Mixed Finite Elements and Stokes Flow

The Stokes equations are a simplified model for the flow of very viscous fluids. The equations

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned} \tag{1}$$

augmented with boundary conditions, for example, $\boldsymbol{u} = 0$ on $\partial\Omega$. Here, $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ represents the fluid **velocity**, and $p : \Omega \to \mathbb{R}$ represents the **pressure**. The condition that the fluid is divergence free ($\nabla \cdot \boldsymbol{u} = 0$) means that the fluid is **incompressible**. These equations are valid in situations where the velocity is very slow and the viscosity is very large (when the non-dimensional Reynolds number is close to zero).

It is our goal currently to use the finite element method to approximate solutions to the Stokes equation. We would like to understand:

- When is problem is well-posed (i.e. when does a unique solution exist)?
- When does the finite element method converge to the true solution?
- What is the rate of convergence of the approximate solution to the true solution?
- How can we efficiently solve the resulting algebraic system of equations?

The main difference between (1) and other equations we have studied (Poisson problem, heat equation, elasticity) is that there are two unknowns and two equations. This means that approximate solutions $\boldsymbol{u}_h$ and p_h need to be drawn from different spaces (at the very least, one needs to be vector-valued and one needs to be scalar-valued). This is an example of using **mixed finite elements** (when we use a "mix" of different finite element spaces to solve one problem). As we will see, choosing these spaces so that the resulting method is stable and convergent is somewhat of a subtle issue. The Stokes problem is just one of many examples of systems of equations that are best treated using mixed finite elements. Other examples include Darcy's equations of flow through porous media and mixed formulations of elasticity (that can be used to avoid the problem of volumetric locking).

1 Well-Posedness

Before proceeding to discretizing the equations, we first consider well-posedness of the problem. For now, we will work in the abstract setting, and then apply these results to (1). Let V and P be two Banach spaces. Let $a(\cdot\,,\cdot)$ and $b(\cdot\,,\cdot)$ be two bilinear forms,

$$a : V \times V \to \mathbb{R}, \qquad b : V \times P \to \mathbb{R}.$$

Note the domain of $b(\cdot\,,\cdot)$ is $V \times P$. Let $F \in V'$ and $G \in P'$ be bounded linear functionals. Consider the variational problem: find $u \in V$ and $p \in P$ such that, for all $v \in V$ and $q \in P$,

$$\begin{aligned} a(u,v) + b(v,p) &= F(v), \\ b(u,q) &= G(q). \end{aligned} \tag{2}$$

Our goal is to understand what conditions on the bilinear forms a and b are required to render this problem well-posed.

We can define linear operators $A : V \to V'$, $B : V \to P'$, and $B' : P \to V'$ by

$$Av(w) = a(v,w), \qquad Bv(q) = B'q(v) = b(v,q).$$

Then, (2) is equivalent to

$$\begin{aligned} Au + B'p &= F, \\ Bu &= G, \end{aligned} \tag{3}$$

or, in block form,

$$\begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}.$$

1.1 Finite-Dimensional Case

By considering the finite-dimensional case, we can gain some intuition into when these equations are solvable. Consider the finite-dimensional problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$. If (u,p) is a solution to (4), then the second equation implies that u must be an element of the nullspace of B. Define

$$Z = \mathcal{N}(B) = \{v \in \mathbb{R}^n : Bv = 0\}.$$

Suppose that $Z \neq \{0\}$, and let (u,p) solve (4). Then, since

$$Au + B^T p = F,$$

we have, for any $v \in Z$,

$$v^T Au + v^t B^T p = v^t F,$$

and since $(Bv)^T = v^T B^T = 0$, this implies that

$$v^T Au = v^t F.$$

One way to ensure that such a $u \in Z$ exists is to assume that the bilinear form $a(\cdot\,,\cdot)$ is coercive on Z, i.e.

$$v^T Av \geq \alpha \|v\|^2.$$

for all $v \in Z$. Under this assumption, the Lax–Milgram theorem implies that there exists a unique $u \in Z$ such that

$$v^T Au = v^T F$$

for all $v \in Z$. Now, it remains to find $p \in \mathbb{R}^m$ that makes the pair (u, p) a solution to (4). The condition that needs to be satisfied is

$$Au + B^T p = F,$$

or

$$B^T p = F - Au. \tag{5}$$

Note that the column space of B^T is orthogonal to the nullspace of B, i.e. $\mathcal{R}(B^T) = Z^\perp$. Since $v^T Au = v^T F$ for all $v \in Z$, we have that $F - Au$ is orthogonal to Z, i.e. $F - Au \in Z^\perp$. This means that $F - Au \in \mathcal{R}(B^T)$, and therefore there exists $p \in \mathbb{R}^m$ that solves (5).

