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Answer 1

a. Assume that $x \in (A \cap B)$.

$$\begin{aligned} x \in (A \cap B) &\rightarrow x \in A \wedge x \in B \\ x \in A \wedge x \in B &\rightarrow (x \in A \vee x \in \overline{B}) \wedge (x \in \overline{A} \vee x \in B) \\ (x \in A \vee x \in \overline{B}) \wedge (x \in \overline{A} \vee x \in B) &\rightarrow x \in (A \cup \overline{B}) \wedge x \in (\overline{A} \cup B) \\ x \in (A \cup \overline{B}) \wedge x \in (\overline{A} \cup B) &\rightarrow x \in ((A \cup \overline{B}) \cap (\overline{A} \cup B)) \end{aligned}$$

For an x such that $x \in A \cap B$, we obtained that $x \in ((A \cup \overline{B}) \cap (\overline{A} \cup B))$. Thus, the following has been proven, $A \cap B \subseteq (A \cup \overline{B}) \cap (\overline{A} \cup B)$.

b. Assume that $x \in (\overline{A} \cap \overline{B})$.

$$\begin{aligned} x \in (\overline{A} \cap \overline{B}) &\rightarrow x \in \overline{A} \wedge x \in \overline{B} \\ x \in \overline{A} \wedge x \in \overline{B} &\rightarrow (x \in \overline{A} \vee x \in B) \wedge (x \in A \vee x \in \overline{B}) \\ (x \in \overline{A} \vee x \in B) \wedge (x \in A \vee x \in \overline{B}) &\rightarrow x \in (\overline{A} \cup B) \wedge x \in (A \cup \overline{B}) \\ x \in (\overline{A} \cup B) \wedge x \in (A \cup \overline{B}) &\rightarrow x \in ((A \cup \overline{B}) \cap (\overline{A} \cup B)) \end{aligned}$$

For an x such that $x \in \overline{A} \cap \overline{B}$, we obtained that $x \in ((A \cup \overline{B}) \cap (\overline{A} \cup B))$. Thus, the following has been proven, $\overline{A} \cap \overline{B} \subseteq (A \cup \overline{B}) \cap (\overline{A} \cup B)$.

Answer 2

In order to prove $f^{-1}((A \cap B) \times C) = f^{-1}(A \times C) \cap f^{-1}(B \times C)$, the following two conditions must be satisfied:

i. $f^{-1}((A \cap B) \times C) \subseteq f^{-1}(A \times C) \cap f^{-1}(B \times C)$

ii. $f^{-1}(A \times C) \cap f^{-1}(B \times C) \subseteq f^{-1}((A \cap B) \times C)$

(Note that, for any two set A and B, if $A \subseteq B$ and $B \subseteq A$, then we say that $A = B$.)
(Also, note that, since f has an inverse f^{-1}, f is a one-to-one and an onto function. Thus, any object x_1, that we pick from X, has a distinct image $f(x_1) = (x_2, x_3)$ from $Y \times Z$ and, similarly, any 2-tuple object (x_2, x_3), that we pick from $Y \times Z$, has a distinct image $f^{-1}(x_2, x_3) = x_1$ from X.)

i. $f^{-1}((A \cap B) \times C) \subseteq f^{-1}(A \times C) \cap f^{-1}(B \times C)$
Assume that $f^{-1}(x_1, x_2) \in f^{-1}((A \cap B) \times C)$.

$$\begin{aligned}
f^{-1}(x_1, x_2) \in f^{-1}((A \cap B) \times C) &\rightarrow (x_1, x_2) \in ((A \cap B) \times C) \\
(x_1, x_2) \in ((A \cap B) \times C) &\rightarrow x_1 \in (A \cap B) \wedge x_2 \in C \\
x_1 \in (A \cap B) \wedge x_2 \in C &\rightarrow x_1 \in A \wedge x_1 \in B \wedge x_2 \in C \\
x_1 \in A \wedge x_1 \in B \wedge x_2 \in C &\rightarrow (x_1 \in A \wedge x_2 \in C) \wedge (x_1 \in B \wedge x_2 \in C) \\
(x_1 \in A \wedge x_2 \in C) \wedge (x_1 \in B \wedge x_2 \in C) &\rightarrow (x_1, x_2) \in (A \times C) \wedge (x_1, x_2) \in (B \times C) \\
(x_1, x_2) \in (A \times C) \wedge (x_1, x_2) \in (B \times C) &\rightarrow f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \\
&\qquad \wedge f^{-1}(x_1, x_2) \in f^{-1}(B \times C) \\
f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \wedge f^{-1}(x_1, x_2) \in f^{-1}(B \times C) &\rightarrow f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \cap f^{-1}(B \times C)
\end{aligned}$$

