Analysis 2: Lebesgue Integration and Fourier Series

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1 Mon January 12

Midterms 2/23 and 4/13, both in class.

1.1 Introduction

We use the **Lebesgue integral** for making sense of the integral when f is not a nice function. We use **Fourier series** for approximating periodic functions by the sum of sine waves. Historically, these two topics are actually intertwined!

Suppose we have a series of functions $f_n : [a, b] \to \mathbb{R}$ all continuous and $f_n \to f$ uniformly. Then we know f is continuous and the integrals converge.

Now suppose f_n Riemann integrable and $0 \le f_{n+1} \le f_n$ and since bounded monotonic sequences converge, both

$$\lim_{n\to\infty} f_n(x)$$

and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x$$

exist. But is it true that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x?$$

For the Riemann Integral, it turns out the answer is no, because the limiting function might not be integrable.

Example 1.1

Fix $[a, b]$. List all rationals in $[a, b]$: $\{q_n\} = [a, b] \cap \mathbb{Q}$. Then let

$$f_n(x) = \begin{cases} 0 & x \in \{q_1, \dots, q_n\} \\ 1 & \text{otherwise} \end{cases}$$

Here, each f_n is Riemann integrable because it only has finitely many discontinuities, but the limiting function f_n has infinitely many discontinuities, so it is not Riemann integrable. The Lebesgue integral fills in the hole by making the integral of the limit the limit of the integrals.

1.2 Riemann Integral

Riemann Integral was given by successive partitions and then lower sums and upper sums. Namely

$$U(P, f) = \sum_{k=1}^{n-1} (p_{k+1} - p_k) \sup(f)$$

where $\sup(f)$ is on the partition. Lower sums increase and upper sums decrease upon refinements. When they are equal, we define the integral

$$\int = \sup_P L(P, f) = \inf_Q U(Q, f).$$

Lemma 1.1

f is Riemann integrable iff for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

Proof. ($\Longrightarrow$): Since sup = inf, for $\varepsilon > 0$ we can find partitions Q and P such that $U(Q, f) - L(P, f) < \varepsilon$. Then let $R = P \cup Q$ and

$$U(R, f) - L(R, f) \le U(Q, f) - L(P, f) < \varepsilon.$$

($\Longleftarrow$): Given $\varepsilon > 0$, choose R such that $U(R, f) - L(R, f) < \varepsilon$. So $\inf U(Q, f) - \sup L(P, f) < \varepsilon$. Since ε is arbitrary, $\sup = \inf$.

This lets us prove

Theorem 1.1

If f is continuous, then f is Riemann integrable.

Proof. Check Real Analysis notes.

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Now to characterize integrability, we want to quantify discontinuity.

Definition 2.1

Given $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$, let

$$\operatorname{osc}(f, x) = \inf_{\delta > 0} \sup\{f(y) - f(z) \mid y, z \in [a, b] \text{ with } |y - x| < \delta \text{ and } |z - x| < \delta\}.$$

Lemma 2.1

f is continuous iff $\operatorname{osc}(f, x) = 0$.

Proof. ($\Longrightarrow$): For any $\varepsilon > 0$, use continuity to choose $\delta > 0$ that implies $|f(y) - f(x)| < \varepsilon$. Then we can bound osc by 2ε.

($\Longleftarrow$): Given $\varepsilon > 0$, use $\operatorname{osc} = 0$ to choose $\delta > 0$ so $\sup \le \varepsilon$, thus $|f(y) - f(x)| \le \varepsilon$ when $|y - x| < \delta$.

Some notation: a closed interval I has length $|I|$ and interior $\mathring{I} = (a, b)$. Now we introduce our first notion of measure.

Definition 2.2

The **Jordan Content** of a bounded $A \subseteq \mathbb{R}$ is

$$J(A) = \inf\{|I_1| + \cdots + |I_n| : I_1 \cup \cdots \cup I_n \supseteq A\}$$

- $J([a, b]) = b - a$
- For a finite set X of points, $J(X) = 0$, since we can make the Is as small as we want.

Lemma 2.2

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is bounded and for $\varepsilon > 0$, $J(\{x \mid \operatorname{osc}(f, x) \ge \varepsilon\}) = 0$. This is saying nearly the same thing as Riemann integrable functions have finitely many discontinuities.

