CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 1

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1. (a) First, let's put value of z form the first equation to the second equation. Notice that the conjugate of z is $\bar{z} = x - yj$. Then, we solve the equation to find values of x and y.

$$\begin{aligned} 3z + 4 &= 2j - \bar{z} \\ 3(x + yj) + 4 &= 2j - (x - yj) \\ 3x + 3yj + 4 &= 2j - x + yj \\ 4x + 2yj - 2j &= -4 \\ 4x + 2j(y - 1) &= -4 \\ x &= -1 \\ y &= 1 \end{aligned} \tag{1}$$

(i) Below, we find $|z|^2$.

$$\begin{aligned} |z|^2 &= x^2 + y^2 \\ |z|^2 &= (-1)^2 + (1)^2 \\ |z|^2 &= 2 \end{aligned} \tag{2}$$

(ii) Below, we plot z on the complex plane.

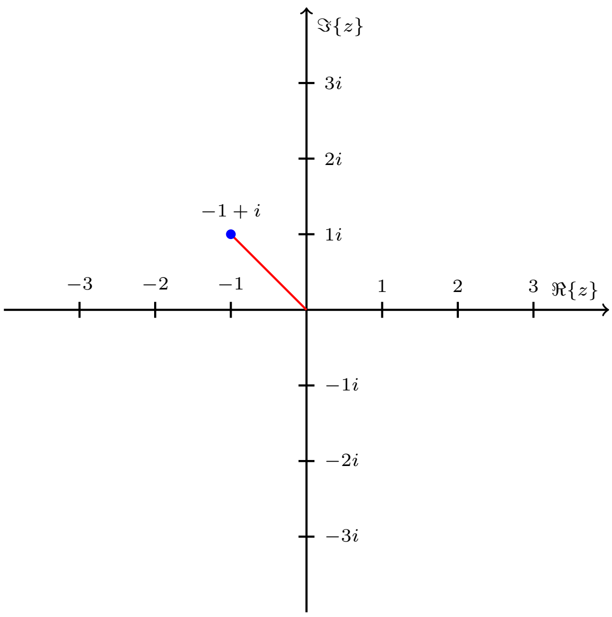

Figure 1: $z = -1 + i$ is plotted on on the Complex Plane

(b) Since $z^3 = 64j$, $z = -4j$. Below, we find z in polar form.

$$\begin{aligned}
\Theta &= \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-4}{0}\right) = \frac{-\pi}{2} \\
r &= \sqrt{a^2+b^2} = \sqrt{0^2+4^2} = 4 \\
z &= 4e^{j\frac{-\pi}{2}} \\
&= 4\left(\cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)\right) \\
&= 4j\sin\left(\frac{-\pi}{2}\right)
\end{aligned} \tag{3}$$

(c) Below, we find the magnitude and angle of z.

$$\begin{aligned}
z &= \frac{(1-j)(1+\sqrt{3}j)}{(1+j)} \\
&= \frac{(1-j)(1-j)(1+\sqrt{3}j)}{(1+j)(1-j)} \\
&= \frac{(-2j)(1+\sqrt{3}j)}{2} \\
&= -j(1+\sqrt{3}j) \\
&= -j+\sqrt{3} \\
r &= \sqrt{a^2+b^2} = \sqrt{1+3} = 2 \\
\Theta &= \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6}
\end{aligned} \tag{4}$$

(d) Below, we write z in polar form.

$$\begin{aligned}
z &= -je^{j\frac{\pi}{2}} \\
z &= -j\left(\cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)\right) \\
z &= 1 \\
z &= \cos(0)
\end{aligned} \tag{5}$$

