Lecture Note #4 **MTH652: Advanced Numerical Analysis**
Winter 2025 Instructor: Will Pazner

Linear Elasticity

1 Introduction

The theory of elasticity is concerned with the deformation of solid bodies when acted on by forces. A material is *elastic* if it returns to its origin (undeformed) configuration when the forces are removed: think of how a rubber ball bounces back to its original shape after you squeeze it. Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$ denote the body. The body Ω is acted on by *body forces* (acting on the volume or bulk of the body), and *surface forces* (acting on the boundary $\partial\Omega$). The most common body force is gravity. External surface forces can result from, for example, an outside object exerting a force on the surface of the body. Given the "reference configuration" of the body (i.e. its un-deformed state), we are interested in computing its *deformation*, and the resulting *strain* and *stress* given body and surface forces.

Let $\Phi : \Omega \to \mathbb{R}^d$ be the domain mapping such that $\Phi(\Omega)$ represents the *deformed domain.* We can write

$$\Phi(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x});$$

in the above, $\boldsymbol{u}(\boldsymbol{x})$ is the **displacement**; it measures how much the body is moved from its reference configuration at point $\boldsymbol{x}$. If the displacement is zero, the body is not moved at all. Note that *displacement* and *deformation* are not the same: a body may be displaced by a rigid motion (translation or rotation), but still be undeformed.

The body forces can be denoted $\boldsymbol{f} : \Omega \to \mathbb{R}^3$. Throughout the body are also *internal surface forces*, $\boldsymbol{t} : \Omega \times \Sigma \to \mathbb{R}^3$, where Σ is the unit sphere; the surface forces within the body depend both on *position* and on *direction.* These internal surface forces result from the resistance of the material to deformation (think of squeezing a rubber ball).

Let ω denote an arbitrary subset of the body $\omega \subseteq \Omega$. The set ω is acted on by *body forces* (acting on the volume of ω) and *surface forces* (acting on the boundary $\partial\omega$).

The surface forces are written in the form $\boldsymbol{\sigma n}$, where $\boldsymbol{\sigma} \in \mathbb{R}^{3\times 3}$, and

$$(\boldsymbol{\sigma n})_i = \sum_{j=1}^{3} \boldsymbol{\sigma}_{ij} n_j.$$

Here, $\boldsymbol{n}$ denotes the normal vector pointing outwards from ω. The matrix $\boldsymbol{\sigma}$ is called the **stress tensor**, and $\boldsymbol{\sigma}_{ij}$ represents the force per unit area in the ith coordinate direction on a surface with normal in the jth coordinate direction. *Cauchy's theorem* is an important theorem that, among other things, states that the stress tensor $\boldsymbol{\sigma}$ is symmetric, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$.

If the body is in *equilibrium*, then the net force at every point needs to be zero (i.e. the internal surface forces from the stresses need to cancel out the external forces). Assuming that $\boldsymbol{f}$ is the sole external force, this can be written as

$$\int_\omega \boldsymbol{f}\, dx + \int_{\partial\omega} \boldsymbol{\sigma n}\, ds = 0$$

for all (sufficiently regular) subsets $\omega \subseteq \Omega$. Conservation of angular momentum gives the additional equation

$$\int_{\omega} \boldsymbol{x} \times \boldsymbol{f}\, dx + \int_{\partial\omega} \boldsymbol{x} \times \boldsymbol{\sigma}\boldsymbol{n}\, ds = 0.$$

Integrating the first equation by parts (i.e. divergence theorem) gives that

$$\int_{\omega} (\boldsymbol{f} + \nabla \cdot \boldsymbol{\sigma})\, dx = 0.$$

Since the domain of integration ω was arbitrary, this means that

$$\boldsymbol{f} + \nabla \cdot \boldsymbol{\sigma} = 0, \tag{1}$$

which is known as *Cauchy's equilibrium equation.* In $\mathbb{R}^3$, this equation really represents three equations for six stress unknowns:

$$\begin{aligned}
f_1 + \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} &= 0,\\
f_2 + \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} &= 0,\\
f_3 + \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} &= 0.
\end{aligned}$$

In order to close the system of equations (make the number of equations and number of unknowns the same), we need to introduce a *constitutive equation* that relates the stress to the body's deformation.

