Exploring mathematics of rolling bridges

Extended essay

RQ: How can catenary curves be used in construction of non-circular "rolling" bridges?

Word count: 1234

Contents

1 Introduction **3**

2 “Smoothly” rolling square **3**

3 Rounding the square **9**

1 Introduction

1. My personal interest - I am interested in architecture and urban planning and this topic combines this with my passion for mathematics
2. I have found this bridge in an internet video - as an interesting mathematical phenomena existing in real life

Figure 1: Bridge photo[1]

2 "Smoothly" rolling square

In order to start, the "smooth rolling" has to be defined. If we e.g. take a circle shape and roll it on a straight surface, then it will roll "smoothly" - its center of mass will move only horizontally and not vertically. However, when a square is rolled on the same surface, its center of mass will move vertically as well and hence its movement will not be "smooth".

Even so, why is it important to roll "smoothly"? The answer is connected to physics, more specifically to the work needed to be done to roll the shape. If the figure is rolled "smoothly", then there is no need to do work in moving the center of mass up and down. Therefore, in the ideal world, the shape will roll without any work needed to be done (in reality, there is always some friction, so some work will always be needed to be done).

Another important property for this "rolling" is that when any point of the shape is in contact with the road surface, its velocity relative to it has to be zero. Otherwise, if the point was moving while touching the road, it would be sliding and not rolling.

[1]Photo source: `https://newatlas.com/architecture/cody-dock-rolling-bridge/`

However, there must exist a surface (other than straight road) on which a square can roll "smoothly". It will be described parametrically as follows:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \tag{1}$$

Similarly, the square shape will also be parametrized. However, it will be parametrized in polar coordinates, as this representation will be easier to work with. The parametrization is as follows:

$$\begin{cases} r = r(t) \\ \theta = \theta(t) \end{cases} \tag{2}$$

To simplify the calculations, the line of movement of the center of mass of the square will be chosen to be the x-axis. Hence, the road will have to be under it, so $y(t) < 0$. Likewise, the center of mass of the square point (axle) will be chosen to be at the origin of its local coordinate system. The parameter t can be thought of as the time, at which there is a point on the square and on the road, which are touching. Therefore, the point (x, y) on the road will be the contact point on the road curve and similarly the point (r, θ) will be the contact point on the square curve.

Now, the relations between those two curves need to be found. If it is assumed that the square is on the road, then it must be touching it at some point. Moreover, the point of contact is not a random point on the curve, but the one directly beneath the center mass point (so the origin point). Therefore, it has to be the $r(t)$ point on the square curve. Hence, the distance from axle to the road must be equal to the road's depth $(-y(t))$:

$$r(t) = -y(t) \tag{3}$$

Another relation can be found by looking at the contact point's velocity. It was earlier defined, that during contact, the velocity of a point relative to the road has to be zero. However, when it is looked at from the perspective of the axle point, all points on the square are constantly moving (while rolling). Therefore, if the point is stationary relative to the ground, but moving relative to the center point, then those speeds have to be equal to each other. The speed relative to road is simply $|\frac{dx}{dt}|$, while the speed relative to the axle can be calculated from its angular speed:

$$\omega = \frac{d\theta}{dt}$$

$$v = r \cdot \omega$$

$$v_{\text{axle}} = r\frac{d\theta}{dt}$$

Thence (the speeds, not the velocities are equal, hence the absolute value):

$$|\frac{dx}{dt}| = |r\frac{d\theta}{dt}|$$

$$\frac{dx}{dt} = \pm\, r\frac{d\theta}{dt} \tag{4}$$

However, it can be assumed that the square will roll to the right (in the positive x direction), so the contact point will have to rotate counter-clockwise. Therefore, both $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$ have to be positive. Hence:

$$\frac{dx}{dt} = r\frac{d\theta}{dt} \tag{5}$$

Therefore, there are two relations between the road and square curve:

$$\begin{cases} r = -y \\ \frac{dx}{dt} = r\frac{d\theta}{dt} \end{cases} \tag{6}$$

The y part of the road curve can easily be found:

$$y = -r \tag{7}$$

Hence:

$$\begin{cases} y = -r \\ \frac{dx}{dt} = r\frac{d\theta}{dt} \end{cases} \tag{8}$$

