CEng 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 4

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1. (a) Below, we give the difference equation of the given system.

$$y[n-2] - 6y[n-1] + 8y[n] = 16x[n] \tag{1}$$

(b) To find the frequency response of the system, i.e., $H(e^{j\omega})$, we will take the Fourier transform of the system and then use the time shifting property of the discrete-time Fourier transform, i.e., $\mathcal{F}\{x[n-n_0]\} = e^{-j\omega n_0}X(e^{j\omega})$.

$$\begin{aligned}
e^{-2j\omega}Y(e^{j\omega}) - 6e^{-j\omega}Y(e^{j\omega}) + 8Y(e^{j\omega}) &= 16X(e^{j\omega}) \\
(e^{-2j\omega} - 6e^{-j\omega} + 8)Y(e^{j\omega}) &= 16X(e^{j\omega}) \\
H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} &= \frac{16}{e^{-2j\omega} - 6e^{-j\omega} + 8} \\
H(e^{j\omega}) &= \frac{16}{(e^{-j\omega} - 4)(e^{-j\omega} - 2)} \\
H(e^{j\omega}) &= \frac{8}{e^{-j\omega} - 4} - \frac{8}{e^{-j\omega} - 2}
\end{aligned} \tag{2}$$

(c) Rewrite the frequency response $H(e^{j\omega})$ as follows.

$$\begin{aligned}
H(e^{j\omega}) &= \frac{8}{e^{-j\omega} - 4} - \frac{8}{e^{-j\omega} - 2} \\
H(e^{j\omega}) &= \frac{4}{1 - \left(\frac{1}{2}\right)e^{-j\omega}} - \frac{2}{1 - \left(\frac{1}{4}\right)e^{-j\omega}}
\end{aligned} \tag{3}$$

To find impulse response of the system, we will use the well know property: $\mathcal{F}\{a^n u[n]\} = \frac{1}{1-ae^{-j\omega}}$ for $|a| < 1$.

$$\begin{aligned}
h[n] &= \left(4\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n\right)u[n] \\
h[n] &= \left(2^{2-n} - 2^{1-2n}\right)u[n]
\end{aligned} \tag{4}$$

(d) First, we take the Fourier transform of $x[n] = \left(\frac{1}{4}\right)^n u[n]$ by using the formula $\mathcal{F}\{a^n u[n]\} = \frac{1}{1-ae^{-j\omega}}$ for $|a| < 1$.

$$\mathcal{F}\left\{\left(\frac{1}{4}\right)^n u[n]\right\} = \frac{1}{1 - \left(\frac{1}{4}\right)e^{j\omega}} = \frac{4}{4 - e^{j\omega}} \tag{5}$$

We know that $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$; hence, we get the following.

$$\begin{aligned}
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
&= \frac{32}{(e^{-j\omega} - 4)(4 - e^{j\omega})} - \frac{32}{(e^{-j\omega} - 2)(4 - e^{j\omega})} \\
&= \frac{32}{(e^{-j\omega} - 2)(e^{j\omega} - 4)} - \frac{32}{(e^{-j\omega} - 4)^2} \\
&= \frac{16}{e^{-j\omega} - 4} - \frac{16}{e^{-j\omega} - 2} + \left(\frac{32}{3}\right)\left(\frac{1}{e^{-j\omega} - 4} + \frac{(1 - e^{-j\omega})}{(e^{-j\omega} - 4)^2}\right) \\
&= (-4)\frac{1}{1 - \frac{1}{4}e^{-j\omega}} + (8)\frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \left(-\frac{8}{3}\right)\frac{1}{1 - \frac{1}{4}e^{-j\omega}} + \left(\frac{2}{3}\right)\frac{(1 - e^{j\omega})}{(1 - \frac{1}{4}e^{-j\omega})^2}
\end{aligned} \tag{6}$$

Now, we will take the inverse Fourier transform of $Y(e^{j\omega})$ by using following pairs: $\mathcal{F}\{a^n u[n]\} = \frac{1}{1-ae^{-j\omega}}$ for $|a| < 1$, $\mathcal{F}\{(n+1)a^n u[n]\} = \frac{1}{(1-ae^{-j\omega})^2}$ for $|a| < 1$, and $\mathcal{F}\{x[n]-x[n-1]\} = (1-e^{-j\omega})X(e^{j\omega})$ for $\mathcal{F}\{x[n]\} = X(e^{j\omega})$.

$$y[n] = (-4)\left(\frac{1}{4}\right)^n u[n] + (8)\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{8}{3}\right)\left(\frac{1}{4}\right)^n u[n] + \left(\frac{2}{3}\right)\left((n+1)\left(\frac{1}{4}\right)^n u[n] - n\left(\frac{1}{4}\right)^{n-1} u[n-1]\right) \tag{7}$$

Since $n\left(\frac{1}{4}\right)^{n-1} u[n-1]$ is 0 at $n = 0$, it is equal to $n\left(\frac{1}{4}\right)^{n-1} u[n]$. Then, we rewrite $y[n]$ as follows.

$$y[n] = \left(2^{3-n} - 2^{-2n}(2n+6)\right) u[n] \tag{8}$$

2. Since two systems are combined in parallel, the overall impulse response of the system is $h[n] = h_1[n] + h_2[n]$. By linearity property of Fourier transform, we can deduce that $H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$. By using the well-known formula $\mathcal{F}\{a^n u[n]\} = \frac{1}{1-ae^{-j\omega}}$ for $|a| < 1$, we can find $H_1(e^{j\omega})$, i.e., $H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}} = \frac{3}{3-e^{-j\omega}}$. Since $H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega})$, we can compute $H_2(e^{j\omega})$ as follows.