2 Rigid Body Motions

A rigid body motion (i.e. a combination of translation and rotation) is not considered a deformation because the shape of Ω doesn't change (but $\boldsymbol{u}(\boldsymbol{x}) \not\equiv 0$). Rigid motions have the form

$$\Phi(\boldsymbol{x}) = \boldsymbol{x}_1 + Q(\boldsymbol{x} - \boldsymbol{x}_0),$$

where $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are fixed points, and Q is a rotation matrix (note $Q^T Q = I$ for such matrices). Let $F(\boldsymbol{x}) = D\Phi(\boldsymbol{x})$ denote the **Jacobian matrix** of the transformation Φ at point $\boldsymbol{x}$,

$$(F(\boldsymbol{x}))_{ij} = \frac{\partial \Phi_i(\boldsymbol{x})}{\partial x_j}.$$

Note that by definition of the derivative, we have

$$\lim_{\boldsymbol{v} \to 0} \frac{\|\Phi(\boldsymbol{x} + \boldsymbol{v}) - \Phi(\boldsymbol{x}) - F(\boldsymbol{x})\boldsymbol{v}\|}{\|\boldsymbol{v}\|} = 0.$$

In other words, consider a point $\boldsymbol{x} \in \Omega$, and another point $\boldsymbol{x} + \boldsymbol{v} \in \Omega$. If these points are close together (i.e. $\|\boldsymbol{v}\|$ is small), then the difference between the original point and the transformed point is well-approximated by $F(\boldsymbol{x})\boldsymbol{v}$,

$$\Phi(\boldsymbol{x} + \boldsymbol{v}) - \Phi(\boldsymbol{x}) \approx F(\boldsymbol{x})\boldsymbol{v}.$$

Therefore,

$$\|\Phi(\boldsymbol{x}+\boldsymbol{v}) - \Phi(\boldsymbol{x})\|^2 \approx \|F(\boldsymbol{x})\boldsymbol{v}\|^2 = \boldsymbol{v}^T F^T F \boldsymbol{v}$$

Assuming Φ is a diffeomorphism, $\det(F) \neq 0$, so $F^T F$ is a symmetric, positive-definite matrix.

For a **rigid motion**, we have $F(\boldsymbol{x}) = D\Phi(\boldsymbol{x}) = Q$ (where Q is a orthogonal matrix), and then $F^T F = I$, which makes sense, since the body Ω is undeformed, and

$$\|\Phi(\boldsymbol{x}+\boldsymbol{v}) - \Phi(\boldsymbol{x})\|^2 \approx \|F(\boldsymbol{x})\boldsymbol{v}\|^2 = \boldsymbol{v}^T F^T F \boldsymbol{v} = \boldsymbol{v}^T I \boldsymbol{v} = \|v\|^2.$$

This motivates the definition of the **strain**

$$E(\boldsymbol{x}) = \frac{1}{2}\left((F^T F)(\boldsymbol{x}) - I\right).$$

For every point $\boldsymbol{x} \in \Omega$, the strain is a $d \times d$ matrix. For rigid-body motions, $E(\boldsymbol{x}) \equiv 0$. For this reason, the strain is a way of describing how much the body is deformed from the original configuration. It is a measure of much a given displacement differs from a rigid body displacement, which is a measure of deformation.