Now, the parametric functions for a square need to be found in order to find the road curve. For the simplicity of this calculation let the side length of the square be 2. To find the equations it is easiest to start with just a single side of the square (starting at $(1, -1)$ and ending at $(1, 1)$). Therefore, x stays constant and y varies from -1 to 1. The parametric equations are as follows:

$$x(t) = 1 \tag{9}$$

$$y(t) = t \tag{10}$$

Those formulas can be converted to polar form using the following equations[1]:

$$r = \sqrt{x^2 + y^2} \tag{11}$$

$$\theta = \arctan \frac{y}{x} \tag{12}$$

Therefore:

$$r(t) = \sqrt{1^2 + t^2} = \sqrt{1 + t^2} \tag{13}$$

$$\theta(t) = \arctan \frac{t}{1} = \arctan(t) \tag{14}$$

These equations may be used to substitute into Equation 8, which is very easy for the y part:

$$y = -r = -\sqrt{1 + t^2} \tag{15}$$

For the x part, the derivative of θ in respect to t has to be found:

$$\frac{d\theta}{dt} = \frac{d}{dt} \arctan(t) \tag{16}$$

$$= \frac{1}{1 + t^2} \tag{17}$$

Then it can be substituted into the equation:

$$\frac{dx}{dt} = r \frac{d\theta}{dt} \tag{18}$$

$$\frac{dx}{dt} = \sqrt{1 + t^2} \frac{1}{1 + t^2} \tag{19}$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 + t^2}} \tag{20}$$

To obtain $x(t)$, the integral of $\frac{dx}{dt}$ has to be found:

$$\int \frac{dx}{dt} dt = \int \frac{1}{\sqrt{1+t^2}}\, dt \tag{21}$$

$$x(t) = \int \frac{1}{\sqrt{1+t^2}}\, dt \tag{22}$$

Which is the integral of the derivative of inverse hyperbolic sine function[2], therefore:

$$x(t) = \operatorname{arsinh}(t) + c \tag{23}$$

Hence, the road functions become (c can be set to 0, as it is just a horizontal shift):

$$\begin{cases} x(t) = \operatorname{arsinh}(t) \\ y(t) = -\sqrt{1+t^2} \end{cases} \tag{24}$$

They can be then turned back into cartesian form:

$$t = \sinh(x) \tag{25}$$

$$y = -\sqrt{1+\sinh^2(x)} \tag{26}$$

$$\tag{27}$$

Using $\cosh^2(x) - \sinh^2(x) = 1$[2], the equation can be simplified:

$$y = -\sqrt{\cosh^2(x)} \tag{28}$$

$$y(x) = -\cosh(x) \tag{29}$$

Which is actually an inverted catenary curve. This is a shape that is formed by a hanging chain, which is a very common shape in architecture. It can be seen graphed below:

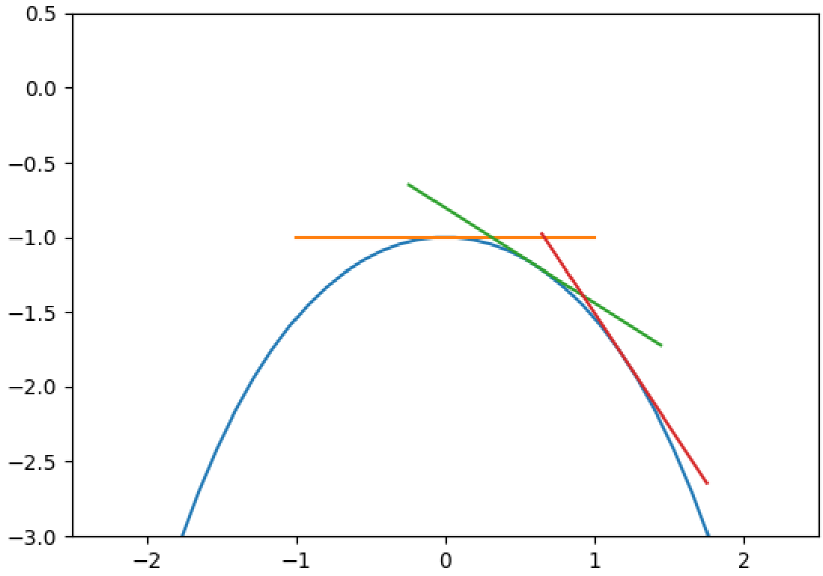

Figure 2: Catenary curve with 3 square sides "rolling" on top of it

It can be observed, that this shape is not the whole road, but only its fragment. It is due to the previous approximation of only one side. However, the road is periodical and needs to repeat every time the magnitude of its slope is equal to 1[3].