For $f^{-1}(x_1, x_2) \in f^{-1}((A \cap B) \times C)$, it has been shown that $f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \cap f^{-1}(B \times C)$. Thus, the following is true, $f^{-1}((A \cap B) \times C) \subseteq f^{-1}(A \times C) \cap f^{-1}(B \times C)$.

ii. $f^{-1}(A \times C) \cap f^{-1}(B \times C) \subseteq f^{-1}((A \cap B) \times C)$
Assume that $f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \cap f^{-1}(B \times C)$.

$$\begin{aligned}
f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \cap f^{-1}(B \times C) &\rightarrow f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \\
&\qquad \wedge f^{-1}(x_1, x_2) \in f^{-1}(B \times C) \\
f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \wedge f^{-1}(x_1, x_2) \in f^{-1}(B \times C) &\rightarrow (x_1, x_2) \in (A \times C) \wedge (x_1, x_2) \in (B \times C) \\
(x_1, x_2) \in (A \times C) \wedge (x_1, x_2) \in (B \times C) &\rightarrow x_1 \in A \wedge x_2 \in C \wedge x_1 \in B \wedge x_2 \in C \\
x_1 \in A \wedge x_2 \in C \wedge x_1 \in B \wedge x_2 \in C &\rightarrow x_1 \in A \wedge x_1 \in B \wedge x_2 \in C \\
x_1 \in A \wedge x_1 \in B \wedge x_2 \in C &\rightarrow x_1 \in (A \cap B) \wedge x_2 \in C \\
x_1 \in (A \cap B) \wedge x \in C &\rightarrow (x_1, x_2) \in ((A \cap B) \times C) \\
(x_1, x_2) \in ((A \cap B) \times C) &\rightarrow f^{-1}(x_1, x_2) \in f^{-1}((A \cap B) \times C)
\end{aligned}$$

For $f^{-1}(x_1, x_2) \in f^{-1}(A \times C) \cap f^{-1}(B \times C)$, it has been shown that $f^{-1}(x_1, x_2) \in f^{-1}((A \cap B) \times C)$. Thus, the following is true, $f^{-1}(A \times C) \cap f^{-1}(B \times C) \subseteq f^{-1}((A \cap B) \times C)$.

As shown above, both conditions i and ii are true. Thus, the following has been proven:

$$f^{-1}((A \cap B) \times C) = f^{-1}(A \times C) \cap f^{-1}(B \times C)$$

Answer 3

a. (i) Assume that $f(x) = ln(x^2 + 5)$ is a one-to-one function, then by definition the following must be true,
$\forall x \forall y (f(x) = f(y) \rightarrow x = y)$
Let's test it for some $x \in R$ and $y \in R$.

$$
\begin{aligned}
f(x) &= f(y) \\
ln(x^2+5) &= ln(y^2+5) \\
x^2+5 &= y^2+5 \\
x^2 &= y^2 \\
x = y \ &or\ x = -y
\end{aligned} \tag{1}
$$

In Equation (1) put $x = 1$, then both $y = 1$ and $y = -1$ are true, so it is not the case that for all x and y if $f(x) = f(y)$, then $x = y$. It is a contradiction with the assumption. The assumption is discharged; thus $f(x) = ln(x^2 + 5)$ is not a one-to-one function.

(ii) For $f(x) = ln(x^2+5)$, $f(R) = [1.609, \infty)$, alternatively, $\forall x(x \in R \rightarrow f(x) \in [1.609, \infty))$, note that the minimum value of x^2 is 0, so the minimum value of $ln(x^2 + 5)$ is $ln(5) = 1.609$. Thus, it can be seen that range of $f(x)$ is $[1.609, \infty)$, it is given that co-domain of the function is R; thus, the following can be concluded, since $R \neq [1.609, \infty)$, $f(x) = ln(x^2 + 5)$ is not an onto function.

As shown above, $f(x) = ln(x^2 + 5)$ is not a one-to-one function and also it is not an onto function.

b. (i) Assume that $f(x) = e^{e^{x^7}}$ is a one-to-one function, then by definition the following must be true,
$\forall x \forall y(f(x) = f(y) \rightarrow x = y)$
Let's test it for some $x \in R$ and $y \in R$.

$$
\begin{aligned}
f(x) &= f(y) \\
e^{e^{x^7}} &= e^{e^{y^7}} \\
e^{x^7} &= e^{y^7} \\
x^7 &= y^7 \\
x &= y
\end{aligned} \tag{2}
$$

In Equation (2) put $x = c$ for any constant $c \in R$, then only $y = c$ is true, so for all x and y if $f(x) = f(y)$, then $x = y$. The assumption is correct; thus $f(x) = e^{e^{x^7}}$ is a one-to-one function.