Recall that the mapping Φ can be written in terms of a displacement $\boldsymbol{u}$, i.e.

$$\Phi(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}).$$

Then,

$$D\Phi(\boldsymbol{x}) = I + D\boldsymbol{u}(\boldsymbol{x}),$$

and so

$$E(\boldsymbol{x}) = \frac{1}{2}\left(D\boldsymbol{u} + (D\boldsymbol{u})^T + (D\boldsymbol{u})^T(D\boldsymbol{u})\right).$$

Linear elasticity is used to model the case of **small deformation gradients**, i.e. when $\|D\boldsymbol{u}\| \ll 1$. In this case, the quadratic term $(D\boldsymbol{u})^T(D\boldsymbol{u})$ is negligible compared with $D\boldsymbol{u}$ and $D\boldsymbol{u}^T$, and so we approximate

$$E \approx \varepsilon := \nabla^s \boldsymbol{u} = \frac{1}{2}\left(D\boldsymbol{u} + (D\boldsymbol{u})^T\right),$$

where ∇^s is called the **symmetric gradient**. The matrix ε is called the **infinitesimal strain tensor**.

Note that while the infinitesimal strain tensor vanishes for all translations, it does not vanish for arbitrary rotations. It only vanishes for **linearized** rotations. To illustrate this, consider a planar rotation of angle θ,

$$R(\boldsymbol{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and set $\boldsymbol{u}(\boldsymbol{x}) = R(\boldsymbol{x}) - \boldsymbol{x} = (R - I)\boldsymbol{x}$. Since this is a linear transformation, $D\boldsymbol{u} = (R - \boldsymbol{I})$, and we have

$$\varepsilon = \frac{1}{2}(D\boldsymbol{u} + (D\boldsymbol{u})^T) = \begin{pmatrix} \cos\theta - 1 & 0 \\ 0 & \cos\theta - 1 \end{pmatrix}.$$

This quantity approaches zero as $\theta \to 0$. For small θ, we have $\cos\theta \approx 1$ and $\sin\theta \approx \theta$, and so the infinitesimal rotations are given by

$$\begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix}.$$

Translations and infinitesimal rotations make up the kernel of ε. These are called the (infinitesimal) rigid body motions or rigid body modes.

The diagonal entries of the strain tensor ε represent the change in length of the body in each of the coordinate directions (stretching or compression) and are called *normal strains*, and the off-diagonal entries represent the change in angle between coordinate directions, and are called *shear strains*.

3 Constitutive Equations

In order to close the equilibrium equations, we need to relate the **stress tensor** $\boldsymbol{\sigma}$ to the **displacement** $\boldsymbol{u}$; this is done through the strain, or, in the case of linear elasticity, through the infinitesimal strain ε. In linear elasticity, a **linear relationship** between stress and strain is assumed; in other words,

$$\sigma_{ij} = \sum_{k\ell} C_{ijk\ell}\varepsilon_{k\ell}$$

for some fourth-order tensor $C_{ijk\ell}$. If the material is **isotropic** (the material properties are independent of direction), then this relationship simplifies to the constitutive equations known as **Hooke's law** for linear elastic isotropic materials,

$$\boldsymbol{\sigma} = 2\mu\varepsilon(\boldsymbol{u}) + \lambda\,\mathrm{tr}(\varepsilon)I = 2\mu\nabla^s\boldsymbol{u} + \lambda(\nabla\cdot\boldsymbol{u})I. \tag{2}$$

This relationship depends on two parameters: μ and λ. These are called the Lamé parameters and they depend on the material properties. They can be related to two well-known material parameters: Young's modulus E and the Poisson ratio ν. Young's modulus measures how much the material resists compression or tension; a low Young's modulus means the material is stretchy, and a high Young's modulus means the material is stiff. Poisson's ratio measures how much the material responds in the direction *perpendicular* to the force. For example, when you stretch a rubber band, it thins out in the perpendicular direction, or when you squeeze a rubber ball, it gets squished outwards in the perpendicular direction. If the Poisson's ratio is low (close to zero), then the material does not deform much in the perpendicular direction. As Poisson's ratio approaches $\frac{1}{2}$, the material becomes more and more *incompressible*, leading to more deformation in the perpendicular direction. The Lamé parameters are related to E and ν by the definitions

