MA113 Notes

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Abstract

Notes for the MA 113 (Mutlivariable Calculus) taught by Dr. Holder.

Contents

1 Coordinate Systems and Polar Functions

1.1 Rectangular Coordinates

- (x, y)
- Every point has only one set of coordinates
- Unique representation!

1.2 Polar Coordinates

- (r, θ)
  - r is the distance from the origin
  - θ is the angle the radius faces
- Lacks unique representation
  - Can add 2π to any angle and get the same point
  - Can make radius negative and add $\frac{\pi}{2}$ to the angle and get the same point

These coordinates are helpful for setting up integrals later.

1.3 Transformations

Definition. $\tan\theta$ is the distance from the intersection of the extended radius and the vertical tangent to the tangent point.

- $x^2 + y^2 = r^2$
- $\tan\theta = \frac{y}{x}$
  - $x = 0 \implies \theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$

- $\theta = \tan^{-1}\frac{y}{x}$
  - $\tan\theta$ is the distance from the intersection of the extended radius and the vertical tangent to the tangent point
  - This is problematic! Range of arctan is $(\frac{-\pi}{2}, \frac{\pi}{2})$, so we do not get all 360 degrees.
  - $\theta = \begin{cases} \tan^{-1}\frac{y}{x} & x \le 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ \frac{3\pi}{2} & x = 0, y < 0 \\ \tan^{-1}\frac{y}{x} + \pi & x < 0 \end{cases}$
- $r\cos\theta = x$
- $r\sin\theta = y$

1.4 Polar Graphs

- Graphs are of the form $r(\theta)$, r is a function of θ.
- $r(\theta) = \cos\theta - 1$ is a cardioid.
- $r(\theta) = \cos(2\theta)$ is a $4-$flower.
  - $\cos(\mathrm{n}\theta)$ has $2n$ petals if n is even, n petals if n is odd.

1.5 Differentiating Polar Functions

- We come to a problem, we wish to find $\frac{dy}{dx}$ but our functions are in terms of r and θ!
- $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ by the Chain Rule.
  - Not because we can cancel out the $d\theta$ terms!

1.5.1 Chain Rule

The default way the Chain Rule is portrayed is

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

Other notation is

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

The Chain Rule allows us to change variables which we do not wish to differentiate by.

1.5.2 Polar Derivatives

Now we can substitute y and x with our earlier transformations to get

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta} r \sin\theta}{\frac{d}{d\theta} r \cos\theta}.$$

1.6 Arc Lengths

In Calculus II, we learned the arclength of a function from a to b as

$$\int_a^b \sqrt{1 + (f'(x))^2} dx$$

This is somewhat related to

$$\frac{d}{dx}\langle x, f(x)\rangle = \langle 1, f'(x)\rangle$$

$$||\langle 1, f'(x)\rangle|| = \sqrt{1 + (f'(x))^2}.$$

We are essentially integrating the magnitude of the derivative of the vector, which gives us the arclength (think about this visually).