Analysis 2: Lebesgue Integration and Fourier Series

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Contents

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Midterms 2/23 and 4/13, both in class.

1.1 Introduction

We use the **Lebesgue integral** for making sense of the integral when f is not a nice function. We use **Fourier series** for approximating periodic functions by the sum of sine waves. Historically, these two topics are actually intertwined!

Suppose we have a series of functions $f_n : [a,b] \to \mathbb{R}$ all continuous and $f_n \to f$ uniformly. Then we know f is continuous and the integrals converge.

Now suppose f_n Riemann integrable and $0 \le f_{n+1} \le f_n$ and since bounded monotonic sequences converge, both

$$\lim_{n\to\infty} f_n(x)$$

and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x$$

exist. But is it true that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x?$$

For the Riemann Integral, it turns out the answer is no, because the limiting function might not be integrable.

Example 1.1

Fix $[a,b]$. List all rationals in $[a,b]$: $\{q_n\} = [a,b] \cap \mathbb{Q}$. Then let

$$f_n(x) = \begin{cases} 0 & x \in \{q_1, \dots, q_n\} \\ 1 & \text{otherwise} \end{cases}$$

Here, each f_n is Riemann integrable because it only has finitely many discontinuities, but the limiting function f_n has infinitely many discontinuities, so it is not Riemann integrable. The Lebesgue integral fills in the hole by making the integral of the limit the limit of the integrals.

1.2 Riemann Integral

Riemann Integral was given by successive partitions and then lower sums and upper sums. Namely

$$U(P,f) = \sum_{k=1}^{n-1} (p_{k+1} - p_k) \sup(f)$$

where $\sup(f)$ is on the partition. Lower sums increase and upper sums decrease upon refinements. When they are equal, we define the integral

$$\int = \sup_P L(P,f) = \inf_Q U(Q,f).$$

Lemma 1.1

f is Riemann integrable iff for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

Proof. ($\implies$): Since sup = inf, for $\varepsilon > 0$ we can find partitions Q and P such that $U(Q,f) - L(P,f) < \varepsilon$. Then let $R = P \cup Q$ and

$$U(R,f) - L(R,f) \le U(Q,f) - L(P,f) < \varepsilon.$$

($\Longleftarrow$): Given $\varepsilon > 0$, choose R such that $U(R, f) - L(R, f) < \varepsilon$. So $\inf U(Q, f) - \sup L(P, f) < \varepsilon$. Since ε is arbitrary, $\sup = \inf$.

This lets us prove

Theorem 1.1

If f is continuous, then f is Riemann integrable.

Proof. Check Real Analysis notes.

Now to characterize integrability, we want to quantify discontinuity.

Definition 1.1

Given $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$, let

$$\operatorname{osc}(f, x) = \inf_{\delta > 0} \sup\{f(y) - f(z) \mid y, z \in [a, b] \text{ with } |y - x| < \delta \text{ and } |z - x| < \delta\}.$$

Lemma 1.2

f is continuous iff $\operatorname{osc}(f, x) = 0$.

Proof. ($\Longrightarrow$): For any $\varepsilon > 0$, use continuity to choose $\delta > 0$ that implies $|f(y) - f(x)| < \varepsilon$. Then we can bound osc by 2ε.

($\Longleftarrow$): Given $\varepsilon > 0$, use osc $= 0$ to choose $\delta > 0$ so $\sup \le \varepsilon$, thus $|f(y) - f(x)| \le \varepsilon$ when $|y - x| < \delta$.

Some notation: a closed interval I has length $|I|$ and interior $\mathring{I} = (a, b)$. Now we introduce our first notion of measure.

Definition 1.2

The **Jordan Content** of a bounded $A \subseteq \mathbb{R}$ is

$$J(A) = \inf\{|I_1| + \cdots + |I_n| : I_1 \cup \cdots \cup I_n \supseteq A\}$$

- $J([a, b]) = b - a$
- For a finite set X of points, $J(X) = 0$, since we can make the Is as small as we want.

Lemma 1.3

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is bounded and for $\varepsilon > 0$, $J(\{x \mid \operatorname{osc}(f, x) \ge \varepsilon\}) = 0$. This is saying nearly the same thing as Riemann integrable functions have finitely many discontinuities.

