Lecture VII - Library Routing Optimization

Applied Optimization with Julia

Dr. Tobias Vlćek
University of Hamburg - Fall 2025

Introduction

Central Libraries

. . .

Question: Anybody an idea what a central library is?

Central Libraries

- Book Delivery to Libraries in Germany
- They supply all local libraries within the same state
- Complex, as number of libraries per state can be large
- Books and media in the libraries change often
- Customers can request books from other libraries

Structure of the Deliveries

- For delivery, central has several employees and cars
- Local libraries differ in size, some receive more items
- Items are collected as well during the tours[1]
- They are transported back to the central library

Potential Decisions

Question: What decisions can the library make for tours?

. . .

- Subdivide set of libraries into several ordered tours
- Decide in which order to visit the libraries
- Evaluate which car to use for each of the tours
- Decide which driver to assign to each of the tours

Impact of the Decisions

Question: What is the impact of the decisions?

. . .

- Longer driving increases the footprint of the deliveries

[1] Due to regulations, the delivery tours cannot exceed a certain duration

- Suboptimal tours can lead to unnecessary costs
- Fuel, personnel, and repairs are increased
- Unhappy customers due to waiting times on books

Have you heard of

this problem before?

Problem Structure

Objective

Question: What could be the objective for central libraries?

. . .

- Lowering costs through improved tours
- Improvement of their footprint through shorter tours
- Faster fulfillment of the deliveries

Modelling

Question: What could we try to model?

. . .

Minimization of the travel time while supplying all libraries in the state and respecting the vehicle capacities and driving time restrictions.

Capacitated

Vehicle Routing

(CVRP)

Vehicle Routing Problem

- CVRP is a subproblem
- Main problem is Vehicle Routing Problem (VRP)
- Problem class about designing routes for vehicle fleets

. . .

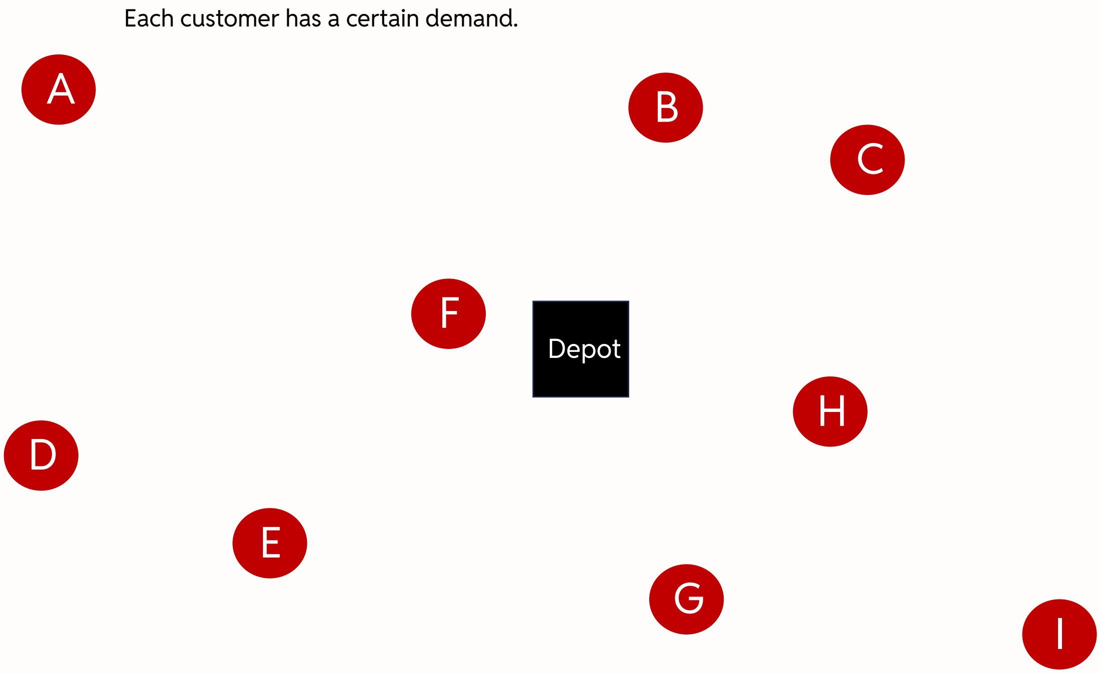
> **i Note**
>
> There are many variants of the VRP! E.g., with time windows, periodic deliveries, allowing for pickups or deliveries, and much more!

Let’s visualize

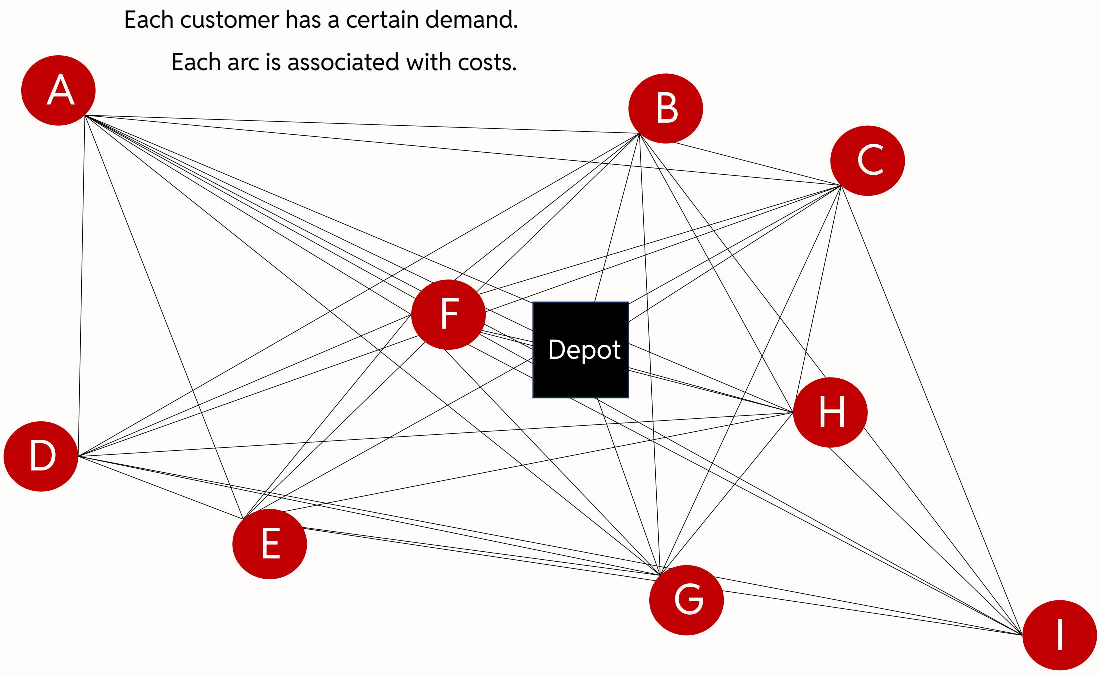
the problem!

