Vector Analysis

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Contents

1 Mon January 12

Midterm on February 25 during class.

1.1 Introduction

We will focus on:

1. Differential Calculus in $\mathbb{R}^n$ (derivatives ∇f are "best linear approximations"). Basically multivariable calculus, implicit function theorem, inverse function theorem.

2. Integral calculus in $\mathbb{R}^n$: on "nice" subsets $S \subset \mathbb{R}^n$, we can define the integral. Also Fubini's theorem (integration on variables one-by-one).

3. **Manifolds**: a subset in $\mathbb{R}^n$ that locally looks like an open set in $\mathbb{R}^k$ for $0 \leq k \leq n$. Also differentiation on maps between manifolds. To help us with integration, we introduce **differential forms** (an algebraic object that can be integrated?), something something exterior algebra, exterior derivatives. And Generalized Stoke's Theorem! $\int_{\partial M} \omega = \int_M d\omega$.

1.2 Topology in the Reals

Some review of vocabulary from Analysis.

Definition 1.1

A **metric space** is a set X equipped with a distance function $d : X \times X \to \mathbb{R}$ such that

- $d(x, y) = d(y, x)$
- $d(x, y) \geq 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$

We can have metrics such as the usual Euclidean metric $||x - y||$ and the sup metric $\max_{1 \leq i \leq n} |x_i - y_i|$. We also commonly use the ε-ball:

$$B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}.$$

In the sup metric d_∞, the ε-ball for $n = 2$ is a square.

Definition 1.2

A set $U \subseteq X$ is **open** if $\forall x \in U$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset U$.
A set $V \subseteq X$ is **closed** if $X \setminus V$ is open.

Here, we define **neighborhoods** of x to just be an open set containing x. We also use

- $\operatorname{Int} A = \{x \in A \mid B_\varepsilon(x) \subset A \text{ for some } \varepsilon > 0\}$.
- Limit points of $A = \{x \in X \mid (B_\varepsilon(x) \setminus \{x\}) \cap A \neq \varnothing \quad \forall \varepsilon > 0.\}$
- $\overline{A} = A \cup$ limit points of A.
- $\operatorname{Bd} A = \overline{A} \setminus \operatorname{Int}(A)$.

Definition 1.3

For metric spaces X and Y and $f : X \to Y$, f is continuous at x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d_y(f(x), f(x_0)) < \varepsilon$ whenever $d_x(x, x_0) < \delta$.

Proposition 1.1

For any dimension n,

$$B_{\infty,\varepsilon/\sqrt{n}}(x) \subset B_{2,\varepsilon}(x) \subset B_{\infty,\varepsilon}(x).$$

Proof. Homework.

Basically this shows us if $\varepsilon \to 0$ in one metric, $\varepsilon \to 0$ in the other. And a set in d_2 is open/closed iff it is open/closed in d_∞. Similarly, a map $f : \mathbb{R}^n \to X$ or $X \to \mathbb{R}^n$ is continuous wrt d_∞ iff it is continuous wrt d_2.

Proof. Homework.

Proposition 1.2

$f : X \to \mathbb{R}^n$ with $f = (f_1, \dots, f_n)$. Then f is continuous iff $f_1 \dots f_n$ are all continuous.

Proof. For $\implies$, we say f is continuous in d_2, and using the definition of d_2 we can put a bound on all of $|f_i(x) - f_i(y)|$.
For $\impliedby$, we choose δ_i such that $d_x(x, y) < \delta_i \implies |f_i(x) - f_i(y)| < \frac{\varepsilon}{\sqrt{n}}$. Then the math just works out when using the minimum of all the δ_i.

2 Wed Jan 14

Differentiation today?

Definition 2.1

We say $f(x)$ approaches y_0 as x approaches x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_y(f(x), y_0) < \varepsilon$ whenever $d_x(x, x_0) < \delta$. In this case, we write $\lim_{x\to x_0} f(x) = y_0$.

Then if x_0 is a limit point of X, f is continuous at x_0 iff $\lim_{x\to x_0} f(x) = f(x_0)$. Also for an open set $U \subset \mathbb{R}$ and $f : U \to \mathbb{R}^n$, f is differentiable if the usual limit exists. Then we can also write

$$\lim_{h\to 0} \frac{f(x+h) - [f(x) + f'(x)h]}{h} = 0.$$

Proposition 2.1

Write $f = (f_1, \dots, f_n)$. Then f is differentiable at x iff all $f_1, \dots, f_n$ are differentiable at x. In this case, $f'(x) = (f_1', \dots, f_n')$.

Proof. Homework. Just use ε-δ?

Definition 2.2

Let $U \subset \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^n$, $u \in \mathbb{R}^m \setminus \{0\}$. The **directional derivative** of f along u is defined to be

$$f'(x; u) = \lim_{t\to 0} \frac{f(x+tu) - f(x)}{t}$$

We can regard the directional derivative as a dot product of a matrix and a vector. E.g. for $f(x_1, x_2) = x_1x_2$ and $u = (u_1, u_2)$,

$$f'(x; u) = \begin{bmatrix} x_2 & x_1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Proposition 2.2
If $f'(x;u)$ exists and $\lambda \neq 0$, $f'(x;\lambda u) = \lambda f'(x;u)$.

Proof. Substitute $t \to t/\lambda$ in the definition.

Definition 2.3

For $x \in U$ and $t \in \{1, 2, \ldots, m\}$, define

$$D_i f(x) = \frac{\partial f}{\partial x_i} := f'(x; e_i)$$

where e_i is a basis vector in $\mathbb{R}^n$.

Now we can extend our definition of the derivative to $\mathbb{R}^m \to \mathbb{R}^n$.

Definition 2.4

Let $U \subset \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^n$, $x \in U$. We say f is differentiable at x if there exists an $n \times m$ matrix $Df(x)$ such that

$$\lim_{h \to 0} \frac{f(x+h) - [f(x) + Df(x) \cdot h]}{\|h\|} = 0$$

where $h \in \mathbb{R}^m$.

Now we wish to show $Df(x)$ is unique if it exists. If we have two matrices A_1 and A_2 that satisfy the property above, we can take the difference, leaving us with

$$\lim_{h \to 0} (A_1 - A_2) \cdot \frac{h}{\|h\|} = 0.$$

We can take $h = te_i$ with $t > 0$, then $\forall i$,

$$(A_1 - A_2) \cdot e_i = 0$$

thus all columns of $A_1 - A_2$ are 0, so $A_1 - A_2 = 0$.

Note:-
$Df(x)$ has to be the matrix that sends $f(x+h) - [f(x) + B \cdot h]$ to 0 as fast as possible otherwise B would also be a derivative of f.

Just like for $\mathbb{R}$, we have differentiable $\implies$ continuous.

Proposition 2.3
If $f(x) = B \cdot x + b$ where B is an $n \times m$ matrix and $b \in \mathbb{R}^n$, then $Df(x) = B$.

Proof. Exercise.

Proposition 2.4
If f differentiable at x, then $\forall u \in \mathbb{R}^m \setminus \{0\}$,

$$f'(x;u) = Df(x) \cdot u.$$

Proof. Substitute $h \to tu$.