Therefore it needs to be truncated when:

$$|\frac{dy}{dx}| = 1 \tag{30}$$

The derivative of $\cosh x$ is $\sinh x$, therefore:

$$\frac{dy}{dx} = -\sinh x$$

$$\Rightarrow |-\sinh x| = 1$$

$$\sinh x = 1 \quad \vee \quad \sinh x = -1$$

Thus, the points at which to truncate are:

$$x = \operatorname{arsinh}(1) \quad \vee \quad x = \operatorname{arsinh}(-1) \tag{31}$$

Due to arsinh properties $\operatorname{arsinh}(-x) = -\operatorname{arsinh}(x)$, so:

$$x = \pm \operatorname{arsinh}(1) \tag{32}$$

It has been found that for any polygon of n-sides, if the road is truncated at $x = \pm \operatorname{arsinh}(\tan(\pi/n))$, then the amount of rotation to get the polygon in the cusp is $\frac{2\pi}{n}$[3]. This statement holds true, as for square $n = 4$ and $\tan(\frac{\pi}{4}) = 1$. Therefore, the cusp angle is $\frac{\pi}{2}$ rad, the road fragments will intersect at a right angle, which makes sense given that a square's vertex needs to fit in between them. Thus, next road fragments should have the equation[3]:

$$y = -\cosh(x + 2k \operatorname{arsinh}(1)) \quad \text{where } k \in \mathbb{Z} \tag{33}$$

A 3-fragment road can be seen below:

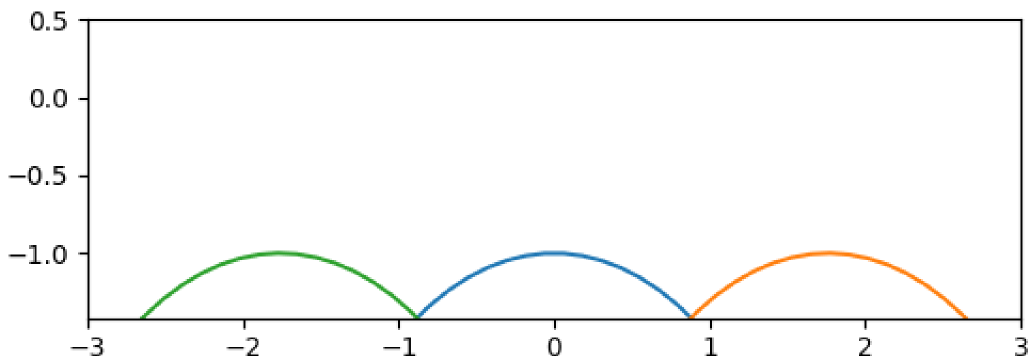

Figure 3: Road for a rolling square consisting of 3 fragments

3 Rounding the square

However, the bridge cannot be a polygon with sharp corners, as standing on just one point (vertex of the square) would be too unstable for such a big structure (it weights 13 tons[4]). Therefore, the corners of the square have to be rounded. This can be done by adding a circle of radius b to each corner of the square. (In the actual bridge the radius is equal to 0.25 when the side length of the square is set to 2). The new shape can be seen on Figure 4 below.

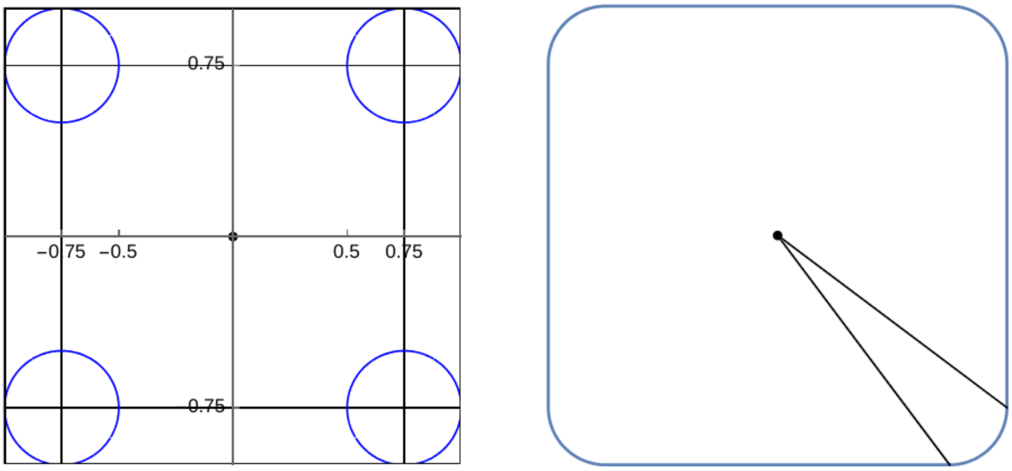

Figure 4: Rounded square[5]

