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Homework 4 - Terms and Predicate Logic

Assigned - Dec 06, 2018, **Due - Dec 14, 2018**

1 Proof Terms

(a) $\cdot \vdash ((p \wedge q) \supset r) \supset (p \supset (q \supset r))$

$u :$	1.	$(p \wedge q) \supset r$	*assumption*
$v :$	2.	p	*assumption*
$w :$	3.	q	*assumption*
$\langle v, w \rangle :$	4.	$p \wedge q$	$\wedge i$ 2, 3
$u(\langle v, w \rangle) :$	5.	r	$\supset e$ 1, 4
$\lambda w.u(\langle v, w \rangle) :$	6.	$q \supset r$	$\supset i$ 3-5
$\lambda v.\lambda w.u(\langle v, w \rangle) :$	7.	$p \supset (q \supset r)$	$\supset i$ 2-6
$\lambda u.\lambda v.\lambda w.u(\langle v, w \rangle) :$	8.	$((p \wedge q) \supset r) \supset (p \supset (q \supset r))$	$\supset i$ 1-7

$$\boxed{\lambda u.\lambda v.\lambda w.u(\langle v, w \rangle)}$$

(b) $\cdot \vdash ((p \vee q) \supset r) \supset ((p \supset r) \wedge (q \supset r))$

$u :$	1.	$(p \vee q) \supset r$	*assumption*
$v :$	2.	p	*assumption*
$\textbf{inl } v :$	3.	$p \vee q$	$\vee i_1$ 2
$u(\textbf{inl } v) :$	4.	r	$\supset e$ 1, 3
$\lambda v.u(\textbf{inl } v) :$	5.	$p \supset r$	$\supset i$ 2-4
$w :$	6.	q	*assumption*
$\textbf{inr } w :$	7.	$p \vee q$	$\vee i_2$ 6
$u(\textbf{inr } w) :$	8.	r	$\supset e$ 1, 7
$\lambda w.u(\textbf{inr } w) :$	9.	$q \supset r$	$\supset i$ 6-8
$\langle \lambda v.u(\textbf{inl } v), \lambda w.u(\textbf{inr } w) \rangle :$	10.	$(p \supset r) \wedge (q \supset r)$	$\wedge i$ 5, 9
$\lambda u.\langle \lambda v.u(\textbf{inl } v), \lambda w.u(\textbf{inr } w) \rangle :$	11.	$((p \vee q) \supset r) \supset ((p \supset r) \wedge (q \supset r))$	$\supset i$ 1-10

$$\boxed{\lambda u.\langle \lambda v.u(\textbf{inl } v), \lambda w.u(\textbf{inr } w) \rangle}$$

(c) $\cdot \vdash \neg(p \supset q) \supset (p \wedge \neg q)$

This sequent is not provable by constructuive logic.

(d) $\cdot \vdash (p \supset (r \supset q)) \supset ((p \wedge \neg q) \supset \neg r)$

$u :$	1.	$p \supset (r \supset q)$	*assumption*
$v :$	2.	$p \wedge \neg q$	*assumption*
$\mathbf{snd}\ v :$	3.	$\neg q$	$\wedge e_2\ 2$
$\mathbf{fst}\ v :$	4.	p	$\wedge e_1\ 2$
$u(\mathbf{fst}\ v) :$	5.	$r \supset q$	$\supset e\ 1, 4$
$w :$	6.	r	*assumption*
$u(\mathbf{fst}\ v)(w) :$	7.	q	$\supset e\ 5, 6$
$\mathbf{snd}\ v(u(\mathbf{fst}\ v)(w)) :$	8.	$\perp$	$\supset e\ 3, 7$
$\lambda w.\mathbf{snd}\ v(u(\mathbf{fst}\ v)(w)) :$	9.	$\neg r$	$\neg i$ 6-8
$\lambda v.\lambda w.\mathbf{snd}\ v(u(\mathbf{fst}\ v)(w)) :$	10.	$(p \wedge \neg q) \supset \neg r$	$\supset i$ 2-9
$\lambda u.\lambda v.\lambda w.\mathbf{snd}\ v(u(\mathbf{fst}\ v)(w)) :$	11.	$(p \supset (r \supset q)) \supset ((p \wedge \neg q) \supset \neg r)$	$\supset i$ 1-10

$$\boxed{\lambda u.\lambda v.\lambda w.\mathbf{snd}\ v(u(\mathbf{fst}\ v)(w))}$$

2 Contraction-free Rules

(a) $(u : p), (v : B \supset C) \vdash M : (p \supset B) \supset C$

$u :$	1.	p	*premise*
$v :$	2.	$B \supset C$	*premise*
$w :$	3.	$p \supset B$	*assumption*
$w(u) :$	4.	B	$\supset e\ 3, 1$
$v(w(u))) :$	5.	C	$\supset e\ 2, 4$
$\lambda w.v(w(u)) :$	6.	$(p \supset B) \supset C$	$\supset i$ 3-5

$$\boxed{M = \lambda w.v(w(u))}$$

(b) $u : (A_1 \supset (A_2 \supset B)) \supset C \vdash M : ((A_1 \wedge A_2) \supset B) \supset C$

