (x, y) on the road will be the contact point on the road curve and similarly the point (r, θ) will be the contact point on the square curve.

Now, the relations between those two curves need to be found. If it is assumed that the square is on the road, then it must be touching it at some point. Moreover, the point of contact is not a random point on the curve, but the one directly beneath the center mass point (so the origin point). Therefore, it has to be the $r(t)$ point on the square curve. Hence, the distance from axle to the road must be equal to the road’s depth ($-y(t)$):

$$r(t) = -y(t) \quad (3)$$

Another relation can be found by looking at the contact point’s velocity. It was earlier

defined, that during contact, the velocity of a point relative to the road has to be zero. However, when it is looked at from the perspective of the axle point, all points on the square are constantly moving (while rolling). Therefore, if the point is stationary relative to the ground, but moving relative to the center point, then those speeds have to be equal to each other. The speed relative to road is simply $|\frac{dx}{dt}|$, while the speed relative to the axle can be calculated from its angular speed:

$$\omega = \frac{d\theta}{dt}$$

$$v = r \cdot \omega$$

$$v_{\text{axle}} = r \frac{d\theta}{dt}$$

Thus (the speeds, not the velocities are equal, hence the absolute value):

$$\begin{aligned} \left| \frac{dx}{dt} \right| &= \left| r \frac{d\theta}{dt} \right| \\ \frac{dx}{dt} &= \pm r \frac{d\theta}{dt} \end{aligned} \tag{4}$$

However, it can be assumed that the square will roll to the right (in the positive x direction), so the contact point will have to rotate counter-clockwise. Therefore, both $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$ have to be positive. Hence:

$$\frac{dx}{dt} = r \frac{d\theta}{dt} \tag{5}$$

Therefore, there are two relations between the road and square curve:

$$\begin{cases} r = -y \\ \frac{dx}{dt} = r \frac{d\theta}{dt} \end{cases} \tag{6}$$

The y part of the road curve can easily be found:

$$y = -r \tag{7}$$

Hence:

$$\begin{cases} y = -r \\ \frac{dx}{dt} = r \frac{d\theta}{dt} \end{cases} \tag{8}$$

3 Finding the road for a rolling square

Now, the parametric functions for a square need to be found in order to find the road curve. For the simplicity of this calculation let the side length of the square be 2. To find the equations it is easiest to start with just a single side of the square (starting at $(1, -1)$ and ending at $(1, 1)$). Therefore, x stays constant and y varies from -1 to 1. The parametric equations are as follows:

$$x(t) = 1 \quad (9)$$

$$y(t) = t \quad (10)$$

Those formulas can be converted to polar form using the following equations(Dawkins, 2023):

$$r = \sqrt{x^2 + y^2} \quad (11)$$

$$\theta = \arctan \frac{y}{x} \quad (12)$$

Therefore:

$$r(t) = \sqrt{1^2 + t^2} = \sqrt{1 + t^2} \quad (13)$$

$$\theta(t) = \arctan \frac{t}{1} = \arctan(t) \quad (14)$$

These equations may be used to substitute into Equation 8, which is very easy for the y part:

$$y = -r = -\sqrt{1 + t^2} \quad (15)$$

For the x part, the derivative of θ in respect to t has to be found:

$$\frac{d\theta}{dt} = \frac{d}{dt} \arctan(t) \quad (16)$$

$$= \frac{1}{1 + t^2} \quad (17)$$

Then it can be substituted into the equation:

$$\frac{dx}{dt} = r \frac{d\theta}{dt} \quad (18)$$

$$\frac{dx}{dt} = \sqrt{1+t^2} \frac{1}{1+t^2} \quad (19)$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1+t^2}} \quad (20)$$

To obtain $x(t)$, the integral of $\frac{dx}{dt}$ has to be found:

$$\int \frac{dx}{dt} dt = \int \frac{1}{\sqrt{1+t^2}} dt \quad (21)$$

$$x(t) = \int \frac{1}{\sqrt{1+t^2}} dt \quad (22)$$

Which is the integral of the derivative of inverse hyperbolic sine function (Earl and Nicholson, 2021), therefore:

$$x(t) = \text{arsinh}(t) + c \quad (23)$$

Hence, the road functions become (c can be set to 0, as it is just a horizontal shift):

$$\begin{cases} x(t) = \text{arsinh}(t) \\ y(t) = -\sqrt{1+t^2} \end{cases} \quad (24)$$