Problem Visualization

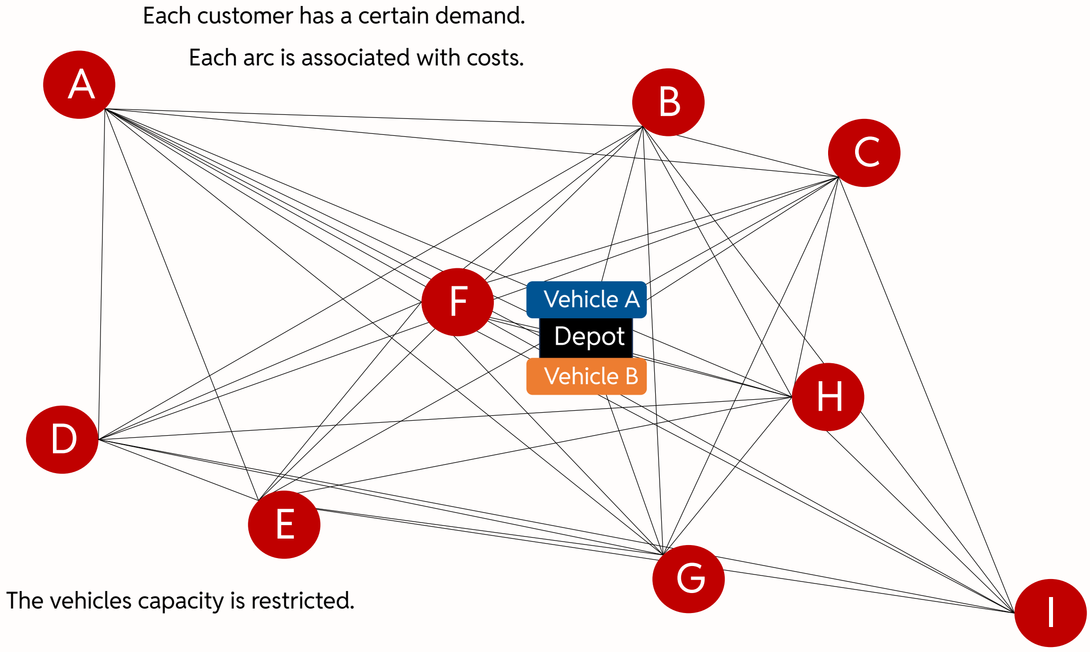
Basic Problem Setting



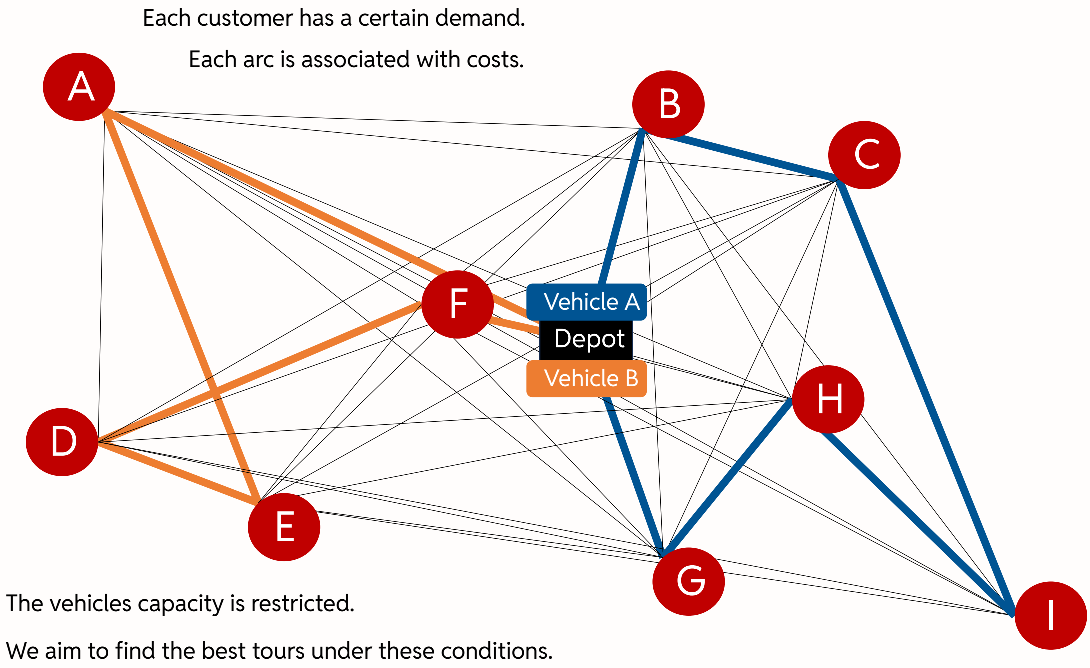
Basic Setting with Arcs



Setting with Vehicles



Basic Setting with Tours



Problem Structure

Available Sets

Question: What could be the sets here?

. . .

- $\mathcal{V}$ - Set of all nodes, index $i \in \{0, 1, 2, ..., n\}$
- $\mathcal{A}$ - Set of all arcs between the nodes, index $(i, j) \in \mathcal{A}$
- $\mathcal{K}$ - Set of vehicles with identical capacity, index $k \in \mathcal{K}$
- $0 \in \mathcal{V}$ - Depot where the vehicles start

Available Parameters

Question: What are possible parameters?

. . .

- b - Capacity per vehicle
- t - Maximal duration of each tour
- d_i - Demand at node i
- $c_{i,j}$ - Travel time on an arc from i to j

. . .

Tip

t is the maximal duration of each tour, not the travel time on an arc or an index!

Decision Variable(s)?

We have the following sets:

- All nodes, including the depot, $i \in \mathcal{V}$
- All arcs between the nodes, $(i, j) \in \mathcal{A}$
- The available vehicles, $k \in \mathcal{K}$

. . .

! Our objective is to:

Minimize the total travel time while supplying all customers and adhering to the vehicle capacities and duration restrictions.

. . .

Question: What could be our decision variable/s?

Decision Variables

- $X_{i,j,k}$ - 1, if k passes between i and j on its tour, 0 otherwise

. . .

i Variable Domain

$X_{i,j,k}$ is a binary variable, as it can only take values 0 or 1. But most likely, you will already have spotted that!

. . .

Question: Why this might make the problem difficult?

Decision Variable/s (again)?

i We have the following sets:

- All nodes, including the depot, $i \in \mathcal{V}$
- All arcs between the nodes, $(i,j) \in \mathcal{A}$
- The available vehicles, $k \in \mathcal{K}$

. . .

! Our objective is to:

Minimization of the travel time (or driving distance), while supplying all customers and adhering to the vehicle capacities and duration restrictions. Hint: Even with many vehicles, each arc can maximally be passed once!

Decision Variables (again)

Question: What could be our decision variable/s?

. . .

- $X_{i,j}$ - 1, if the arc between i and j is part of a tour, else 0

. . .

! Only possible under certain conditions!

Only possible, if time and capacity constraints are equal for all vehicles!

Model Formulation

Objective Function?

! Our objective is to:

Minimization of the travel time (or driving distance), while supplying all customers and adhering to the vehicle capacities and duration restrictions.

. . .

Question: What could be our objective function?

. . .

i We need the following variable:

- $X_{i,j}$ - 1, if the arc between i and j is part of a tour, else 0

Objective Function

i We need the following parameters:

- $c_{i,j}$ - travel time on an arc from i to j

. . .

$$\text{minimize} \quad \sum_{(i,j)\in\mathcal{A}} c_{i,j} \times X_{i,j}$$

. . .

Question: What does $(i,j) \in \mathcal{A}$ under the sum mean?

