MA113 Notes

Brady Exoo

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Abstract

Notes for MA113 (Mutlivariable Calculus) taught by Dr. Holder.

Contents

1 Coordinate Systems and Polar Functions

1.1 Rectangular Coordinates

- (x, y)
- Every point has only one set of coordinates
- Unique representation!

1.2 Polar Coordinates

- (r, θ)
  - r is the distance from the origin
  - θ is the angle the radius faces
- Lacks unique representation
  - Can add 2π to any angle and get the same point
  - Can make radius negative and add π to the angle and get the same point

These coordinates are helpful for setting up integrals later.

1.3 Transformations

Definition. $\tan\theta$ is the distance from the intersection of the extended radius and the vertical tangent to the tangent point.

- $x^2 + y^2 = r^2$
- $\tan\theta = \frac{y}{x}$
  - $x = 0 \implies \theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$

- $\theta = \tan^{-1}\frac{y}{x}$
  - $\tan\theta$ is the distance from the intersection of the extended radius and the vertical tangent to the tangent point
  - This is problematic! Range of arctan is $(\frac{-\pi}{2}, \frac{\pi}{2})$, so we do not get all 360 degrees.
  - $\theta = \begin{cases} \tan^{-1}\frac{y}{x} & x \leq 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ \frac{3\pi}{2} & x = 0, y < 0 \\ \tan^{-1}\frac{y}{x} + \pi & x < 0 \end{cases}$
- $r\cos\theta = x$
- $r\sin\theta = y$

1.4 Polar Graphs

- Graphs are of the form $r(\theta)$, r is a function of θ.
- $r(\theta) = \cos\theta - 1$ is a cardioid.
- $r(\theta) = \cos(2\theta)$ is a $4-$flower.
  - $\cos(\mathrm{n}\theta)$ has $2n$ petals if n is even, n petals if n is odd.
  - $r = \theta$ is the Archimedes spiral.
    - This can be generalized by $r = a\theta$.

1.5 Differentiating Polar Functions

- We come to a problem, we wish to find $\frac{dy}{dx}$ but our functions are in terms of r and θ!
- $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ by the Chain Rule.
  - Not because we can cancel out the $d\theta$ terms!

1.5.1 Chain Rule

The default way the Chain Rule is portrayed is

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Other notation is

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

The Chain Rule allows us to change variables which we do not wish to differentiate by.

1.5.2 Polar Derivatives

Now we can substitute y and x with our earlier transformations to get

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta} r \sin\theta}{\frac{d}{d\theta} r \cos\theta}.$$

1.6 Arc Lengths

1.6.1 Review

In Calculus II, we learned the arclength of a function from a to b as

$$\int_a^b \sqrt{1 + (f'(x))^2} dx$$

This is somewhat related to

$$\frac{d}{dx} \langle x, f(x) \rangle = \langle 1, f'(x) \rangle$$

$$||\langle 1, f'(x) \rangle|| = \sqrt{1 + (f'(x))^2}.$$

Here we are essentially integrating the magnitude of the derivative of the vector, which gives us the arclength (think about this visually with the Pythagorean Theorem). We can also think of this as "vectorizing" the derivative.

1.6.2 Polar Arc Length

We want to find the arc length of a function in terms of $r(\theta)$.

$$\begin{aligned}
s &= \int_{\theta_1}^{\theta_2} \|\frac{d}{d\theta}\langle r\cos(\theta), r\sin(\theta)\rangle\| d\theta \\
&= \int_{\theta_1}^{\theta_2} ||\langle r'\cos(\theta) - r\sin(\theta), r'\sin(\theta) + r\cos(\theta)\rangle|| d\theta \\
&= \int_{\theta_1}^{\theta_2} \sqrt{(r'\cos(\theta) - r\sin(\theta))^2 + (r'\sin(\theta) + r\cos(\theta))^2} d\theta \\
&= \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (r')^2} d\theta
\end{aligned}$$

This is the same as the previous arc length formula, just with substitutions.

1.7 Polar Area

We integrate all the sectors of the function, each of which approximates the sector of a circle, each of which has angle $\Delta\theta$ and radius $r(\theta)$. The area of each sector is

$$(\pi r^2)(\frac{\Delta\theta}{2\pi}) = \frac{1}{2}r^2\Delta\theta.$$

Thus, the total area is

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r(\theta)^2 d\theta.$$

We can subtract these integrals as needed to find areas between two curves:

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r_2(\theta)^2 - r_1(\theta)^2 d\theta.$$

2 Vectors

2.1 Fundamentals

- The point $A(1,2)$ to the point $B(6,5)$ is denotated as the vector $\overrightarrow{AB} = \langle 6-1, 5-2\rangle = \langle 5,3\rangle$.
- Vectors have magnitude and direction.
- Vectors in angle bracket notation are typically assumed to start at the origin.