(ii) For $f(x) = f(x) = e^{e^{x^7}}$, $f(R) = (1, \infty)$, alternatively, $\forall x(x \in R \rightarrow f(x) \in (1, \infty))$, note that, for $x \rightarrow -\infty$, $e^{x^7} \rightarrow 0$ and for $e^{x^7} \rightarrow 0$, $e^{e^{x^7}} \rightarrow 1$. Thus, it can be seen that range of $f(x)$ is $(1, \infty)$, it is given that co-domain of the function is R; thus, the following can be concluded, since $R \neq (1, \infty)$, $f(x) = ln(x^2 + 5)$ is not an onto function.

As shown above, $f(x) = e^{e^{x^7}}$ is a one-to-one function, but it is not an onto function.

Answer 4

a. There are four cases that must be considered.

(i) A and B are both finite sets.
Assume that $|A| = n$ and $|B| = m$.
$A = \{a_1, a_2, a_3, a_4, ..., a_n\}$
$B = \{b_1, b_2, b_3, b_4, ..., b_m\}$
$A \times B = \{(a_1, b_1), (a_1, b_2), ..., (a_1, b_m), (a_2, b_1), (a_2, b_2), ..., (a_2, b_m), ..., (a_n, b_1), (a_n, b_2), ..., (a_n, b_m)\}$
Since both A and B are finite sets, $A \times B$ is also a finite set with the cardinality $|A \times B| = |A||B| = nm$. Thus, $A \times B$ is a countable set.

(ii) A is a countable infinite set and B is a finite set.
Assume that $|B| = m$.
$A = \{a_1, a_2, a_3, a_4, ..., a_m, ...\}$
$B = \{b_1, b_2, b_3, b_4, ..., b_m\}$

Table 1:

	a_1	a_2	a_3	a_4	...	a_m	...
b_1	(a_1, b_1)	(a_2, b_1)	(a_3, b_1)	(a_4, b_1)	...	(a_m, b_1)	...
b_2	(a_1, b_2)	(a_2, b_2)	(a_3, b_2)	(a_4, b_2)	...	(a_m, b_2)	...
b_3	(a_1, b_3)	(a_2, b_3)	(a_3, b_3)	(a_4, b_3)	...	(a_m, b_3)	...
b_4	(a_1, b_4)	(a_2, b_4)	(a_3, b_4)	(a_4, b_4)	...	(a_m, b_4)	...
...	...	...	...	...	...	...	...
b_m	(a_1, b_m)	(a_2, b_m)	(a_3, b_m)	(a_4, b_m)	...	(a_m, b_m)	...

$A \times B = \{(a_1, b_1), (a_1, b_2), ..., (a_1, b_m), (a_2, b_1), (a_2, b_2), ..., (a_2, b_m), ...\}$
$A \times B$ can be counted column by column in Table 1. Thus, $A \times B$ is a countable set.

(iii) A is a finite set and B is a countable infinite set.
Same proof as (ii), define A = B and B = A.

(iv) A and B are both countable infinite sets.
$A = \{a_1, a_2, a_3, a_4, ...\}$
$B = \{b_1, b_2, b_3, b_4, ...\}$

Table 2:

	a_1	a_2	a_3	a_4	...
b_1	(a_1, b_1)	(a_2, b_1)	(a_3, b_1)	(a_4, b_1)	...
b_2	(a_1, b_2)	(a_2, b_2)	(a_3, b_2)	(a_4, b_2)	...
b_3	(a_1, b_3)	(a_2, b_3)	(a_3, b_3)	(a_4, b_3)	...
b_4	(a_1, b_4)	(a_2, b_4)	(a_3, b_4)	(a_4, b_4)	...
...	...	...	...	...	...

$A \times B = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_2), (a_3, b_1), (a_4, b_1), (a_3, b_2), (a_2, b_3), (a_1, b_4), ...\}$

We can enumarate every element in Table 2 by the reverse diagonal of the table, and by doing that we can count $A \times B$. Thus, $A \times B$ is a countable set.

In conclusion, it has been shown that if A and B are two countable sets, $A \times B$ is also a countable set.

b. Assume that B is a countable set. It is known that any subset of a countable set must be also a countable set (Since any subset S' of a countable set S has at most same number of elements with S, so S being countable makes S' a countable set too.), so $A \subseteq B$ is not true for uncountable set A, if B is a countable set. It is a contradiction with the assumption. The assumption is discharged; thus, B is an uncountable set.

c. Assume that A is an uncountable set. For countable set B, any subset of B is also a countable set (Since any subset S' of a countable set S has at most same number of elements with S, so S being countable makes S' a countable set too.), so $A \subseteq B$ is not true for countable set B, if A is an uncountable set. It is a contradiction with the assumption. The assumption is discharged; thus, A is a countable set.