From Math to Code

The objective function in JuMP looks very similar:

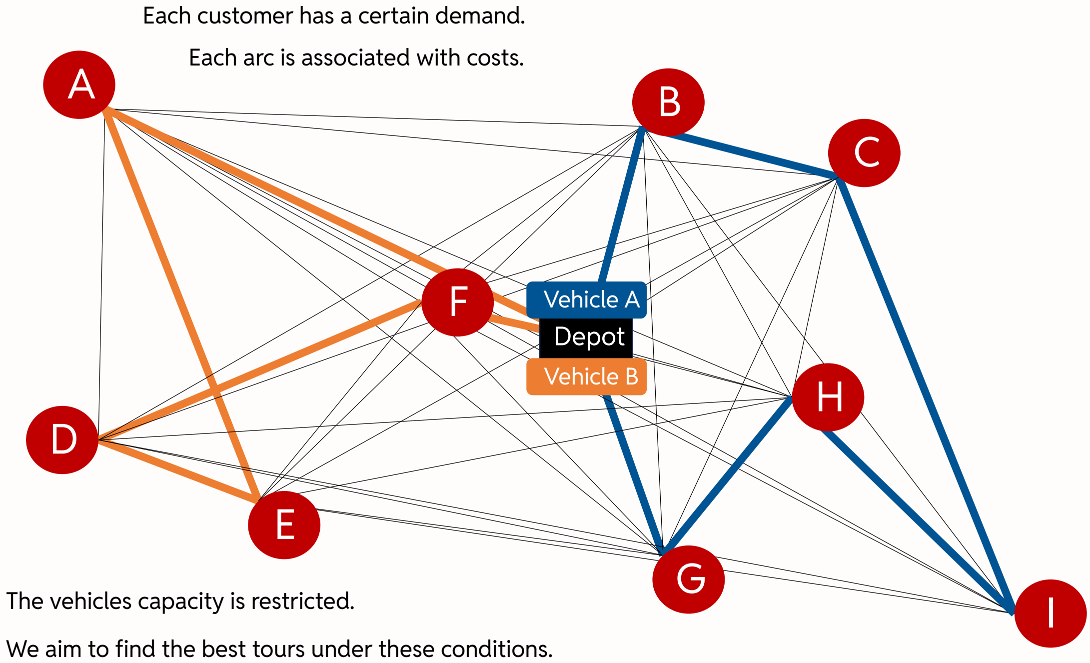
```julia
using JuMP, HiGHS

# Simple example with 4 nodes
nodes = ["depot", "A", "B", "C"]
arcs = [(i,j) for i in nodes, j in nodes if i != j]
c = Dict(("depot","A") => 10, ("depot","B") => 15, ("depot","C") => 20,
         ("A","depot") => 10, ("A","B") => 12, ("A","C") => 8,
         ("B","depot") => 15, ("B","A") => 12, ("B","C") => 5,
         ("C","depot") => 20, ("C","A") => 8, ("C","B") => 5)

model = Model(HiGHS.Optimizer)
@variable(model, X[arcs], Bin)
@objective(model, Min, sum(c[i,j] * X[(i,j)] for (i,j) in arcs))
```

. . .

i Note

The JuMP syntax `sum(c[i,j] * X[(i,j)] for (i,j) in arcs)` directly mirrors our mathematical notation $\sum_{(i,j)\in\mathcal{A}} c_{i,j} \times X_{i,j}$!

Problem Constraints



Constraints?

Question: What constraints do we need?

. . .

- Each customer has to be visited once
- The depot has to be entered and left $|\mathcal{K}|$ times
- We have to enforce the capacity of our vehicles
- We have to ensure the maximal duration of each tour

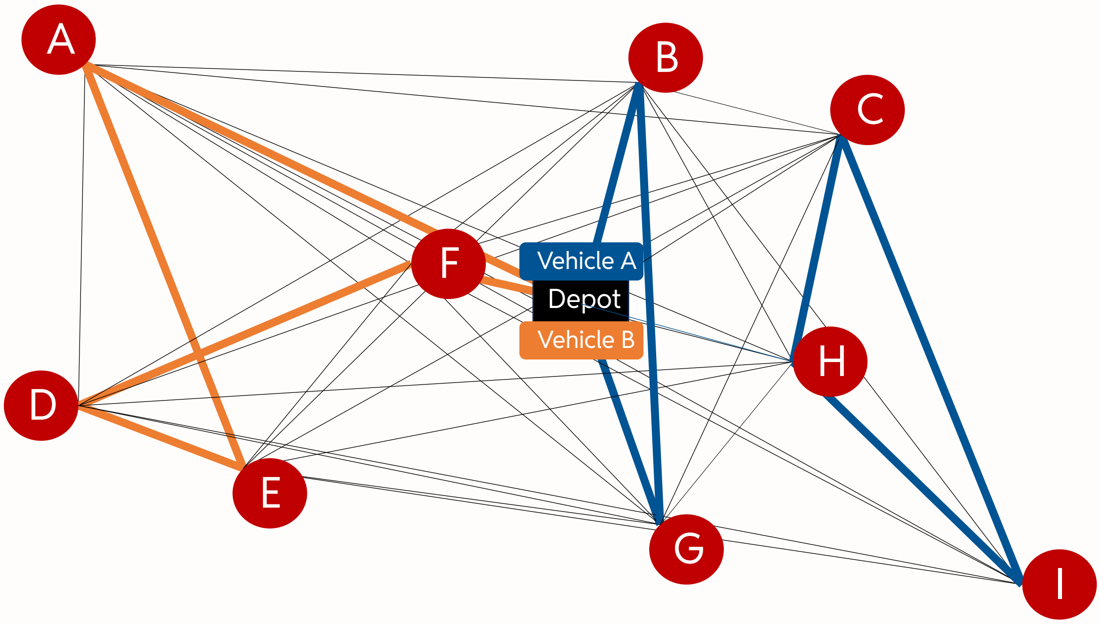
. . .

> ! Subtours
>
> In addition, we have to prevent subtours!

What is a

subtour?

Subtours



Constraints

Visit Each Customer Once?

> ! The goal of these constraints is to:
>
> Ensure that each customer is visited exactly once. Essentially, we could also say that each node has to be entered and left exactly once.

. . .

> i We need the following sets and variables:
>
> - $\mathcal{V}$ - Set of all nodes, index $i \in \{0, 1, 2, ..., n\}$
> - $X_{i,j}$ - 1, if the arc between i and j is part of a tour, 0 otherwise

Visit Each Customer Once

Question: What could the constraint look like?

. . .

$$\sum_{i \in \mathcal{V}} X_{i,j} = 1 \quad \forall j \in \mathcal{V} \setminus \{0\}, i \neq j$$

$$\sum_{j \in \mathcal{V}} X_{i,j} = 1 \quad \forall i \in \mathcal{V} \setminus \{0\}, i \neq j$$

. . .

Question: Why for all nodes except the depot?

. . .

The depot is the only node that is visited multiple times!

Depot Entry/ Exit Constraints?

! The goal of these constraints is to:

Ensure that each vehicle enters and leaves the depot exactly $|\mathcal{K}|$ times, as we have $|\mathcal{K}|$ vehicles and each vehicle has to return to the depot.

. . .

i We need the following sets and variables:

- $\mathcal{V}$ - Set of all nodes, index $i \in \{0, 1, 2, ..., n\}$
- $|\mathcal{K}|$ - Number of vehicles
- $X_{i,j}$ - 1, if the arc between i and j is part of a tour, 0 otherwise

. . .

Question: What could the constraint look like?

Depot Entry/ Exit Constraints

$$\sum_{i \in \mathcal{V} \setminus \{0\}} X_{i,0} = |\mathcal{K}|$$

$$\sum_{j \in \mathcal{V} \setminus \{0\}} X_{0,j} = |\mathcal{K}|$$

. . .

i Are all constraints necessary?

No, theoretically we could also say that we only have to leave or enter the depot exactly $|\mathcal{K}|$ times, as the other constraint is already enforced by the “visit each customer once constraint”.

Capacity and Subtour Elimination

The next ones are

a little bit tricky.

MTZ Formulation

- Miller-Tucker-Zemlin (MTZ) Constraints
- Formulation by I. Kara, G. Laporte, and T. Bektas [1]
- Prevent subtours and track routes and capacity utilization
- First, we need an additional variable!
- U_i - Capacity utilization at i of vehicle on its tour with $i \in \mathcal{I}$

...