2. You can see the graph in Figure 2.

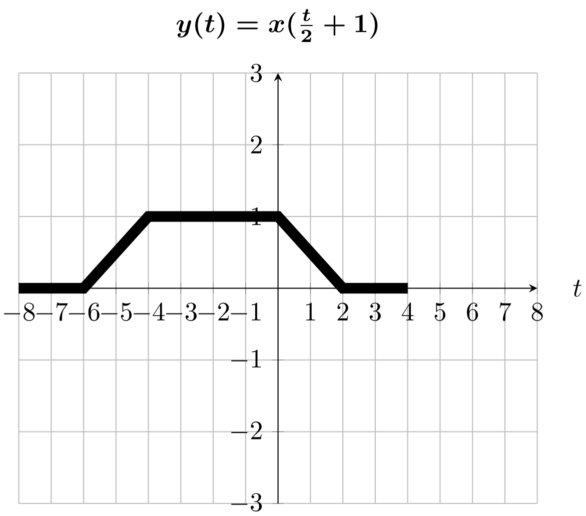

Figure 2: t vs. $y(t) = x(\frac{t}{2}+1)$.

3. (a) The graph can be seen in Figure 3.

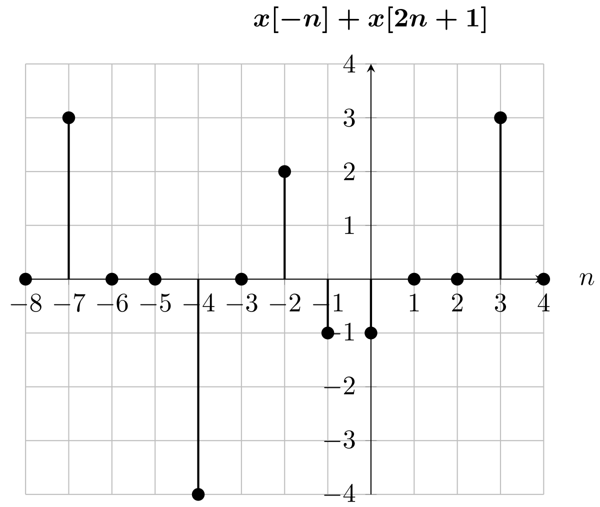

Figure 3: n vs. $x[-n] + x[2n+1]$.

(b) Below, we present $x[-n] + x[2n+1]$ in terms of the unit impulse function.

$$
\begin{aligned}
x[n] &= -\delta[n-1] + 2\delta[n-2] - 4\delta[n-4] + 3\delta[n-7] \\
x[-n] &= -\delta[-n-1] + 2\delta[-n-2] - 4\delta[-n-4] + 3\delta[-n-7] \\
x[2n+1] &= -\delta[2n] + 2\delta[2n-1] - 4\delta[2n-3] + 3\delta[2n-6] \\
x[-n] + x[2n+1] &= -\delta[-n-1] + 2\delta[-n-2] - 4\delta[-n-4] + 3\delta[-n-7] \\
&\quad - \delta[2n] + 2\delta[2n-1] - 4\delta[2n-3] + 3\delta[2n-6] \\
x[-n] + x[2n+1] &= -\delta[n+1] + 2\delta[n+2] - 4\delta[n+4] + 3\delta[n+7] - \delta[n] + 3\delta[n-3]
\end{aligned}
\tag{6}
$$