The corners have the polar form[5] [Here maybe also add derivation?]:

$$r = (1-b)\sqrt{2}\cos(\frac{\pi}{4}+\theta) + \sqrt{b^2 - 2(1-b)^2\sin^2(\frac{\pi}{4}+\theta)} \tag{34}$$

$$-\operatorname{arccot}(1-b) \leq \theta \leq \operatorname{arccot}(1-b) - \frac{\pi}{2}$$

Substituting $b = 0.25$, the polar form of the rounded square is found:

$$r(\theta) = \frac{3\sqrt{2}}{4}\cos(\frac{\pi}{4}+\theta) + \sqrt{\frac{1}{16} - \frac{9}{8}\sin^2(\frac{\pi}{4}+\theta)} \tag{35}$$

$$-\operatorname{arccot}(\frac{3}{4}) \leq \theta \leq \operatorname{arccot}(\frac{3}{4}) - \frac{\pi}{2} \qquad \approx \qquad -0.927 \leq \theta \leq -0.644$$

Now, the road for the corners has to be found. From Equation 8 it is easy to find y:

$$y(\theta) = -\frac{3\sqrt{2}}{4}\cos(\frac{\pi}{4}+\theta) - \sqrt{\frac{1}{16} - \frac{9}{8}\sin^2(\frac{\pi}{4}+\theta)} \tag{36}$$

This way, we find y in terms of the angle θ, but this form will be easier for this part of the road. Hence, x should also be found in terms of θ.

From Equation 8 it is known that:

$$\frac{dx}{dt} = r(\theta)\,\frac{d\theta}{dt} \tag{37}$$

Thus:

$$\int \frac{dx}{dt}\ dt = \int r(\theta)\ \frac{d\theta}{dt}\ dt \tag{38}$$

$$\int dx = \int r(\theta)\ d\theta \tag{39}$$

$$x(\theta) = \int r(\theta)\ d\theta \tag{40}$$

Substituting r:

$$x(\theta) = \int \frac{3\sqrt{2}}{4}\cos(\frac{\pi}{4}+\theta) + \sqrt{\frac{1}{16}-\frac{9}{8}\sin^2(\frac{\pi}{4}+\theta)}\ d\theta \tag{41}$$

$$= \frac{3\sqrt{2}}{4}\int \cos(\frac{\pi}{4}+\theta)\ d\theta + \int \sqrt{\frac{1}{16}-\frac{9}{8}\sin^2(\frac{\pi}{4}+\theta)}\ d\theta \tag{42}$$

Let:

$$I_1 = \frac{3\sqrt{2}}{4}\int \cos(\frac{\pi}{4}+\theta)\ d\theta \quad , \quad I_2 = \int \sqrt{\frac{1}{16}-\frac{9}{8}\sin^2(\frac{\pi}{4}+\theta)}\ d\theta \tag{43}$$

Using compound angle identity:

$$I_1 = \frac{3\sqrt{2}}{4}\int \cos(\frac{\pi}{4})\cos(\theta) - \sin(\frac{\pi}{4})\sin(\theta)\ d\theta \tag{44}$$

$$= \frac{3\sqrt{2}}{4}\int \frac{1}{\sqrt{2}}\cos(\theta) - \frac{1}{\sqrt{2}}\sin(\theta)\ d\theta \tag{45}$$

$$= \frac{3\sqrt{2}}{4}\cdot\frac{1}{\sqrt{2}}\int \cos(\theta) - \sin(\theta)\ d\theta \tag{46}$$

$$I_1 = \frac{3}{4}\cdot\frac{\sin(\theta)+\cos(\theta)}{\sqrt{2}} \tag{47}$$

However, the I_2 integral is way trickier than the I_1 one and cannot be solved using simple methods learned during the IB mathematics, as it is an elliptic integral of the second kind[6]. Therefore, it will be instead solved numerically using the Euler's method.

