

# Lecture VII - Library Routing Optimization

## Applied Optimization with Julia

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University of Hamburg - Fall 2025

### Introduction

#### Central Libraries

...

Question: Anybody an idea what a central library is?

#### Central Libraries

- Book Delivery to Libraries in Germany
- They supply all local libraries within the same state
- Complex, as the number of libraries per state can be large
- Books and media in the libraries change often
- Customers can request books from other libraries

#### Structure of the Deliveries

- For delivery, central has several employees and cars
- Local libraries differ in size, some receive more items
- Items are collected as well during the tours<sup>1</sup>
- They are transported back to the central library

#### Potential Decisions

Question: What decisions can the library make for tours?

...

- Subdivide their set of libraries into several ordered tours
- Decide in which order to visit the libraries
- Evaluate which car to use for each of the tours
- Decide which driver to assign to each of the tours

#### Impact of the Decisions

Question: What is the impact of the decisions?

...

- Longer driving routes increase the footprint of the deliveries

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<sup>1</sup>Due to regulations, the delivery tours cannot exceed a certain duration

- Suboptimal tours can lead to unnecessary costs
- Fuel, personnel, and repairs are increased
- Unhappy customers due to waiting times on ordered books

Have you heard of  
this problem before?

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## Problem Structure

### Objective

Question: What could be the objective for central libraries?

...

- Lowering costs through improved tours
- Improvement of their footprint through shorter tours
- Faster fulfillment of the deliveries

### Modelling

Question: What could we try to model?

...

Minimization of the travel time while supplying all libraries in the state and adhering to the vehicle capacities and driving time restrictions.

Capacitated

Vehicle Routing  
(CVRP)

### Vehicle Routing Problem

- CVRP is a subproblem
- Main problem is Vehicle Routing Problem (VRP)
- Problem class about designing routes for vehicle fleets

...

#### Note

There are many variants of the VRP! E.g., with time windows, periodic deliveries, allowing for pickups or deliveries, and much more!

Let's visualize

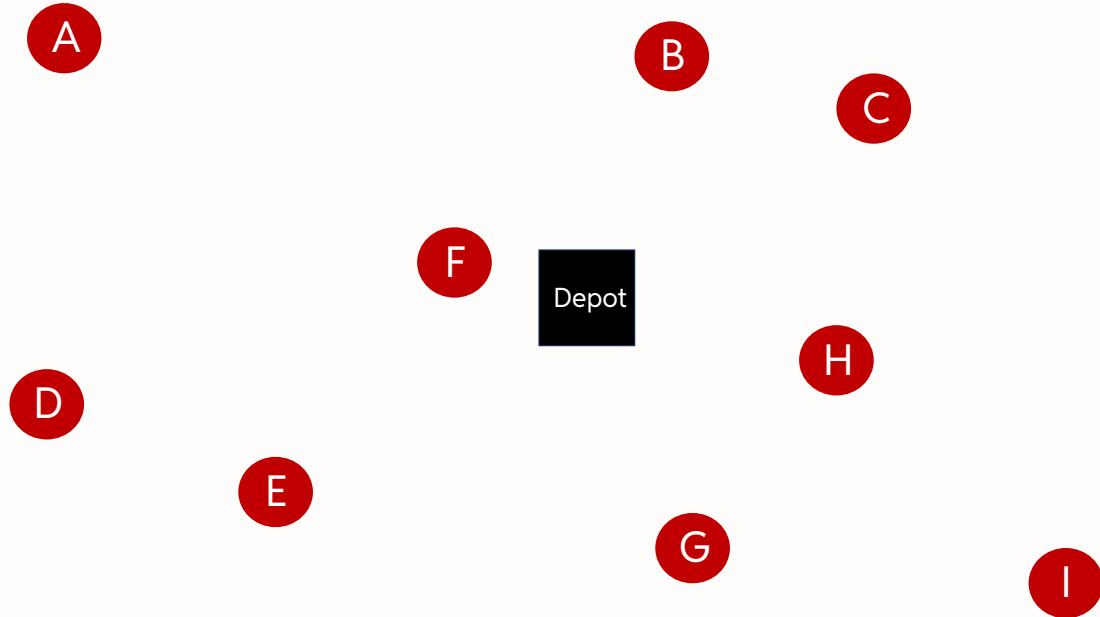
the problem!

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## Problem Visualization

### Basic Problem Setting

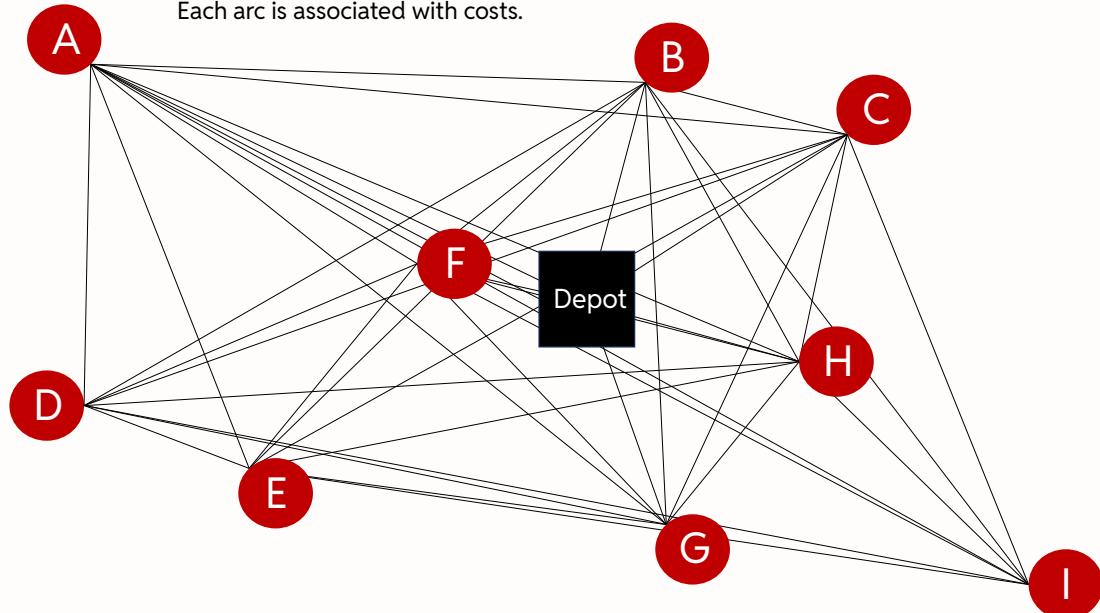
Each customer has a certain demand.



### Basic Problem Setting with Arcs

Each customer has a certain demand.

