CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 2

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1. (a) The differential equation represented by the given system is presented below. In the second step, we further simplify the expression for the sake of presentation.

$$\begin{gathered}\int_{-\infty}^{t} x(\tau) - 4y(\tau)d\tau = y(t) \\ \frac{dy(t)}{dt} + 4y(t) = x(t)\end{gathered} \tag{1}$$

(b) For the input $x(t) = (e^{-t} + e^{-2t})u(t)$ we have the following differential equation

$$y'(t) + 4y(t) = e^{-t} + e^{-2t} \quad \text{for } t > 0 \tag{2}$$

Since it is not a homogeneous differential equation the solution should be in the from of $y(t) = y_H(t) + y_P(t)$. To obtain $y_H(t)$ part of the solution we need to solve the following equation

$$y'(t) + 4y(t) = 0 \tag{3}$$

We hypothesize the solution as $y_H(t) = Ae^{st}$ and plug this into the differential equation.

$$\begin{aligned} Ase^{st} + 4Ase^{st} &= 0 \\ Ae^{st}(s+4) &= 0 \\ s &= -4 \\ y_H(t) &= Ae^{-4t} \quad \text{for } t > 0 \end{aligned} \tag{4}$$

To obtain $y_P(t)$ we first should divide it into two for simplicity as $y_P(t) = y_{P_1}(t) + y_{P_2}(t)$.

$y_{P_1}(t)$ is a particular solution for

$$y'(t) + 4y(t) = e^{-t} \quad \text{for } t > 0 \tag{5}$$

$y_{P_2}(t)$ is a particular solution for

$$y'(t) + 4y(t) = e^{-2t} \quad \text{for } t > 0 \tag{6}$$

$y_P(t) = y_{P_1}(t) + y_{P_2}(t)$ is a particular solution for

$$y'(t) + 4y(t) = e^{-t} + e^{-2t} \quad \text{for } t > 0 \tag{7}$$

We hypothesize $y_{P_1}(t) = Be^{-t}$ and plug it into

$$\begin{aligned} y'(t) + 4y(t) &= e^{-t} \quad \text{for } t > 0 \\ -Be^{-t} + 4Be^{-t} &= e^{-t} \\ B &= \frac{1}{3} \end{aligned} \tag{8}$$

We hypothesize $y_{P_2}(t) = Ce^{-2t}$ and plug it into

$$\begin{aligned} y'(t) + 4y(t) &= e^{-t2} \quad \text{for } t > 0 \\ -2Ce^{-2t} + 4Ce^{-2t} &= e^{-2t} \\ C &= \frac{1}{2} \end{aligned} \tag{9}$$

We found $y_P(t) = y_{P_1}(t) + y_{P_2}(t) = \frac{e^{-t}}{3} + \frac{e^{-2t}}{2}$ for $t > 0$ therefore,

$$y(t) = Ae^{-4t} + \frac{e^{-t}}{3} + \frac{e^{-2t}}{2} \tag{10}$$

We assume that the system is initially at rest therefore we say that $y(0) = 0$ which yields to

$$\begin{aligned} y(0) &= A + \frac{1}{3} + \frac{1}{2} = 0 \\ A &= -\frac{5}{6} \\ y(t) &= -\frac{5e^{-4t}}{6} + \frac{e^{-t}}{3} + \frac{e^{-2t}}{2} \quad \text{for } t > 0 \\ y(t) &= \left(-\frac{5e^{-4t}}{6} + \frac{e^{-t}}{3} + \frac{e^{-2t}}{2}\right) u(t) \end{aligned} \tag{11}$$

2. (a) Below, we give the solution and the graph for $y[n]$.

$$\begin{aligned} &x[n] = \delta[n-1] - 3\delta[n-2] + \delta[n-3] \\ &h[n] = \delta[n+1] + 2\delta[n] - 3\delta[n-1] \\ &y[n] = x[n] * h[n] = \sum_{-\infty}^{+\infty} x[k]h[n-k] \\ &y[0] = \ldots + x[1]h[0-1] + x[2]h[0-2] + x[3]h[0-3] + \ldots = \ldots + (1)(1) + (-3)(0) + (1)(0) = 1 \\ &y[1] = \ldots + x[1]h[1-1] + x[2]h[1-2] + x[3]h[1-3] + \ldots = \ldots + (1)(2) + (-3)(1) + (1)(0) = -1 \\ &y[2] = \ldots + x[1]h[2-1] + x[2]h[2-2] + x[3]h[2-3] + \ldots = \ldots + (1)(-3) + (-3)(2) + (1)(1) = -8 \\ &y[3] = \ldots + x[1]h[3-1] + x[2]h[3-2] + x[3]h[3-3] + \ldots = \ldots + (1)(0) + (-3)(-3) + (1)(2) = 11 \\ &y[4] = \ldots + x[1]h[4-1] + x[2]h[4-2] + x[3]h[4-3] + \ldots = \ldots + (1)(0) + (-3)(0) + (1)(-3) = -3 \end{aligned} \tag{12}$$