They can be then turned back into cartesian form:

$$t = \sinh(x) \quad (25)$$

$$y = -\sqrt{1+\sinh^2(x)} \quad (26)$$

$$(27)$$

Using $\cosh^2(x) - \sinh^2(x) = 1$ (Earl and Nicholson, 2021), the equation can be simplified:

$$y = -\sqrt{\cosh^2(x)} \quad (28)$$

$$y(x) = -\cosh(x) \quad (29)$$

The obtained road function is actually an inverted catenary curve ($\cosh(x)$ is the catenary curve function). What is interesting is that the catenary is a curve formed by a hanging chain supported at its ends (even its name “catenary” is derived from latin word for “chain”).(Weisstein, a) The curve is also a very common shape in architecture. It can be seen graphed below along with some square sides:

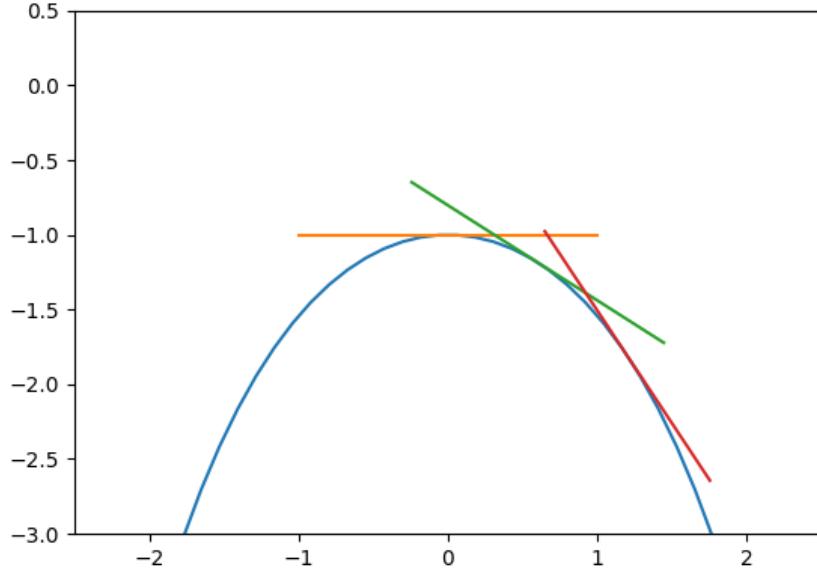


Figure 3: Catenary curve with 3 square sides “rolling” on top of it³

It can be observed, that this shape is not the whole road, but only its fragment. It is due to the previous approximation of only one side. However, the road is periodical and needs to repeat every time the magnitude of its slope is equal to 1(Hall and Wagon, 1992).

Therefore it needs to be truncated when:

$$\left| \frac{dy}{dx} \right| = 1 \quad (30)$$

The derivative of $\cosh x$ is $\sinh x$, therefore:

$$\begin{aligned} \frac{dy}{dx} &= -\sinh x \\ \Rightarrow |-\sinh x| &= 1 \end{aligned}$$

$$\sinh x = 1 \quad \vee \quad \sinh x = -1$$

³All plots were created using the *Matplotlib*(Hunter, 2007) Python package, unless stated otherwise

Thus, the points at which to truncate are:

$$x = \text{arsinh}(1) \quad \vee \quad x = \text{arsinh}(-1) \quad (31)$$

Due to arsinh properties $\text{arsinh}(-x) = -\text{arsinh}(x)$, so:

$$x = \pm \text{arsinh}(1) \quad (32)$$

It has been found that for any polygon of n -sides, if the road is truncated at $x = \pm \text{arsinh}(\tan(\pi/n))$, then the amount of rotation to get the polygon in the cusp is $\frac{2\pi}{n}$ (Hall and Wagon, 1992). This statement holds true, as for square $n = 4$ and $\tan(\frac{\pi}{4}) = 1$. Therefore, the cusp angle is $\frac{\pi}{2}$ rad, the road fragments will intersect at a right angle, which makes sense given that a square's vertex needs to fit in between them. Thus, next road fragments should have the equation(Hall and Wagon, 1992):

$$y = -\cosh(x + 2k \text{arsinh}(1)) \quad \text{where } k \in \mathbb{Z} \quad (33)$$