$u :$	1.	$(A_1 \supset (A_2 \supset B)) \supset C$	*premise*
$v :$	2.	$(A_1 \wedge A_2) \supset B$	*assumption*
$w :$	3.	A_1	*assumption*
$t :$	4.	A_2	*assumption*
$\langle w, t \rangle :$	5.	$A_1 \wedge A_2$	$\wedge i\ 3, 4$
$v(\langle w, t \rangle) :$	6.	B	$\supset e\ 2, 5$
$\lambda t.v(\langle w, t \rangle) :$	7.	$A_2 \supset B$	$\supset i$ 4-6
$\lambda w.\lambda t.v(\langle w, t \rangle) :$	8.	$A_1 \supset (A_2 \supset B)$	$\supset i$ 3-7
$u(\lambda w.\lambda t.v(\langle w, t \rangle)) :$	9.	C	$\supset e\ 1, 8$
$\lambda v.u(\lambda w.\lambda t.v(\langle w, t \rangle)) :$	10.	$((A_1 \wedge A_2) \supset B) \supset C$	$\supset i$ 2-9

$$\boxed{M = \lambda v.u(\lambda w.\lambda t.v(\langle w, t \rangle))}$$

(c) $u : B \supset C \vdash M : (\top \supset B) \supset C$

$u:$	1.	$B \supset C$	*premise*
$v:$	2.	$\top \supset B$	*assumption*
$\top I:$	3.	$\top$	$\top i$
$v(\top I):$	4.	B	$\supset e\ 2,3$
$u(v(\top I)):$	5.	C	$\supset e\ 1,4$
$\lambda v.u(v(\top I)):$	6.	$(\top \supset B) \supset C$	$\supset i$ 2-5

$$\boxed{M = \lambda v.u(v(\top I))}$$

(d) $u : (A_1 \supset B) \supset ((A_2 \supset B) \supset C) \vdash M : ((A_1 \vee A_2) \supset B) \supset C$

$u:$	1.	$(A_1 \supset B) \supset ((A_2 \supset B) \supset C)$	*premise*
$v:$	2.	$((A_1 \vee A_2) \supset B)$	*assumption*
$w:$	3.	A_1	*assumption*
$\textbf{inl}\ w:$	4.	$A_1 \vee A_2$	$\vee i_1\ 3$
$v(\textbf{inl}\ w):$	5.	B	$\supset e\ 2,4$
$\lambda w.v(\textbf{inl}\ w):$	6.	$A_1 \supset B$	$\supset i$ 3-5
$u(\lambda w.v(\textbf{inl}\ w)):$	7.	$(A_2 \supset B) \supset C$	$\supset e\ 1,6$
$t:$	8.	A_2	*assumption*
$\textbf{inr}\ t:$	9.	$A_1 \vee A_2$	$\vee i_2\ 8$
$v(\textbf{inr}\ t):$	10.	B	$\supset e\ 2,9$
$\lambda t.v(\textbf{inr}\ t):$	11.	$A_2 \supset B$	$\supset i$ 8-10
$u(\lambda w.v(\textbf{inl}\ w))(\lambda t.v(\textbf{inr}\ t)):$	12.	C	$\supset e\ 7,11$
$\lambda v.u(\lambda w.v(\textbf{inl}\ w))(\lambda t.v(\textbf{inr}\ t)):$	13.	$((A_1 \vee A_2) \supset B) \supset C$	$\supset i$ 2-12

$$\boxed{M = \lambda v.u(\lambda w.v(\textbf{inl}\ w))(\lambda t.v(\textbf{inr}\ t))}$$

(e) $u : C \vdash M : (\bot \supset B) \supset C$

$u:$	1.	C	*premise*
$v:$	2.	$\bot \supset B$	*assumption*
$u:$	3.	C	*copy* 1
$\lambda v.u:$	6.	$(\bot \supset B) \supset C$	$\supset i$ 2-3

$$\boxed{M = \lambda v.u}$$

(f) $(u : (A_2 \supset B) \supset (A_1 \supset A_2)), (v : B \supset C) \vdash M : ((A_1 \supset A_2) \supset B) \supset C$