> **i Note**
>
> You don't need to guess these constraints, as they are quite tricky!

MTZ Constraints

> **i We need the following sets, parameters, and variables:**
>
> - $\mathcal{V}$ - Set of all nodes, index $i \in \{0, 1, 2, ..., n\}$
> - $X_{i,j}$ - 1, if the arc between i and j is part of a tour, 0 otherwise
> - U_i - Capacity utilization at i of vehicle on its tour with $i \in \mathcal{I}$
> - b - Capacity per vehicle (all are identical!)
> - d_i - Demand at node i

...

$$U_i - U_j + b \times X_{i,j} \leq b - d_j \quad \forall i, j \in \mathcal{V} \setminus \{0\}, i \neq j$$

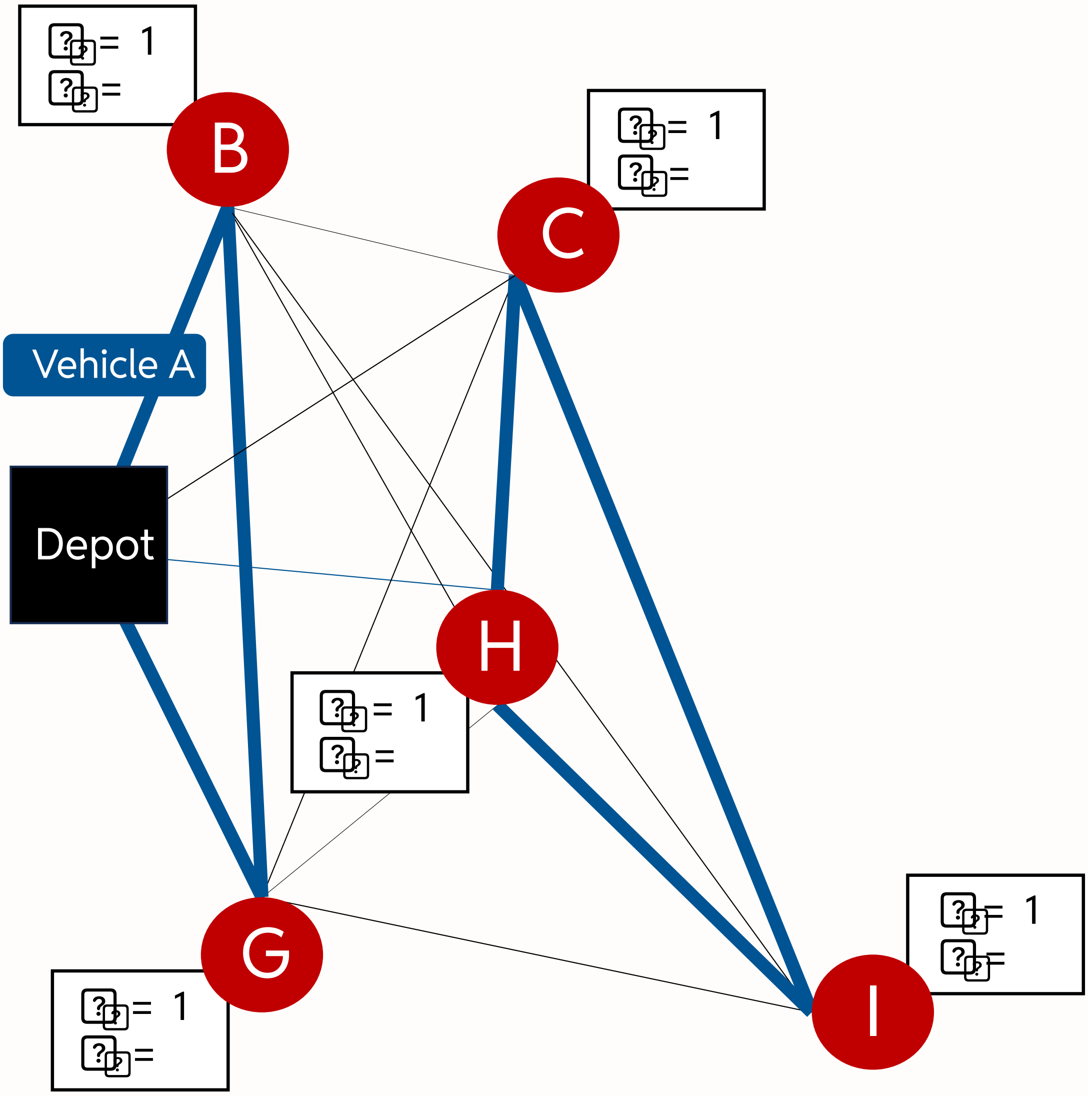
...

$$d_i \leq U_i \leq b \quad \forall i \in \mathcal{V} \setminus \{0\}$$

Too

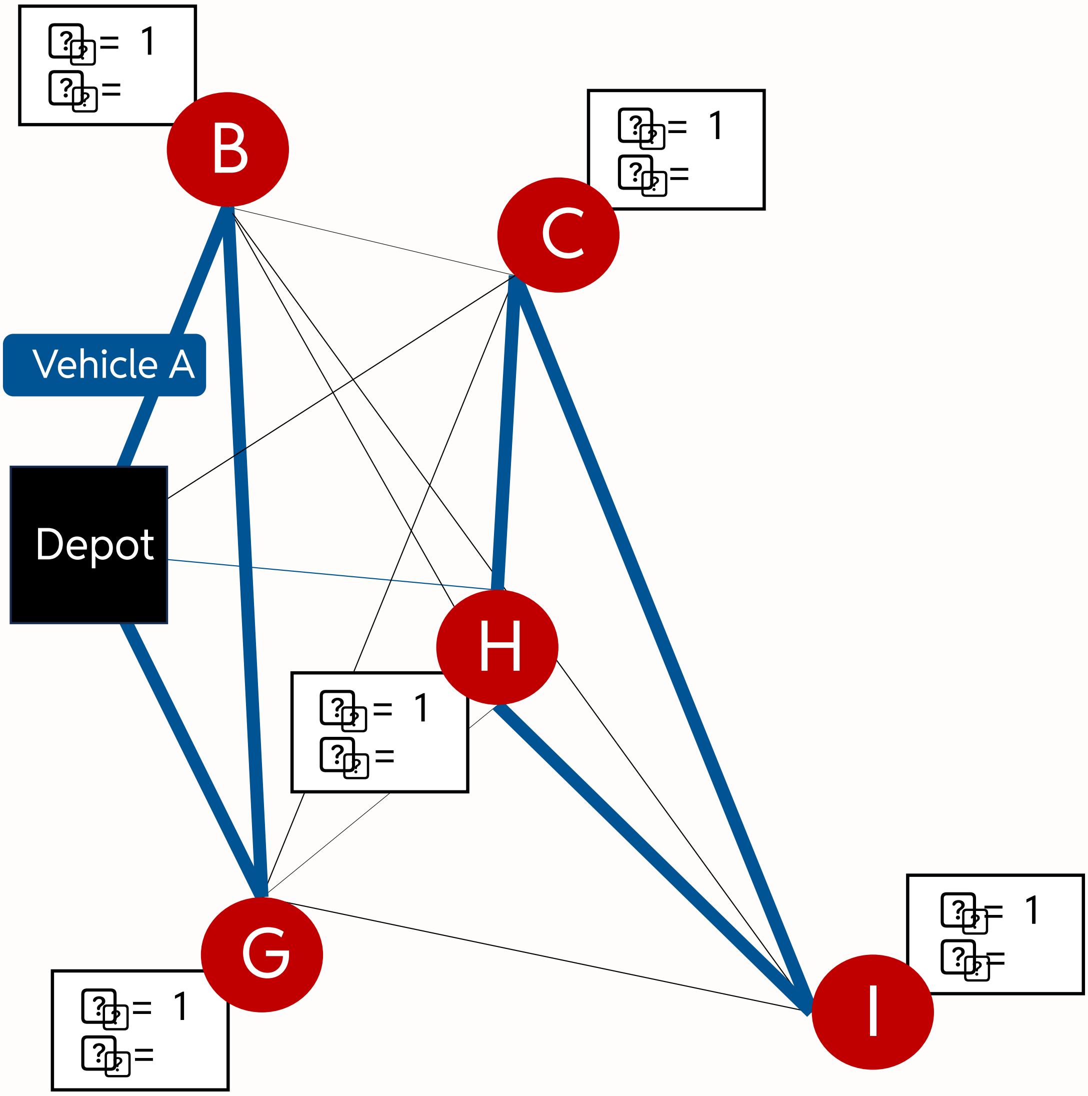
complicated?