4. (a) In this question, we will find the periods of the sin and cos functions separately and then we will find the period of the whole function.
A signal in the form of $A\cos[\Omega_0 n + c]$ where c is a real number is periodic if $\frac{\Omega_0}{2\pi} = \frac{m}{N_0}$ and m and N_0 are integers. Moreover, the smallest integer value of N_0 is the fundamental period. For the given signal $3\cos[\frac{13\pi}{10}n]$, we have $\frac{\frac{13\pi}{10}}{2\pi} = \frac{13}{20} = \frac{m}{N_0}$. For $m = 13$, $N_0 = 20$. Notice that $m = 13$ is the smallest integer value of m that makes N_0 an integer. Hence, the signal is periodic with the fundamental period $N_0 = 20$.
A signal in the form of $A\sin[\Omega_0 n + c]$ where c is a real number is periodic if $\frac{\Omega_0}{2\pi} = \frac{m}{N_0}$ and m and N_0 are integers. Moreover, the smallest integer value of N_0 is the fundamental period. For the given signal $5\sin[\frac{7\pi}{3}n - \frac{2\pi}{3}]$, we have $\frac{\frac{7\pi}{3}}{2\pi} = \frac{7}{6} = \frac{m}{N_0}$. For $m = 7$, $N_0 = 6$. Notice that $m = 7$ is the smallest integer value of m that makes N_0 an integer. Hence, the signal is periodic with the fundamental period $N_0 = 6$.
Now, we have two signals with fundamental periods 20 and 6, i.e., the first signal repeats itself for $n = 20, 40, \underline{60}, 80, \ldots$ and the second signal repeats itself for $n = 6, 12, 18, 24, 30, 36, 40, 44, 48, 52, 56, \underline{60}, \ldots$. The summation of these signals is a new signal which repeats itself for $n = \underline{60}, 120, \ldots$. Hence, the signal $3\cos[\frac{13\pi}{10}n] + 5\sin[\frac{7\pi}{3}n - \frac{2\pi}{3}]$ is periodic with the fundamental period $N_0 = 60$.

(b) A signal in the form of $A\sin[\Omega_0 n + c]$ where c is a real number is periodic if $\frac{\Omega_0}{2\pi} = \frac{m}{N_0}$ and m and N_0 are integers. Moreover, the smallest integer value of N_0 is the fundamental period. For the given signal $5\sin[3n - \frac{\pi}{4}]$, we have $\frac{3}{2\pi} = \frac{m}{N_0}$. There is no integer value of m that makes N_0 an integer. Hence, the given signal is not periodic.

(c) A signal in the form of $A\cos(\omega_0 t + c)$ where c is a real number is periodic with the fundemental period $T_0 = \frac{2\pi}{\omega_0}$ where T_0 is a real number. For the given signal $2\cos(3\pi t - \frac{2\pi}{5})$, we have $T_0 = \frac{2}{3}$. Hence, the given signal is periodic with the fundamental period $T_0 = \frac{2}{3}$.

(d) In this question, we solve the equation $x(t) = x(t+T)$ in order to find the period T.

$$
\begin{aligned}
x(t) &= x(t+T) \\
-je^{j5t} &= -je^{j5(t+T)} \\
e^{j5t} &= e^{j5(t+T)} \\
e^{j5t} &= e^{j5t}e^{j5T} \\
1 &= e^{j5T} = \cos(5T) + \sin(5T) \\
2\pi &= 5T \\
T_0 &= \frac{2\pi}{5}
\end{aligned}
\tag{7}
$$

Notice that, since $5T = 0$ is the trivial solution, we pick the next smallest value $5T = 2\pi$ to find the fundamental period. Hence, the given signal is periodic with the fundamental period $T_0 = \frac{2\pi}{5}$

5. First, we check whether $x[n]$ is odd or even or none. For oddness, we check whether $x[n] = -x[-n]$ holds and for evenness, we check whether $x[n] = x[-n]$ holds.

   (i) Oddness:

$$-\delta[n-1] + 2\delta[n-2] - 4\delta[n-4] + 3\delta[n-7] \neq \delta[-n-1] - 2\delta[-n-2] + 4\delta[-n-4] - 3\delta[-n-7] \tag{8}$$

   (ii) Evenness:

$$-\delta[n-1] + 2\delta[n-2] - 4\delta[n-4] + 3\delta[n-7] \neq -\delta[-n-1] + 2\delta[-n-2] - 4\delta[-n-4] + 3\delta[-n-7] \tag{9}$$

   Hence, $x[n]$ is neither an odd nor an even function. Now, find the even and odd decompositions of $x[n]$ and give the graphs of both.

