Computational Fluid Dynamics EML4722 **Name:** *Benjamin Jones*
Homework 1, Introduction
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1. What is CFD?

   Computational fluid dynamics is a method of simulating fluid flow by using iterative solution methods to solve equations representing the fluid's characteristics.

2. What is a fluid?

   A fluid is a liquid or gas which strains continuously under shear.

3. What are the three major steps of CFD? Describe what occurs during each step.

   The three major steps are Pre-processing, Solving, and Post-processing. Pre-processing involves setting up the mesh, fluid, and boundary conditions. This is the most important step to ensure accurate outcomes. Solving involves plugging in your setup configuration into a CFD solver. Post-processing the process of analyzing the results of the solver.

4. Why do we use CFD?

   CFD is a method of predicting fluid flow in real-world applications. It is used as an alternative to wind-tunnel testing when wind-tunnel testing is too expensive or impossible to complete for some applications. It may also be used in conjunction with wind-tunnel testing to cross-check both methods.

5. What is a computational grid? What is computational domain (hint Ω in Mathematics)? What is the purpose of a computational domain and a computational grid?

   A computational domain is a network of discrete points in the computational space for which the states are calculated. A computational grid is a grid generated around a body which is located within the computational domain. Both of these are used to generate solutions for the state of the points within the computational domain.

6. The time independent heat equation is solved in three spatial dimensions. The number of grid points within the computational domain is 386,000. How many discretized equations are solved?

   At each point the equation is solved in 3 dimensions, and as a result 386,000*3 equations are solved.

7. An incompressible flow of an ideal gas moves through a computational domain. The two-dimensional incompressible steady Navier-Stokes equations are discretized onto 110,123 grid points. How many discretized equations are solved to find the numerical solution?

   With 4 equations of state (Mass, momentum, energy, ideal gas), the number of equations will be 110,123*4 equations.

8. Define stability, consistency, and convergence. Also, can a CFD code be convergent without stability?

   Stability of a solver can be determined by checking to see that the error is decreasing between each iteration of the solver. Consistency is the consistency to converge in varying test conditions. Convergence is when the solver reaches the error criteria defined by the user.

9. Find the vorticity of the velocity field given by $u = 23\sin[x]\hat{i} + 3\cos[y]\hat{j} + \exp[z]\hat{k}$

   Vorticity ω is given by the following equation:

$$\omega = \nabla \times \mathbf{V} \tag{1}$$

   Expanding this, all terms factor to zero. As a result, the vorticity at all points in this flow field is zero.

10. Give one example of a specific analytical solution of the Navier-Stokes equations for a particular flow with boundary conditions.

    Channel flow is an example of an exact solution to the Navier-stokes equations. Channel flow assumes steady8, parallel, and incompressible flow through a straight channel. Analysis leads to the following equation for fluid velocity as a function of position within the channel, where H is the height of the channel.

$$u(z) = \frac{1}{2\mu}\frac{dp}{dx}(z^2 - Hz) \tag{2}$$

11. Why are $\frac{\partial^2 u_i}{\partial x_i \partial x_i}$ and $\frac{\partial^2 u_i}{\partial x_i^2}$ not equivalent (hint-expand)?

    Within the denominator, taking the partial twice is different from taking the partial of a squared term.

12. Expand the following to their full forms in Cartesian coordinates

    (a) $\rho\frac{\partial u_i}{\partial x_i}$

$$\rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \tag{3}$$

    (b) $\frac{\partial \rho u_i u_j}{\partial x_j}$

$$\frac{\partial \rho}{\partial x}(\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial x} + \frac{\partial uw}{\partial x})$$
$$\frac{\partial \rho}{\partial y}(\frac{\partial (uv)}{\partial y} + \frac{\partial v^2}{\partial y} + \frac{\partial (vw)}{\partial y})$$

$$\frac{\partial \rho}{\partial z}(\frac{\partial (uw)}{\partial z} + \frac{\partial (vw)}{\partial z} + \frac{\partial w^2}{\partial z})$$

(c) $\frac{\partial p}{\partial x_j}\delta_{ij}$ for $i = 1$

$$\frac{\partial p}{\partial x}\delta_{xx} + \frac{\partial p}{\partial y}\delta_{xy} + \frac{\partial p}{\partial z}\delta_{xz}$$

(d) $\mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ for $i = 1$

$$\mu(2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})$$

13. The solution of the diffusion equation is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \exp{[-\frac{(x-y)^2}{4kt}]}\phi(y)\,dy \tag{4}$$

Given that $\phi(y) = \exp{[-y^2]}$ simplify the solution. What does the diffusion equation model? Is it possible for a solution at time $t > t_0$ to posses a higher absolute value at any point x_0?

