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Answer 1

$$For\ all\ n \geq 1, \left(\sum_{k=1}^{n} k\right)^2 \geq \sum_{k=1}^{n} k^2 \tag{1}$$

Basis Step:

For $n = 1$:

$$\begin{gathered}\left(\sum_{k=1}^{1} k\right)^2 \geq \sum_{k=1}^{1} k^2 \\ (1)^2 \geq 1^2 \\ 1 \geq 1\end{gathered} \tag{2}$$

For $n = 1$, $\left(\sum_{k=1}^{n} k\right)^2 \geq \sum_{k=1}^{n} k^2$ is true. This completes the basis step.

Inductive Step:

Assume that $\left(\sum_{k=1}^{c} k\right)^2 \geq \sum_{k=1}^{c} k^2$ is true for an arbitrary fixed integer $c \geq 1$. Under this assumption we need to find if $\left(\sum_{k=1}^{c+1} k\right)^2 \geq \sum_{k=1}^{c+1} k^2$ is true.

$$\begin{aligned}\left(\sum_{k=1}^{c} k\right)^2 &\geq \sum_{k=1}^{c} k^2 \qquad (By\ inductive\ hypothesis) \\ \left(\sum_{k=1}^{c} k\right)^2 &\geq \sum_{k=1}^{c} k^2 - 2(c+1)\left(\sum_{k=1}^{c} k\right) \qquad \left(Since\ 2(c+1)\left(\sum_{k=1}^{c} k\right) \geq 0\right) \\ \left(\sum_{k=1}^{c} k\right)^2 + 2(c+1)\left(\sum_{k=1}^{c} k\right) &\geq \sum_{k=1}^{c} k^2 \\ \left(\sum_{k=1}^{c} k\right)^2 + 2(c+1)\left(\sum_{k=1}^{c} k\right) + (c+1)^2 &\geq \sum_{k=1}^{c} k^2 + (c+1)^2 \\ \left(\sum_{k=1}^{c+1} k\right)^2 &\geq \sum_{k=1}^{c+1} k^2\end{aligned} \tag{3}$$

By Inductive Hypothesis, $\left(\sum_{k=1}^{c+1} k\right)^2 \geq \sum_{k=1}^{c+1} k^2$ is true. This completes the inductive step.

Both basis and inductive step have been completed; thus, by mathematical induction, $\left(\sum_{k=1}^{n} k\right)^2 \geq \sum_{k=1}^{n} k^2$ is true for all $n \geq 1$.

Answer 2

1.

Let's define the set of positive integers up to 41 as set S, then $S = \{1, 2, 3, ..., 20, 21, 22, ..., 39, 40, 41\}$. S can be divided into its subsets as following; $\{1, 41\}$, $\{2, 40\}$, $\{3, 39\}$, ..., $\{20, 22\}$. Note that, sum of two elements of each subset is 42.

Because players are playing with their best strategies, at one point of the game 21 will be picked by one of the players, since it does not have a pair such that their sum is 42.

Since there are 20 pairs of the form $\{x_1, x_2\}$ such that $x_1 + x_2 = 42$ and there is number 21, which is guaranteed to be picked at some point of the game, there will be 22 picks at most and the $22nd$ pick will be the losing pick, it can proven by the Pigeonhole Principle as following:

Since 21 has no pair, we do not consider it for now. Then there will be 21 picks (excluding number 21) and there are 20 subsets to pick from. Then, by Pigeonhole Principle, (pigeons are defined as number of picks and pigeonholes are defined as number of pairs) $\lceil 21/20 \rceil = 2$, meaning there will be at least one pair whose both elements will be picked.

Now we include the pick for 21, so there will be 22 picks in the game and the $22nd$ pick will be the losing pick. Since Alice started to game, she will always pick at the odd numbered turns and Bob will pick at even numbered turns. Since 22 is an even number, it will be Bob's turn; thus, Bob will lose the game. Alice will win the game.

2.

Since the order does not matter, we need to find different sets with 3 elements and the summation of the elements must be 5. Since 5 is not a big number it is trivial to find such sets. The followings are the all of them: $\{5, 0, 0\}$, $\{4, 1, 0\}$, $\{3, 2, 0\}$, $\{3, 1, 1\}$, $\{2, 2, 1\}$. Thus, there are 5 ways to pick three such integers.

3.

