CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 3

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April 17, 2019

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1. (a) The signal is periodic with the period $N = 4$. We can find the Fourier series coefficients by using the well-known analysis equation, i.e., $a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$.

$$\begin{aligned}
a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n} \\
&= \frac{1}{4}\left(x[0] + x[1]e^{-jk(2\pi/4)} + x[2]e^{-jk(2\pi/4)2} + x[3]e^{-jk(2\pi/4)3}\right) \\
&= \frac{1}{4}\left(0 + e^{-jk(2\pi/4)} + 2e^{-jk(2\pi/4)2} + e^{-jk(2\pi/4)3}\right) \\
&= \frac{1}{4}\left(e^{-jk\pi/2} + 2e^{-jk\pi} + e^{-jk3\pi/2}\right) \\
&= \frac{1}{4}\left((-j)^k + 2(-1)^k + (j)^k\right)
\end{aligned} \tag{1}$$

We know that $a_k = a_{k+N}$, i.e., if we consider more than N sequential values of k, the values a_k repeat periodically with period N. In particular, since there are only N distinct complex exponentials that are periodic with period N, the discrete-time Fourier series representation is a finite series with N terms. Hence, it is sufficient for us to find only a_0, a_1, a_2, and a_3 by using the Eqn. 1 and for the other coefficients we have $a_k = a_{k+N}$.

$$\begin{aligned}
a_0 &= \frac{1}{4}(1 + 2 + 1) = 1 \\
a_1 &= \frac{1}{4}(-j - 2 + j) = -\frac{1}{2} \\
a_2 &= \frac{1}{4}(-1 + 2 - 1) = 0 \\
a_3 &= \frac{1}{4}(j - 2 - j) = -\frac{1}{2}
\end{aligned} \tag{2}$$

Below, we plot the Fourier series coefficient of $x[n]$.

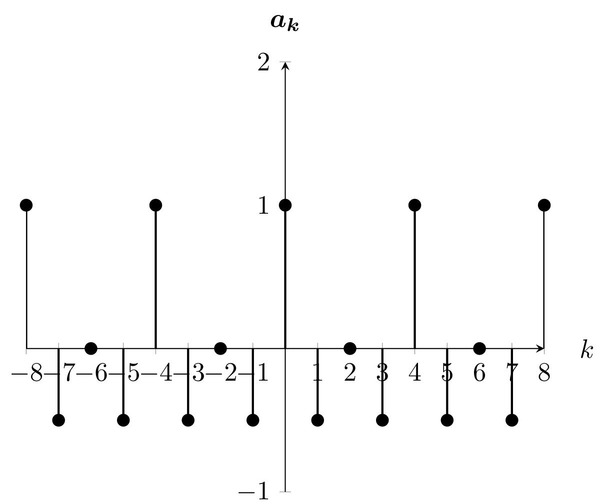

Figure 1: k vs a_k.

(b) (i) Observe that $y[n]$ is almost the same signal as $x[n]$; however, for $n = \dots, -5, -1, 3, 7, \dots$, $y[n] = 0$ whereas $x[n] = 1$. We can define this difference between signals as an impulse train signal and if we subtract this impulse train signal from the $x[n]$, then we get $y[n]$. Below, we define $y[n]$ in terms of $x[n]$.

$$y[n] = x[n] - \sum_{k=-\infty}^{+\infty} \delta[n+1-4k] \tag{3}$$

(ii) To find the Fourier series coefficients of $y[n]$ (name it c_k), we will use the Fourier series coefficients of $x[n]$ (name it a_k) and the Fourier series coefficients of $\sum_{k=-\infty}^{+\infty} \delta[n+1-4k]$ (name it b_k). From part (a), we know a_k and we can easily find b_k. Since the period of both signals are the same, i.e., $N = 4$, by the linearity property of discrete-time Fourier series, we have $c_k = a_k - b_k$ for $y[n] = x[n] - \sum_{k=-\infty}^{+\infty} \delta[n+1-4k]$.
First, we find the Fourier series coefficients of $\sum_{k=-\infty}^{+\infty} \delta[n+1-4k]$, i.e., b_k, by using the analysis equation.