Proof. Assume f is integrable, and fix $\varepsilon > 0$. For $\delta > 0$ to be determined, choose a partition P so $U(P, f) - L(P, f) < \delta$. So we have finitely many intervals on which $\sup - \inf$ of f is $< \infty$. Then write $I_k = [p_k, p_{k+1}]$. Call I_k "good" if

$$\sup_{I_k} f - \inf_{I_k} f < \varepsilon.$$

So if $x \in \mathring{I}_k$ and I_k good, $\operatorname{osc}(f, x) < \varepsilon$. Thus the bad intervals and the endpoints of all intervals cover

$\{x \mid \operatorname{osc}(f, x) \geq \varepsilon\}$. Observe

$$\begin{aligned}\delta &> U(P, f) - L(P, f) \\ &= \sum_k |I_k| \left(\sup_{I_k} f - \inf_{I_k} f\right) \\ &\geq \sum_{\text{bad}} \cdots \\ &\geq \sum_{\text{bad}} |I_k| \varepsilon\end{aligned}$$

so

$$\sum_{\text{bad}} |I_k| < \delta/\varepsilon.$$

Since we can choose δ after ε, we can make this as small as we want, so

$$J(\{x \mid \operatorname{osc}(f, x) \geq \varepsilon\}) = 0.$$

Lemma 2.3

If f is bounded and for $\varepsilon > 0$, $J(\{x \mid \operatorname{osc}(f, x) \geq \varepsilon\}) = 0$, then f is Riemann integrable.

Proof. $B = \sup_{[a,b]} |f| < \infty$. Fix $\varepsilon > 0$. Choose $I_1, \ldots, I_n$ so $\operatorname{osc}(f, x) \geq \varepsilon \implies x \in I_1 \cup \cdots \cup I_n$ and $|I_1| + \cdots + |I_n| < \varepsilon$. We can fatten the intervals slightly so we get

$$\operatorname{osc}(f, x) \geq \varepsilon \implies x \in \mathring{I}_1 \cup \cdots \cup \mathring{I}_n.$$

We can also merge overlapping Is, and now assume all the I_k are disjoint.

Now if $x \in [a, b] \setminus (\mathring{I}_1 \cup \cdots \cup \mathring{I}_n)$, then $\operatorname{osc}(f, x) < \varepsilon$. Choose δ_x such that $\sup f - \inf f < \varepsilon$ on $\tilde{I}_k = [a, b] \cap [x - \delta_x, x + \delta_x]$. Since $A = [a, b] \setminus (\mathring{I}_1 \cup \cdots \cup \mathring{I}_n)$, we can choose $x_1, \ldots, x_m$ so

$$A \subset (x_1 - \delta_{x_1}, x_1 + \delta_{x_1}) \cup \cdots.$$

Now let P be the union of the endpoints of I_k and $\tilde{I}_k$. Each $[p_k, p_{k+1}]$ is either contained in one of the I_j or $\tilde{I}_j$. Ok now we are introducing the normal intervals $I'_k = [p_k, p_{k+1}]$. Then

$$U(P, f) - L(P, f) = \sum_k |I'_k| (\sup_{I'_k} f - \inf_{I'_k} f).$$

Splitting this up into the bad and good intervals, the bad cases have are bounded by $\varepsilon \cdot 2B$, and the good ones are bounded by $(b - a) \cdot \varepsilon$. So both go to 0 as $\varepsilon \to 0$. Thus f is Riemann integrable.

Theorem 2.1

$f : [a, b] \to \mathbb{R}$ is Riemann integrable iff f is bounded and $\forall \varepsilon > 0$, $J(\{x \mid \operatorname{osc}(f, x) \geq \varepsilon\}) = 0$.

But Jordan content is kind of difficult to work with. In particular, it doesn't play nice with infinite sums. So we introduce the Lebesgue measure.

Definition 2.3

The **Lebesgue outer measure** of a set $A \subset \mathbb{R}$ is

$$m^*(A) = \inf\{\sum_{k=1}^{\infty} |I_k| : \cup_{k=1}^{\infty} I_k \supseteq A\}.$$

The Lebesgue measure of the rationals is 0, since if we let

$$I_k = [q_k - \varepsilon 2^{-k}, q_k + \varepsilon 2^{-k}]$$

the sum of all the lengths is 2ε.

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Lemma 3.1

$J(A) \geq m^*(A)$.

Proof. We can convert a finite cover to an infinite cover by "padding", adding intervals of length $\varepsilon 2^{-k}$, adding at most $\varepsilon > 0$ total length.

Lemma 3.2

If $K \subseteq \mathbb{R}$ is compact, then $m^*(K) = J(K)$.