Don't worry!



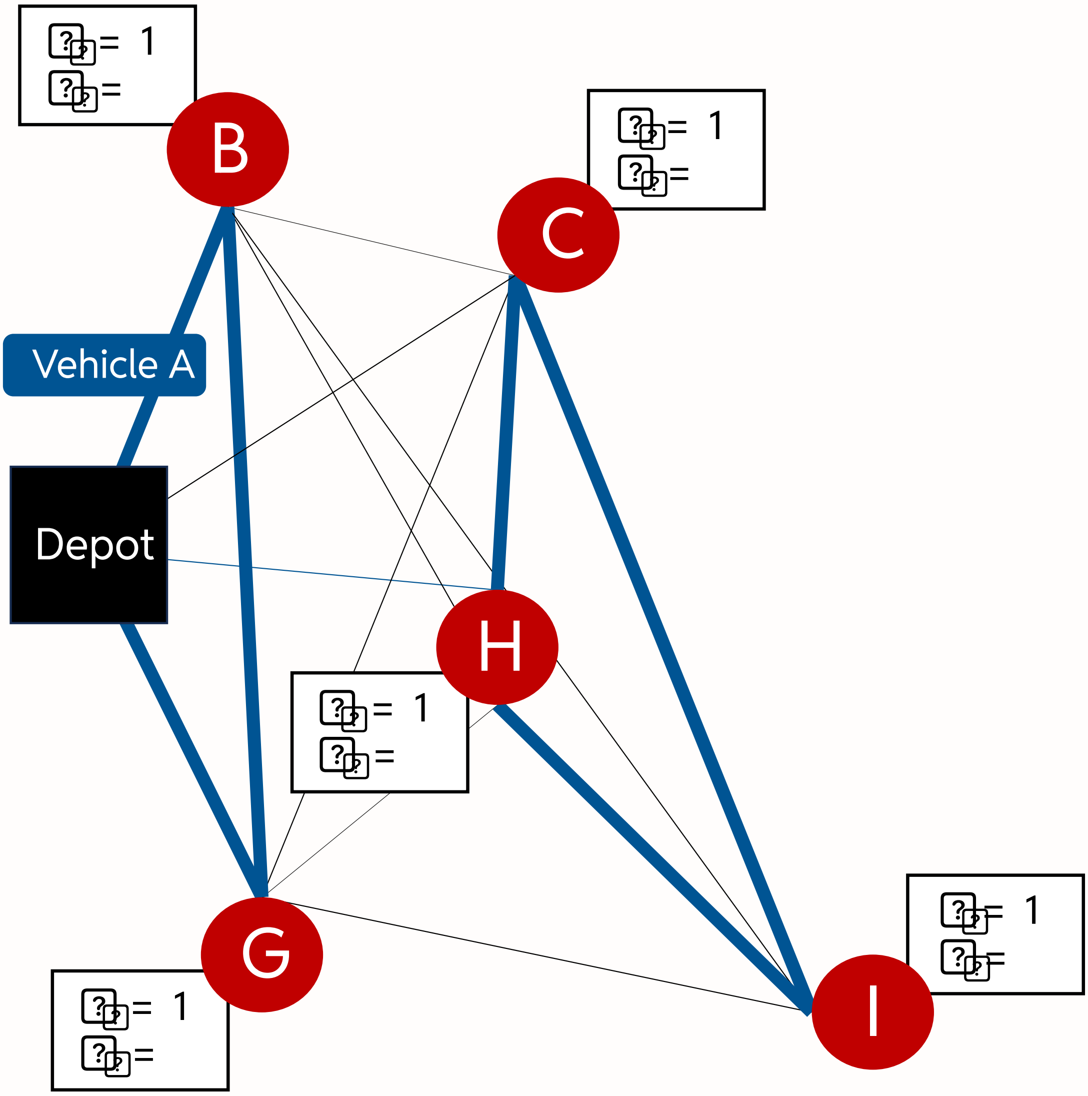
- Let's break it down!
- d_i for all customers is 1
- Capacity b per vehicle is 5
- U_i is the current capacity utilization at node $i \in \mathcal{I}$

No connection between nodes



- In case $X_{ij} = 0$:
  - $U_i - U_j \leq b - d_j$
  - Non-binding for relation between two nodes
- Following is perfectly fine:
  - $U_i \leq b$ and $U_j \geq d_j$

Connection between two nodes



- In case $X_{ij} = 1$:
  - $$U_i - U_j + b \leq b - d_j$$
  - Binding for relation between two nodes
- Can be summarized to:
  - $U_j \geq d_j + U_i$

Connection in more detail

Question: Why is it binding?

. . .

- Binding as U_j has to be at least as large as $d_i + U_i$
- Hence, fullfiled if the demand of i is added to the vehicle