$$\begin{aligned} b_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n} \\ &= \frac{1}{4}\left(x[0] + x[1]e^{-jk(\pi/2)} + x[2]e^{-jk\pi} + x[3]e^{-jk(3\pi/2)}\right) \\ &= \frac{1}{4}\left(e^{-jk(3\pi/2)}\right) \\ &= \frac{j^k}{4} \end{aligned} \tag{4}$$

Now, we find b_0, b_1, b_2, and b_3 by using the Eqn. 4.

$$\begin{aligned} b_0 &= \frac{j^0}{4} = \frac{1}{4} \\ b_1 &= \frac{j^1}{4} = \frac{j}{4} \\ b_2 &= \frac{j^2}{4} = -\frac{1}{4} \\ b_3 &= \frac{j^3}{4} = -\frac{j}{4} \end{aligned} \tag{5}$$

By using the linearity property, below, we compute the Fourier series coefficients of $y[n]$, i.e., $c_k = a_k - b_k$.

$$\begin{aligned} c_0 &= \frac{3}{4} \\ c_1 &= -\frac{1}{2} - \frac{j}{4} \\ c_2 &= \frac{1}{4} \\ c_3 &= -\frac{1}{2} + \frac{j}{4} \end{aligned} \tag{6}$$

Below, we plot the Fourier series coefficient of $y[n]$ for both of its real and imaginary parts.

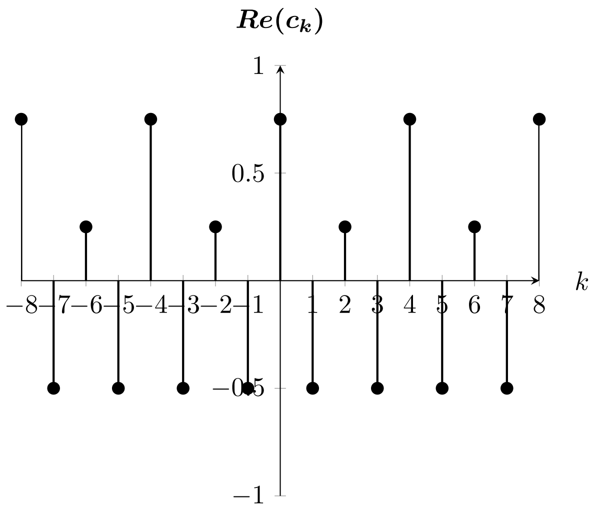

Figure 2: k vs $Re(c_k)$.

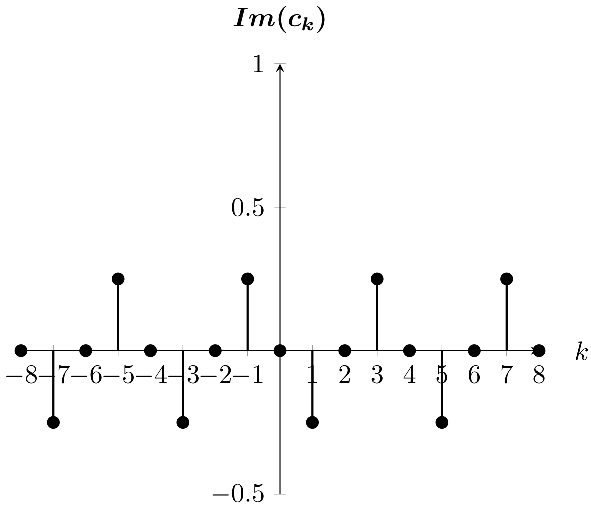

Figure 3: k vs $Im(c_k)$.