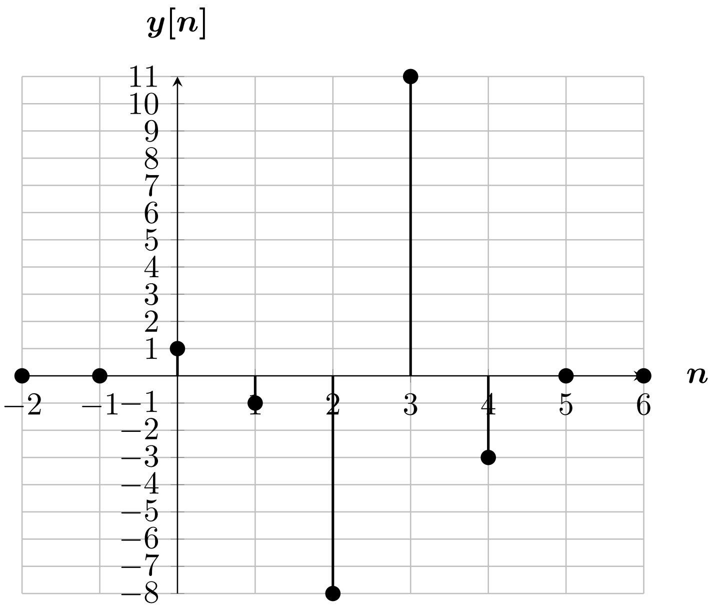

Figure 1: n vs $y[n]$.

(b) Below, we give the solution for $y(t)$.

$$
\begin{aligned}
x(t) &= u(t) + u(t-1) \\
h(t) &= e^{-2t}\cos(t)u(t) \\
y(t) &= \frac{dx(t)}{dt} * h(t) \\
&= \frac{d(u(t)+u(t-1))}{dt} * h(t) \\
&= \left(\frac{du(t)}{dt} + \frac{du(t-1)}{dt}\right) * h(t) \\
&= (\delta(t) + \delta(t-1)) * h(t) \\
&= (\delta(t) * h(t)) + (\delta(t-1) * h(t)) \\
&= h(t) + h(t-1) \\
&= e^{-2t}\cos(t)u(t) + e^{-2(t-1)}\cos(t-1)u(t-1)
\end{aligned} \tag{13}
$$

3. (a) Below, we give the solution for $y(t)$.

$$
\begin{aligned}
x(t) &= e^{-t}u(t) \\
h(t) &= e^{-3t}u(t) \\
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{+\infty} e^{-\tau}u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau \\
&= \int_{0}^{+\infty} e^{-\tau}e^{-3(t-\tau)}u(t-\tau)d\tau \\
&= \int_{0}^{t} e^{-\tau}e^{-3(t-\tau)}d\tau \\
&= \int_{0}^{t} e^{-\tau}e^{-3t+3\tau}d\tau \\
&= \int_{0}^{t} e^{-3t}e^{2\tau}d\tau \\
&= e^{-3t}\int_{0}^{t} e^{2\tau}d\tau \\
&= e^{-3t}\left(\frac{e^{2t}}{2} - \frac{1}{2}\right) \\
&= \frac{e^{-t}}{2} - \frac{e^{-3t}}{2}
\end{aligned} \tag{14}
$$

(b) In this question, we need to consider three cases for the values of t while computing the convolution, i.e., $t < 1$, $1 \le t \le 2$, and $2 < t$.

$$
y(t) = \begin{cases} 0 & \text{if } 1 < t \\ \int_1^t e^{t-\tau}d\tau & \text{if } 1 \le t \le 2 \\ \int_1^2 e^{t-\tau}d\tau & \text{if } 2 < t \end{cases} \tag{15}
$$

When we evaluate the integrals, we get the following result.

