Exponential Map in SE(2)

bexultan-rakhim

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1 How Exponential map works in a Circle

First of all, the idea of exponential map comes from the fact that rotation operation has certain constraints:

$$R^T R = I \tag{1}$$

If you differentiate above equation, you will get the following equation:

$$R^T \dot{R} + \dot{R}^T R = 0 \tag{2}$$

Given the above equation, we can try to find what is the value of $\dot{R}$. Good guess is to substitute $\dot{R} = RX$, where X - some unknown matrix. Therefore, you will get the following equation:

$$\begin{aligned} R^T RX + X^T R^T R = 0 \\ X + X^T = 0 \end{aligned} \tag{3}$$

if, a, b, c, d are elements of X, then following is the condition for X:

$$\begin{aligned} X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \begin{bmatrix} 2a & b + c \\ c + b & 2d \end{bmatrix} = 0 \end{aligned} \tag{4}$$

Therefore, we infer that a, d is zero and $c = -b = \theta$:

$$X = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \tag{5}$$

Keep in mind, that now X is tangent space to R that depends only on one parameter θ. Therefore, we can try to find the family of all solutions to $R(\theta)$ that satisfy constraint equation (1) by integrating $\dot{R} = RX$. Now, imagine that the a, b, c, d are elements of the R, then we can try to find the equation that satisfies above differential equation:

$$\begin{bmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} = \theta \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} \tag{6}$$

It is very easy to find the equations $\ddot{b} = -\theta^2 b$ and $\ddot{d} = -\theta^2 d$. Solution, therefore should be a sine and cosine functions. If you put the constraints, then the full solutions come out to be:

$$R(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \tag{7}$$

There is alternative way to solve this equation. Remind yourself that $e^{i\theta} = cos(\theta) + isin(\theta)$. If we take a matrix exponent of the X we will have the same result. Because of the Taylor series expansion:

$$Exp(X) = I + \frac{1}{1!}X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + ... \tag{8}$$

also, observe that $X^2 = -\theta^2 I$ and $X^3 = -\theta^2 X$ and $X^4 = \theta^4 I$ and so on. We can divide each part of the equation into separate parts and achieve the same results. as equation (7). Equation (7) is exponential map given by constraint (1). Therefore, we can use the same technique to define any other constraint.

2 Exponential map in SE(2)

SE(2) defines two translations and one rotation. We can define the solution in the following form:

$$M = \begin{bmatrix} cos(\theta) & -sin(\theta) & x \\ sin(\theta) & cos(\theta) & y \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Tangent space is differentiation of this matrix M for three values $dx, dy, d\theta$, therefore is defined by the following equation:

$$\delta = \begin{bmatrix} 0 & -d\theta & dx \\ d\theta & 0 & dy \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

The rotation part is defined by the equation (5) because rotation rotation should satisfy constraint (1). In SE(2) there are no other constraints except rotation, so dx, dy can have any form. Therefore, integrating this equation is same as finding exponential map:

$$Exp(\delta) = I + \frac{1}{1!}\delta + \frac{1}{2!}\delta^2 + \frac{1}{3!}\delta^3 + ... \tag{11}$$

it is very easy to show that the rotation part of the exponential map will still be the same as above equation. However, translation part will have quite interesting solution. We can define solution in the following form:

$$Exp(\delta) = \begin{bmatrix} R(\theta) & V(\theta)p \\ 0 & 1 \end{bmatrix} \tag{12}$$

Here$V(\theta) = sin(\theta)/\theta I + (1 - cos(\theta))/\theta[1]_x$. so that $[1]_x = [0, -1; 1, 0]$. Alternative form of the solution is:

$$V(\theta) = I + \frac{1}{2!}X + \frac{1}{3!}X^2 + ... \tag{13}$$

Here X is same as in equation (5) and I is identity 2x2 matrix. This is very useful, form, because for very small angles, we might not need high accuracy. At the same time, we have equation to solve the exponential map in SE(2) precisely.

How this is useful? If you want to know how the values change from the given M_1 with changing values $\delta_2 = (x_2, y_2, \theta_2)$, you can use the following equation:

$$M_2 = M_1 Exp(\delta_2) \tag{14}$$

Moreover, this equation (14) generalizes to many class of solutions, not only in SE(2).

References

[1] Solà, J. & Deray, J. & Atchuthan, D. (2018), A micro Lie theory for state estimation in robotics, link: https://arxiv.org/pdf/1812.01537.pdf

[2] Ethan Eade (2017), Lie Groups for 2D and 3D Transformations. link: https://www.ethaneade.com/lie.pdf