- Arrow hats represent vectors, some books boldface instead.
  - Column vectors are also used: $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$
- Adding vectors is the same as adding their components.
  - Geometrically, we add tail of one vector to the head of another.
  - Can only add vectors in the same number of dimensions.
  - $\langle 1, 2, 0 \rangle + \langle -1, 4, 8 \rangle = \langle 0, 6, 3 \rangle$
- In subtraction we add the negative of the vector.
- Scalar multiplication of a vector is the same as multiplying each component by that scalar.
  - $3\langle 1, 2, 0 \rangle = \langle 3, 6, 0 \rangle$

2.2 Norms

- Norms are the length or magnitude of a vector
  - Notated as $||\overrightarrow{v}||$.
- $||\overrightarrow{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + r_n^2}$.
- $\langle 1, 3, 2 \rangle = \sqrt{14}\langle \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \rangle$.
  - Generally, this is useful because we separate the norm from the direction (unit vector).
  - General form is $||\vec{v}||(\frac{\vec{v}}{||\vec{v}||})$.
  - This is called normalizing a vector.
- Special Vectors
  - $\vec{i} = \langle 1, 0, 0 \rangle$
  - $\vec{j} = \langle 0, 1, 0 \rangle$
  - $\vec{k} = \langle 0, 0, 1 \rangle$
  - Always assume we are in three dimensions (every vector has $0\overrightarrow{k}$).

2.3 Dot Product

The dot product is defined as

$$\overrightarrow{v} \cdot \overrightarrow{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

This has a lot of cool properties, for one:

$$\overrightarrow{v} \cdot \overrightarrow{w} = ||\overrightarrow{v}||\,||\overrightarrow{w}||.$$

Also, the dot product is commutative and can be distributed.

Example 2.1. $\langle 1, 3, 2\rangle \cdot \langle 3, 0, 2\rangle = 3 + 0 + 2 = 7.$

2.3.1 Angles in the Dot Product

Additionally, by the law of cosines on our vector subtraction example, we get:

$$\begin{aligned} ||\overrightarrow{v}||^2 + ||\overrightarrow{u}||^2 - 2||\overrightarrow{u}||||\overrightarrow{v}||\cos\theta &= ||\overrightarrow{v} - \overrightarrow{u}||^2 \\ &= (\overrightarrow{v} - \overrightarrow{u}) \cdot (\overrightarrow{v} - \overrightarrow{u}) \\ &= \overrightarrow{v} \cdot \overrightarrow{v} - 2\overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{u} \end{aligned}$$

Simplifying, we get

$$\cos\theta = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{||\overrightarrow{u}||\,||\overrightarrow{v}||}$$

This may also be represented as

$$||\overrightarrow{u}||\,||\overrightarrow{v}||\cos\theta = \overrightarrow{u} \cdot \overrightarrow{v}$$

So, the dot product gives us:

- Norms
- Angles
- Distance (metric)

2.3.2 Projections

We wish to find the component of one vector $\overrightarrow{u}$ onto another vector $\overrightarrow{v}$. This is written as $\text{proj}_{\overrightarrow{v}}\overrightarrow{u}$, which is the projection of $\overrightarrow{u}$ onto $\overrightarrow{v}$. This projection will be a scalar multiple of $\overrightarrow{v}$ (the vector we project onto), which can be represented as $\text{proj}_{\overrightarrow{v}}\overrightarrow{u} = \alpha\overrightarrow{v}$. Recall that

$$\cos\theta = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{||\overrightarrow{u}||\,||\overrightarrow{v}||}.$$

In our case $\theta = 90°$, so $\overrightarrow{u} \cdot \overrightarrow{v} = 0$. Note that $\overrightarrow{u} \perp \overrightarrow{v} \iff \overrightarrow{u} \cdot \overrightarrow{v} = 0$. We know that

$$\begin{aligned} \alpha\overrightarrow{v} \cdot (\overrightarrow{u} - \alpha\overrightarrow{v}) &= 0 \\ \overrightarrow{u} &= \alpha\overrightarrow{v} \cdot \overrightarrow{v} \\ \alpha &= \frac{\overrightarrow{v} \cdot \overrightarrow{u}}{\overrightarrow{v} \cdot \overrightarrow{v}}. \end{aligned}$$

Therefore,

$$\text{proj}_{\overrightarrow{v}}\overrightarrow{u} = (\frac{\overrightarrow{v} \cdot \overrightarrow{u}}{\overrightarrow{v} \cdot \overrightarrow{v}})\overrightarrow{v}.$$

A way to remember this is that $\overrightarrow{v}$ is the vector we are projecting onto.

2.4 Three Dimensions

We orient our axes by the right hand rule, where the x axis is coming towards us.

