CENG 384 - Signals and Systems for Computer Engineers
Spring 2020
Written Assignment 2

Aydın, Onur
e217126@metu.edu.tr

Bütün, Beyza
e2171411@ceng.metu.edu.tr

March 13, 2020

---

1. (a) **Memory:**

   The output of the system must not depend on an input which is from past or future time if it is memoryless. Therefore, $y[n] = h(x[n-n_0]) for \exists n_0 \neq 0$ breaks the memoryless property.

   In this question, $n_0 \in [1, \infty)$. Hence, the system has memory.

   Stability:

   A system must be able to generate bounded output with a given bounded input (BIBO).
   Let a boundary B which is a finite number, with $|x[n-k]| \leq B, k \in Z$
   When $B \neq 0$, $\sum_{k=1}^{\infty} x[n-k]$, diverges to indefinite number. Hence, the system is not stable.

   Causality:

   A system shouldn't depend on the future inputs in order to be causal.
   Therefore, $y[n] = h(x[n-n_0]), \exists n_0 \leq 0$ breaks the causality.

   In this system,
   $\sum_{k=1}^{\infty} x[n-k]$, k starts at 1. Hence output only depends on the past time so the system is causal.

   Linearity: We need to check the superposition property to verify linearity.
   $y_1[n] = \sum_{k=1}^{\infty} x_1[n-k]$
   $y_2[n] = \sum_{k=1}^{\infty} x_2[n-k]$

   $a_1y_1[n] = \sum_{k=1}^{\infty} a_1x_1[n-k] = a_1 \sum_{k=1}^{\infty} x_1[n-k]$
   $a_2y_2[n] = \sum_{k=1}^{\infty} a_2x_2[n-k] = a_2 \sum_{k=1}^{\infty} x_2[n-k]$
   If we add them, we get
   $a_1y_1[n] + a_2y_2[n] = \sum_{k=1}^{\infty}(a_1x_1[n-k] + a_2x_2[n-k]) = a_1 \sum_{k=1}^{\infty} x_1[n-k] + a_2 \sum_{k=1}^{\infty} x_2[n-k]$

   As you can see, this system holds superposition property, hence, the system is linear.

   Invertibility:

   If the system is invertible, it shall map all the distinct inputs to distinct outputs (1-1 and onto). We need to manipulate the system equation so that we can map inputs of y's to output x[n].
   The upper boundary of the system goes to infinity.Therefore, If we iterate the $y[n]$ one more step, we get,
   $y[n+1] = \sum_{k=1}^{\infty} x[n+1-k]$
   $y[n+1] = \sum_{k=0}^{\infty} x[n-k]$
   $y[n+1] = x[n] + \sum_{k=1}^{\infty} x[n-k]$
   $y[n+1] = x[n] + y[n]$
   $x[n] = y[n+1] - y[n]$
   Hence, the system is invertible.

   Time invariance:

   We need to shift both inputs and outputs independently and verify they are equal.
   Let $x_1[n] = x[n-n_0]$ then

   $y_1[n] = \sum_{k=1}^{\infty} x_1[n-k] = \sum_{k=1}^{\infty} x[n-n_0-k] =?\ y[n-n_0] = \sum_{k=1}^{\infty} x[n-n_0-k]$
   As you can see, this holds the equality. Therefore, this system is time invariant.

   (b) **Memory:**

   The output of the system must not depend on an input which is from past or future time if it is memoryless.

$y(t) = tx(2t+3)$ depends on future time when $t > -3$, and dependent on past time when $t < -3$. Hence, it has memory.

Stability:

The system has t value as coefficient of x(t). Therefore, as time goes to infinity, we can bound it to a finite interval because time coefficient will multiply any finite output to either positive or negative infinity. Hence, the system is unstable.

Causality:

The system is causal, when it doesn't depend on the any future input.
$y(t) = h(x(t - t_0))$, if we place any $t_0 > -3$ such as 1, we get
$y(1) = h(x(5))$, which breaks the causality. Hence, the system is not causal.