Answer 5

a. By definition $f_1(x)$ is $\mathcal{O}(f_2(x))$ means, for some c_1 and k, $|f_1(x)| \leq c_1|f_2(x)|$ where $x > k$. Then the following can be done.
For some c_1, k and $x > k$,

$$\begin{aligned}
|f_1(x)| &\leq c_1|f_2(x)| \\
ln|f_1(x)| &\leq ln(c_1|f_2(x)|) \\
ln|f_1(x)| &\leq ln(c_1) + ln|f_2(x)| \\
ln|f_1(x)| &\leq c_2 + ln|f_2(x)| \leq c_3 ln|f_2(x)| \quad (for\ some\ c_2 = ln(c_1)\ and\ c_3) \\
ln|f_1(x)| &\leq c_3 ln|f_2(x)| \\
ln|f_1(x)| &\ is\ \mathcal{O}(ln|f_2(x)|)
\end{aligned} \tag{3}$$

As shown above if $f_1(x)$ is $\mathcal{O}(f_2(x))$, then $ln|f_1(x)|$ is $\mathcal{O}(ln|f_2(x)|)$.

b. Let say $f_1(x) = 2x^2 + 10x$ and $f_2(x) = x^2$. Note that $2x^2 + 10x$ and x^2 are both increasing functions and $f_1(x)$ is $\mathcal{O}(f_2(x))$ since they satisfy the condition $|f_1(x)| \leq 7|f_2(x)|$ for $x > 2$. Assume that $3^{f_1(x)}$ is $\mathcal{O}(3^{f_2(x)})$, then the following can be done for some c, k and $x > k$,

$$\begin{aligned}
|3^{f_1(x)}| &\leq c|3^{f_2(x)}| \\
|3^{2x^2+10x}| &\leq c|3^{x^2}| \\
3^{2x^2+10x} &\leq c3^{x^2} \qquad (Since\ both\ are\ positive\ increasing\ functions) \\
3^{2x^2+10x-x^2} &\leq c \\
3^{x^2+10x} &\leq c
\end{aligned} \tag{4}$$

Such c cannot be found, so we get a contradiction, assumption is discharged, $3^{f_1(x)}$ is not $\mathcal{O}(3^{f_2(x)})$. Thus, for increasing functions $f_1(x)$ and $f_2(x)$, it has been proven that $3^{f_1(x)}$ is not $\mathcal{O}(3^{f_2(x)})$ with a counter example $f_1(x) = 2x^2 + 10x$ and $f_2(x) = x^2$.

Answer 6

a.

$$\begin{aligned} (3^x - 1)\ mod\ (3^y - 1) &= 3^{x\ mod\ y} - 1 \\ (3^x - 3^{x\ mod\ y})\ mod\ (3^y - 1) &= 0 \qquad (if\ a \equiv b(mod\ n),\ then\ a - b \equiv 0(mod\ n)) \\ (3^y - 1)&|(3^x - 3^{x\ mod\ y}) \end{aligned} \tag{5}$$

There are three possibilities for x and y that need to be considered using the information from Equation (5).

(i) if $x = y$, then $x\ mod\ y = 0$.

$$\begin{aligned} &(3^y - 1)|(3^x - 3^{x\ mod\ y}) \\ &(3^y - 1)|(3^y - 3^0) \\ &(3^y - 1)|(3^y - 1) \quad (This\ is\ true\ for\ all\ x\ and\ y) \end{aligned} \tag{6}$$

(ii) if $x < y$, then $x\ mod\ y = x$.

$$\begin{aligned} &(3^y - 1)|(3^x - 3^{x\ mod\ y}) \\ &(3^y - 1)|(3^x - 3^x) \\ &(3^y - 1)|0 \quad (This\ is\ true\ for\ all\ x\ and\ y) \end{aligned} \tag{7}$$

(iii) if $x > y$, then $x\ mod\ y = k$ for some positive integer c and k satisfying $x = cy + k$.

$$\begin{aligned} &(3^y - 1)|(3^x - 3^{x\ mod\ y}) \\ &(3^y - 1)|(3^{cy+k} - 3^k) \\ &(3^y - 1)|(3^k(3^{cy} - 1)) \\ &(3^y - 1)|(3^{cy} - 1) \quad (3^k\ is\ odd\ and\ (3^y - 1)\ is\ even,\ (3^y - 1)|3^k\ is\ impossible) \\ &(3^y - 1)|(3^{cy} - 1) \quad (This\ is\ true\ for\ all\ x,\ y\ and\ c) \end{aligned} \tag{8}$$

Thus, it is has been proven that, $(3^x - 1) \mod (3^y - 1) = 3^{x \mod y} - 1$ is true for $x,\ y \in Z^+$.

b.

$$\begin{aligned} gcd(123, 277) &= gcd(277, 123) \\ &= gcd(123, 31) \\ &= gcd(31, 30) \\ &= gcd(30, 1) \\ &= gcd(1, 0) \\ &= 1 \end{aligned} \tag{9}$$