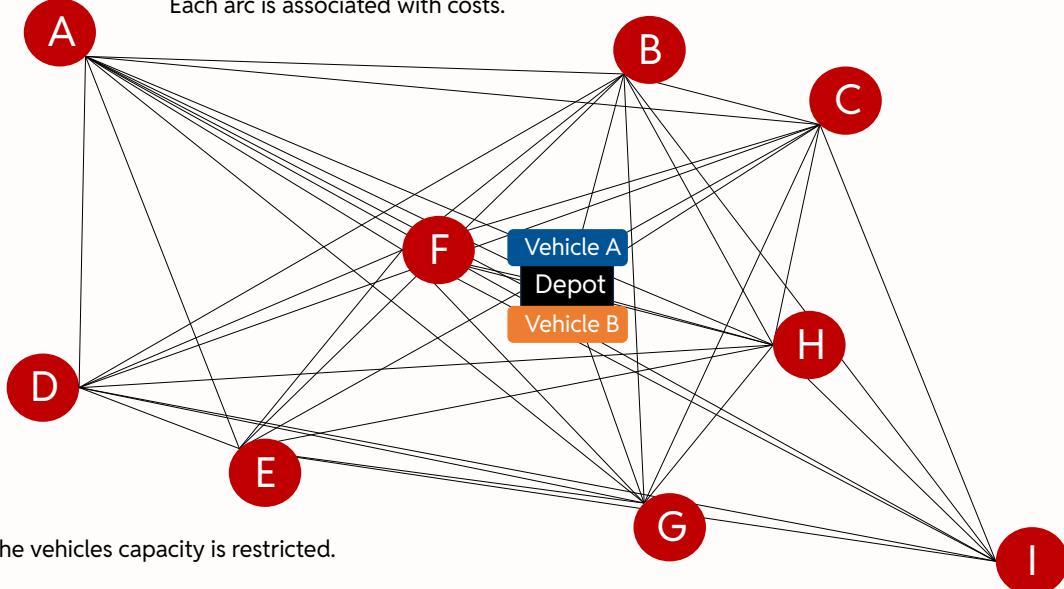
Each arc is associated with costs.



## Setting with Vehicles

Each customer has a certain demand.

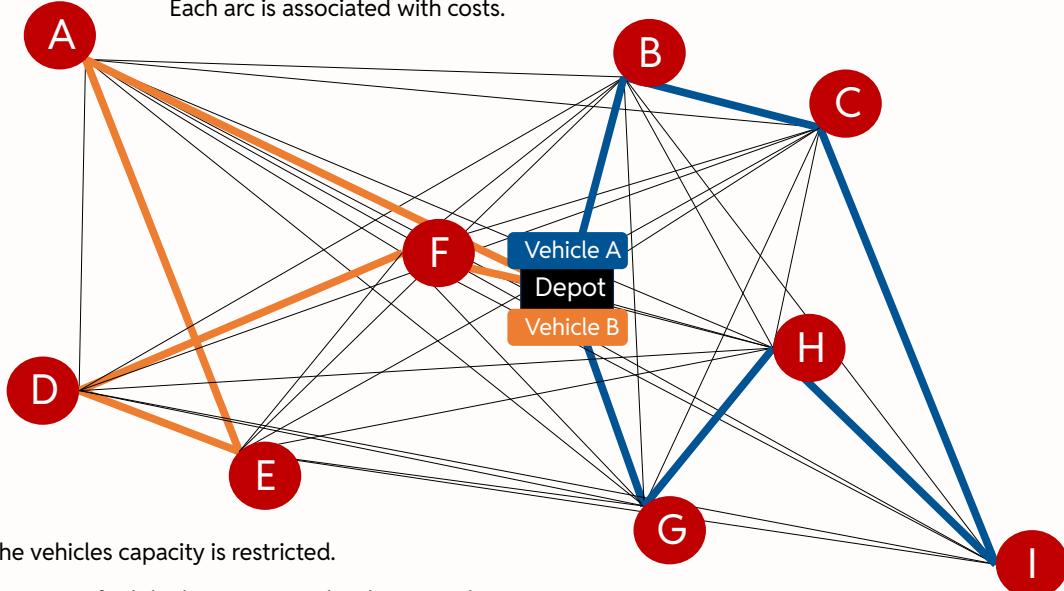
Each arc is associated with costs.



## Basic Problem Setting with Tours

Each customer has a certain demand.

Each arc is associated with costs.



## Problem Structure

### Available Sets

Question: What could be the sets here?

...

- $\mathcal{V}$  - Set of all nodes, index  $i \in \{0, 1, 2, \dots, n\}$
- $\mathcal{A}$  - Set of all arcs between the nodes, index  $(i, j) \in \mathcal{A}$
- $\mathcal{K}$  - Set of vehicles with identical capacity, index  $k \in \mathcal{K}$
- $0 \in \mathcal{V}$  - Depot where the vehicles start

## Available Parameters

Question: What are possible parameters?

...

- $b$  - Capacity per vehicle
- $t$  - Maximal duration of each tour
- $d_i$  - Demand at node  $i$
- $c_{i,j}$  - Travel time on an arc from  $i$  to  $j$

...



### Tip

$t$  is the maximal duration of each tour, not the travel time on an arc or an index!

## Decision Variable(s)?

**i** We have the following sets:

- All nodes, including the depot,  $i \in \mathcal{V}$
- All arcs between the nodes,  $(i, j) \in \mathcal{A}$
- The available vehicles,  $k \in \mathcal{K}$

...

**!** Our objective is to:

Minimize the total travel time while supplying all customers and adhering to the vehicle capacities and duration restrictions.

...

Question: What could be our decision variable/s?

## Decision Variables

- $X_{i,j,k}$  - 1, if  $k$  passes between  $i$  and  $j$  on its tour, 0 otherwise

...

### **i** Variable Domain

$X_{i,j,k}$  is a binary variable, as it can only take values 0 or 1. But most likely, you will already have spotted that!

...

Question: Why this might make the problem difficult?

Decision Variable/s (again)?

### **i** We have the following sets:

- All nodes, including the depot,  $i \in \mathcal{V}$
- All arcs between the nodes,  $(i, j) \in \mathcal{A}$
- The available vehicles,  $k \in \mathcal{K}$

...

! Our objective is to:

Minimization of the travel time (or driving distance), while supplying all customers and adhering to the vehicle capacities and duration restrictions. Hint: Even with many vehicles, each arc can maximally be passed once!

Decision Variables (again)

Question: What could be our decision variable/s?

...

- $X_{i,j} = 1$ , if the arc between  $i$  and  $j$  is part of a tour, else 0

...

! Only possible under certain conditions!

Only possible, if time and capacity constraints are equal for all vehicles!

---

## Model Formulation

Objective Function?

! Our objective is to:

Minimization of the travel time (or driving distance), while supplying all customers and adhering to the vehicle capacities and duration restrictions.