2.4.1 Set Notation

- A sphere would be $\{\overrightarrow{u} : \|\overrightarrow{u}\| = k\}$.
  - $x^2 + y^2 + z^2 = k^2$
- A sphere centered at another point would be $\{\overrightarrow{u} : \|\overrightarrow{u} - \overrightarrow{w}\| = k\}$, where $\overrightarrow{w}$ is the vector from the origin to the center of the sphere.
  - $(x - a)^2 + (y - b)^2 + (z - c)^2 = k^2$

2.5 Cross Product

Definition. $\overrightarrow{v} \times \overrightarrow{w} = \|\overrightarrow{v}\|\,\|\overrightarrow{w}\| \sin\theta\,\overrightarrow{n}$, where $\overrightarrow{n}$ is a unit vector which satisfies the right-hand rule.

n must be perpendicular to both vectors $\overrightarrow{v}$ and $\overrightarrow{w}$. Alternatively,

$$\|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{v}\|\,\|\overrightarrow{w}\|\,|\sin\theta|$$

2.5.1 Properties

- Not commmuative, can be clearly seen by the right-hand rule, which gives opposite values if we reverse the order.
  - $\overrightarrow{v} \times \overrightarrow{w} = -(\overrightarrow{w} \times \overrightarrow{v})$
- The magnitude of the cross product is the area of the parallelogram formed by the two vectors.
- $\overrightarrow{v} \times (\alpha\overrightarrow{v}) = \langle 0, 0, 0\rangle$
- $\|\overrightarrow{v}\|\,\|\overrightarrow{w}\| = \overrightarrow{v} \times \overrightarrow{w} \iff \overrightarrow{v} \perp \overrightarrow{w}$.
  - This means $\overrightarrow{v} \perp \overrightarrow{w} \iff \overrightarrow{v} \cdot \overrightarrow{w} = 0$.

2.6 Matrices

2.6.1 2 × 2

Given a matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The determinant of the matrix is $AD - BC$. This is the area of the parallelogram formed by the vectors $\langle A, B\rangle$ and $\langle C, D\rangle$.

2.6.2 3 × 3

Given a matrix

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$$

We first take A and multiply it by the determinant of the matrix formed by crossing out its row and column (any place a rook could be placed and not hit A).

$$A(EI - FH)$$

Every other term is negative, so we subtract the B term and add the C term, giving us

$$A(EI - FH) - B(DI - FG) + C(DH - GE)^1$$

[1] We can calulate this easier by expanding by a row (or column) with 0s, we don't always need to do the top row! The process will be the same, and the +s and −s form a checkerboard pattern.

as the determinant of this matrix. This is the volume of the parallelepiped formed by the vectors $\langle A, B, C\rangle$, $\langle D, E, F\rangle$ and $\langle G, H, I\rangle$. This is also known as the triple product, but it is really just

$$\langle A, B, C\rangle \times \langle D, E, F\rangle \cdot \langle G, H, I\rangle$$

2.6.3 Calculating the Cross Product With Matrices

To calculate the cross product of $\overrightarrow{v} = \langle 0, 1, 2\rangle$ and $\overrightarrow{w} = \langle 1, 1, 3\rangle$ we use the determinant.

$$\overrightarrow{v} \times \overrightarrow{w} = \det{}^2\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}\right)$$

We can dot the resulting vector with any of our original vectors and we should get 0, this verifies that our calculation is correct.

3 Parametric Equations

3.1 Introduction

Generally, a parametric function is one that takes a variable t and maps it to a point (x, y, z). They are usually used to model the position of moving objects. It can also be thought of as mapping a straight line onto a flexible line in space.

3.2 Some notation things

Usually we write functions as $y(x) = x^2$, however a more specific notation would be

$$y : \mathbb{R} \to \mathbb{R}_+ : x \mapsto x^2$$

This means we have a function y such that it maps real numbers onto positive real numbers, and it maps a value x onto x^2. In this way, we can write a parametric function as

$$\overrightarrow{r} : \mathbb{R} \to \mathbb{R}^3 : t \mapsto \langle x(t), y(t), z(t)\rangle$$

[2] This is slightly a misuse of notation, usually we should not have a vector in this matrix.

3.2.1 Crossing Sets

We take the Cartesian product (similar to the cross product) of sets like this.

$$\{1, 2, 3\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, c)\}$$

In this same way,

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}.$$

Also,

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

3.3 Lines in 3 Dimensions

Defining lines as $y = mx + b$ does not work in three dimensions, since a line in three dimensions does not have a slope.. Instead, we define lines with a point and a direction.

$$\overrightarrow{r}(t) = \overrightarrow{A} + t\overrightarrow{v}$$

where $\overrightarrow{A}$ is the origin point of the line and $\overrightarrow{v}$ is the direction of the line, which we can think of as taking the place of the slope. Usually we combine these two terms into a single vector.