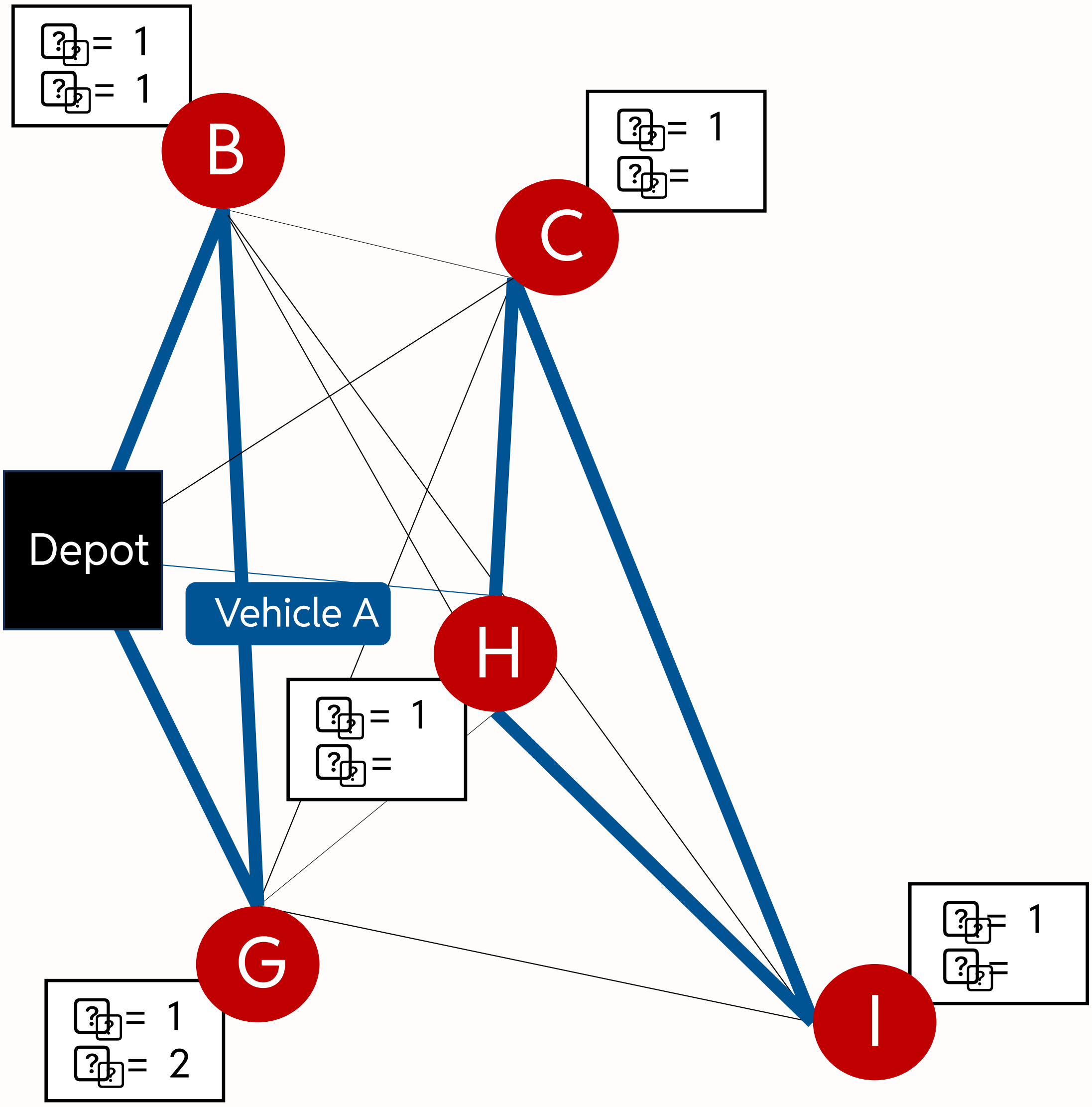
. . .

Question: Do you get the idea?

. . .

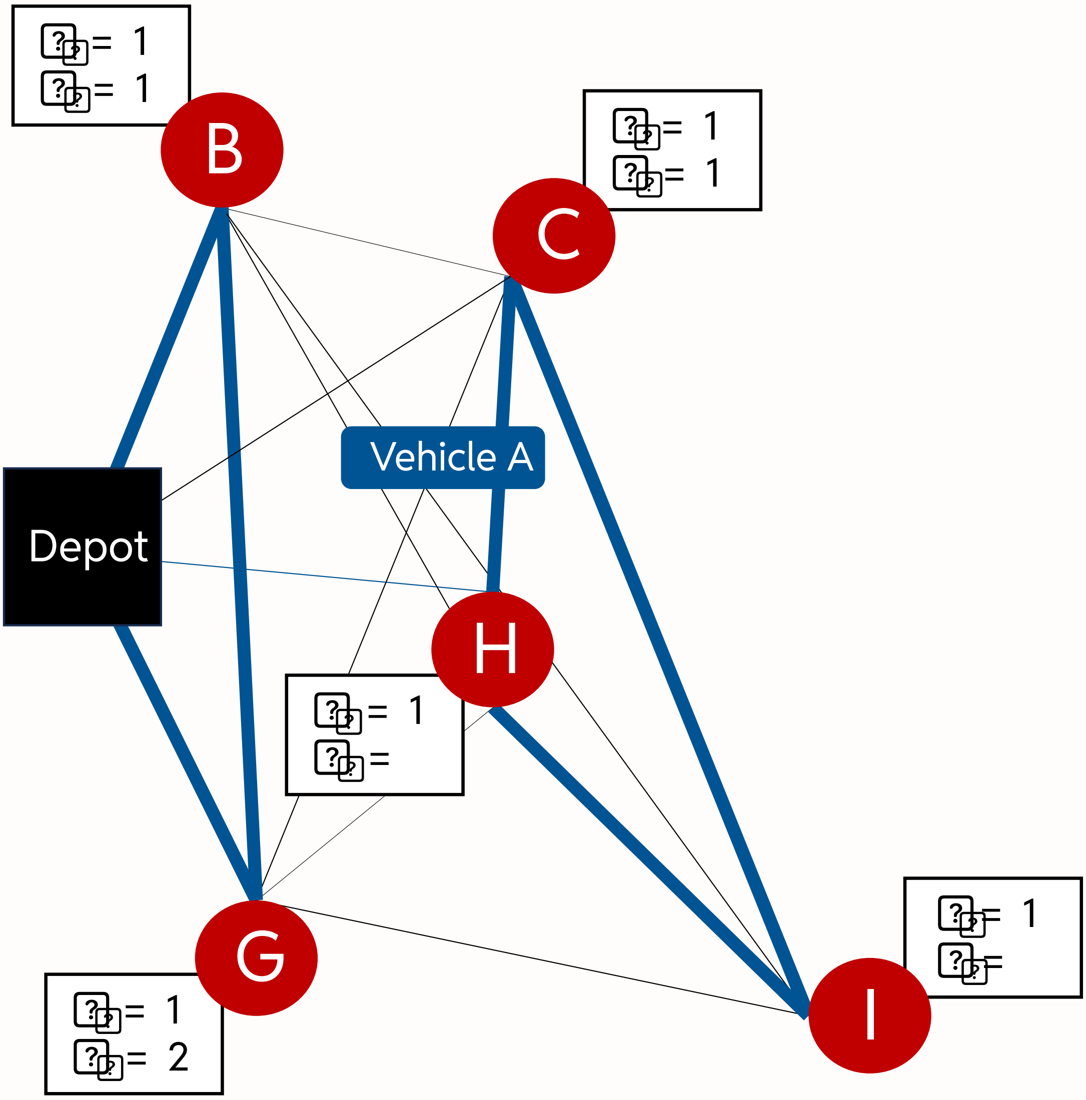
- If $X_{ij} = 1$, then U_j has to be at least as large as $d_i + U_i$
- If $X_{ij} = 0$, then $U_i \leq b$ and $U_j \geq d_j$