Linearity:

We need to check the superposition property to verify linearity.
$y_1(t) = tx_1(2t+3)$
$y_2(t) = tx_2(2t+3)$

$a_1y_1(t) = a_1tx_1(2t+3)$
$a_2y_2(t) = a_2tx_2(2t+3)$
If we sum, we get

$a_1y_1(t) + a_2y_2(t) = t(a_1x_1(2t+3) + a_2x_2(2t+3)) = a_1tx_1(2t+3) + a_2tx_2(2t+3)$ Thus, this system is linear.

Invertibility:

Given $y(t) = h(x(t))$, if $\exists$ a unique h^{-1} such that $x(t) = h^{-1}(y(t))$ then the system is invertible. This system is invertible since $x(t) = \dfrac{2}{t-3}y((t-3)/2)$.

Time invariance:

Assume that $x_1(2t+3) = x(2t - 2t_0 + 3)$ then

$y_1(t) = tx_1(2t+3) = tx(2t - 2t_0 + 3) =?\ y(t - t_0) = (t - t_0)x(2t - 2t_0 + 3)$
Since $tx(2t - 2t_0 + 3) \neq (t - t_0)x(2t - 2t_0 + 3)$, this system is not time invariant.

2. (a) $y'(t) = x(t) - 5y(t)$
$y'(t) + 5y(t) = x(t)$

(b) Because the system is initially at rest, y(0) = $y'(0) = ... = 0$
y(t) = $y_p(t) + y_h(t)$

Let's start with homogeneous equation.
$y_h(t) = Ae^{\lambda t}$ If we replace the $y_h(t)$ in the given equation, we will have
$\lambda Ae^{\lambda t} + 5Ae^{\lambda t} = 0$
$Ae^{\lambda t}(\lambda + 5) = 0$
$\lambda + 5 = 0 \implies \lambda = -5$
$y_h(t) = Ae^{-5t}$

Finding the particular solution;
Since we will give input to the system after $t = 0$, we can discard the $u(t)$ part because it is equal to 1 in the given time interval.
$x(t) = e^{-t} + e^{-3t}$, for $t > 0$
Let $y_p(t) = Ke^{-t} + Me^{-3t}$.
$-Ke^{-t} - 3Me^{-3t} + 5Ke^{-t} + 5Me^{-3t} = e^{-t} + e^{-3t}$
$(-K + 5K)e^{-t} + (5M - 3M)e^{-3t} = e^{-t} + e^{-3t}$
$4Ke^{-t} = e^{-t} \implies 4K = 1 \implies K = \dfrac{1}{4}$
$2Me^{-3t} = e^{-3t} \implies 2M = 1 \implies M = \dfrac{1}{2}$
$y_p(t) = \dfrac{1}{4}e^{-t} + \dfrac{1}{2}e^{-3t} \quad ,t > 0$
$y(t) = y_h(t) + y_p(t)$
$y(t) = Ae^{-5t} + \dfrac{1}{4}e^{-t} + \dfrac{1}{2}e^{-3t} \quad ,t > 0$
Stating that the system is initially at rest means there are no inputs at the beginning. Therefore, $y(0) = 0$.
$y(0) = A + \dfrac{1}{4} + \frac{1}{2} = 0 \implies A = \frac{-3}{4}$
$y(t) = \frac{-3}{4}e^{-5t} + \frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t} \quad ,t > 0$
When we add the $u(t)$ that we omitted before,
$y(t) = [\frac{-3}{4}e^{-5t} + \frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t}]u(t)$

3. (a) $y[n] = \int_{-\infty}^{\infty} x[k]h[n-k] = \int_{-\infty}^{\infty} x[n-k]h[k]$ = x[n]*h[n]

   y[n] = h[1]x[n-1] + h[-1]x[n+1] (zero, for other values of k)
   y[n] = x[n-1]+2x[n+1]
   And we know that x[n] = $2\delta[n] + \delta[n+1]$
   so we get
   y[n] = $2\delta[n-1] + \delta[n] + 2(2\delta[n+1] + \delta[n+2])$
   y[n] = $2\delta[n-1] + \delta[n] + 4\delta[n+1] + 2\delta[n+2]$