$$
y(t) = \begin{cases} 0 & \text{if } 1 < t \\ e^{t-1} - 1 & \text{if } 1 \le t \le 2 \\ e^{t-1} - e^{t-2} & \text{if } 2 < t \end{cases} \tag{16}
$$

4. (a) The characteristic equation of this equation is the following

$$
\begin{aligned}
r^2 - 15r + 26 &= 0 \\
(r-2)(r-13) &= 0 \\
r_1 = 2,\ r_2 &= 13
\end{aligned} \tag{17}
$$

Therefore,

$$\begin{aligned} y[n] &= A(2^n) + B(13^n) \\ y[0] &= A + B = 10 \\ y[1] &= 2A + 13B = 42 \\ A &= 8 \\ B &= 2 \end{aligned} \tag{18}$$

So the solution is

$$y[n] = 2^{n+3} + 2(13^n) \tag{19}$$

(b) The characteristic equation of this equation is the following

$$\begin{aligned} r^2 - 3r + 1 = 0 \\ b^2 - 4ac = 9 - 4 = 5 > 0 \\ r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{3 + \sqrt{5}}{2} \\ r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{3 - \sqrt{5}}{2} \end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned} y[n] &= A\left(\frac{3+\sqrt{5}}{2}\right)^n + B\left(\frac{3-\sqrt{5}}{2}\right)^n \\ y[0] &= A + B = 1 \\ y[1] &= A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right) = 2 \end{aligned} \tag{21}$$

When we solve these equations,

$$\begin{aligned} \frac{3}{2}(A+B) + \frac{\sqrt{5}}{2}(A-B) = 2 \\ \frac{\sqrt{5}}{2}(A-B) = \frac{1}{2} \\ A - B = \frac{\sqrt{5}}{5} \\ A = \frac{5+\sqrt{5}}{10} \\ B = \frac{5-\sqrt{5}}{10} \end{aligned} \tag{22}$$

So the solution is

$$y[n] = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n \tag{23}$$

5. (a) Below, we find the impulse response $h(t)$. First, we find the unit step response $s(t)$, then differentiate it to find $h(t)$ since $\frac{ds(t)}{dt} = h(t)$.

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t) \tag{24}$$

Assume the homogenous solution is of the form $y_h(t) = Ke^{\alpha t}$. From this, we find the homogenous solution $y_h(t)$.

$$\begin{aligned} \frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 0 \\ \alpha^2 Ke^{\alpha t} + 6\alpha Ke^{\alpha t} + 8Ke^{\alpha t}) = 0 \\ e^{\alpha t}(\alpha^2 + 6\alpha + 8) = 0 \\ e^{\alpha t}(\alpha + 4)(\alpha + 2) = 0 \\ \alpha_1 = -4, \ \alpha_2 = -2 \\ y_h(t) = c_1 e^{-4t} + c_2 e^{-2t} \end{aligned} \tag{25}$$

Assume the particular solution is of the form $y_p(t) = H(\lambda)e^{\lambda t}$ and we know that $H(\lambda) = \frac{\sum_0^M b_k\lambda^k}{\sum_0^N a_k\lambda^k}$; hence, $H(\lambda) = \frac{2}{\lambda^2+6\lambda+8}$. From this, we find the particular solution $y_p(t)$. Notice that, since we are trying to find unit step response $s(t)$, we given the unit step function as the input, i.e., $x(t) = u(t)$, equivalently $x(t) = 1$ for $t > 0$.

$$
\begin{aligned}
\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) &= 2u(t) \\
\frac{2\lambda^2 e^{\lambda t}}{\lambda^2+6\lambda+8} + \frac{12\lambda e^{\lambda t}}{\lambda^2+6\lambda+8} + \frac{16e^{\lambda t}}{\lambda^2+6\lambda+8} &= 2 \text{ for } t > 0 \\
\frac{\lambda^2 e^{\lambda t}}{\lambda^2+6\lambda+8} + \frac{6\lambda e^{\lambda t}}{\lambda^2+6\lambda+8} + \frac{8e^{\lambda t}}{\lambda^2+6\lambda+8} &= 1 \text{ for } t > 0 \\
\lambda^2 e^{\lambda t} + 6\lambda e^{\lambda t} + 8e^{\lambda t} &= \lambda^2 + 6\lambda + 8 \text{ for } t > 0 \\
e^{\lambda t}(\lambda^2 + 6\lambda + 8) &= \lambda^2 + 6\lambda + 8 \;\text{ for } t > 0 \\
e^{\lambda t} &= 1 \text{ for } t > 0 \\
\lambda &= 0 \\
y_p(t) &= \frac{1}{4}
\end{aligned}
\tag{26}
$$