Proof. We already know $m^*(K) \leq J(K)$ from the previous lemma. Now fix $\varepsilon > 0$. Choose $K \subseteq \bigcup_{k=1}^{\infty} I_k$ so $\sum_{k=1}^{\infty} |I_k| \leq m^*(K) + \varepsilon$. Let $\hat{I}_k = (a_k - \varepsilon 2^{-k}, b_k + \varepsilon 2^{-k})$. Since K is compact, we can create a finite subcover. We eventually get

$$\sum_{k=1}^{N} \left|\hat{I}_k\right| \leq 3\varepsilon + m^*(K).$$

Lemma 3.3

If $A \subset B$, then $J(A) \subseteq J(B)$ and $m^*(A) \leq m^*(B)$.

Proof. Exercise. Should be simple? If I_k is an open cover of B, then it is an open cover of A.

Lemma 3.4

For $A_k \subseteq \mathbb{R}$,

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

Proof. Dovetail it!

Fix $\varepsilon > 0$. For each k, choose cover for A_k that achieves $m^*(A_k)$ with at most $\varepsilon 2^{-k}$ error. Choose intervals $I_{k,1}, \ldots$ so

$$A_K \subseteq \bigcup_{j}^{\infty} I_{k,j}$$

and

$$\sum_{j=1}^{\infty} \left|I_{k,j}\right| \leq m^*|A_k| + \varepsilon 2^{-k}.$$

Let $\{\hat{I}_l\} = \{I_{k,j}\}$. Observe:

$$\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} I_{k,j} = \bigcup_{l=1}^{\infty} \hat{I}_l.$$

so

$$m^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{l=1}^{\infty} \left|\hat{I}_l\right| \leq \sum_{k=1}^{\infty} \left(m^*(A_k) + \varepsilon 2^{-k}\right).$$

Lemma 3.5

If $f : [a, b] \to \mathbb{R}$ and $\varepsilon > 0$, the

$$Z_\varepsilon = \{x \in [a, b] \mid \operatorname{osc}(f, x) \geq \varepsilon\}$$

is compact.

Proof. Z_ε is bounded. So we just need to show it is also closed by showing it contains all its limit points.

For each $n \geq 1$, choose y_n and $z_n \in [a, b]$ so $|y_n - x_n| < 2^{-n}$ and same for z_n. Then $f(y_n) - f(z_n) \geq \varepsilon - 2^{-n}$. Using $\operatorname{osc}(f, x_n) \geq \varepsilon$, since $x_n \to x$, we have $y_n \to x$ and $z_n \to x$. Conclude $\operatorname{osc}(f, x) \geq \varepsilon$.

Theorem 3.1

$f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if f is bounded and $m^*(Z) = 0$ where $Z = \{x \in [a, b] \mid f \text{ discontinuous at } x\}$.

Proof. This follows from:

1. $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff f is bounded and, for $n \geq 1$, $J(Z_{1/n}) = 0$.
2. $Z = \bigcup_{n=1}^{\infty} Z_{1/n}$.
3. $m^*(Z) = 0$ iff for all n $J(Z_{1/n}) = 0$.

4 Fri Jan 23

We continue the discussion of Cantor sets.

Lemma 4.1

C is nonempty and compact.

Lemma 4.2

$J(C_n) = \left(\frac{2}{3}\right)^n$.

Proof. By induction, we have 2^n disjoint intervals of length 3^{-n}.

Let $2^{\mathbb{N}}$ denote the set of sequences of 0 or 1. Given $x \in 2^{\mathbb{N}}$, define $b(x) = \sum_{k=0}^{\infty} 2 \cdot 3^{-k-1} \cdot x_k$.

Lemma 4.3

$b : 2^{\mathbb{N}} \to C$ is a bijection.

Proof. The possible values of $\sum_{k=0}^{n-1} 2 \cdot 3^{-k-1} \cdot x_k$ are the left endpoints of the intervals of C_n. Also $0 \leq \sum_{k=n}^{\infty} 2 \cdot 3^{-k-1} \cdot x_k \leq 3^{-n}$. Also $b(2^{\mathbb{N}}) \subseteq C_n$, and is dense in C. Also if $x_n \neq \bar{x}_n$, then $|b(x) - b(\bar{x})| \geq 3^{-n}$. This together somehow yields b is a bijection.

Let $\mathbb{1}_C(x)$ be the indicator function of C. Then

Lemma 4.4

$\operatorname{osc}(\mathbb{1}_C, x) = \mathbb{1}_C(x)$.

Proof. If $x \notin C$, ther eis an open interval (a, b) with $x \in (a, b)$ and $(a, b) \cap C = \varnothing$. So if $x \notin C$, then $\operatorname{osc}(\mathbb{1}_C, x) = 0$. On the other hand, if $x \in C$, the complement of C is dense at x. Every neighborhood of x has a point $y \notin C$. So if $x \in C$, then $\operatorname{osc} = 1$. Thus, $\mathbb{1}_C$ is Riemann integrable.