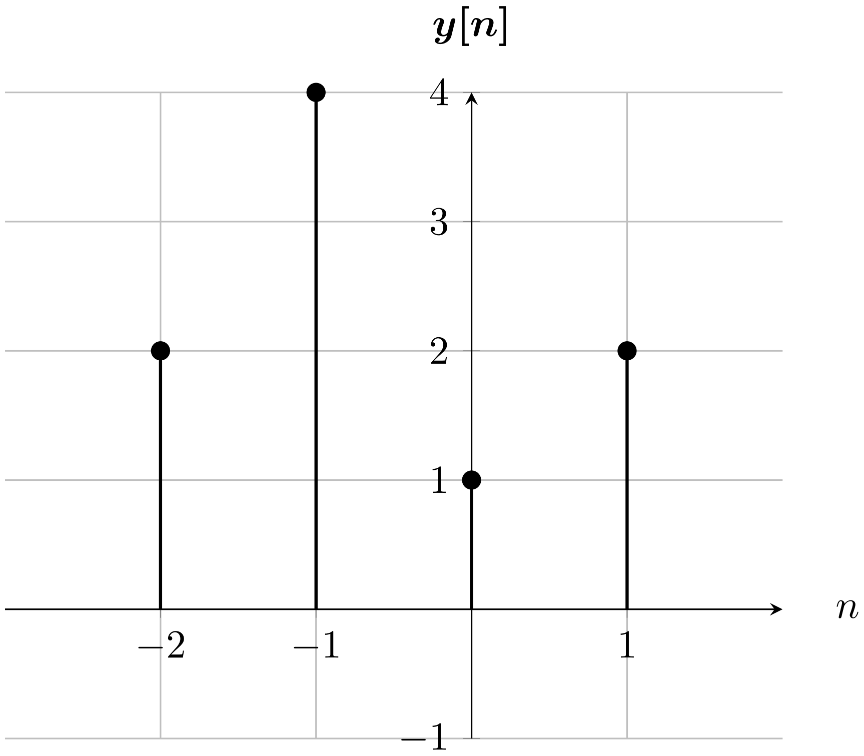

Figure 1: n vs. $y[n]$.

(b) $\frac{dx(t)}{dt} = \delta(t-1) + \delta(t+1)$
   y(t) = $\int_0^{\infty} e^{-\tau} \sin\tau\delta(t-1-\tau)d\tau$ + $\int_0^{\infty} e^{-\tau} \sin\tau\delta(t+1-\tau)d\tau$
   y(t) = h(t-1)+h(t+1)
   y(t) = $(e^{-t+1}\sin(t-1))u(t-1) + (e^{-t-1}\sin(t+1))u(t+1)$

4. (a) $y(t) = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{\tau-t}u(t-\tau)d\tau$ where $u(\tau)u(t-\tau) = u(t)$,

   and we can omit it from the equation by changing boundaries accordingly

   $= \int_0^t e^{-\tau-t}d\tau = e^{-t}\int_0^t e^{-\tau}d\tau = e^{-t}(1-e^{-t})u(t) = (e^{-t} - e^{-2t})u(t)$

(b) y(t) = x(t)*h(t)
   y(t) = $\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$
   y(t) = $\int_0^{\infty} e^{3\tau}(u(t-\tau) - u(t-1-\tau))d\tau$

   For $t \leq 0$, y(t)=0 because x(t) and h(t) don't intersect at that area.

   For $0 \leq t \leq 1$, x(t)=1
   y(t) = $\int_0^t e^{3\tau}d\tau = e^{3t}/3 - 1/3$

   For $t > 1$, x(t)=1
   y(t) = $\int_{t-1}^t e^{3\tau}d\tau = e^{3t}/3 - e^{3(t-1)}/3$

   Combining all three cases we get

$$x(t)\begin{cases} 0 & t \leq 0 \\ e^{3t}/3 - 1/3 & 0 < t \leq 1 \\ (e^{3t} - e^{3(t-1)})/3 & t > 1 \end{cases}$$

5. (a) First we should shift the equation by 2 in order to get y[n] in terms of its predecessors. $2y[n] - 3y[n-1] + 1 = 0$, in order to get a solution in terms of $c\lambda^n$, we replace $y[n]$ with λ.