2. (a) Using the given information we can deduce that,

$$\begin{aligned} N &= 4 \\ a_k &= a^*_{-k} \\ a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n} \end{aligned} \tag{7}$$

(b) Since $x[n]$ is periodic and the period is 4 we can say that $x[n] = x[n+4]$ for any $n \in Z$.

$$\begin{aligned} 8 &= x[-3] + x[-2] + x[-1] + x[0] + x[1] + x[2] + x[3] + x[4] \\ 8 &= x[1] + x[2] + x[3] + x[0] + x[1] + x[2] + x[3] + x[0] \\ 4 &= x[0] + x[1] + x[2] + x[3] \end{aligned} \tag{8}$$

(c) Since it is a discrete-time signal, coefficients will repeat periodically after the period, i.e., $a_k = a_{k+N}$, we can say the following.

$$\begin{aligned} a_{-3} &= a^*_{15} \\ a_{-3} &= a_1 \\ a^*_{15} &= a^*_3 \\ a_1 &= a^*_3 \\ \text{Assume that } a_1 &= x + yj, \\ a_3 &= x - yj \\ |a_1 - a_{11}| &= 1 \\ a_{11} &= a_3 \\ |a_1 - a_3| &= 1 \\ |2yj| &= 1 \\ y &= +\frac{1}{2} \text{ or } y = -\frac{1}{2} \end{aligned} \tag{9}$$

Therefore, there are two cases,

$$\begin{aligned} a_1 = x + \frac{j}{2} &\text{ and } a_3 = x - \frac{j}{2} \\ &\text{or} \\ a_1 = x - \frac{j}{2} &\text{ and } a_3 = x + \frac{j}{2} \end{aligned} \tag{10}$$

(d) Let's skip this one for now.

(e) The first information from this is $x[0]-x[2]=2$ which is derived as

$$\begin{aligned}
4 &= \sum_{k=0}^{3} x[k]\left(e^{-j\pi k/2}+e^{-j\pi 3k/2}\right)\\
4 &= 2x[0]+\left(e^{-j\pi/2}+e^{-j\pi 3/2}\right)x[1]+\left(e^{-j\pi}+e^{-j\pi 3}\right)x[2]+\left(e^{-j\pi 3/2}+e^{-j\pi 9/2}\right)x[3]\\
4 &= 2x[0]+(j-j)\,x[1]+(-1-1)\,x[2]+(-j+j)\,x[3]\\
4 &= 2x[0]-2x[2]\\
2 &= x[0]-x[2]
\end{aligned} \tag{11}$$

The second information derived from this is $a_1+a_3=1$,

$$\begin{aligned}
4 &= \sum_{k=0}^{3} x[k]\left(e^{-j\pi k/2}+e^{-j\pi 3k/2}\right)\\
4 &= \sum_{n=0}^{3} x[n]\left(e^{-j\pi n/2}\right)+\sum_{n=0}^{3} x[n]\left(e^{-j\pi 3n/2}\right)\\
1 &= \frac{1}{4}\sum_{n=0}^{3} x[n]\left(e^{-j\pi n/2}\right)+\frac{1}{4}\sum_{n=0}^{3} x[n]\left(e^{-j\pi 3n/2}\right)\\
a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n}\\
1 &= a_1+a_3
\end{aligned} \tag{12}$$

Now using the a_1 and a_3 values from the Eqn. 10,

$$\begin{aligned}
a_1 &= x+\frac{j}{2} \text{ and } a_3 = x-\frac{j}{2}\\
1 &= a_1+a_3 = 2x\\
x &= \frac{1}{2}
\end{aligned} \tag{13}$$

a_1 and a_3 have now become,

$$\begin{aligned}
a_1 &= \frac{1}{2}+\frac{j}{2} \text{ and } a_3 = \frac{1}{2}-\frac{j}{2}\\
&\text{or}\\
a_1 &= \frac{1}{2}-\frac{j}{2} \text{ and } a_3 = \frac{1}{2}+\frac{j}{2}
\end{aligned} \tag{14}$$