Proof. Assume f is integrable, and fix $\varepsilon > 0$. For $\delta > 0$ to be determined, choose a partition P so $U(P, f) - L(P, f) < \delta$. So we have finitely many intervals on which $\sup - \inf$ of f is $< \infty$. Then write $I_k = [p_k, p_{k+1}]$. Call I_k "good" if

$$\sup_{I_k} f - \inf_{I_k} f < \varepsilon.$$

So if $x \in \mathring{I}_k$ and I_k good, $\operatorname{osc}(f, x) < \varepsilon$. Thus the bad intervals and the endpoints of all intervals cover

$\{x \mid \operatorname{osc}(f,x) \geq \varepsilon\}$. Observe

$$\begin{aligned}\delta &> U(P,f) - L(P,f)\\ &= \sum_k |I_k| \left(\sup_{I_k} f - \inf_{I_k} f\right)\\ &\geq \sum_{\text{bad}} \cdots\\ &\geq \sum_{\text{bad}} |I_k| \varepsilon\end{aligned}$$

so

$$\sum_{\text{bad}} |I_k| < \delta/\varepsilon.$$

Since we can choose δ after ε, we can make this as small as we want, so

$$J(\{x \mid \operatorname{osc}(f,x) \geq \varepsilon\}) = 0.$$

Lemma 1.4

If f is bounded and for $\varepsilon > 0$, $J(\{x \mid \operatorname{osc}(f,x) \geq \varepsilon\}) = 0$, then f is Riemann integrable.

Proof. $B = \sup_{[a,b]} |f| < \infty$. Fix $\varepsilon > 0$. Choose $I_1, \ldots, I_n$ so $\operatorname{osc}(f,x) \geq \varepsilon \implies x \in I_1 \cup \cdots \cup I_n$ and $|I_1| + \cdots + |I_n| < \varepsilon$. We can fatten the intervals slightly so we get

$$\operatorname{osc}(f,x) \geq \varepsilon \implies x \in \mathring{I}_1 \cup \cdots \cup \mathring{I}_n.$$

We can also merge overlapping Is, and now assume all the I_k are disjoint.

Now if $x \in [a,b] \setminus (\mathring{I}_1 \cup \cdots \cup \mathring{I}_n)$, then $\operatorname{osc}(f,x) < \varepsilon$. Choose δ_x such that $\sup f - \inf f < \varepsilon$ on $\tilde{I}_k = [a,b] \cap [x - \delta_x, x + \delta_x]$. Since $A = [a,b] \setminus (\mathring{I}_1 \cup \cdots \cup \mathring{I}_n)$, we can choose $x_1, \ldots, x_m$ so

$$A \subset (x_1 - \delta_{x_1}, x_1 + \delta_{x_1}) \cup \cdots.$$

Now let P be the union of the endpoints of I_k and $\tilde{I}_k$. Each $[p_k, p_{k+1}]$ is either contained in one of the I_j or $\tilde{I}_j$. Ok now we are introducing the normal intervals $I'_k = [p_k, p_{k+1}]$. Then

$$U(P,f) - L(P,f) = \sum_k |I'_k| (\sup_{I'_k} f - \inf_{I'_k} f).$$

Splitting this up into the bad and good intervals, the bad cases have are bounded by $\varepsilon \cdot 2B$, and the good ones are bounded by $(b-a) \cdot \varepsilon$. So both go to 0 as $\varepsilon \to 0$. Thus f is Riemann integrable.

Theorem 1.2

$f : [a,b] \to \mathbb{R}$ is Riemann integrable iff f is bounded and $\forall \varepsilon > 0$, $J(\{x \mid \operatorname{osc}(f,x) \geq \varepsilon\}) = 0$.

But Jordan content is kind of difficult to work with. In particular, it doesn't play nice with infinite sums. So we introduce the Lebesgue measure.

Definition 1.3

The **Lebesgue outer measure** of a set $A \subset \mathbb{R}$ is

$$m^*(A) = \inf\{\sum_{k=1}^{\infty} |I_k| : \cup_{k=1}^{\infty} I_k \supseteq A\}.$$

The Lebesgue measure of the rationals is 0, since if we let

$$I_k = [q_k - \varepsilon 2^{-k}, q_k + \varepsilon 2^{-k}]$$

the sum of all the lengths is 2ε.