...

Question: What could be our objective function?

...

**i** We need the following variable:

- $X_{i,j}$  - 1, if the arc between  $i$  and  $j$  is part of a tour, else 0

## Objective Function

**i** We need the following parameters:

- $c_{i,j}$  - travel time on an arc from  $i$  to  $j$

...

$$\text{minimize} \quad \sum_{(i,j) \in \mathcal{A}} c_{i,j} \times X_{i,j}$$

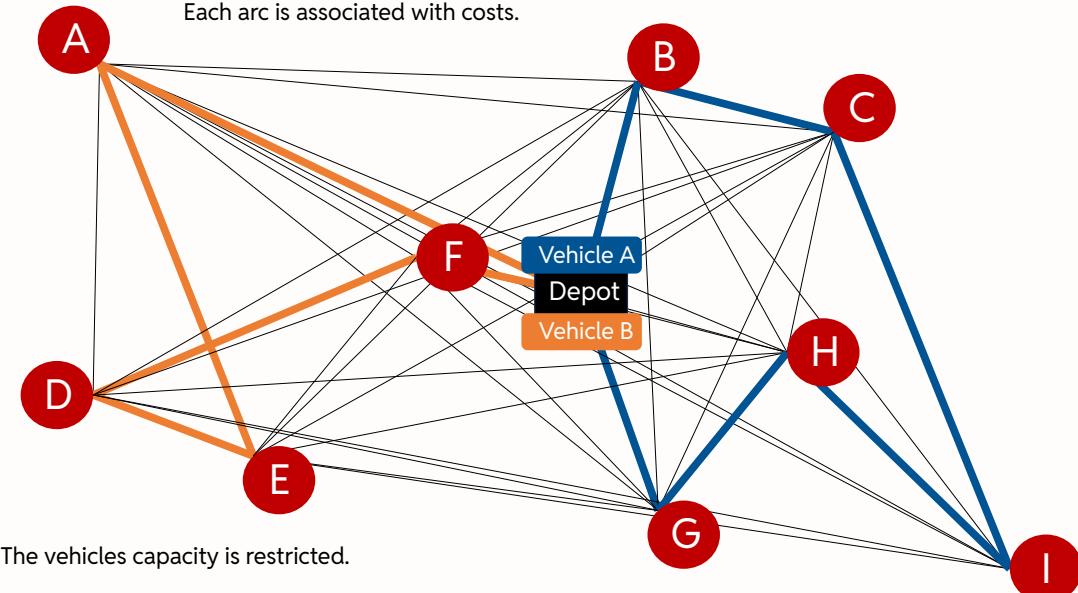
...

Question: What does  $(i, j) \in \mathcal{A}$  under the sum mean?

## Problem Constraints

Each customer has a certain demand.

Each arc is associated with costs.



## Constraints?

Question: What constraints do we need?

...

- Each customer has to be visited once
- The depot has to be entered and left  $|\mathcal{K}|$  times
- We have to enforce the capacity of our vehicles
- We have to ensure the maximal duration of each tour

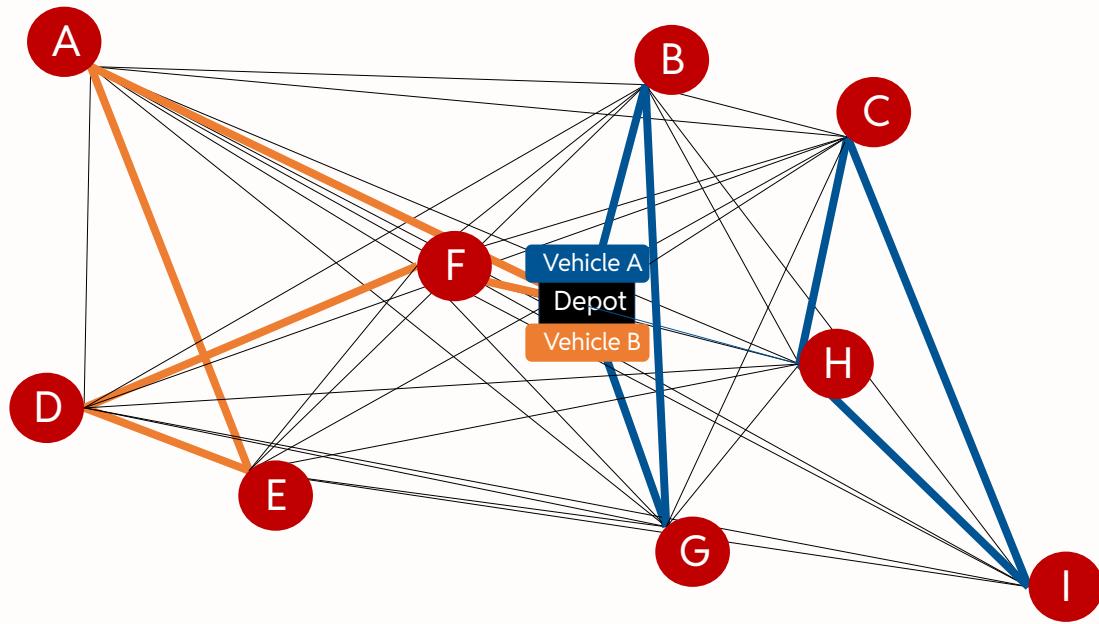
...

### ! Subtours

In addition, we have to prevent subtours!

What is a  
subtour?

### Subtours



### Constraints

Visit Each Customer Once?

! The goal of these constraints is to:

Ensure that each customer is visited exactly once. Essentially, we could also say that each node has to be entered and left exactly once.

...

**i** We need the following sets and variables:

- $\mathcal{V}$  - Set of all nodes, index  $i \in \{0, 1, 2, \dots, n\}$
- $X_{i,j}$  - 1, if the arc between  $i$  and  $j$  is part of a tour, 0 otherwise

## Visit Each Customer Once

Question: What could the constraint look like?

...

$$\sum_{i \in \mathcal{V}} X_{i,j} = 1 \quad \forall j \in \mathcal{V} \setminus \{0\}, i \neq j$$

$$\sum_{j \in \mathcal{V}} X_{i,j} = 1 \quad \forall i \in \mathcal{V} \setminus \{0\}, i \neq j$$

...

Question: Why for all nodes except the depot?

...

The depot is the only node that is visited multiple times!

---

## Depot Entry and Exit Constraints?

**!** The goal of these constraints is to:

Ensure that each vehicle enters and leaves the depot exactly  $|\mathcal{K}|$  times, as we have  $|\mathcal{K}|$  vehicles and each vehicle has to return to the depot.

...