Hence, the general solution is given below.

$$
\begin{aligned}
y(t) &= y_h(t) + y_p(t) \\
y(t) &= c_1e^{-4t} + c_2e^{-2t} + \frac{1}{4}
\end{aligned}
\tag{27}
$$

Since the system is initially at rest, we have $y(0) = 0$ and $y'(0) = 0$. We use them to find c_1 and c_2.

$$
\begin{aligned}
y(0) = c_1 + c_2 + \frac{1}{4} &= 0 \\
c_1 + c_2 &= -\frac{1}{4} \\
y'(0) = -4c_1 - 2c_2 &= 0 \\
4c_1 + 2c_2 &= 0 \\
c_1 = \frac{1}{4},\; c_2 &= -\frac{1}{2} \\
y(t) = s(t) &= \frac{1}{4}e^{-4t} - \frac{1}{2}e^{-2t} + \frac{1}{4}
\end{aligned}
\tag{28}
$$

Since $\frac{ds(t)}{dt} = h(t)$, we find $h(t)$ as follows.

$$
\begin{aligned}
h(t) &= \frac{ds(t)}{dt} \\
h(t) &= (-e^{-4t} + e^{-2t})u(t)
\end{aligned}
\tag{29}
$$

An alternative way to solve this problem is as follows.
This differential equation can be written as,

$$
\begin{aligned}
Q(D)y(t) &= P(D)x(t) \\
\text{where} \quad Q(D) &= (D^2 + 6D + 8) \\
P(D) &= 2
\end{aligned}
\tag{30}
$$

Then the impulse response is,

$$
h(t) = b_n\delta(t) + (P(D)y_h(t))u(t) \tag{31}
$$

Since the order of $P(D)$ is less than of $Q(D)$, $b_n = 0$. Therefore the impulse response can be written as,

$$
h(t) = (2y_h(t))u(t) \tag{32}
$$

The corresponding homogeneous differential equation is,

$$
y''(t) + 6y'(t) + 8y(t) = 0 \quad \text{subject to } y_h'(0) = 1,\, y_h(0) = 0 \tag{33}
$$

The characteristic equation of the differential equation above is,

$$
s^2 + 6s + 8 = 0 \tag{34}
$$

Which yields to two different roots $s_1 = -2$ and $s_2 = -4$. Therefore the homogeneous solution is,

$$\begin{aligned}
y_h(t) &= Ae^{-2t} + Be^{-4t} \\
y_h(0) &= A + B = 0 \\
y_h'(0) &= -2A - 4B = 1 \\
A &= \frac{1}{2} \\
B &= \frac{-1}{2} \\
y_h(t) &= \frac{1}{2}e^{-2t} + \frac{-1}{2}e^{-4t}
\end{aligned} \tag{35}$$

Therefore the impulse response is,

$$h(t) = (e^{-2t} - e^{-4t})u(t) \tag{36}$$

(b)
- ***Causality:*** The system is causal because $h(t) = 0$ for $t < 0$. $u(t)$ makes it causal.
- ***Memory:*** The system is not memoryless since $h(t)$ cannot be written in the form of $K\delta(t)$ where K is a constant. In other words, the system has memory.
- ***Stability:*** If the system is stable then,

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(\tau)| d\tau &< \infty \\
-\infty < \int_{-\infty}^{\infty} h(\tau) d\tau &< \infty
\end{aligned} \tag{37}$$

  We plug in the impulse response,

$$\begin{aligned}
\int_{-\infty}^{\infty} h(\tau) d\tau &= \int_{-\infty}^{\infty} (e^{-2\tau} - e^{-4\tau})u(\tau) d\tau \\
&= \int_{0}^{\infty} (e^{-2\tau} - e^{-4\tau}) d\tau \\
&= \lim_{R\to\infty} \int_{0}^{R} (e^{-2\tau} - e^{-4\tau}) d\tau \\
&= \lim_{R\to\infty} (\frac{e^{-4R}}{4} - \frac{e^{-2R}}{2} - \frac{1}{4} + \frac{1}{2}) \\
&= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}
\end{aligned} \tag{38}$$

  Since $\frac{1}{4}$ is finite, the system is stable.
- ***Invertibility:*** The system is not invertible since there no system $h_1(t)$ that makes $h(t) * h_1(t) = \delta(t)$.