   (i) Oddness:

$$\begin{aligned} Odd\{x[n]\} &= \frac{1}{2}\{x[n] - x[-n]\} \\ &= \frac{1}{2}\{-\delta[n-1] + 2\delta[n-2] - 4\delta[n-4] + 3\delta[n-7] + \delta[-n-1] - 2\delta[-n-2] + 4\delta[-n-4] - 3\delta[-n-7]\} \end{aligned} \tag{10}$$

   (ii) Evenness:

$$\begin{aligned} Ev\{x[n]\} &= \frac{1}{2}\{x[n] + x[-n]\} \\ &= \frac{1}{2}\{-\delta[n-1] + 2\delta[n-2] - 4\delta[n-4] + 3\delta[n-7] - \delta[-n-1] + 2\delta[-n-2] - 4\delta[-n-4] + 3\delta[-n-7]\} \end{aligned} \tag{11}$$

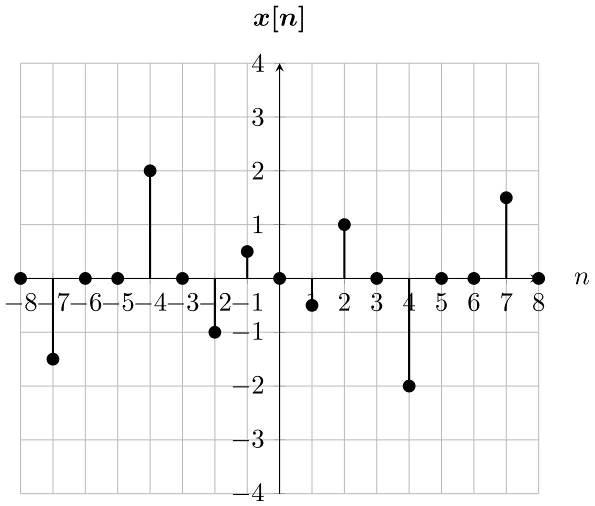

Figure 4: n vs. $Odd\{x[n]\}$.

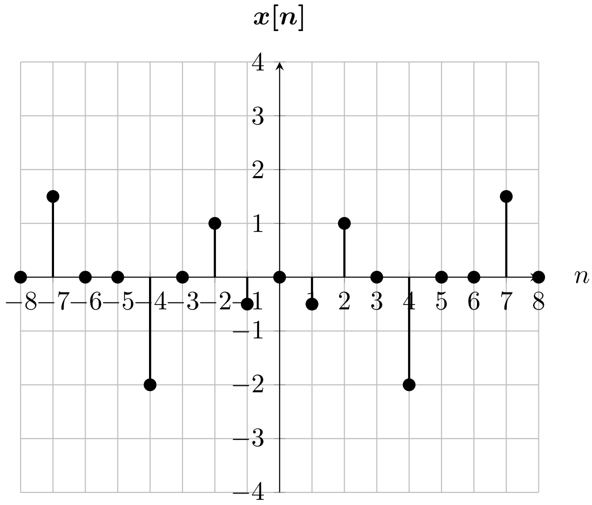

Figure 5: n vs. $Ev\{x[n]\}$.

6. (a)
- ***Memory:*** The system requires memory since the output value of the system depends on the value of $x(t)$ in a different time instance, e.g. $y(1) = x(-1)$.
- ***Stability:*** The system is bounded because for bounded $x(t)$, $y(t)$ is bounded.

$$\begin{aligned} &-B < x(t) < B \\ &-B < x(2t-3) < B \\ &-B < y(t) < B \end{aligned} \tag{12}$$

- ***Causality:*** For $t = 4$, $y(4) = x(5)$ which violates the causality property; hence, the system is not causal.
- ***Linearity:*** Consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$\begin{aligned} x_1(t) &\to y_1(t) = x_1(2t-3) \\ x_2(t) &\to y_2(t) = x_2(2t-3) \end{aligned} \tag{13}$$

Then, consider a linear combination of $x_1(t)$ and $x_2(t)$, i.e., $x_3(t) = \alpha x_1(t) + \beta x_2(t)$

$$\begin{aligned} y_3(t) &= x_3(2t-3) \\ &= \alpha x_1(2t-3) + \beta x_2(2t-3) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned} \tag{14}$$