**i** We need the following sets and variables:

- $\mathcal{V}$  - Set of all nodes, index  $i \in \{0, 1, 2, \dots, n\}$
- $|\mathcal{K}|$  - Number of vehicles
- $X_{i,j}$  - 1, if the arc between  $i$  and  $j$  is part of a tour, 0 otherwise

...

Question: What could the constraint look like?

## Depot Entry and Exit Constraints

$$\sum_{i \in \mathcal{V} \setminus \{0\}} X_{i,0} = |\mathcal{K}|$$

$$\sum_{j \in \mathcal{V} \setminus \{0\}} X_{0,j} = |\mathcal{K}|$$

...

### **i** Are all constraints necessary?

No, theoretically we could also say that we only have to leave or enter the depot exactly  $|\mathcal{K}|$  times, as the other constraint is already enforced by the “visit each customer once constraint”.

## Capacity and Subtour Elimination

The next ones are

a little bit tricky.

### MTZ Formulation

- Miller-Tucker-Zemlin (MTZ) Constraints
- Formulation by I. Kara, G. Laporte, and T. Bektas [1]
- Prevent subtours and track routes and capacity utilization
- First, we need an additional variable!
- $U_i$  - Capacity utilization at  $i$  of vehicle on its tour with  $i \in \mathcal{J}$

...

### **i** Note

You don't need to guess these constraints, as they are quite tricky!

## MTZ Constraints

### **i** We need the following sets, parameters, and variables:

- $\mathcal{V}$  - Set of all nodes, index  $i \in \{0, 1, 2, \dots, n\}$
- $X_{i,j}$  - 1, if the arc between  $i$  and  $j$  is part of a tour, 0 otherwise
- $U_i$  - Capacity utilization at  $i$  of vehicle on its tour with  $i \in \mathcal{J}$
- $b$  - Capacity per vehicle (all are identical!)
- $d_i$  - Demand at node  $i$

...

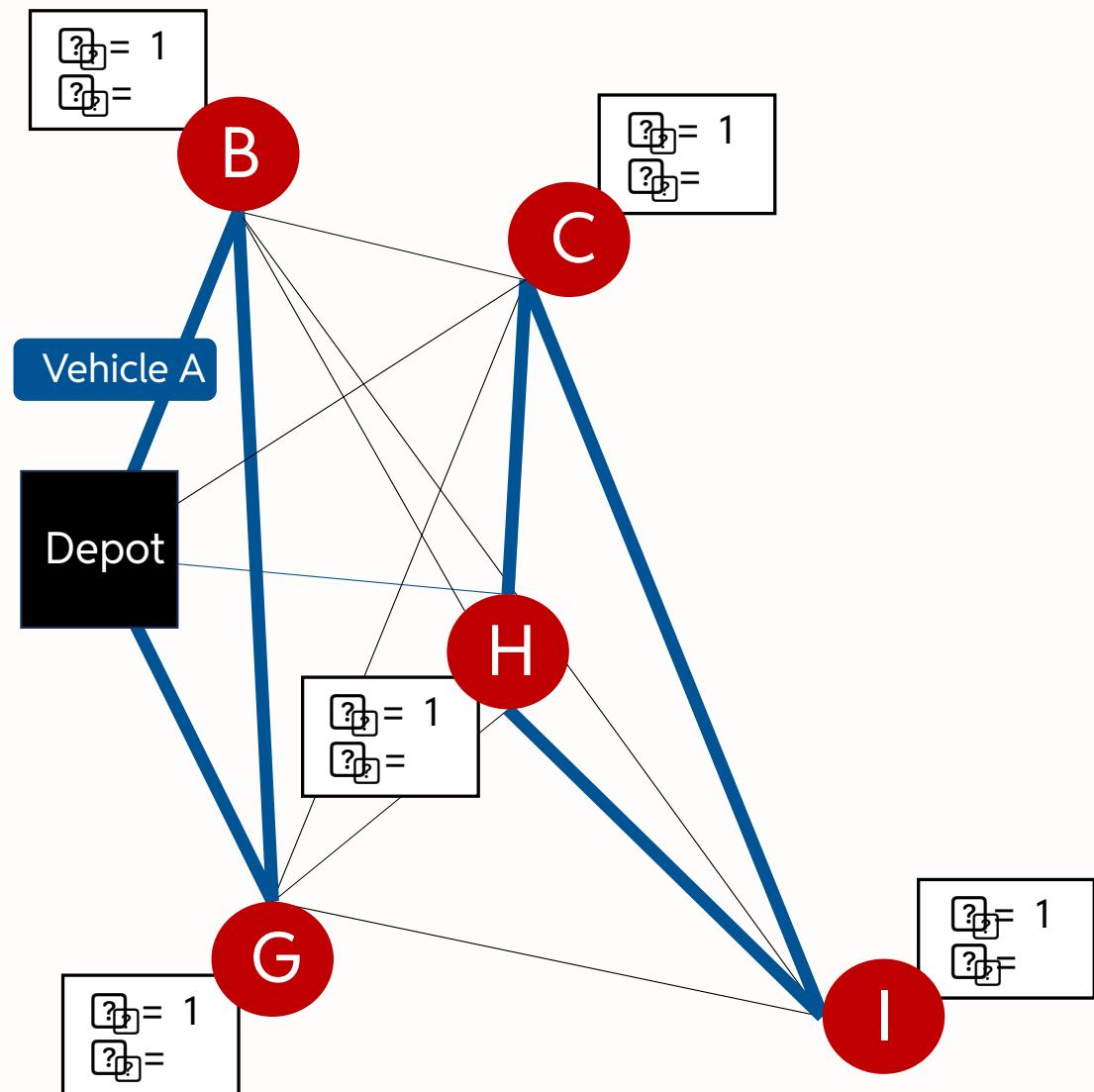
$$U_i - U_j + b \times X_{i,j} \leq b - d_j \quad \forall i, j \in \mathcal{V} \setminus \{0\}, i \neq j$$

...

$$d_i \leq U_i \leq b \quad \forall i \in \mathcal{V} \setminus \{0\}$$

Too  
complicated?

Don't worry!