We have shown so far that coercivity of A on the kernel of B can guarantee the existence of a solution, and that u is unique. We have not yet shown that p is unique. This requires that the operator B^T is injective, since if there are two solutions $p_1, p_2 \in \mathbb{R}^m$, defining $p_* = p_1 - p_2$, we have

$$B^T p_* = 0.$$

Note that if we require that, for any nonzero $q \in \mathbb{R}^m$, there exists some $v \in \mathbb{R}^n$ such that

$$v^T B^T q > 0, \tag{6}$$

then B^T must be injective (it is impossible for such a p_* to exist). Under this condition, p is unique, and there exists a unique solution $(u, p) \in \mathbb{R}^n \times \mathbb{R}^m$. Notice that (6) is equivalent to

$$\frac{v^T B^T q}{\|v\| \|q\|} > 0,$$

which in finite dimensions is equivalent to existence of a constant β such that, for all $q \in \mathbb{R}^m$, there exists $v \in \mathbb{R}^n$ such that

$$\frac{v^T B^T q}{\|v\| \|q\|} \geq \beta > 0.$$

This can be written in the equivalent form

$$\inf_{0 \neq q \in \mathbb{R}^m} \sup_{0 \neq v \in \mathbb{R}^n} \frac{v^T B^T q}{\|v\| \|q\|} \geq \beta > 0, \tag{7}$$

which is known as the **inf–sup condition** or the **Ladyzhenskaya–Babuška–Brezzi** (LBB) condition.

1.2 Infinite-Dimensional Case

Now we extend the theory to the infinite-dimensional case. First first assume that both bilinear forms are continuous, i.e.

$$a(u, v) \leq \gamma_1 \|u\|_V \|v\|_V,$$
$$b(u, p) \leq \gamma_2 \|u\|_V \|p\|_P.$$

As before, consider the (easier) case where $G = 0$; we will then generalize to $G \neq 0$. Consider the closed subspace $Z \subseteq V$ defined by

$$Z := \{v \in V : b(v,q) = 0 \text{ for all } q \in P\}.$$

Now consider the bilinear form $a(\cdot\,,\cdot)$ restricted to $Z \times Z$. Clearly, $a(\cdot\,,\cdot)$ is still continuous. Assume further that it is coercive, i.e.

$$a(v,v) \geq \alpha \|v\|_V^2 \qquad \text{for all } v \in Z.$$

Lax–Milgram implies that there exists a unique $u \in Z$ such that

$$a(u,v) = F(v) \qquad \text{for all } v \in Z.$$

By the definition of Z, it also holds that $b(u,q) = 0$ for all $q \in P$. Given such $u \in Z$, it remains to find $p \in P$ such that

$$a(u,v) + b(v,p) = F(v).$$

If such a (unique) p exists, then $(u,p) \in V \times P$ will solve (2).

The question is what conditions must $b(\cdot\,,\cdot)$ satisfy in order to guarantee existence and uniqueness. By analogy to the inf–sup condition (7) in the finite-dimensional case, suppose that there exists a constance $\beta > 0$ such that

$$\inf_{0 \neq q \in P} \sup_{0 \neq v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_P} \geq \beta. \tag{8}$$

Assuming this condition holds, we would like to find (unique) $p \in P$ such that

$$b(v,p) = L(v) \qquad \text{for all } v \in V,$$

where $L(v) := F(v) - a(u,v)$ (note that u is fixed). Note further that $L(v) = 0$ for all $v \in Z$ by definition of u. Then, existence and uniqueness of p is given by the following lemma.

Lemma 1. *Let $b(\cdot\,,\cdot) : V \times P \to \mathbb{R}$ be a continuous bilinear form satisfying the inf–sup condition* (8)*. Let $Z \subseteq V$ be the closed subspace $Z = \{v : b(v,q) = 0\}$. Let $L : V \to \mathbb{R}$ be a bounded linear functional such that $F(v) = 0$ for all $v \in Z$. Then, there exists a unique $p \in P$ such that*

$$b(v,p) = L(v)$$

for all $v \in V$.

Before proving this lemma, we require some auxiliary results. We will state (without proving) the following result from functional analysis.