4.1 Generalized Cantor Sets

Fix $\tau : \mathbb{N} \to (0, 1)$. Build Cantor set by removing middle τ_nth fraction at step n.

Lemma 4.5

$J(C^\tau) = \prod_{n=1}^{\infty} (1 - \tau_n)$.

Proof. By induction, similar to the previous proof that $J(C_n) = (2/3)^n$.

4.2 Lebesgue Measure

We wish to replace the Riemann sum. Replace the lower sum with

$$\lim_{\delta \to 0} \sum_{k=1}^{\infty} \delta m^*(\{x \mid f(x) \geq k\delta\}).$$

But we are missing countable additivity when A_k are disjoint, which is false in general.

5 Jan 28

Last time:

- $[a, b]$ measurable
- IF A and B are measurable, then so is $A \cup B$.
- If A measurable, $\mathbb{R} \setminus A$ is measurable.
- If A_k measurable and disjoint, then $\cup A_k$ is measurable and

$$m^*(\bigcup_k A_k) = \sum_k m^*(A_k).$$

Now onto today. We want to show

Lemma 5.1

If an infinite set A_k are all measurable, then $\bigcup_k A_k$ and $\bigcap_k A_k$ are measurable.

Proof. Note we don't have A_k disjoint. Using 2 and 3, any finite combination of A_k is measurable. Basically we want to subtract all the previous $A_1, \ldots A_{k-1}$ from A_k to create disjoint sets, which we know an infinite union of which is measurable.

Lemma 5.2

If A_k measurable, then

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} m^*(A_1 \cup \cdots \cup A_k).$$

And if we also have $m^*(A_1) < \infty$, then we can say the same for intersections.

Lemma 5.3

Open and closed sets are measurable

Proof. Open sets are countable unions of intervals. And closed sets have open complements.

Lemma 5.4

If A measurable and $\varepsilon > 0$, there are $K \subseteq A \subseteq O$ with K closed, O open, and $m^*(O \setminus K) < \varepsilon$.

Theorem 5.1

A is measurable iff for every $\varepsilon > 0$, there are $K \subseteq A \subseteq O$ with K closed, O open and $m^*(O \setminus K) \leq \varepsilon$.

6 Mon Feb 2

Vitali sets are not measurable, created with a construction using the axiom of choice. Define an equivalence relation $\sim$ on $\mathbb{R}$ such that $a \sim b \iff a - b \in \mathbb{Q}$. Using the axiom of choice, we choose one point from every equivalence class intersected with $(1, -1)$. This is a Vitali set E.

Definition 6.1

aIf $a \in B$ and $B \subset \mathbb{R}$, then

$$a + B = \{a + b \mid b \in B\}$$

is the translation of B by a.

Let $q_k = \mathbb{Q} \cap (-2, 2)$. We claim that $q_k + E$ are disjoint, and

$$(-1, 1) \subseteq \bigcup_k (q_k + E) \subseteq (-3, 3).$$

Proof. (1): If there is an element in multple $q_k + E$, then using the fact E only has one element from each equivalence class, we get $q_K = q_j$. (2): $E \subseteq (-1, 1)$ and $q_k \in (-2, 2)$, so $\bigcup_k (q_k + E) \subseteq (-3, 3)$. Now supply $a \in (-1, 1)$. Since E contains an element from each equivalence class, there is an $e \in E$ such that $a - e \in \mathbb{Q} \cap (-2, 2)$, there is k so $a = q_k + e$. Since a is arbitrary, $(-1, 1) \subseteq \bigcup_k (q_k + E)$.

Lemma 6.1

Lebesgue outer measure is invariant under translation.

Theorem 6.1

A is not measurable if and only if there is a $B \subseteq \mathbb{R}$ such that

$$m^*(B) < m^*(B \cap A) + m^*(B \cap A^c).$$

Corollary 6.1

There is no function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such that

1. $m([a, b]) = b - a$
2. $A \subseteq B \implies m(A) \leq m(B)$
3. $m(a + B) = m(B)$.
4. A_k disjoint $\implies m(\bigcup_k A_k) = \sum_k m(A_k)$.

It turns out (3) is not necessary, and the above corollary is still true. We can use 1, 2, and 4 to prove regularity and then deduce (3).

We can just define the **Lebesgue Measure** as the restriction of the outer measure to only measurable sets.

Proposition 6.1

$f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if, for every open set $O \subseteq \mathbb{R}$, $f^{-1}(O) \subseteq \mathbb{R}$ is measurable.