- Let's break it down!
- $d_i$  for all customers is 1
- Capacity  $b$  per vehicle is 5
- $U_i$  is the current capacity utilization at node  $i \in \mathcal{J}$

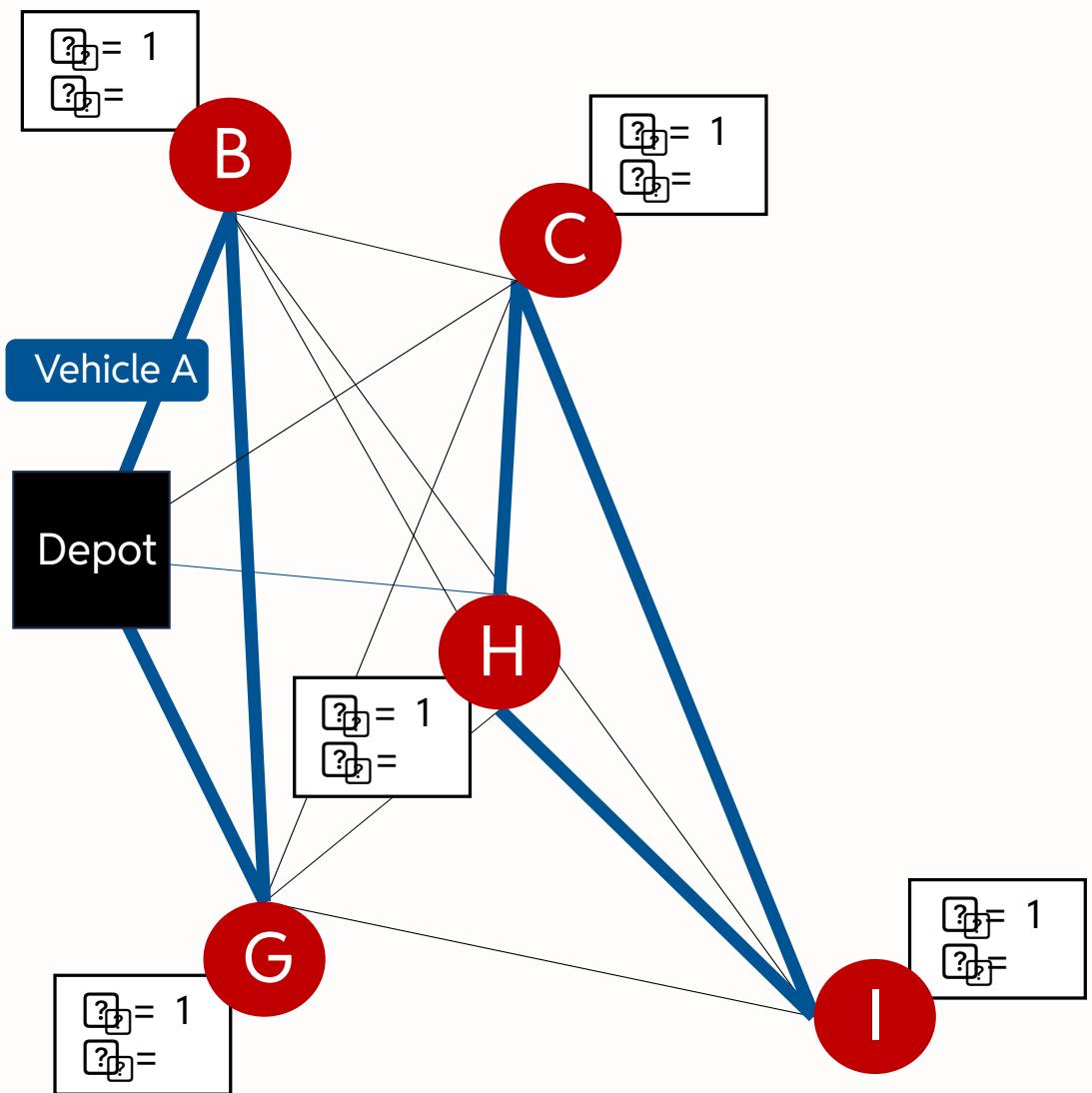
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No connection between nodes



- In case  $X_{ij} = 0$ :
  - $U_i - U_j \leq b - d_j$
  - Non-binding for relation between two nodes
- Following is perfectly fine:
  - $U_i \leq b$  and  $U_j \geq d_j$

Connection between two nodes



- In case  $X_{ij} = 1$ :
  - ▶  $U_i - U_j + b \leq b - d_j$
  - ▶ Binding for relation between two nodes
- Can be summarized to:
  - ▶  $U_j \geq d_j + U_i$

Connection in more detail

Question: Why is it binding?

...

- Binding as  $U_j$  has to be at least as large as  $d_i + U_i$
- Hence, fulfilled if the demand of  $i$  is added to the vehicle

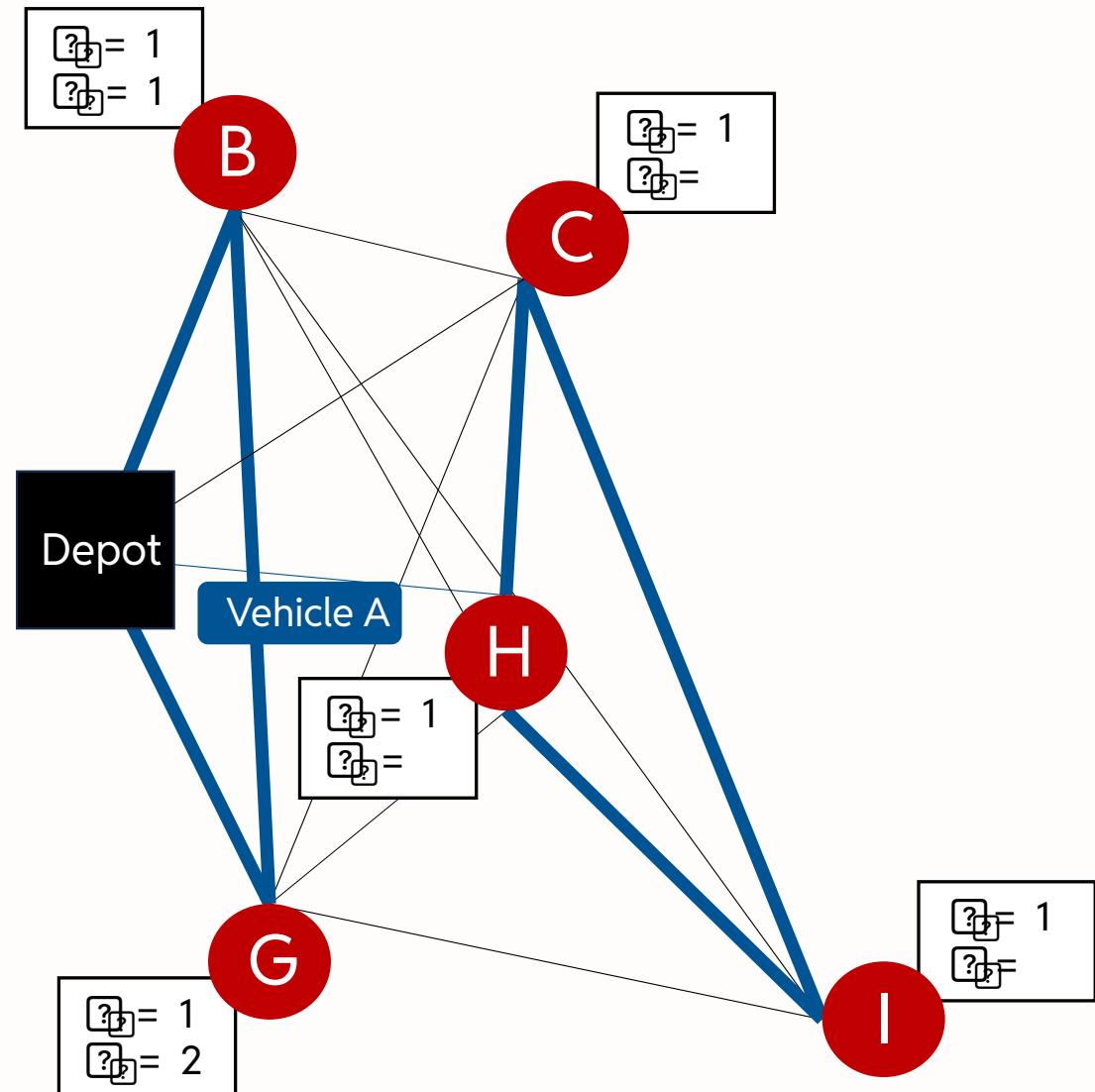
...