$$\begin{aligned} H_2(e^{j\omega}) &= H(e^{j\omega}) - H_1(e^{j\omega}) \\ &= \frac{5e^{-j\omega} - 12}{e^{-2j\omega} - 7e^{-j\omega} + 12} - \frac{3}{3 - e^{-j\omega}} \\ &= \frac{8}{e^{-j\omega} - 4} - \frac{3}{e^{-j\omega} - 3} - \frac{3}{3 - e^{-j\omega}} \\ &= \frac{8}{e^{-j\omega} - 4} + \frac{3}{3 - e^{-j\omega}} - \frac{3}{3 - e^{-j\omega}} \\ &= \frac{8}{e^{-j\omega} - 4} \end{aligned} \tag{9}$$

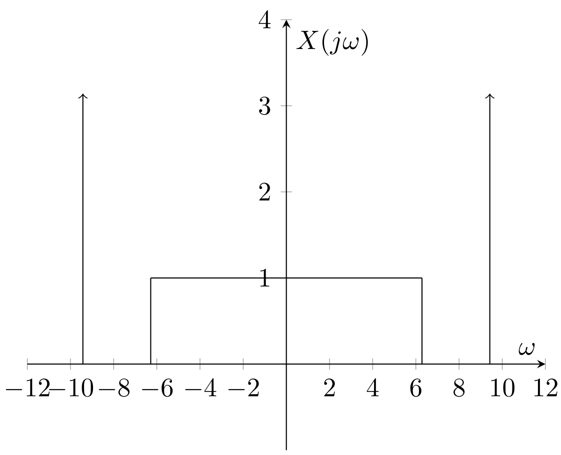
By using the formula $\mathcal{F}\{a^n u[n]\} = \frac{1}{1-ae^{-j\omega}}$ for $|a| < 1$, $h_2[n]$ is as follows.

$$h_2[n] = (-2^{1-2n})u[n] \tag{10}$$

3. (a) Below, we give the Fourier transform of $x(t)$ by using well-know Fourier transform pairs for the sinc function and cosine.

$$X(j\omega) = \begin{cases} 1, & \text{if } |\omega| < 2\pi \\ \pi\delta(0), & \text{if } |\omega| = 3\pi \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

Below, we plot $X(j\omega)$.

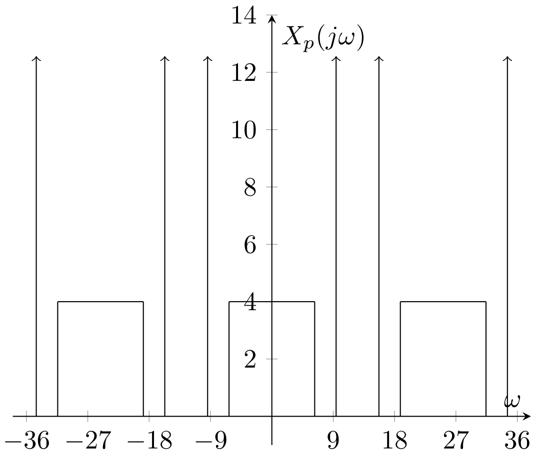


(b) Nyquist frequency, i.e., ω_M, of the system is 3π and Nyquist period is $\frac{2}{3}$.

(c) By sampling theorem, we know that the sampling fruquency shall be more than Nyquist rate, i.e., $\omega_s > 2\omega_M$. Nyquist rate ($2\omega_M$) is two times Nyquist frequency (ω_M); hence, Nyquist rate is 6π. Therefore, we pick $\omega_s = 8\pi$. Then, we have $T = \frac{1}{4}$. From these, below, we find the Fourier transform of $x_p(t)$, i.e., $X_p(j\omega)$.

$$\begin{aligned} X_p(j\omega) &= \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s)) \\ &= 4\sum_{k=-\infty}^{+\infty} X(j(\omega - 8\pi k)) \end{aligned} \tag{12}$$

Below, we plot $X(j\omega)$.



4. (a) Since, $\omega_s = \pi$, $T = 2$. Below, we find $X_d(e^{j\omega})$.

$$\begin{aligned} X_d(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - 2\pi k)/T) \\ &= \frac{1}{2} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - 2\pi k)/2) \end{aligned} \tag{13}$$

We can rewrite $X_d(e^{j\omega})$ as follows.

$$X_d(e^{j\omega}) = \begin{cases} \frac{\omega - 2\pi k}{\pi}, & \text{if } 2\pi k - \frac{\pi}{2} \leq \omega \leq 2\pi k + \frac{\pi}{2} \text{ for } k = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

(b) Below, we find the $H(e^{j\omega})$ by using the well-know Fourier transform pair for cosine.

$$H(e^{j\omega}) = \pi \sum_{l=-\infty}^{+\infty} (\delta(\omega - \pi - 2\pi l) + \delta(\omega + \pi - 2\pi l)) \tag{15}$$

(c) By using the multiplication property of discrete-time Fourier transform, we can find $Y_d(e^{j\omega})$ as follows.

$$\begin{aligned} Y_d(e^{j\omega}) &= \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta \\ &= 1 \end{aligned} \tag{16}$$