A 3-fragment road can be seen below:

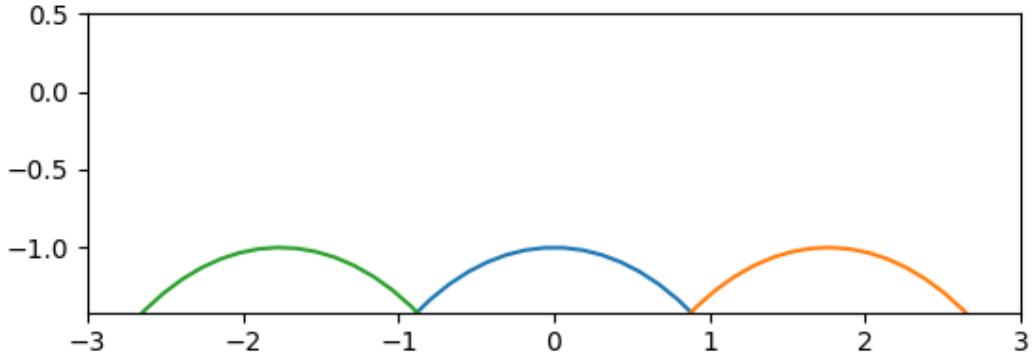


Figure 4: Road for a rolling square consisting of 3 fragments

4 Rounding the square

However, the bridge cannot be a polygon with sharp corners, as standing on just one point (vertex of the square) would be too unstable for such a big structure (it weights 13 tons(Williams, 2023)). Therefore, the corners of the square have to be rounded. This can be done by adding a circle of radius b to each corner of the square. (In the actual bridge the radius is equal to 0.25 when the side length of the square is set to 2). The new shape can be seen on Figure 5 below.

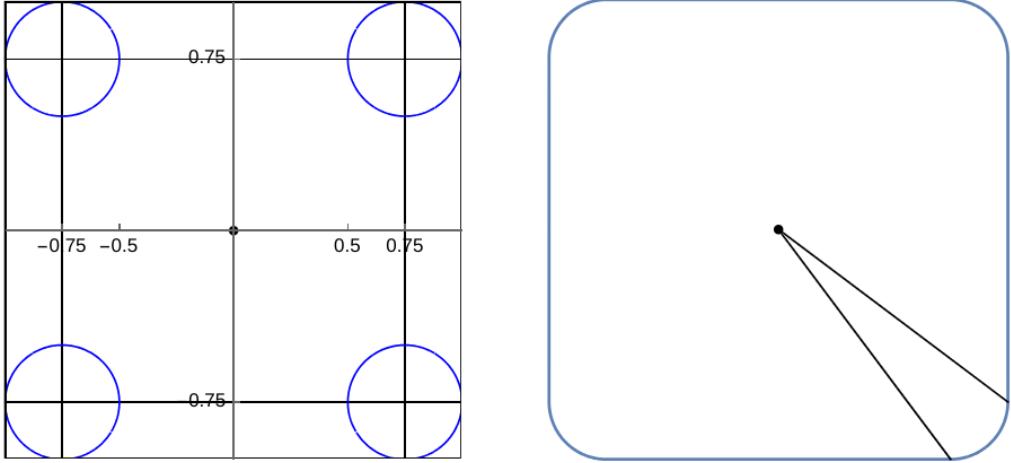


Figure 5: Rounded square(Wagon and Slavík, 2023)

The corners have the polar form(Wagon and Slavík, 2023) [Here maybe also add derivation?]:

$$r = (1 - b)\sqrt{2} \cos\left(\frac{\pi}{4} + \theta\right) + \sqrt{b^2 - 2(1 - b)^2 \sin^2\left(\frac{\pi}{4} + \theta\right)}$$

$$-\operatorname{arccot}(1 - b) \leq \theta \leq \operatorname{arccot}(1 - b) - \frac{\pi}{2}$$
(34)