$u:$	1.	$(A_2 \supset B) \supset (A_1 \supset A_2)$	*premise*
$v:$	2.	$B \supset C$	*premise*
$w_1:$	3.	$(A_1 \supset A_2) \supset B$	*assumption*
$w_2:$	4.	A_1	*assumption*
$w_3:$	5.	A_2	*assumption*
$w_4:$	6.	A_1	*assumption*
$w_3:$	7.	A_2	*copy* 5
$\lambda w_4.w_3:$	8.	$A_1 \supset A_2$	$\supset i$ 6-7
$w_1(\lambda w_4.w_3):$	9.	B	$\supset e\ 3,8$
$\lambda w_3.w_1(\lambda w_4.w_3):$	10.	$A_2 \supset B$	$\supset i$ 5-9
$u(\lambda w_3.w_1(\lambda w_4.w_3)):$	11.	$A_1 \supset A_2$	$\supset e\ 1,10$
$u(\lambda w_3.w_1(\lambda w_4.w_3))(w_2):$	12.	A_2	$\supset e\ 11,4$
$\lambda w_2.u(\lambda w_3.w_1(\lambda w_4.w_3))(w_2):$	13.	$A_1 \supset A_2$	$\supset i$ 4-12
$w_1(\lambda w_2.u(\lambda w_3.w_1(\lambda w_4.w_3))(w_2)):$	14.	B	$\supset e\ 3,13$
$v(w_1(\lambda w_2.u(\lambda w_3.w_1(\lambda w_4.w_3))(w_2))):$	15.	C	$\supset e\ 2,14$
$\lambda w_1.v(w_1(\lambda w_2.u(\lambda w_3.w_1(\lambda w_4.w_3))(w_2))):$	16.	$((A_1 \supset A_2) \supset B) \supset C$	$\supset i$ 3-15

$$\boxed{M = \lambda w_1.v(w_1(\lambda w_2.u(\lambda w_3.w_1(\lambda w_4.w_3))(w_2)))}$$

3 New Connectives

(a)

p	q	$p * q$
T	T	T
T	F	F
F	T	F
F	F	T

(b)

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \phi\end{array}}}{\phi * \psi}\ {*i} \qquad \frac{\phi * \psi \qquad \psi}{\phi}\ {*e_1} \qquad \frac{\phi * \psi \qquad \phi}{\psi}\ {*e_2}$$

(c)

$$p \ * \ q \vdash (p \supset q) \wedge (q \supset p)$$

1.	$p * q$	*premise*
2.	p	*assumption*
3.	q	$*e_2\ 1,2$
4.	$p \supset q$	$\supset i$ 2-3
5.	q	*assumption*
6.	p	$*e_1\ 1,5$
7.	$q \supset p$	$\supset i$ 5-6
8.	$(p \supset q) \wedge (q \supset p)$	$\wedge i\ 4,7$

$$(p \supset q) \wedge (q \supset p) \vdash p \ * \ q$$

1.	$(p \supset q) \wedge (q \supset p)$	*premise*
2.	$p \supset q$	$\wedge e_1\ 1$
3.	$q \supset p$	$\wedge e_2\ 1$
4.	p	*assumption*
5.	q	$\supset e\ 2,4$
6.	q	*assumption*
7.	p	$\supset e\ 3,6$
8.	$p * q$	$*i$ 4-5, 6-7

(d)

$$\frac{\boxed{\begin{array}{r} u : A \\ \vdots \\ M : B \end{array}} \qquad \boxed{\begin{array}{r} v : B \\ \vdots \\ N : A \end{array}}}{M \equiv N : A * B} *I \qquad \frac{M : \phi * \psi \qquad N : \psi}{\textbf{left } M : \phi} *E_L \qquad \frac{M : \phi * \psi \qquad N : \phi}{\textbf{right } M : \psi} *E_R$$

$u_1 :$	1.	$p * q$	*assumption*
$u_2 :$	2.	$q \supset r$	*assumption*
$u_3 :$	3.	$r \supset p$	*assumption*
$u_4 :$	4.	p	*assumption*
$\textbf{right } u_1 :$	5.	q	$*e_2\ 1,4$
$u_2(\textbf{right } u_1) :$	6.	r	$\supset e\ 2,5$
$u_5 :$	7.	r	*assumption*
$u_3(u_5) :$	8.	p	$\supset e\ 3,7$
$u_2(\textbf{right } u_1) \equiv u_3(u_5) :$	9.	$p * r$	$*i$ 4-6, 7-8
$\lambda u_3.u_2(\textbf{right } u_1) \equiv u_3(u_5) :$	10.	$(r \supset p) \supset (p * r)$	$\supset i$ 3-9
$\lambda u_2.\lambda u_3.u_2(\textbf{right } u_1) \equiv u_3(u_5) :$	11.	$(q \supset r) \supset ((r \supset p) \supset (p * r))$	$\supset i$ 2-10
$\lambda u_1.\lambda u_2.\lambda u_3.u_2(\textbf{right } u_1) \equiv u_3(u_5) :$	12.	$(p * q) \supset ((q \supset r) \supset ((r \supset p) \supset (p * r)))$	$\supset i$ 1-11

$$\boxed{\lambda u_1.\lambda u_2.\lambda u_3.u_2(\textbf{right } u_1) \equiv u_3(u_5)}$$