Abstract Algebra

Brady Exoo

Spring 2026

Contents

1 Tues January 13

One midterm one final. Midterm on groups, final mostly on rings.

1.1 Introduction

Motivating question: consider a triangle. How many symmetries are there?

The answer turns out to be 6 through rotations and reflections, just the permutations of the vertices. We call this set D_6, with composition as the operation. Together, this makes a **group**. Note that all elements of D_6 can be written as compositions of rotations and reflections. We can then say that rotation and reflection **generate** the group D_6.

Definition 1.1

A **group** is a set G equipped with a binary operation $\cdot : G \times G \longrightarrow G$ satisfying the following axioms:

1. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. Identity: $\exists 1 \in G$ s.t. $a \cdot 1 = a$.
3. Inverse: $\forall a \in G$, $\exists a^{-1}$ s.t. $a^{-1} \cdot a = 1$
4. Closed: $\forall a, b \in G$, $a \cdot b \in G$. (This is kind of implicit in our definition of $\cdot$).

Some other groups/rings:

- $(\mathbb{Z}, +)$
- $(\mathbb{Q} \setminus 0, \times)$
- $(\mathbb{Z}, +, \times)$: Ring
- $(\mathbb{Z}/12, +)$
- S_3: all the permutations of 3 elements

2 Thu Jan 15

Today we will prove some basic properties of groups.

Proposition 2.1

Let $(G, \cdot)$ be a group.

1. The identity is unique.
2. For any $g \in G$, g^{-1} is unique.
3. For any $g \in G$, $(g^{-1})^{-1} = g$.
4. $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$.
5. Associativity works for any n

Proof. Just do the obvious.

Definition 2.1

The **order** of an element $g \in G$ is the smallest $n \in \mathbb{Z}_{>0}$ such that $g^n = e$.
The **order** of a group G is its cardinality $|G|$.

Definition 2.2

A subset $S \subseteq G$ **generates** the group G if

$$\{\text{iterative products of elements of } S \text{ and their inverses}\} = G.$$

For example, $\{1\}$ generates $\mathbb{Z}$. A word is just an expression like $r^2 s$. A relation in G is a word whose underlying element is the identity. We write $G =< S \mid R >$ if G is generated by S and has relations R.

Definition 2.3

The dihedral group D_{2n} is the set of symmetries of a regular n-gon.

D_{2n} has presentation $< r, s \mid r^n = e, s^2 = e, rs = sr^{-1} >$.

Definition 2.4

A group is **Abelian** if $\forall x, y \in G$, $x \cdot y = y \cdot x$.