Substituting $b = 0.25$ (which is the real value used for the Cody Dock bridge(Wagon and Slavík, 2023)), the polar form of the rounded square is found:

$$r(\theta) = \frac{3\sqrt{2}}{4} \cos\left(\frac{\pi}{4} + \theta\right) + \sqrt{\frac{1}{16} - \frac{9}{8} \sin^2\left(\frac{\pi}{4} + \theta\right)}$$

$$-\operatorname{arccot}\left(\frac{3}{4}\right) \leq \theta \leq \operatorname{arccot}\left(\frac{3}{4}\right) - \frac{\pi}{2} \approx -0.927 \leq \theta \leq -0.644$$
(35)

5 Finding the road for a rounded square

The rounded square is very similar to the normal one, apart from its corners. Hence, the road for the corners has to be found, as the one for the sides will be the same. From Equation 8 it is easy to find y :

$$y(\theta) = -\frac{3\sqrt{2}}{4} \cos\left(\frac{\pi}{4} + \theta\right) - \sqrt{\frac{1}{16} - \frac{9}{8} \sin^2\left(\frac{\pi}{4} + \theta\right)}$$
(36)

This way, we find y in terms of the angle θ , but this form will be easier for this part of the road. Hence, x should also be found in terms of θ .

From Equation 8 it is known that:

$$\frac{dx}{dt} = r(\theta) \frac{d\theta}{dt} \quad (37)$$

Thus:

$$\int \frac{dx}{dt} dt = \int r(\theta) \frac{d\theta}{dt} dt \quad (38)$$

$$\int dx = \int r(\theta) d\theta \quad (39)$$

$$x(\theta) = \int r(\theta) d\theta \quad (40)$$

Substituting r :

$$x(\theta) = \int \frac{3\sqrt{2}}{4} \cos\left(\frac{\pi}{4} + \theta\right) + \sqrt{\frac{1}{16} - \frac{9}{8} \sin^2\left(\frac{\pi}{4} + \theta\right)} d\theta \quad (41)$$

$$= \frac{3\sqrt{2}}{4} \int \cos\left(\frac{\pi}{4} + \theta\right) d\theta + \int \sqrt{\frac{1}{16} - \frac{9}{8} \sin^2\left(\frac{\pi}{4} + \theta\right)} d\theta \quad (42)$$

Let:

$$I_1 = \frac{3\sqrt{2}}{4} \int \cos\left(\frac{\pi}{4} + \theta\right) d\theta , \quad I_2 = \int \sqrt{\frac{1}{16} - \frac{9}{8} \sin^2\left(\frac{\pi}{4} + \theta\right)} d\theta \quad (43)$$

Using compound angle identity:

$$I_1 = \frac{3\sqrt{2}}{4} \int \cos\left(\frac{\pi}{4}\right) \cos(\theta) - \sin\left(\frac{\pi}{4}\right) \sin(\theta) d\theta \quad (44)$$

$$= \frac{3\sqrt{2}}{4} \int \frac{1}{\sqrt{2}} \cos(\theta) - \frac{1}{\sqrt{2}} \sin(\theta) d\theta \quad (45)$$

$$= \frac{3\sqrt{2}}{4} \cdot \frac{1}{\sqrt{2}} \int \cos(\theta) - \sin(\theta) d\theta \quad (46)$$

$$I_1 = \frac{3}{4} \cdot \frac{\sin(\theta) + \cos(\theta)}{\sqrt{2}} \quad (47)$$

However, the I_2 integral is way trickier than the I_1 one and cannot be solved using simple methods learned during the IB Mathematics course, as it is an elliptic integral of the second kind(Weisstein, b). Therefore, it will be instead solved numerically using the

Euler's method.

Euler's method states that:

$$x_{n+1} = x_n + h \cdot x'(\theta_n) , \quad \theta_{n+1} = \theta_n + h \quad (48)$$

Given that:

$$x = \frac{3}{4} \cdot \frac{\sin(\theta) + \cos(\theta)}{\sqrt{2}} + \int \sqrt{\frac{1}{16} - \frac{9}{8} \sin^2(\frac{\pi}{4} + \theta)} d\theta \quad (49)$$