Now let's try to find a_0,

$$\begin{aligned}
a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n}\\
a_0 &= \frac{1}{4}\sum_{n=0}^{3} x[n]\\
a_0 &= \frac{1}{4}\left(x[0]+x[1]+x[2]+x[3]\right)\\
a_0 &= \frac{4}{4} = 1 \text{ using the Eqn. 8}
\end{aligned} \tag{15}$$

a_0, a_1 and a_3 are non-zero and we know from (d) that one of the coefficients is zero therefore $a_2=0$. Until now what we know about $x[n]$ is as follows,

$$\begin{aligned}
4 &= x[0]+x[1]+x[2]+x[3]\\
x[0] &= x[2]+2
\end{aligned} \tag{16}$$

To find $x[n]$ we need to obtain more equations,

$$\begin{aligned}
a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(\pi/2)n}\\
a_k &= \frac{1}{4}\left(x[0]+x[1](-j)^k+x[2](-1)^k+x[3](j)^k\right)
\end{aligned} \tag{17}$$

Let's write the equation for the coefficient a_1,

$$\begin{aligned} a_1 &= \frac{1}{4}\left(x[0] + x[1](-j)^1 + x[2](-1)^1 + x[3](j)^1\right) \\ a_1 &= \frac{1}{4}\left(x[0] + -jx[1] - x[2] + jx[3]\right) = \frac{1}{2} + \frac{j}{2} \text{ assume we chose this case} \end{aligned} \tag{18}$$

Now we can obtain a new equation but it is not yet enough to find $x[n]$,

$$\begin{aligned} \frac{1}{4}\left(2 + j(x[3] - x[1])\right) &= \frac{1}{2} + \frac{j}{2} \text{ assume we chose this case} \\ \frac{1}{2} + \frac{j}{4}(x[3] - x[1]) &= \frac{1}{2} + \frac{j}{2} \\ x[3] &= x[1] + 2 \end{aligned} \tag{19}$$

Now let's write the equation for coefficient a_2,

$$\begin{aligned} a_2 &= \frac{1}{4}\left(x[0] + x[1](-j)^2 + x[2](-1)^2 + x[3](j)^2\right) \\ a_2 &= \frac{1}{4}\left(x[0] - x[1] + x[2] - x[3]\right) = 0 \end{aligned} \tag{20}$$

We know from Eqn. 8 that,

$$\begin{aligned} 4 &= x[0] + x[1] + x[2] + x[3] \\ 0 &= x[0] - x[1] + x[2] - x[3] \\ 4 &= 2x[0] + 2x[2] \\ 2 &= x[0] + x[2] \\ 2 &= x[0] - x[2] \\ x[0] &= 2 \end{aligned} \tag{21}$$

Now the equation reduces to,

$$\begin{aligned} 2 &= x[3] + x[1] \\ 2 &= x[3] - x[1] \\ x[3] &= 2 \\ x[1] &= 0 \end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned} x[0] &= 2 \\ x[1] &= 0 \\ x[2] &= 0 \\ x[3] &= 2 \end{aligned} \tag{23}$$

Below, we plot $x[n]$.

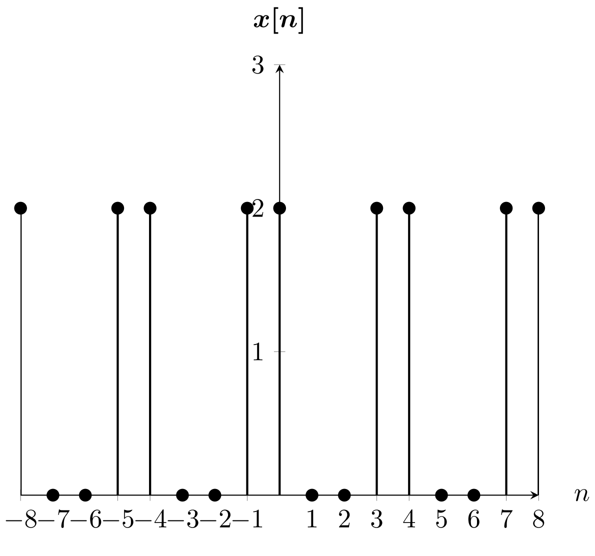

Figure 4: n vs $x[n]$.