Theorem 1 (Closed Range Theorem)**.** *Let U and V be Banach spaces, and let $T : U \to V$ be a continuous linear operator. Its adjoint is the map $T' : V' \to U'$ defined by $T'G(u) = G(Tu)$. Then, the following are equivalent:*

(a) $\mathcal{R}(T)$ is closed in V.

(b) $\mathcal{R}(T')$ *is closed in* U'*.*

(c) $\mathcal{R}(T') = (\mathcal{N}(T))^\circ := \{F \in U' : F(u) = 0$ *for all* $u \in \mathcal{N}(T)\}$*.*

(d) $\mathcal{R}(T) = (\mathcal{N}(T'))^\circ := \{v \in V : G(v) = 0$ *for all* $G \in \mathcal{N}(T')\}$*.*

Using the closed range theorem, we can prove the following lemma.

Lemma 2. *Let* $b(\cdot\,,\cdot) : V \times P \to \mathbb{R}$ *be a continuous bilinear form, and let* $B : V \to P'$ *and* $B' : P \to V'$ *be bounded linear operators defined by*

$$Bv(q) = B'q(v) = b(v, q).$$

Let $Z^\circ := \{G \in V' : G(v) = 0$ *for all* $v \in Z\}$*. Then, the following are equivalent:*

(a) There exists $\beta > 0$ *such that*

$$\inf_{0\neq q\in P} \sup_{0\neq v\in V} \frac{b(v,q)}{\|v\|_V \|q\|_P} \geq \beta. \tag{9}$$

(b) B' *is an isomorphism from* P *onto* Z° *and*

$$\|B'q\|_{V'} \geq \beta \|q\|_P. \tag{10}$$

(c) B *is an isomorphism from* $Z^\perp$ *onto* P' *and*

$$\|Bv\|_{P'} \geq \beta \|v\|_V. \tag{11}$$

Proof. We first show that (a) is equivalent to (b), and then we will show that (b) is equivalent to (c). The inf–sup condition (9) is equivalent to existence of $\beta > 0$ such that

$$\sup_{0\neq v\in V} \frac{B'q(v)}{\|v\|_V} \geq \beta \|q\|_P.$$

This is equivalent to (10). It remains to show that B' is an isomorphism. If this condition holds, it is clear that B' is injective. Since B' is bounded and (10), B' is an isomorphism from P onto $\mathcal{R}(B')$. By the closed range theorem, $\mathcal{R}(B') = (\mathcal{N}(B))^\circ = Z^\circ$. Therefore, (a) is equivalent to (b).

We now show that (b) is equivalent to (c). We identify the dual of the space $Z^\perp$ with Z°. Suppose $G \in (Z^\perp)'$. Define $\tilde{G} \in V'$ by

$$\tilde{G}(v) = G(v^\perp),$$

where $v^\perp$ denotes orthogonal projection of v onto $Z^\perp$. For all $v \in Z$, $v^\perp = 0$, and so $\tilde{G}(v) = 0$. Therefore, $\tilde{G} \in Z^\circ$. It is easy to check that $G \mapsto \tilde{G}$ is an isometry from $(Z^\perp)'$ onto Z°. So, the spaces Z° and $(Z^\perp)'$ can be identified, and (b) and (c) are equivalent. $\square$

Using these results, we now prove Lemma 1.

Proof. The hypotheses of the theorem imply that $L \in Z^{\circ}$. The inf–sup condition implies condition (a) from Lemma 2, which in turn implies (b). This implies the existence of a unique $p \in P$ such that $B'p = L$, equivalently

$$b(v, p) = L(v)$$

for all $v \in V$. $\square$

Now suppose that $G \neq 0$ in (2). Suppose there exists some $u_0 \in Z^{\perp}$ such that $b(u_0, q) = G(q)$ for all $q \in P$. We then look for $\tilde{u} \in V$ such that

$$\begin{aligned} a(\tilde{u}, v) + b(v, p) &= F(v) - a(u_0, v), \\ b(\tilde{u}, q) &= 0. \end{aligned}$$

Since the right-hand side of the second equation is zero, the above theory implies that a unique $\tilde{u}$ exists. Then, $u = \tilde{u} + u$ satisfies

$$\begin{aligned} a(u, v) + b(v, p) &= F(v), \\ b(u, q) &= G(q), \end{aligned}$$

as desired. If the inf–sup condition holds, then so does condition (c) from Lemma 2. Since $G \in P'$, there exists a unique $u_0 \in Z^{\perp}$ such that $Bu_0 = G$. This means

$$b(u_0, q) = G(q)$$

for all q, and existence of (unique) u_0 is confirmed.