The diffusion equation models the natural dispersion of a fluid through a medium. It is not possible for it to posses a higher absolute value without an external force, as that would violate the second law of thermodynamics.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \exp{[-\frac{(x-y)^2}{4kt}]}\exp{(-y^2)}\,dy$$

Removing the integral for now:

$$\begin{aligned}
&= \exp{(-\frac{(x-y)^2}{4kt} - y^2)} \\
&= \exp{(-\frac{x^2}{4kt} + \frac{2xy}{4kt} - \frac{y^2}{4kt} - y^2)} \\
&= \exp{(-\frac{x^2}{4kt} + \frac{2xy}{4kt} - y^2(\frac{1}{4kt} + 1))} \\
&= \exp{(-(y^2(\frac{1}{4kt} + 1) - \frac{2xy}{4kt} + \frac{x^2}{4kt}))} \\
&= \exp{(-(\frac{1}{4kt} + 1)(y^2 - \frac{2xy}{4kt+1} + \frac{x^2}{4kt+1}))}
\end{aligned}$$

Completing the square:

$$\exp{(-(\frac{1}{4kt} + 1)(y - \frac{x}{4kt+1})^2)}$$

This is now in the form of a Gaussian function:

$$\int_{-\infty}^{\infty} \exp{(-a(x+b)^2)}\,dx = \sqrt{\frac{\pi}{a}} \tag{5}$$

Plugging the exponent back into the original function:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{1}{4kt}+1\right)\left(y-\frac{x}{4kt+1}\right)^2\right) dy$$
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \sqrt{\frac{4\pi kt}{4kt+1}}$$

$$\boxed{u(x,t) = \sqrt{\frac{1}{4kt+1}}}$$

14. Assume that a wave is governed by the one-dimensional wave equation. The initial condition is a triangular function of height b and width a centered about the origin and is zero elsewhere (see class powerpoint). Draw the solution on the $u-x$ plane at time $t = a/c$. Also, at what speeds and directions are the waves propagating.

   The wave is propagating at a speed of c in both directions.

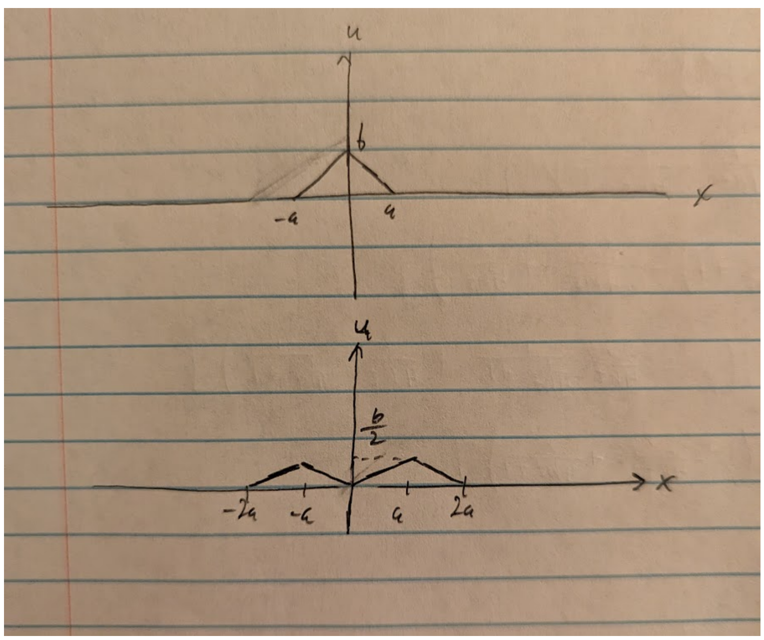

Figure 1: Graph of wave propagation

15. The wave equation is a partial differential equation that governs propagation of linear disturbances. Do solutions of the wave equation exist that permit propagation beyond the speed of sound within the wave equation? Is it possible for waves in a fluid to travel faster than the speed of sound?

   It is possible for waves to travel faster than the speed of sound, however they are not governed by the wave equation as they would be non-linear disturbances.

16. What are some advantages and disadvantages of using a research versus commercial CFD code?

Commercial CFD code costs money in operation, but it may provide better results for one's specific application. Open source CFD code is more readily available and easier to edit if desired, but may come at the downside of less intuitive use or worse performance for one's application.

17. What is a shock wave? A definiton in words in adequate.

    A shock wave is a non-linear wave in a flow induced by the flow or a body moving faster than the speed of sound.

18. What is a Prandtl-Meyer expansion wave? A definition in words is adequate.

    An expansion wave is a slowing of the flow as the fluid expands around a body.