Tour from the Depot



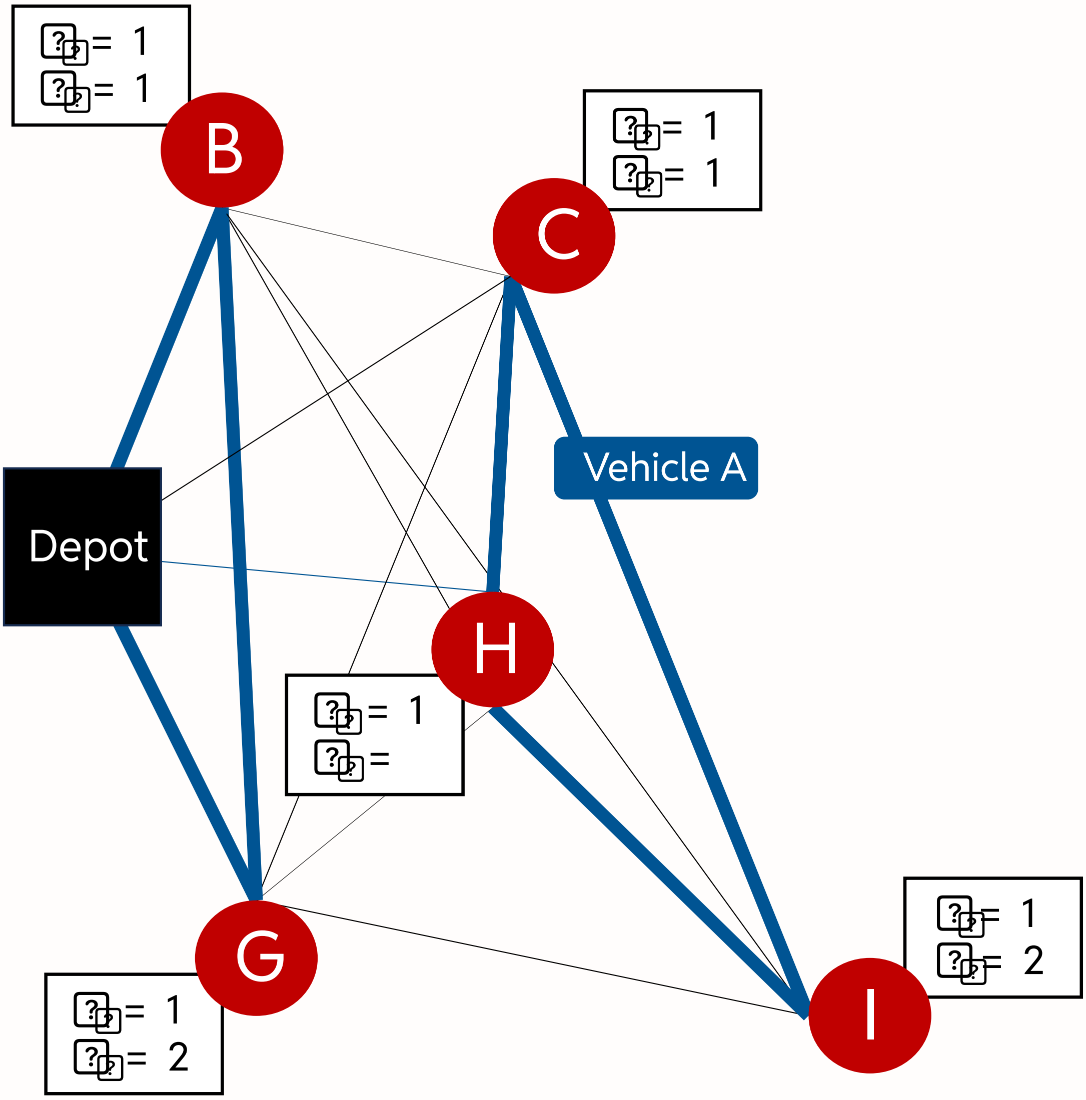
- Tour of vehicle A ok
- Depot is the only node visited multiple times
- But the constraints are not applied here!

Tour on its Own



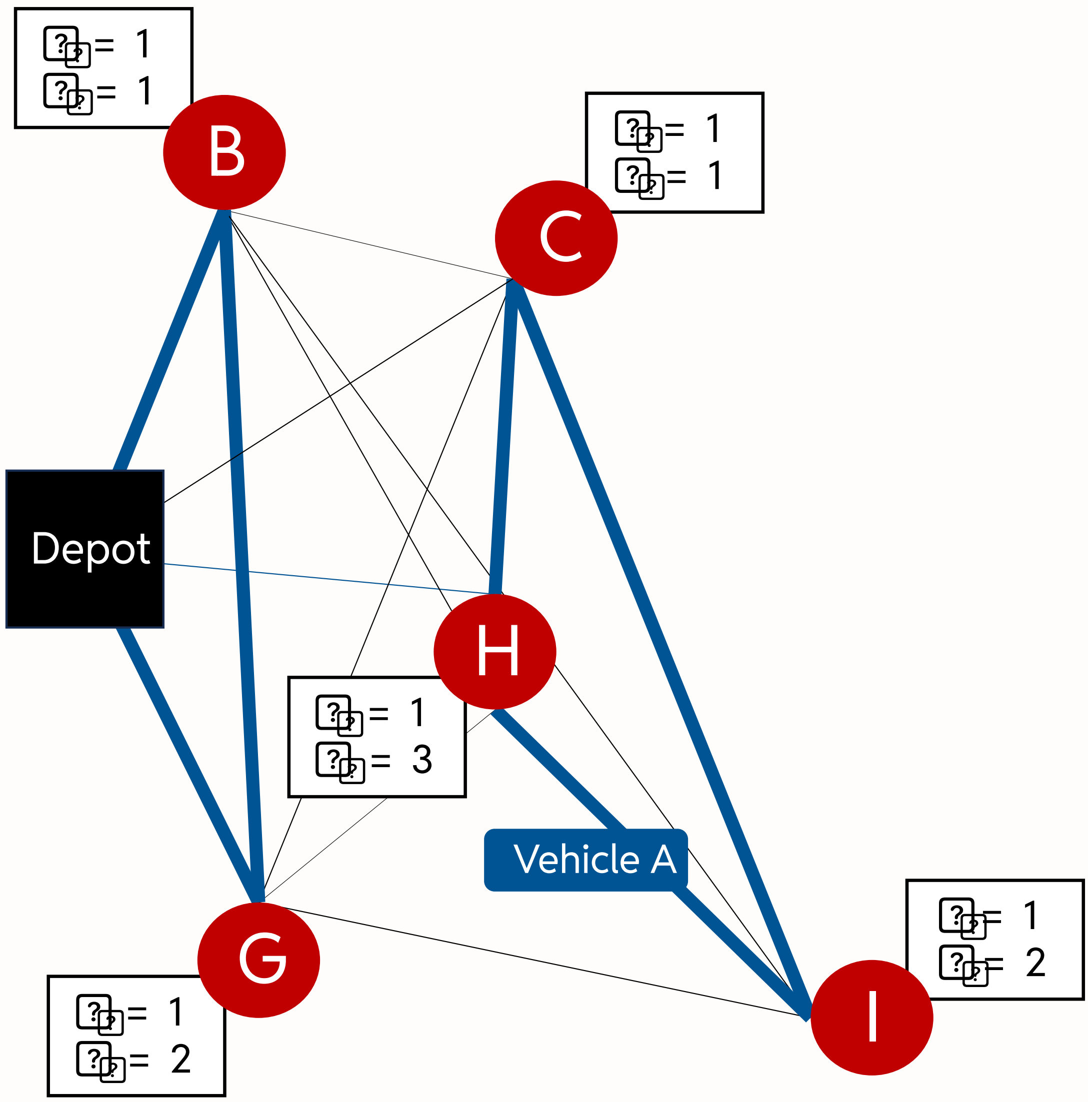
- Let's start at node H
- We drive to node C
- $U_C = 1$

Tour on its Own II



- We continue to node I
- $U_I = 2$

Tour on its Own III



- We continue to node H
- $U_H = 3$
- Connection from H to C
- U_H is greater than U_C!
- Infeasible solution!

Subtour Elimination

- Connection in the other direction wouldn't work as well
- Only depot as "reset" , as constraints are not applied here

Question: What about the capacity?

...

- Remember variable domain of U_i?
- $d_i \leq U_i \leq b \longrightarrow$ Overall capacity limit enforced!

Last Constraint

Ensure time limit?

Question: Anybody an idea?

. . .

- Constraints basically follow the same idea!
- First, we again need an additional variable
- T_i - Time spent on tour at node i of a vehicle with $i \in \mathcal{I}$

Ensure time limit

i We need the following sets and variables:

- $\mathcal{V}$ - Set of all nodes, index $i \in \{0, 1, 2, ..., n\}$
- $X_{i,j}$ - 1, if the arc between i and j is part of a tour, 0 otherwise
- T_i - Time spent on tour at the node i of a vehicle with $i \in \mathcal{I}$
- t - Maximal duration of a tour
- $c_{i,j}$ - Travel time on an arc from i to j

. . .

$$T_i - T_j + t \times X_{i,j} \leq t - c_{i,j} \quad \forall i, j \in \mathcal{V} \setminus \{0\}, i \neq j$$

. . .

$$0 \leq T_i \leq t \quad \forall i \in \mathcal{V} \setminus \{0\}$$

Any questions?