3. We need to use a low pass filter to get rid of the noise $r(t)$. We define the low pass filter as follows.

$$H(j\omega) = \begin{cases} 1 & \text{if } |\omega| \leq 2K\pi/T \\ 0 & \text{if } |\omega| > 2K\pi/T \end{cases} \tag{24}$$

We will use the well-know formula,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(j\omega) e^{j\omega t} d\omega \tag{25}$$

in order to find the impulse response of such a system. Below, we find the impulse response $h(t)$.

$$\begin{aligned} h(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(j\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-2K\pi/T}^{+2K\pi/T} e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \left(\frac{e^{jt2K\pi/T}}{jt} - \frac{e^{-jt2K\pi/T}}{jt} \right) \\ &= \frac{e^{jt2K\pi/T} - e^{-jt2K\pi/T}}{2\pi jt} \\ &= \frac{sin\left(\frac{t2K\pi}{T}\right)}{\pi t} \end{aligned} \tag{26}$$

4. (a) The differential equation of the given system is,

$$y''(t) + 5y'(t) + 6y(t) = x(t) + 4x'(t) \tag{27}$$

Note that the differential property of Fourier Transform is as follows,

$$f\{x(t)\} = X(j\omega) \rightarrow^{FT} f\{x'(t)\} = jwX(j\omega) \tag{28}$$

Using this property and the linearity of Fourier Transform we can write the differential equation of the system above as follows,

$$(j\omega)^2 Y(j\omega) + 5jwY(j\omega) + 6Y(j\omega) = X(j\omega) + 4jwX(j\omega) \tag{29}$$

Because of $f\{y(t) = x(t) * h(t)\} = Y(j\omega) = X(j\omega)H(j\omega)$, the Eqn. 29. yields to,

$$(j\omega)^2 H(j\omega) + 5jwH(j\omega) + 6H(j\omega) = 1 + 4jw \tag{30}$$

Therefore the frequency response is,

$$H(j\omega) = \frac{1 + 4jw}{(j\omega + 3)(j\omega + 2)} = \frac{A}{(j\omega + 3)} + \frac{B}{(j\omega + 2)} \tag{31}$$

Solving this equation yields to $A = 11$ and $B = -7$ therefore the frequency response is,

$$H(j\omega) = \frac{11}{(j\omega + 3)} - \frac{7}{(j\omega + 2)} \tag{32}$$

(b) We know that,

$$f\{e^{-at}u(t)\} = \frac{1}{j\omega + a} \tag{33}$$

Therefore the impulse response of the frequency response from the part a is,

$$h(t) = (11e^{-3t} - 7e^{-2t})u(t) \tag{34}$$

(c) Note that $f\{y(t) = x(t) * h(t)\} = Y(j\omega) = X(j\omega)H(j\omega)$ and we also know that,

$$f\{e^{-at}u(t)\} = \frac{1}{j\omega + a} \tag{35}$$

Therefore,

$$f\{x(t) = \frac{1}{4} e^{-t/4} u(t)\} = \frac{1}{4(j\omega + 1/4)} = \frac{1}{1 + 4jw} \tag{36}$$

Now we simply multiply this Fourier Transformation of the input with the impulse response we found in the part a to find the transformed output,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{1+4jw} . \frac{1+4jw}{(j\omega+3)(j\omega+2)} = \frac{1}{(j\omega+3)(j\omega+2)} = \frac{A}{(j\omega+3)} + \frac{B}{(j\omega+2)} \tag{37}$$

Solving this equation $A = -1$ and $B = 1$,

$$Y(j\omega) = \frac{1}{(j\omega+2)} - \frac{1}{(j\omega+3)} \tag{38}$$

Using the Eqn. 35.

$$y(t) = (e^{-2t} - e^{-3t})u(t) \tag{39}$$