In this problem we can think that we have 5 beans and we need to divide them into three, so we can visualize it as following: $* * * * * \mid \mid$ (where asterisks are beans and pipes are separators). Thus, we need to place 2 separators. We can find it as following:

$$\begin{aligned} C(7, 2) &= \frac{7!}{5!2!} \\ &= 21 \end{aligned} \tag{4}$$

Note that, we can also directly use the formula for r-combinations where n is defined as 3 and r is defined as 5:

$$\begin{aligned} C(n+r-1, n-1) &= C(7,2) \\ &= \frac{7!}{5!2!} \\ &= 21 \end{aligned} \tag{5}$$

Thus, the equation $x_1 + x_2 + x_3 = 5$ has 21 different solutions.

Answer 3

$$\begin{aligned} (1-x^3)^n &= \sum_{k=0}^{n} a_k x^k (1-x)^{3n-2k} \\ \frac{(1-x^3)^n}{(1-x)^n} &= \frac{1}{(1-x)^n} \sum_{k=0}^{n} a_k x^k (1-x)^{3n-2k} \\ \frac{(1-x)^n((1-x)^2+3x)^n}{(1-x)^n} &= \sum_{k=0}^{n} a_k x^k \frac{(1-x)^{3n-2k}}{(1-x)^n} \\ ((1-x)^2+3x)^n &= \sum_{k=0}^{n} a_k x^k (1-x)^{2n-2k} \\ \sum_{k=0}^{n} \binom{n}{k} (3x)^k ((1-x)^2)^{n-k} &= \sum_{k=0}^{n} a_k x^k ((1-x)^2)^{n-k} \qquad \textit{(By Binomial Theorem)} \\ \sum_{k=0}^{n} \binom{n}{k} 3^k x^k ((1-x)^2)^{n-k} &= \sum_{k=0}^{n} a_k x^k ((1-x)^2)^{n-k} \\ a_k &= 3^k \binom{n}{k} \\ \textit{Thus, } a_r &= 3^r \binom{n}{r}. \end{aligned} \tag{6}$$

Answer 4

We need to find homogeneous solution $a_n^{(h)}$ and particular solution $a_n^{(p)}$, then for the solution a_n we need to find $a_n = a_n^{(h)} + a_n^{(p)}$. Let's first write characteristic equation and find characteristic roots of the equation.

$$\begin{aligned} a_n &= 4a_{n-1} - a_{n-2} - 6a_{n-3} + n - 2 \\ a_n - 4a_{n-1} + a_{n-2} + 6a_{n-3} &= n - 2 \\ r^3 - 4r^2 + r + 6 &= 0 \qquad \textit{(Characteristic Equation)} \\ (r-2)(r-3)(r+1) &= 0 \end{aligned} \tag{7}$$

Thus, characteristic roots are $r = 2$, $r = 3$ and $r = -1$. Then, $a_n^{(h)}$ is of the form $A(2)^n + B(3)^n + C(-1)^n$ and since characteristic roots do not appear as multipliers of $(n-2)$, $a_n^{(p)}$ is of the form $Dn + E$. Now, we can find A, B, C, D and E as following.

$$
\begin{aligned}
a_n &= 4a_{n-1} - a_{n-2} - 6a_{n-3} + n - 2 \\
Dn + E &= 4(D(n-1) + E) - (D(n-2) + E) - 6(D(n-3) + E) + n - 2 \\
n - 2 &= n(4D) + (4E - 16D) \\
D = 0.25 &\ and\ E = 0.50 \\
a_n &= a_n^{(h)} + a_n^{(p)} \\
a_n &= A(2)^n + B(3)^n + C(-1)^n + (0.25)n + 0.50 \qquad (8) \\
a_0 &= A + B + C + 0.50 = 3.5 \\
a_1 &= 2A + 3B - C + 0.25 + 0.50 = 4.75 \\
a_2 &= 4A + 9B + C + 2(0.25) + 0.50 = 13 \\
A &= 1.67,\ B = 0.50\ and\ C = 0.83 \\
a_n &= (1.67)(2)^n + (0.50)(3)^n + (0.83)(-1)^n + (0.25)n + 0.50
\end{aligned}
$$

Hence, the solution of the given recurrence relation is

$$a_n = (1.67)(2)^n + (0.50)(3)^n + (0.83)(-1)^n + (0.25)n + 0.50$$