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

We see that for nearly incompressible materials ($\nu \approx \frac{1}{2}$) we have $\lambda \to \infty$. Combining (1) with (2), we arrive at three equations posed solely in terms of the three displacement unknowns. The equations are written

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma} &= \boldsymbol{f}, \\ \boldsymbol{\sigma} &= 2\mu \nabla^s \boldsymbol{u} + \lambda(\nabla \cdot \boldsymbol{u}) I. \end{aligned}$$

For this problem to be well-posed, we need to augment the equations with appropriate boundary conditions. Just like the Poisson problem had Dirichlet conditions (for the scalar unknown u) and Neumann conditions (for the normal derivatives $\partial u / \partial \boldsymbol{n}$), the equations of elasticity have *displacement boundary conditions* (Dirichlet) $\boldsymbol{u} = \boldsymbol{g}_D$ and *traction boundary conditions* (Neumann) $\boldsymbol{\sigma n} = \boldsymbol{g}_N$. The displacement conditions can be enforced on part of the boundary Γ_D, and the traction boundary conditions on $\Gamma_T = \partial\Omega \setminus \Gamma_D$.

4 Weak Formulation

We will consider the space of **vector-valued functions**, with each component in $H^1(\Omega)$, with zero trace on Γ_D, i.e. define the space $\boldsymbol{V}$ by

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}|_{\Gamma_D} = 0\},$$

where $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ means that $v_i \in H^1(\Omega)$ for all i, where $\boldsymbol{v} = (v_1, v_2, \ldots, v_d)$. Suppose we multiply by a (vector-valued) test function $\boldsymbol{v}$ and integrate over Ω. Then,

$$\begin{aligned} \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx &= -\int_\Omega \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{v} \, dx \\ &= \int_\Omega (2\mu \nabla^s \boldsymbol{u} + \lambda \nabla \cdot \boldsymbol{u} I) : \nabla \boldsymbol{v} \, dx + \{\text{boundary terms}\} \end{aligned} \tag{3}$$

Since the test function $\boldsymbol{v}$ vanishes on Γ_D, the boundary term is given by

$$\int_{\Gamma_T} \boldsymbol{\sigma n} \cdot \boldsymbol{v} \, ds.$$

The way (3) is written, the fact that this form is symmetric is a little bit hidden. To see the symmetry, notice that

$$(\nabla \cdot \boldsymbol{u} I) : \nabla \boldsymbol{v} = \sum_{i=1}^d \frac{\partial v_i}{\partial x_i} \sum_{j=1}^d \frac{\partial u_j}{\partial x_j} = \sum_{i,j=1}^d \frac{\partial v_i}{\partial x_i} \frac{\partial u_j}{\partial x_j},$$

which shows symmetry of the second term in the volume integral. The symmetry of the first term is seen by decomposing $\nabla \boldsymbol{v}$ into $\nabla^s \boldsymbol{v}$ and an anti-symmetric part. The dot-product of a symmetric matrix and an anti-symmetric matrix is zero, and so

$$\nabla^s \boldsymbol{u} : \nabla \boldsymbol{v} = \nabla^s \boldsymbol{u} : \nabla^s \boldsymbol{v}.$$

The weak formulation is given by: find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{u}|_{\Gamma_D} = \boldsymbol{g}$ and

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx + \int_{\Gamma_T} \boldsymbol{t} \cdot \boldsymbol{v} \, ds$$

where $a(\cdot, \cdot)$ is the symmetric bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \left(2\mu \nabla^s \boldsymbol{u} : \nabla^s \boldsymbol{v} + \lambda (\nabla \cdot \boldsymbol{u})(\nabla \cdot \boldsymbol{v})\right) \, dx$$

Our next task in showing well-posedness of the problem is continuity and coercivity of the symmetric bilinear form $a(\cdot\,, \cdot)$.