Question: Do you get the idea?

...

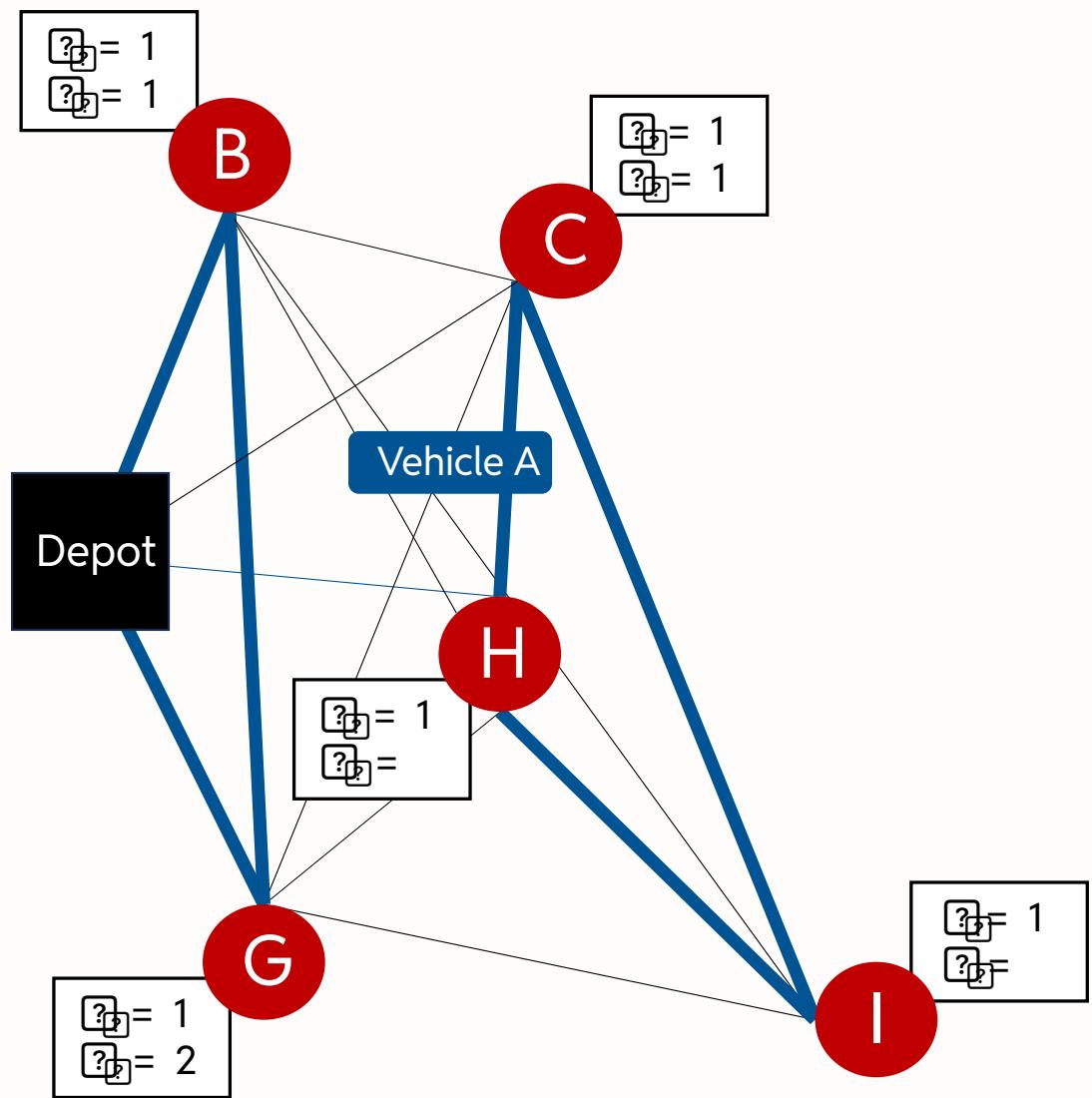
- If  $X_{ij} = 1$ , then  $U_j$  has to be at least as large as  $d_i + U_i$
- If  $X_{ij} = 0$ , then  $U_i \leq b$  and  $U_j \geq d_j$