Euler's method states that:

$$x_{n+1} = x_n + h\cdot x'(\theta_n) \quad , \quad \theta_{n+1} = \theta_n + h \tag{48}$$

Given that:

$$x = \frac{3}{4} \cdot \frac{\sin(\theta) + \cos(\theta)}{\sqrt{2}} + \int \sqrt{\frac{1}{16} - \frac{9}{8}\sin^2(\frac{\pi}{4} + \theta)}\ d\theta \tag{49}$$

Hence, the road for the rounded corner can be written as:

$$x_{n+1} = x_n + \frac{3}{4} \cdot \frac{\sin(\theta) + \cos(\theta)}{\sqrt{2}} + h \cdot \sqrt{\frac{1}{16} - \frac{9}{8}\sin^2(\frac{\pi}{4} + \theta)} \tag{50}$$

$$y = -\frac{3\sqrt{2}}{4}\cos(\frac{\pi}{4} + \theta) - \sqrt{\frac{1}{16} - \frac{9}{8}\sin^2(\frac{\pi}{4} + \theta)} \tag{51}$$

$$\theta_{n+1} = \theta_n + h \quad , \quad -\text{arccot}(\frac{3}{4}) \le \theta \le \text{arccot}(\frac{3}{4}) - \frac{\pi}{2} \tag{52}$$

If we let $x_0 = 0$ and $h = 0.001$, the road can be seen plotted on Figure 5 below:

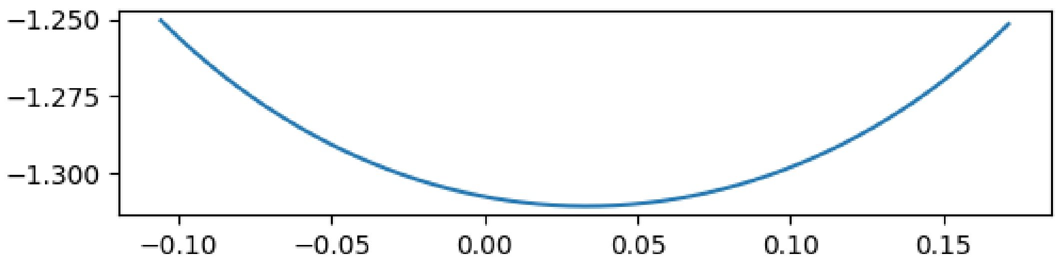

Figure 5: Road for the rounded corner

Thanks to this approach, the minimum and maximum x value can also be found easily:

$$x_{\min} = -0.106 \quad , \quad x_{\max} = 0.172$$

Thus, the "center" of this road piece (which is also its lowest point) has x value:

$$x = \frac{x_{\min} + x_{\max}}{2} = 0.0661 \tag{53}$$

Hence, to have the center of the piece at $x = 0$, the road has to be shifted left by 0.0661 and thus it can be seen on Figure 6 below:

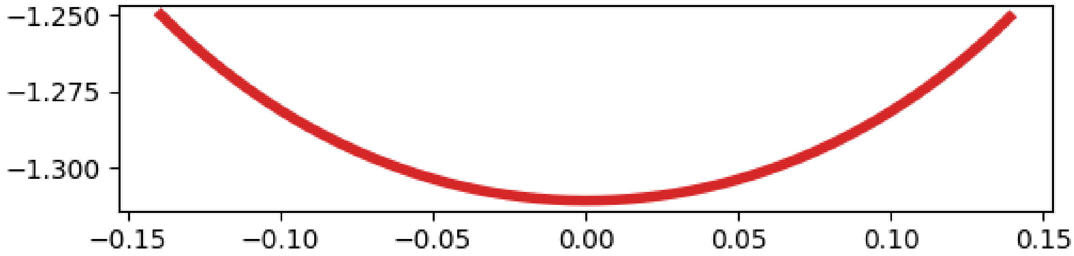

Figure 6: Centered rounded corner road

Now, to insert the new road piece into the whole road, the center of the corner piece has to be where the catenary curves intersect. Thus, the first such point will be at $x = \operatorname{arsinh}(1)$ (from Equation 31). Therefore, the corner road part has to be shifted horizontally by additional $\operatorname{arsinh}(1) \approx 0.881$ to the right. Combining this with the 0.0661 shift to the left, we get in total a shift to the right by approximately 0.815. Hence, a fragment of the final road can be seen on Figure 7 below:

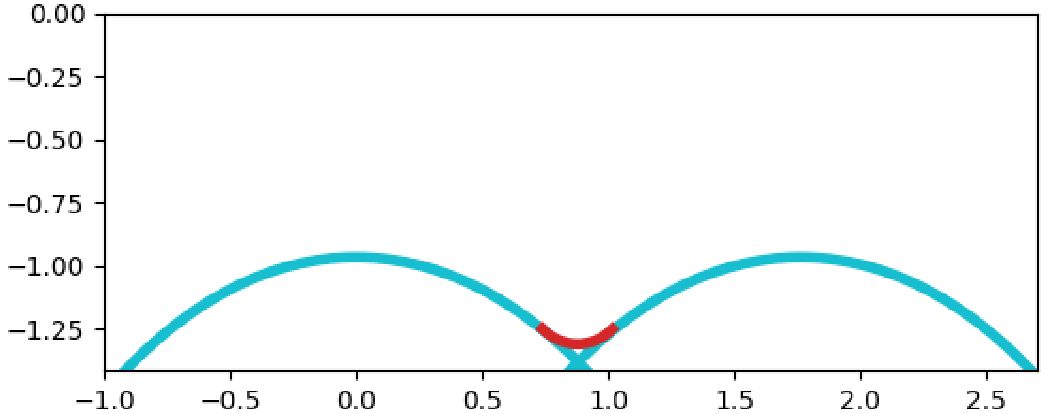

Figure 7: Road for the square with rounded corners

However, to make a 180 degree rotation, the square has to "tumble over" two times. Therefore, the catenaries and the "corner parts" have to repeat three and two times respectively. Hence, the full road with the rounded square can be seen on the Figure 8:

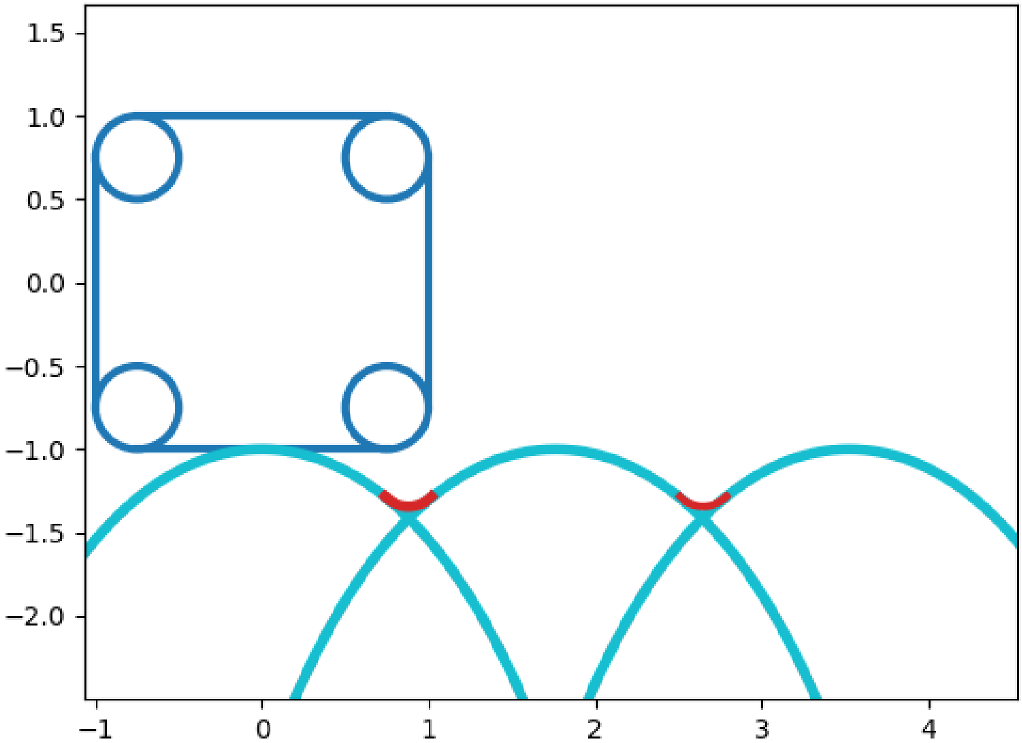

Figure 8: Final road plotted along with the square

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