Therefore the system is linear.
- ***Invertibility:*** When $x(t)$ is given to the system as the input, $y(t) = x(2t-3)$ is the output. The inverse system takes $y(t)$ as the input and produces $w(t)$ as the output which should be equal to $x(t)$ in order system to be invertible. To ensure that $w(t) = y(\frac{t+3}{2})$ which produces $x(t)$. To prove that say $k = \frac{t+3}{2}$ then $y(k) = x(2k-3)$. Hence, the system is invertible since the invertible system exists.
- ***Time-Invariance:*** Shift the input signal $x(t)$ by t_0, i.e., $x(t-t_0)$. When the shifted input is applied to the system, we get the following output signal $y(t-t_0) = x(2t-2t_0-3)$. A time shift in the input signal does not result in an identical time shift in the output signal, i.e., $x(2t-2t_0-3) \neq x(2t-t_0-3)$. Hence, the system is not time invariant.

(b)
- ***Memory:*** The system is memoryless since the output value of the system depends on the value of $x(t)$ in the current time instance for all values of t.
- ***Stability:*** To disprove stability of this system we observe the following: a constant input $x(t) = 1$ yields to $y(t) = t$ which is unbounded. Therefore, the system is not stable.
- ***Causality:*** The system is causal since $y(t)$ only depends on the values of $x(t)$ at the current time instance.
- ***Linearity:*** Consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$\begin{aligned} x_1(t) &\to y_1(t) = tx_1(t) \\ x_2(t) &\to y_2(t) = tx_2(t) \end{aligned} \tag{15}$$

Then, consider a linear combination of $x_1(t)$ and $x_2(t)$, i.e., $x_3(t) = \alpha x_1(t) + \beta x_2(t)$

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(\alpha x_1(t) + \beta x_2(t)) \\ &= t\alpha x_1(t) + t\beta x_2(t) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned} \tag{16}$$

Therefore the system is linear.

- ***Invertibility:*** When $x(t)$ is given to the system as the input, $y(t) = tx(t)$ is the output. The inverse system takes $y(t)$ as the input and produces $w(t)$ as the output which should be equal to $x(t)$ in order system to be invertible. To ensure that $w(t) = \frac{y(t)}{t}$ which produces $x(t)$. To prove that observe $w(t) = \frac{y(t)}{t} = \frac{tx(t)}{t} = x(t)$. Hence, the system is invertible since the invertible system exists.
- ***Time-Invariance:*** Shift the input signal $x(t)$ by t_0, i.e., $x(t - t_0)$. When the shifted input is applied to the system, we get the following output signal $y(t - t_0) = (t - t_0)x(t - t_0)$. A time shift in the input signal does not result in an identical time shift in the output signal, i.e., $(t - t_0)x(t - t_0) \neq tx(t - t_0)$. Hence, the system is not time invariant.

(c)

- ***Memory:*** The system requires memory since the output value of the system depends on the value of $x[n]$ in a different time instance, e.g. $y[1] = x[-1]$.
- ***Stability:*** The system is bounded because for bounded $x[n]$, $y[n]$ is bounded.

$$\begin{aligned} &-B < x[n] < B \\ &-B < x[2n-3] < B \\ &-B < y[n] < B \end{aligned} \tag{17}$$

- ***Causality:*** For $n = 4$, $y[4] = x[5]$ which violates the causality property; hence, the system is not causal.
- ***Linearity:*** Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$\begin{aligned} x_1[n] &\to y_1[n] = x_1[2n-3] \\ x_2[n] &\to y_2[n] = x_2[2n-3] \end{aligned} \tag{18}$$

Then, consider a linear combination of $x_1[n]$ and $x_2[n]$, i.e., $x_3[n] = \alpha x_1[n] + \beta x_2[n]$

$$\begin{aligned} y_3[n] &= x_3[2n-3] \\ &= \alpha x_1[2n-3] + \beta x_2[2n-3] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned} \tag{19}$$

Therefore the system is linear.