The road equations for the rounded corner can be written as:

$$\begin{cases} x_{n+1} = \frac{3}{4} \cdot \frac{\sin(\theta) + \cos(\theta)}{\sqrt{2}} + x_n + h \cdot \sqrt{\frac{1}{16} - \frac{9}{8} \sin^2(\frac{\pi}{4} + \theta)} \\ y = -\frac{3\sqrt{2}}{4} \cos(\frac{\pi}{4} + \theta) - \sqrt{\frac{1}{16} - \frac{9}{8} \sin^2(\frac{\pi}{4} + \theta)} \\ \theta_{n+1} = \theta_n + h , \quad -\text{arccot}(\frac{3}{4}) \leq \theta \leq \text{arccot}(\frac{3}{4}) - \frac{\pi}{2} \end{cases} \quad (50)$$

If we let $x_0 = 0$ and $h = 0.001$, the x and y values can be calculated using the code seen in Appendix A. By plotting those points, the road can be seen in Figure 6 below:

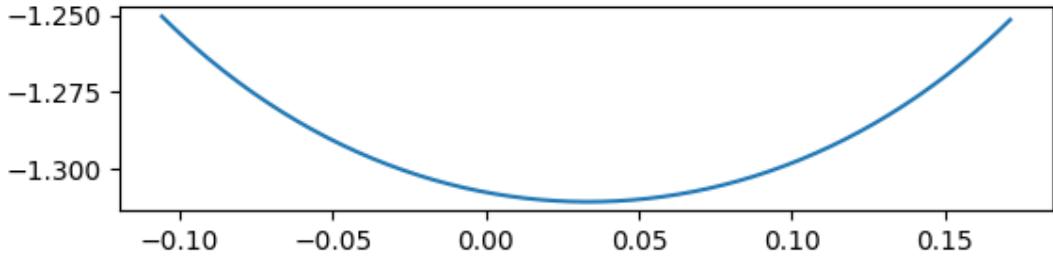


Figure 6: Road for the rounded corner

It can be seen that the road is not centered - its lowest point is not at $x = 0$. Thanks to the programmatic approach, the minimum and maximum x value can be found easily:

$$x_{\min} = -0.106 , \quad x_{\max} = 0.172$$

Thus, the “center” of this road piece (which is also its lowest point) has x value:

$$x = \frac{x_{\min} + x_{\max}}{2} = 0.0661 \quad (51)$$

Hence, to have the center of the piece at $x = 0$, the road has to be shifted left by 0.0661 and thus it can be seen on Figure 7 below:

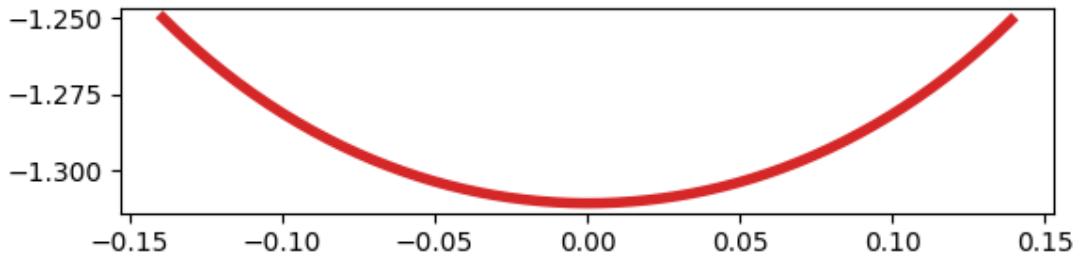


Figure 7: Centered rounded corner road

To insert the new road piece into the whole road, its center has to be where the catenary curves intersect. The first such point is at $x = \text{arsinh}(1)$ (from Equation 31). Therefore, the corner road has to be shifted horizontally by additional $\text{arsinh}(1) \approx 0.881$ to the right. Combining this with the 0.0661 shift to the left, we get in total a shift to the right by approximately 0.815. Hence, a fragment of the final road can be seen on Figure 8 below:

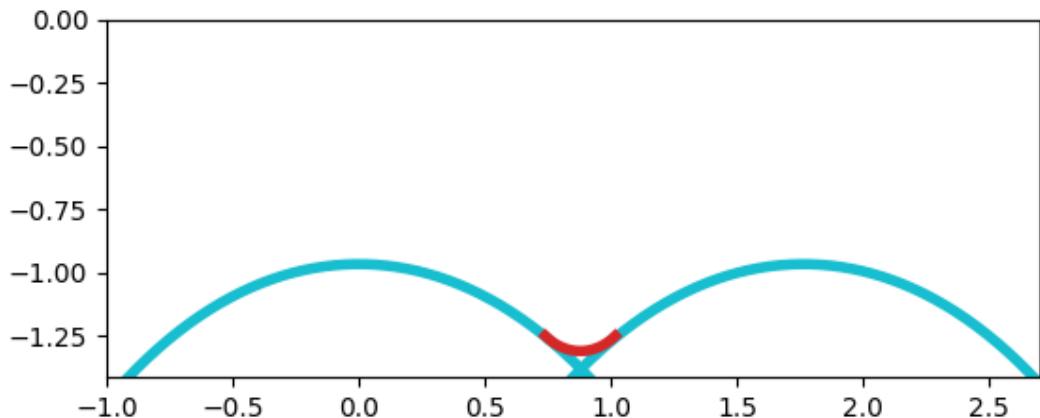


Figure 8: Road for the square with rounded corners

For the bridge to make a 180 degree rotation, the square has to “tumble over” two times. Therefore, the catenaries and the “corner parts” have to repeat three and two times respectively. Given that from the center of the catenary to the intersection is the distance $\text{arsinh}(1)$, the second corner part will be at $x = 3\text{arsinh}(1)$. Hence, the full road with the rounded square can be seen on the Figure 9:

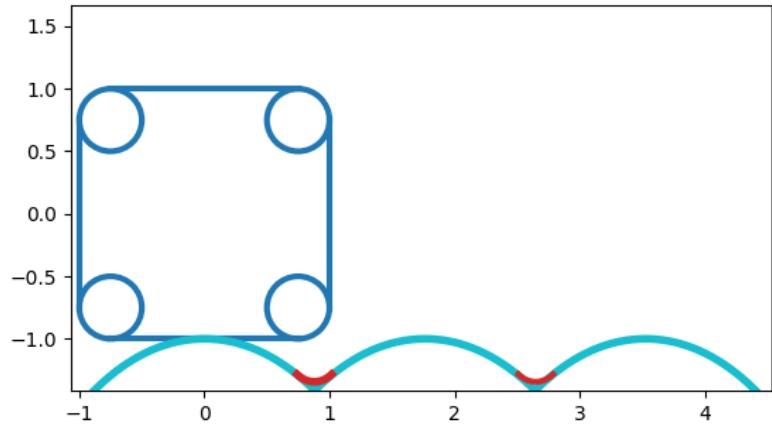


Figure 9: Full road plotted along with the bridge

Additionally, since the bridge should stand still on the beginning and end of the road, the catenary curves can be restricted to $x \in [0, 4\text{arsinh}(1)]$. A $y = -1$ line can be thus added for $x \in [-1, 0] \cup [4\text{arsinh}(1), 4\text{arsinh}(1) + 1]$. Given that the first catenary curve intersects the corner road at $x \approx \text{arsinh}(1) - 0.15$ (The center of the road piece is at $x = \text{arsinh}(1)$ and the width of this piece is around 0.3), the catenaries were restricted to $y \in [-\cosh(\text{arsinh}(1) - 0.15), -1] \approx y \in [-1.28, -1]$. Hence, the final road can be seen in Figure 10 below:

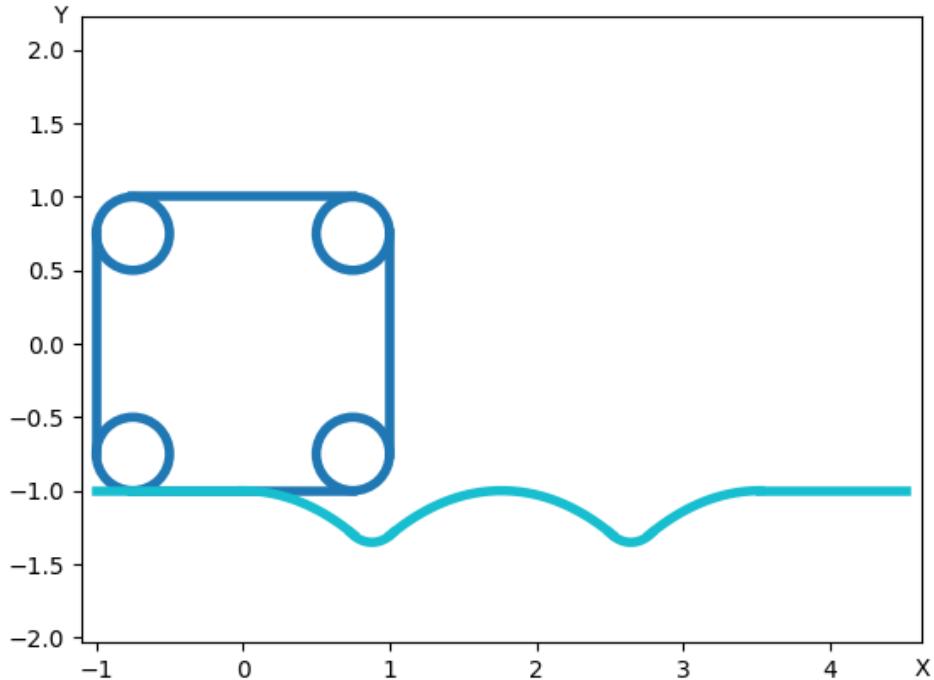


Figure 10: Final road for the rounded square

Thus, by using finding the polar form of the rounded corner and then approximating its road fragment, a full road (which should have the same shape as the one currently in London) was calculated. This can be seen visually in the Figure 11 below:

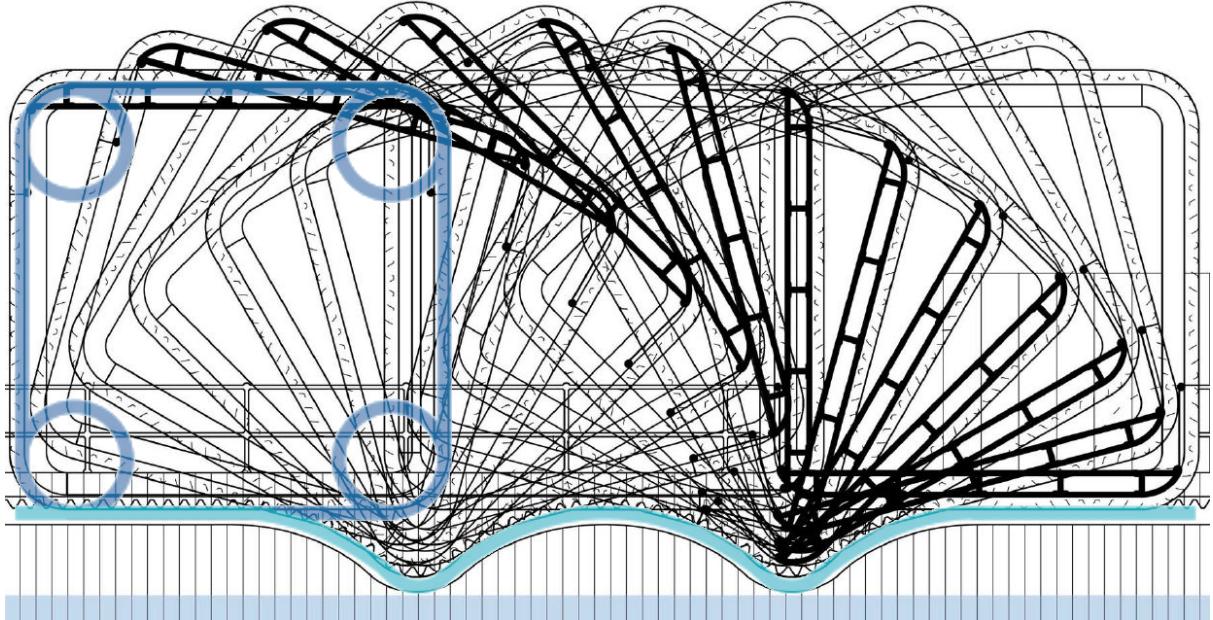


Figure 11: Obtained road overlayed on top of the official masterplan⁴

6 Finding roads for different polygonal wheels

It may also be interesting to find the roads for polygons other than a simple square. Thus, it may be interesting to start with the triangle, as a normal (not-rounded) triangle cannot physically have a “perfect road”, as the road is so steep, that the triangle crashes into it. This situation can be seen in Figure 12 below:

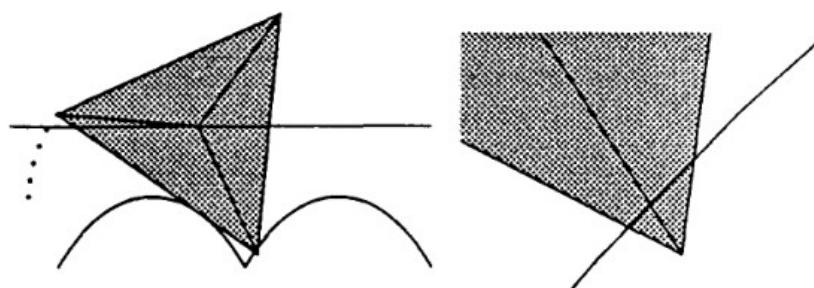


Figure 12: Triangle crashing into its own road(Hall and Wagon, 1992)

⁴Masterplan photo source: <https://codydock.org.uk/cody-dock-masterplan/>

However, when the triangle is rounded at the corners enough, it can successfully roll.

6.1 Conclusions

Catenaries

References

- Billington, D. P., Billington, P. N., and Shirley-Smith, H. (2024). Encyclopedia Britannica - “bridge”. <https://www.britannica.com/technology/bridge-engineering>. [Online; Accessed 5th February 2024].
- Dawkins, P. (2023). Pauls Online Notes - Polar Coordinates. <https://tutorial.math.lamar.edu/Classes/CalcII/PolarCoordinates.aspx>. [Online; Accessed 18th December 2023].
- Earl, R. and Nicholson, J. (2021). *The concise oxford dictionary of mathematics*. Oxford University Press, 6th edition.
- Hall, L. and Wagon, S. (1992). Roads and wheels. *Mathematics Magazine*, 65(5):283–301.
- Hunter, J. D. (2007). Matplotlib: A 2d graphics environment. *Computing in Science & Engineering*, 9(3):90–95.
- Wagon, S. and Slavík, A. (2023). A rolling square bridge: Reimagining the wheel. <https://community.wolfram.com/groups/-/m/t/2917199>. [Online; Accessed 27th December 2023].
- Weisstein, E. ”Catenary.” From MathWorld - A Wolfram Web Resource. <https://mathworld.wolfram.com/Catenary.html>. [Online; Accessed 5th February 2024].
- Weisstein, E. ”Elliptic Integral of the Second Kind” From MathWorld - A Wolfram Web Resource. <https://mathworld.wolfram.com/EllipticIntegraloftheSecondKind.html>. [Online; Accessed 3rd February 2024].
- Williams, A. (2023). One-of-a-kind footbridge tumbles over to let boats pass. <https://newatlas.com/architecture/cody-dock-rolling-bridge/>. [Online; Accessed 2nd February 2024].

7 Appendix

A Euler's approximation Python code

```
1 from numpy import arctan, sqrt, sin, cos, pi
2
3 def x_2_prim(theta): # derivative of 2nd part of x function
4     return sqrt( 1/16 - 9/8 * (sin(pi/4 + theta) )**2 )
5
6 def x_1(theta): # 1st part of x function
7     return 3/4 * (sin(theta) + cos(theta))/sqrt(2)
8
9 def y(theta): # y function
10    return -(3*sqrt(2))/4 * cos(theta + pi/4) - sqrt( 1/16 - 9/8 * (sin(pi/4 + theta) )**2 )
11
12 theta_start = - arctan(4/3)
13 theta_stop = arctan(4/3) - pi/2
14
15 h = 0.001
16 x_0 = 0
17
18 cur_theta = theta_start
19 cur_x2 = x_0
20
21 x_list = []
22 y_list = []
23
24 while cur_theta < theta_stop:
25     cur_x_2_prim = x_2_prim(cur_theta)
26     cur_x2 += h * cur_x_2_prim
27
28     cur_x = x_1(cur_theta) + cur_x2
29
30     x_list.append(cur_x)
31     y_list.append(y(cur_theta))
32
33     cur_theta += h
34
35 print("min x", min(x_list))
36 print("max x", max(x_list))
```