Tour from the Depot



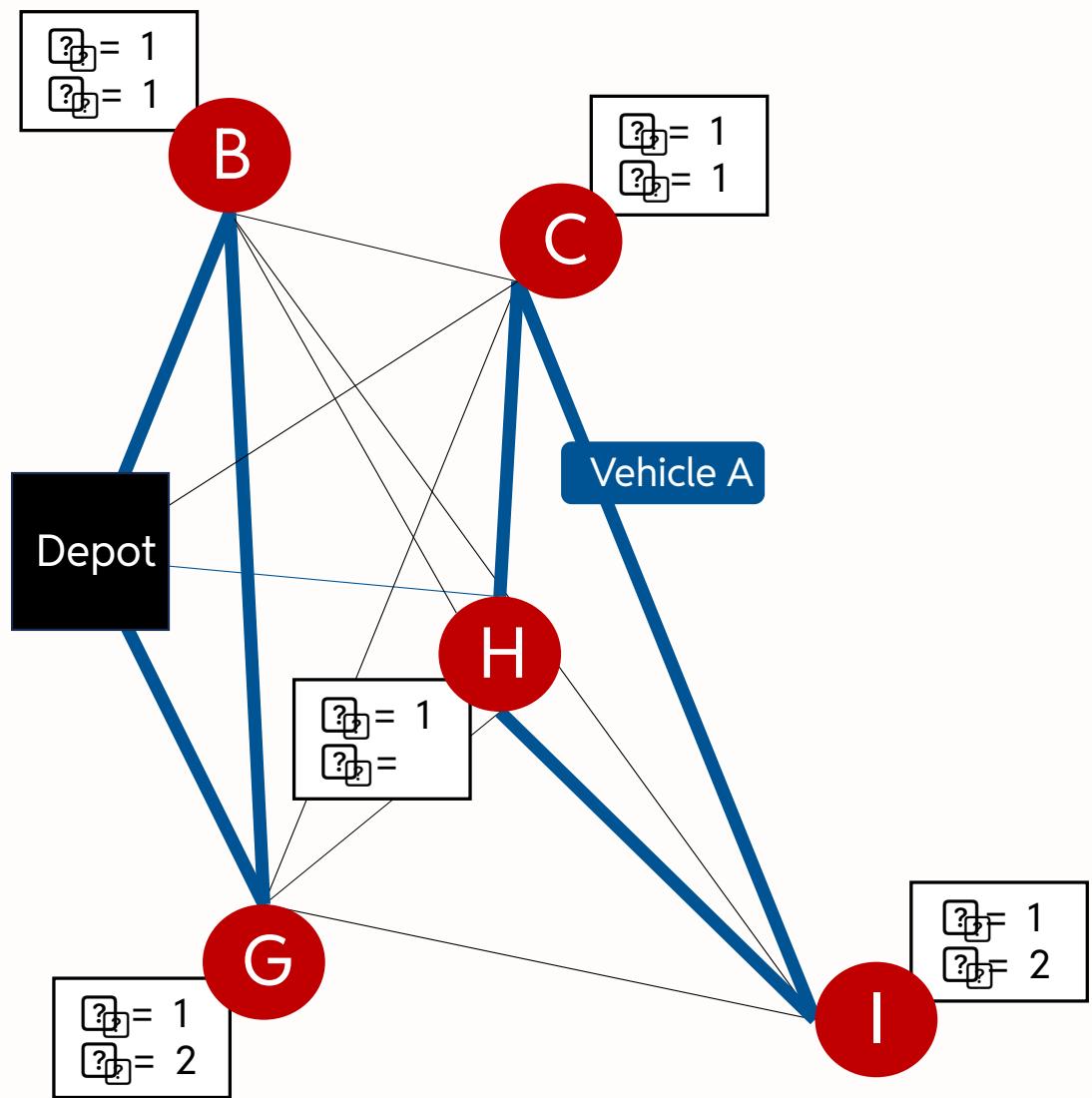
- Tour of vehicle A ok
- Depot is the only node visited multiple times
- But the constraints are not applied here!

Tour on its Own



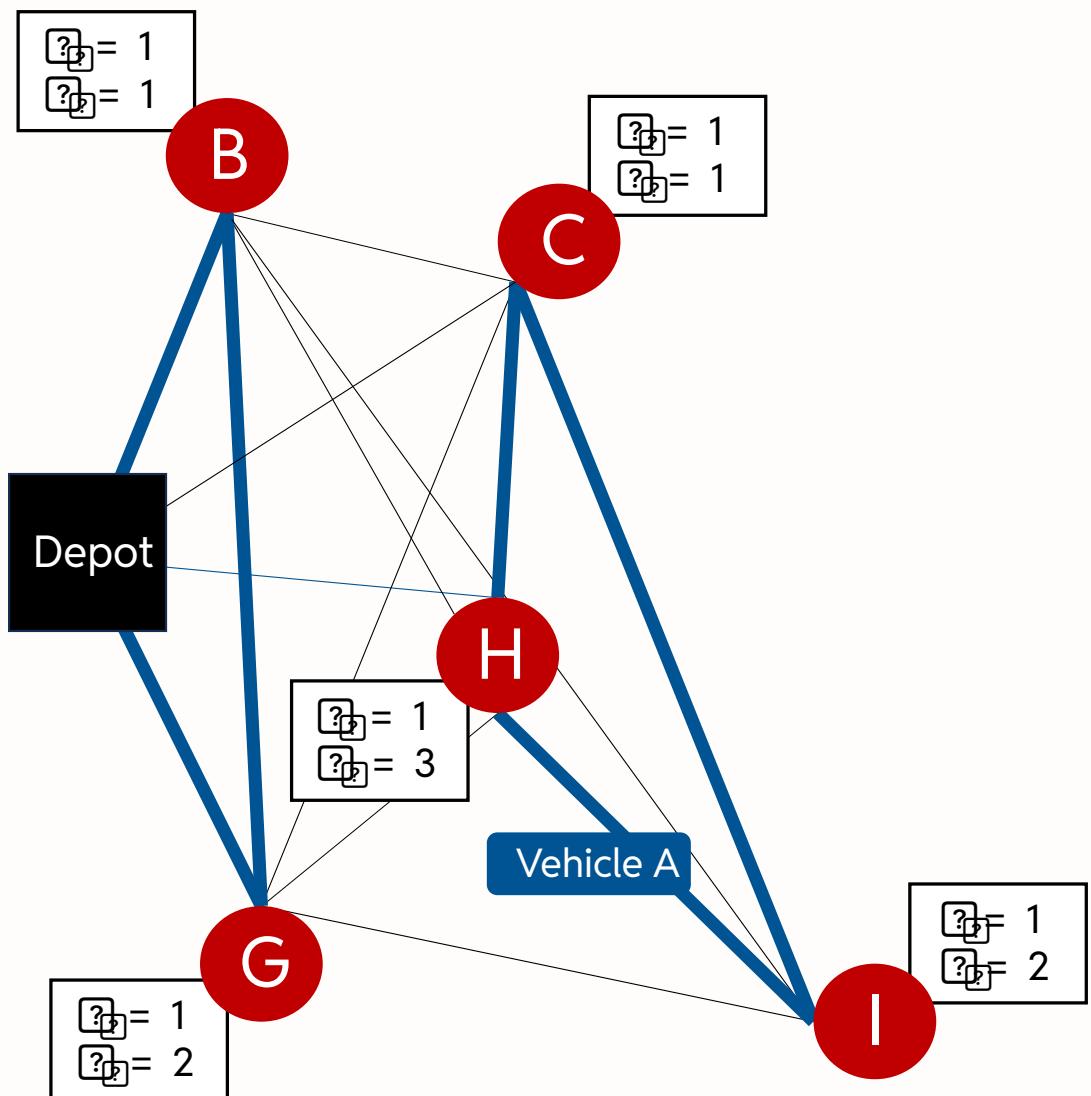
- Let's start at node  $H$
- We drive to node  $C$
- $U_C = 1$

Tour on its Own II



- We continue to node  $I$
- $U_I = 2$

Tour on its Own III



- We continue to node  $H$
- $U_H = 3$
- Connection from  $H$  to  $C$
- $U_H$  is greater than  $U_C$ !
- Infeasible solution!

