Math Stats Proofs

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0.1 About this document

This document does *not* contain all the proofs required for STA3041F, as we have not been given a list of proofs to learn. When I have found a proof used in a past paper, I've listed the past paper in the margins with the year, and a code like T2 for Test 2 or E for Exam.

Example note

0.1.1 PGF of a Branching Process

Work in progress

0.1.2 Expectation of a Branching Process

Note that:

- $\mathbb{G}_X(1) = 1$ is true for any random variable X.
- $\mathbb{G}'_X(1) = \mathbb{E}[X]$ is true for any random variable X.

$$
\begin{aligned}
\mathbb{E}\left[Z_m\right] &= \left.\frac{d}{ds}\mathbb{G}_{Zm}(s)\right|_{s=1} \\
&= \left.\frac{d}{ds}\mathbb{G}_X\left(\mathbb{G}_{Z_{m-1}}(s)\right)\right|_{s=1} \\
&= \mathbb{G}'_X\left(\mathbb{G}_{Zm-1}(1)\right)\cdot{\mathbb{G}_{Zm-1}}'(1) \quad \text{(Chain Rule)} \\
&= \mathbb{G}'_X\left(1\right){\mathbb{G}_{Zm-1}}'(1) \\
&= \mathbb{E}\left[X\right]\mathbb{E}\left[Z_{m-1}\right] \\
&= \mathbb{E}\left[X\right]\mathbb{E}\left[X\right]\mathbb{E}\left[Z_{m-2}\right] \\
&\vdots \\
\mathbb{E}\left[Z_m\right] &= \mathbb{E}\left[X\right]^m \\
&= \mu^m
\end{aligned}
$$

0.1.3 Variance of a Branching Process

Work in progress

0.1.4 Probability of Eventual Extinction of a Branching Process

There's two parts to this,

1. The part that $G(s) - s = 0$ is the probability of eventual extinction.
2. The part that the probability of eventual extinction is the smallest positive root of s.

Proof of $G(s) - s = 0$

Note that this isn't exactly how it's laid out in the notes. I've tried to simplify the terminology a bit.

$$\begin{aligned}
\eta &= \mathbb{P}\left[\bigcup_{m=1}^{\infty} Z_m = 0\right] \\
&= \lim_{m\to\infty} \mathbb{P}\left[Z_m = 0\right] \\
&= \lim_{m\to\infty} \mathbb{G}_{Z_m}(0) \\
&= \lim_{m\to\infty} \mathbb{G}_X(\mathbb{G}_{Z_{m-1}}(0)) \\
&= \lim_{m\to\infty} \mathbb{G}_X(\mathbb{P}\left[Z_{m-1} = 0\right]) \\
&= \mathbb{G}_X(\lim_{m\to\infty} \mathbb{P}\left[Z_{m-1} = 0\right]) \\
&= \mathbb{G}_X\left(\mathbb{P}\left[\bigcup_{m=1}^{\infty} Z_m = 0\right]\right) \\
\eta &= \mathbb{G}_X(\eta)
\end{aligned}$$

Proof that the probability of eventual extinction is the smallest positive root

Given that the probability of eventual extinction η is a root of $\mathbb{G}_X(s) - s = 0$, we now prove that η is the smallest non-negative root, note that:

- We define w as a solution to $\mathbb{G}_X(w) = w$
- The process, by definition starts out with positive population: $Z_0 > 0 \implies \mathbb{P}\left[Z_0 = 0\right] = 0$
- $\eta_m = \mathbb{G}_X(\eta_{m-1}) = \mathbb{G}_X(\mathbb{G}_X(\eta_{m-2})) = \ldots$
- The generating function has infinite positive terms, so is non-decreasing: $\mathbb{G}_X(s) \leq \mathbb{G}_X(t) \implies s \leq t$

So, to actually do the proof:

$$\begin{aligned}
\eta_0 &= \mathbb{P}\left[Z_0 = 0\right] = 0 \leq w \\
\eta_0 &\leq w \\
\mathbb{G}(\eta_0) &\leq \mathbb{G}(w) = w \quad \text{(by definition of } w\text{)} \\
\eta_1 &\leq w \\
\mathbb{G}(\eta_1) &\leq \mathbb{G}(w) \\
\eta_2 &\leq w \\
\mathbb{G}(\eta_2) &\leq \mathbb{G}(w) \\
\eta_3 &\leq w \\
&\vdots \\
\text{So } \eta_n &\leq w \quad \text{for all } n \\
\eta &= \lim_{m\to\infty} \eta_m \leq w
\end{aligned}$$

So η will always be smaller than any other possible root w to the equation $\mathbb{G}(s) = s$

0.1.5 MGFs for First Passage and First Return

Work in progress

0.1.6 Chapman-Kolmogorov Equations

2019 T1

Given that a chain has the Markov property, the equations are, for all $0 \leq v \leq t$:

$$\begin{aligned}
p_{ij}(s,t) :&= \mathbb{P}\left[X_{s+t} = j | X_s = i\right] \\
&= \sum_{k\in\mathcal{U}} p_{ik}(s,v) \cdot p_{kj}(s+v, t-v)
\end{aligned}$$

The proof is due to the law of total probability and the Markov property. It might help to first recall that:

$$\mathbb{P}\left[A|B\right] = \frac{\mathbb{P}\left[A\cap B\right]}{\mathbb{P}\left[B\right]}$$

$$\iff$$

$$\mathbb{P}\left[A\cap B\right] = \mathbb{P}\left[A|B\right]\cdot\mathbb{P}\left[B\right]$$

So the proof is:

$$\begin{aligned}
p_{ij}(s,t) &= \mathbb{P}\left[X_{s+t} = j | X_s = i\right] \\
&= \sum_{k \in \mathcal{U}} \mathbb{P}\left[X_{s+t} = j, X_{s+v} = k | X_s = i\right] \\
&= \sum_{k \in \mathcal{U}} \mathbb{P}\left[X_{s+v} = k | X_s = i\right] \cdot \mathbb{P}\left[X_{s+t} = j | X_{s+v} = k, X_s = i\right] \\
&= \sum_{k \in \mathcal{U}} \mathbb{P}\left[X_{s+v} = k | X_s = i\right] \cdot \mathbb{P}\left[X_{s+t} = j | X_{s+v} = k\right] \\
&= \sum_{k \in \mathcal{U}} p_{ik}(s,v) \cdot p_{kj}(s+v, t-v)
\end{aligned}$$

0.1.7 Probability of one Exponential RV being less than another

2019 T1

Where $X_i \overset{\text{iid}}{\sim} Exp(\lambda_i)$ for $i \in \{1, 2\}$:

$$\begin{aligned}
\mathbb{P}\left[X_1 < X_2\right] &= \int_0^\infty \int_0^{x_2} \left(\lambda_1 e^{-\lambda_1 x_1}\right) \cdot \left(\lambda_2 e^{-\lambda_2 x_2}\right) \, dx_1 \, dx_2 \\
&= \int_0^\infty \int_0^{x_2} \lambda_1 e^{-\lambda_1 x_1} \, dx_1 \, \lambda_2 e^{-\lambda_2 x_2} \, dx_2 \\
&= \int_0^\infty \left(1 - e^{-\lambda_1 x_2}\right) \cdot \left(\lambda_2 e^{-\lambda_2 x_2}\right) \, dx_2 \\
&= 1 - \lambda_2 \int_0^\infty e^{-(\lambda_1 + \lambda_2)} \, dx_2 \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$