   $2\lambda^2 - 3\lambda + 1 = 0$
   $(2\lambda - 1)(\lambda - 1) = 0$
   $\lambda_1 = 1/2$ and $\lambda_2 = 1$
   $y[n] = c_1 * (1/2)^n + c_2 * 1^n$
   $y[0] = c_1 + c_2 = 1$
   $y[1] = 1/2c_1 + c_2 = 0$
   $c_1 = 2$
   $c_2 = -1$
   $y[n] = 2(1/2)^n - 1 = 2^{1-n} - 1$

   (b) From given equation, we will find homogenous solution.
   y(t) = $Ke^{\lambda t}$
   $y'(t) = K\lambda e^{\lambda t}$
   $y''(t) = K\lambda^2 e^{\lambda t}$
   $y^{(3)}(t) = K\lambda^3 e^{\lambda t}$
   From these equations, we get
   $e^{\lambda t}K(\lambda^3 - 3\lambda^2 + 4\lambda - 2)$=0, then
   $\lambda^3 - 3\lambda^2 + 4\lambda - 2 = 0$
   After solving equation, we get $\lambda_1 = 1, \lambda_2 = 1 + i$ and $\lambda_3 = 1 - i$

   $y(t) = K_1e^t + K_2e^{(1+i)t} + K_3e^{(1-i)t}$
   $y'(t) = K_1e^t + (1+i)K_2e^{(1+i)t} + (1-i)K_3e^{(1-i)t}$
   $y''(t) = K_1e^t + (1+i)^2K_2e^{(1+i)t} + (1-i)^2K_3e^{(1-i)t}$

   Then
   y(0) = $K_1 + K_2 + K_3 = 3$
   $y'(0) = K_1 + (1+i)K_2 + (1-i)K_3 = 1$
   $y''(0) = K_1 + (1+i)^2K_2 + (1-i)^2K_3 = 2$
   From these equations, we got 2 different equations as
   $-iK_2 + iK_3 = 2 \rightarrow K_3 - K_2 =$ -2i
   $(1-i)K_2 + (1+i)K_3 =$ -1 $\rightarrow iK_3 + K_2 =$ -(1+i)/2
   $K_3 = \frac{-3}{2} - i$, $K_2 = \frac{-3}{2} + i$ and $K_1 = 6$

   Finally the equation will be like
   $y(t) = K_1e^t + K_2e^{(1+i)t} + K_3e^{(1-i)t}$
   y(t) = $6e^t + (\frac{-3}{2} + i)e^{(1+i)t} + (\frac{-3}{2} - i)e^{(1-i)t}$

6. (a) First the system is initially at rest, so we can conclude that x[n] = 0 and h[n] = 0 for $n < 0$.
   x[n]=w[n]-1/2w[n-1]
   w[n]=x[n]*$h_0[n]$

   Lets assume x[n]=$\delta[n]$. Then we get w[n]=$h_0[n]$
   So $h_0[n]$-$1/2h_0[n-1] = \delta[n]$
   $h_0[n] = 1/2h_0[n-1] + \delta[n]$
   From this equation, we can find a pattern as

   $h_0[0] = 1/2h_0[-1]+\delta[0] = 1$
   $h_0[1] = 1/2h_0[0]+\delta[1] = 1/2$
   $h_0[2] = 1/2h_0[1]+\delta[2] = 1/4$
   .
   .
   .
   $h_0[n] = (1/2)^n$u[n]

   (b) Assume two serial LTI systems have impulse responses $h_1[n]$ and $h_2[n]$ respectively. The overall impulse response $h[n]$ will be equal to $h_1[n] * h_2[n]$.

   Therefore, the overall impulse response of the given discrete time LTI system can be specified as:

   $h[n] = h_0[n] * h_0[n]$ Then,
   $h[n] = ((1/2)^n * (1/2)^n)$u[n] $= (1/4)^n$u[n]

(c) From the first response we have
x[n] + 1/2w[n-1] = w[n]

From the second response we have
w[n] + 1/2y[n-1] = y[n]
w[n] = y[n] - 1/2y[n-1]

If we solve these equations we have
x[n] + 1/2(y[n-1] - 1/2y[n-2]) = y[n] - 1/2y[n-1]
x[n] = y[n] - y[n-1] + 1/4y[n-2]