### Subtour Elimination

- Connection in the other direction wouldn't work as well
- Only depot as “reset”, as constraints are not applied here

Question: What about the capacity?

...

- Remember variable domain of  $U_i$ ?
- $d_i \leq U_i \leq b \rightarrow$  Overall capacity limit enforced!

## Last Constraint

Ensure time limit?

Question: Anybody an idea?

...

- Constraints basically follow the same idea!
- First, we again need an additional variable
- $T_i$  - Time spent on tour at the node  $i$  of a vehicle with  $i \in \mathcal{J}$

Ensure time limit

**i** We need the following sets and variables:

- $\mathcal{V}$  - Set of all nodes, index  $i \in \{0, 1, 2, \dots, n\}$
- $X_{i,j}$  - 1, if the arc between  $i$  and  $j$  is part of a tour, 0 otherwise
- $T_i$  - Time spent on tour at the node  $i$  of a vehicle with  $i \in \mathcal{J}$
- $t$  - Maximal duration of a tour
- $c_{i,j}$  - Travel time on an arc from  $i$  to  $j$

...

$$T_i - T_j + t \times X_{i,j} \leq t - c_{i,j} \quad \forall i, j \in \mathcal{V} \setminus \{0\}, i \neq j$$

...

$$0 \leq T_i \leq t \quad \forall i \in \mathcal{V} \setminus \{0\}$$

Any questions?

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## Asymmetric Vehicle Routing Problem

Objective

$$\text{minimize} \quad \sum_{(i,j) \in \mathcal{A}} c_{i,j} \times X_{i,j}$$

**!** The goal of the objective function is to:

Minimize the total travel distance.

Each customer is visited once

$$\sum_{i \in \mathcal{V}} X_{i,j} = 1 \quad \forall j \in \mathcal{V} \setminus \{0\}, i \neq j$$

$$\sum_{j \in \mathcal{V}} X_{i,j} = 1 \quad \forall i \in \mathcal{V} \setminus \{0\}, i \neq j$$

! Our constraints ensure:

Each customer is visited exactly once.

### Depot entry and exit

$$\sum_{i \in \mathcal{V} \setminus \{0\}} X_{i,0} = |\mathcal{K}|$$

$$\sum_{j \in \mathcal{V} \setminus \{0\}} X_{0,j} = |\mathcal{K}|$$

! Our constraints ensure:

The depot is visited by exactly  $|\mathcal{K}|$  vehicles. Note, that we could remove one of the constraints and the solution would still be optimal.

### Capacity and subtour elimination

$$U_i - U_j + b \times X_{i,j} \leq b - d_j \quad \forall i, j \in \mathcal{V} \setminus \{0\}, i \neq j$$

$$d_i \leq U_i \leq b \quad \forall i \in \mathcal{V} \setminus \{0\}$$

! Our constraints ensure:

The capacity limit is respected and subtours are eliminated.

### Time constraints

$$T_i - T_j + t \times X_{i,j} \leq t - c_{i,j} \quad \forall i, j \in \mathcal{V} \setminus \{0\}, i \neq j$$

$$0 \leq T_i \leq t \quad \forall i \in \mathcal{V} \setminus \{0\}$$

! Our constraints ensure:

The time limit is respected (and subtours are eliminated).

### Variables

$$X_{i,j} \in \{0, 1\} \quad \forall i, j \in \mathcal{V}$$

$$d_i \leq U_i \leq b \quad \forall i \in \mathcal{V} \setminus \{0\}$$

$$0 \leq T_i \leq t \quad \forall i \in \mathcal{V} \setminus \{0\}$$

! The variable domains make sure that:

The binary setup variable is either 0 or 1 and the new variables are below the time and capacity limit.

---

## Model Characteristics

### Characteristics

Questions: On model characteristics

- Is the model formulation linear/ non-linear?
  - What kind of variable domains do we have?
  - What do you think, can the model be solved quickly?
- 

### Model Assumptions

Questions: On model assumptions

- What assumptions have we made?
  - What are likely issues that can arise if the model is applied?
  - Have we considered service times?
- 

### Extensions of the CVRP

Questions: What extensions do you know?

...

- time windows (TW)
  - soft time windows (STW)
  - multiple depots (MD)
  - heterogeneous fleet (HF)
  - backhauls (B)
  - pickup and delivery (PD)
- 

## Implementation and Impact

### Case Study in Schleswig-Holstein

...

- 165 libraries
- 119 visited biweekly
- Up to 9 different tours
- Time-Limit of 8 hours for each tour

Can our formulation  
solve the problem  
on a real-world instance?

No, although we can find  
solutions within one hour,  
the gap is still very large  
with 40%-45%.

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### Problem is NP-hard

- We have already seen that a problem can be NP-hard
- Likely, that there are no polynomial-time algorithms
- Doesn't mean that it can't be solved!

...

#### Note

We won't go deeper into this, but feel free to ask me!

Can we do  
anything to solve  
the model?

### Heuristics

- We can still solve the problem with a heuristic
- Likely not the optimal solution, but a lot of research goes into efficient algorithms to solve these problems
- In our case study we applied Hybrid Genetic Search for the CVRP (HGS-CVRP) by T. Vidal [2]

...

#### Tip

For problems with 100+ locations, heuristics are often the only practical choice.

## Applications Beyond Libraries

### Logistics & Delivery

- Package delivery
- Food delivery services
- Grocery delivery
- Mail distribution

### Service Industries

- Maintenance crews
- Home healthcare visits
- Waste collection
- School bus routing

...

Question: Can you think of other applications?

### Conclusion

- Standard problem that occurs in many different places
- Solving the problem with a mathematical model is difficult
- Nowadays, there are many good heuristics
- Many companies are working on the problem

...

 And that's it for todays lecture!

We now have covered the Capacitated Vehicle Routing Problem and are ready to start solving some tasks in the upcoming tutorial.

Questions?

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## Literature

### Literature I

For more interesting literature to learn more about Julia, take a look at the [literature list](#) of this course.

### Literature II

## Bibliography

- [1] I. Kara, G. Laporte, and T. Bektas, “A note on the lifted Miller–Tucker–Zemlin subtour elimination constraints for the capacitated vehicle routing problem,” European

Journal of Operational Research, vol. 158, no. 3, pp. 793–795, 2004, doi: [https://doi.org/10.1016/S0377-2217\(03\)00377-1](https://doi.org/10.1016/S0377-2217(03)00377-1).

- [2] T. Vidal, “Hybrid genetic search for the CVRP: Open-source implementation and SWAP\* neighborhood,” Computers & Operations Research, vol. 140, p. 105643, Apr. 2022, doi: [10.1016/j.cor.2021.105643](https://doi.org/10.1016/j.cor.2021.105643).