Asymmetric Vehicle Routing Problem

Objective

$$\text{minimize} \quad \sum_{(i,j) \in \mathcal{A}} c_{i,j} \times X_{i,j}$$

! The goal of the objective function is to:

Minimize the total travel distance.

Each customer is visited once

$$\sum_{i \in \mathcal{V}} X_{i,j} = 1 \quad \forall j \in \mathcal{V} \setminus \{0\}, i \neq j$$

$$\sum_{j \in \mathcal{V}} X_{i,j} = 1 \quad \forall i \in \mathcal{V} \setminus \{0\}, i \neq j$$

! Our constraints ensure:

Each customer is visited exactly once.

Depot entry and exit

$$\sum_{i \in \mathcal{V} \setminus \{0\}} X_{i,0} = |\mathcal{K}|$$

$$\sum_{j \in \mathcal{V} \setminus \{0\}} X_{0,j} = |\mathcal{K}|$$

! Our constraints ensure:

The depot is visited by exactly $|\mathcal{K}|$ vehicles. Note, that we could remove one of the constraints and the solution would still be optimal.

Capacity/ subtour elimination

$$U_i - U_j + b \times X_{i,j} \leq b - d_j \quad \forall i, j \in \mathcal{V} \setminus \{0\}, i \neq j$$

$$d_i \leq U_i \leq b \quad \forall i \in \mathcal{V} \setminus \{0\}$$

! Our constraints ensure:

The capacity limit is respected and subtours are eliminated.

Time constraints

$$T_i - T_j + t \times X_{i,j} \leq t - c_{i,j} \quad \forall i, j \in \mathcal{V} \setminus \{0\}, i \neq j$$

$$0 \leq T_i \leq t \quad \forall i \in \mathcal{V} \setminus \{0\}$$

! Our constraints ensure:

The time limit is respected (and subtours are eliminated).

Variables

$$X_{i,j} \in \{0, 1\} \quad \forall i, j \in \mathcal{V}$$

$$d_i \leq U_i \leq b \quad \forall i \in \mathcal{V} \setminus \{0\}$$

$$0 \leq T_i \leq t \quad \forall i \in \mathcal{V} \setminus \{0\}$$

! The variable domains make sure that:

The binary setup variable is either 0 or 1 and the new variables are below the time and capacity limit.

Model Characteristics

Characteristics

Questions: On model characteristics

- Is the model formulation linear/ non-linear?
- What kind of variable domains do we have?
- What do you think, can the model be solved quickly?

Model Assumptions

Questions: On model assumptions

- What assumptions have we made?
- What are issues that can arise if the model is applied?
- Have we considered service times?

Model Limitations

Our formulation makes several simplifying assumptions:

- No service time at customer locations
- Homogeneous fleet (all vehicles are identical)
- Deterministic demand (known exactly in advance)
- Single depot (all vehicles start and end at same location)

Real-World Complications

Operational Challenges

- Traffic variability
- Driver breaks and regulations
- Vehicle breakdowns
- Weather conditions

Planning Challenges

- Dynamic customer requests
- Uncertain demand
- Last-minute cancellations
- Multi-day horizons

...

> Tip
>
> These complications often require extensions of the basic CVRP model or robust optimization approaches!

Extensions of the CVRP

Questions: What extensions do you know?

. . .

- time windows (TW)
- soft time windows (STW)
- multiple depots (MD)
- heterogeneous fleet (HF)
- backhauls (B)
- pickup and delivery (PD)

Implementation and Impact

Case Study in Schleswig-Holstein

. . .

- 165 libraries
- 119 visited biweekly
- Up to 9 different tours
- Time-Limit of 8 hours for each tour

Can our formulation

solve the problem

on a real-world instance?

No, although we can find

solutions within one hour,

the gap is still very large

with 40%-45%.

Problem is NP-hard

- We have already seen that a problem can be NP-hard
- Likely, that there are no polynomial-time algorithms
- Doesn't mean that it can't be solved!

. . .

> **Note**
>
> The CVRP is a generalization of the Traveling Salesman Problem (TSP), which is itself NP-hard!

Why Does This Matter?

The solution space grows explosively with problem size:

Nodes	Possible Tour Orders	Approx. Computation
10	3.6 million	Seconds
15	1.3 trillion	Minutes
20	2.4 × 10^18	Hours to days
50	3 × 10^64	Infeasible

. . .

Question: Why is the 40-45% gap after 1 hour expected?

Understanding Optimality Gap

- The optimality gap measures:

$$\frac{\text{Best Solution} - \text{Lower Bound}}{\text{Lower Bound}} \times 100\%$$

- A 40% gap means our solution could be up to 40% worse than optimal
- For 165 libraries, even state-of-the-art solvers struggle
- This is why we need heuristics!

Can we do

anything to solve

the model?

Heuristics

- We can still solve the problem with a heuristic
- Likely not the optimal solution, but a lot of research goes into efficient algorithms to solve these problems

. . .

> **Tip**
>
> For problems with 100+ locations, heuristics are often the only practical choice.

Heuristics for VRP

Construction

Build a solution from scratch

- Nearest neighbor
- Savings algorithm
- Sweep method

Improvement

Refine existing solutions

- 2-opt (swap edges)
- Or-opt (relocate)
- Exchange moves

Metaheuristics

Intelligent search strategies

- Genetic algorithms
- Simulated annealing
- Tabu search

. . .

> Tip
>
> Interested in more details? Check the lecture Management Science

HGS-CVRP: State-of-the-Art

In our case study we applied Hybrid Genetic Search for the CVRP (HGS-CVRP) by T. Vidal [2]

- Maintains diverse population of solutions
- Applies intensive local search to improve offspring
- Achieves near-optimal solutions in seconds to minutes

. . .

> Tip
>
> For real-world applications, also consider OR-Tools by Google, it is open-source and production-ready!

Case Study Results

Using HGS-CVRP instead of exact optimization:

- Solution found in < 5 minutes (vs. hours with MIP)
- Total driving distance reduced by ~20% compared to manual planning

- CO_2 emissions reduced by approximately 12-15 tonnes/year

. . .

> ! Important
>
> The heuristic solution is better than what the MIP solver found in 1 hour!

Applications Beyond Libraries

Logistics & Delivery

- Package delivery
- Food delivery services
- Grocery delivery
- Mail distribution

Service Industries

- Maintenance crews
- Home healthcare visits
- Waste collection
- School bus routing

. . .

Question: Can you think of other applications?

Conclusion

- Standard problem that occurs in many different places
- Solving the problem with a mathematical model is difficult
- Nowadays, there are many good heuristics
- Many companies are working on the problem

. . .

> i And that's it for todays lecture!
>
> We now have covered the Capacitated Vehicle Routing Problem and are ready to start solving some tasks in the upcoming tutorial.

Questions?

Literature

Literature I

For more interesting literature to learn more about Julia, take a look at the literature list of this course.

Literature II

Bibliography

[1] I. Kara, G. Laporte, and T. Bektas, "A note on the lifted Miller–Tucker–Zemlin subtour elimination constraints for the capacitated vehicle routing problem," European Journal of Operational Research, vol. 158, no. 3, pp. 793–795, 2004, doi: https://doi.org/10.1016/S0377-2217(03)00377-1.

[2] T. Vidal, "Hybrid genetic search for the CVRP: Open-source implementation and SWAP* neighborhood," Computers & Operations Research, vol. 140, p. 105643, Apr. 2022, doi: 10.1016/j.cor.2021.105643.