- ***Invertibility:*** Consider the inputs $x_1[n] = u[n]$ and $x_2[n] = u[n-1]$ where $u[n]$ is the unit step function. These inputs yield to $y_1[n] = x_1[2n-3] = u[2n-3]$ and $y_2[n] = x_2[2n-3] = u[2n-4]$ which implies $u[2n-3] = u[2n-4] = u[n-2]$ for all integer values of n. Hence, $y_1[n] = y_2[n]$ even though $x_1[n] \neq x_2[n]$, therefore the system is not invertible since distinct inputs yield to the same outputs.
- ***Time-Invariance:*** Shift the input signal $x[n]$ by n_0, i.e., $x[n - n_0]$. When the shifted input is applied to the system, we get the following output signal $y[n - n_0] = x[nt - nt_0 - 3]$. A time shift in the input signal does not result in an identical time shift in the output signal, i.e., $x[2n - 2n_0 - 3] \neq x[2n - n_0 - 3]$. Hence, the system is not time invariant.

(d)

- ***Memory:*** The system requires memory since the output value of the system depends on the value of $x[n]$ in different time instances, e.g. $y[1] = x[0] + x[-1] + \ldots$.
- ***Stability:*** To disprove stability of the system observe the following: a constant input $x[n] = 1$ yields to $y[n] = \sum_{k=1}^{\infty} 1$ which does not converge to value; hence, is unbounded. Therefore, the system is not stable.
- ***Causality:*** The system is causal since the output only depends on the values of the input in the past.
- ***Linearity:*** Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$\begin{aligned} x_1[n] &\to y_1[n] = \sum_{k=1}^{\infty} x_1[n-k] \\ x_2[n] &\to y_2[n] = \sum_{k=1}^{\infty} x_2[n-k] \end{aligned} \tag{20}$$

Then, consider a linear combination of $x_1[n]$ and $x_2[n]$, i.e., $x_3[n] = \alpha x_1[n] + \beta x_2[n]$

$$\begin{aligned} y_3[n] &= \sum_{k=1}^{\infty} x_3[n-k] \\ y_3[n] &= \sum_{k=1}^{\infty} (\alpha x_1[n-k] + \beta x_2[n-k]) \\ y_3[n] &= \sum_{k=1}^{\infty} \alpha x_1[n-k] + \sum_{k=1}^{\infty} \beta x_2[n-k] \\ &= \alpha \sum_{k=1}^{\infty} x_1[n-k] + \beta \sum_{k=1}^{\infty} x_2[n-k] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned} \tag{21}$$

Therefore the system is linear.

- ***Invertibility:*** When $x[n]$ is given to the system as the input, $y[n] = \sum_{k=1}^{\infty} x[n-k]$ is the output. The inverse system takes $y[n]$ as the input and produces $w[n]$ as the output which should be equal to $x[n]$ in order system to be invertible. To ensure that $w[n] = y[n+1] - y[n]$ which produces $x[n]$. To prove that observe the following $y[n+1] = \sum_{k=1}^{\infty} x[n-k+1]$ and $y[n] = \sum_{k=1}^{\infty} x[n-k] = \sum_{k=2}^{\infty} x[n-k+1]$ so $y[n+1] - y[n] = x[n]$. Hence, the system is invertible since the invertible system exists.
- ***Time-Invariance:*** Shift the input signal $x[n]$ by n_0, i.e., $x[n-n_0]$. When the shifted input is applied to the system, we get the following output signal $\sum_{k=1}^{\infty} x[n-n_0-k]$. To prove time invariance observe the following argument $\sum_{k=1}^{\infty} x[n-n_0-k] = \sum_{l=1+n_0}^{\infty} x[n-l] = y[n-n_0]$. Hence, the system is time invariant.