Lecture Note #1 **MTH652: Advanced Numerical Analysis**
Winter 2025 Instructor: Will Pazner

Parabolic and Time-Dependent Problems

1 Terminology

First, we discuss some notation and terminology related to the classification of PDEs. Consider a linear second-order differential operator $\mathcal{L}$,

$$\mathcal{L}u := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

The matrix of coefficients $A = (a_{ij})$ can be taken to be symmetric. The operator $\mathcal{L}$ can be classified as:

- **Elliptic** if A is positive (or negative) definite
- **Parabolic** if A is semidefinite (with one zero eigenvalue)
- **Hyperbolic** if A is indefinite and nonsingular

These three cases correspond to conic sections in the case that $n = 2$. In that case, we can write the matrix A as

$$A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

and

$$\mathcal{L}u = au_{xx} + bu_{xy} + c_{yy}.$$

The three cases can be classified according to the determinant of A,

- Elliptic $\iff \det(A) > 0$
- Parabolic $\iff \det(A) = 0$
- Hyperbolic $\iff \det(A) < 0$

Note that $\det(A) = ac - \frac{b^2}{4}$, and so

$$-4\det(A) = b^2 - 4ac =: \Delta,$$

which is the **discriminant** $\Delta(P)$ of the quadratic polynomial

$$P(x, y) = ax^2 + bxy + cy^2.$$

The sign of Δ determines whether the zero set $\{(x, y) : P(x, y) = 0\}$ defines an ellipse, a parabola, or a hyperbola. This explains the terminology for second-order partial differential equations.

The reason we care about classifying PDEs in these cases is because equations from the same category share important similarities. For example, elliptic equations are the most easily amenable to solution through finite element methods, as we saw in the first term of this sequence. Parabolic equations are typically time-dependent equations with infinite speed of propagation. Hyperbolic equations are typically time-dependent equations with finite speed of propagation. The "prototypical" equations of each class ares

- Elliptic: Laplace equation
- Parabolic: heat equation
- Hyperbolic: wave equation

2 Heat Equation

Just as our model elliptic problem was the Poisson problem, our model parabolic problem is the heat equation. The Poisson problem can be thought of as the steady-state (time-independent) version of the heat equation. As before, we have a spatial domain $\Omega \subseteq \mathbb{R}^d$. Additionally, we have a temporal domain $[0, T]$ for some $T > 0$. We look for a time-dependent solution $u(\boldsymbol{x}, t) : \Omega \times [0, T] \to \mathbb{R}$ that satisfies

$$\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= f && \text{in } \Omega \times [0, T], && (1)\\
u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}) && \text{(initial conditions)}, && (2)\\
u &= 0 && \text{on } \partial\Omega \times [0, T] \text{ (boundary conditions)}. && (3)
\end{aligned}$$

(A more general form of this equation may be considered by introducing coefficients, more general boundary conditions, etc.). In the case with zero forcing term ($f = 0$), we have the following stability results:

$$\|u(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} \tag{4}$$

$$\|\partial u/\partial t(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{t}\|u_0\|_{L^2(\Omega)} \tag{5}$$

for some constant C independent of u and t.

2.1 Semi-discretization

A very common approach for time-dependent problems is **semi-discretization**. This means that we first discretize the spatial variables (e.g. with the finite element method), and as a result, obtain a system of ordinary differential equations (ODEs). The system of ODEs can then be discretized using standard methods (forward Euler, backward Euler, Runge–Kutta, etc.).

The variational form of (1) is obtained in the usual way: multiplying by a test function v and integrating by parts (in the spatial variables only).

$$\begin{aligned}\int_\Omega \frac{\partial u}{\partial t} v\,dx - \int_\Omega (\Delta u) v\,dx &= \int_\Omega \frac{\partial u}{\partial t} v\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx \\ &= \int_\Omega f v\,dx\end{aligned}$$

Let $V_h \subseteq H_0^1(\Omega)$ be a finite element space (for example, piecewise linear functions defined on a mesh $\mathcal{T}$ of Ω). We look for a function $u_h(\boldsymbol{x}, t)$ with $u_h(\cdot, t) \in V_h$ for all $t \in [0, T]$ satisfying

$$\int_\Omega \frac{\partial u_h}{\partial t} v_h\,dx + \int_\Omega \nabla u_h \cdot \nabla v_h\,dx = \int_\Omega f v_h\,dx \tag{6}$$

for all test functions $v_h \in V_h$.

Let $\{\phi_i\}_{i=1}^N$ denote a basis for the finite-dimensional space V_h. Since $u_h(\cdot, t) \in V_h$, there are **time-dependent coefficients** $\{u_i(t)\}_{i=1}^N$ such that

$$u_h(\boldsymbol{x}, t) = \sum_{j=1}^N u_j(t) \phi_j(\boldsymbol{x}). \tag{7}$$

Inserting (7) into (6), and setting $v_h = \phi_i$, we obtain

$$\begin{aligned}&\int_\Omega \frac{\partial}{\partial t} \left(\sum_{j=1}^N u_j(t) \phi_j \right) \phi_i\,dx + \int_\Omega \nabla \left(\sum_{j=1}^N u_j(t) \phi_j \right) \cdot \nabla \phi_i\,dx \\ &= \sum_{j=1}^N \left(\int_\Omega \phi_i \phi_j\,dx \right) \frac{du_j(t)}{dt} + \sum_{j=1}^N \left(\int_\Omega \nabla \phi_i \cdot \nabla \phi_j\,dx \right) u_j(t) = \int_\Omega f \phi_i\,dx\end{aligned} \tag{8}$$

We recall the definition of the mass and stiffness matrices,

$$\begin{aligned}M_{ij} &= \int_\Omega \phi_i \phi_j\,dx \\ A_{ij} &= \int_\Omega \nabla \phi_i \cdot \nabla \phi_j\,dx.\end{aligned}$$

Let $\boldsymbol{u} \in \mathbb{R}^N$ denote the vector

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix},$$

and let $d\boldsymbol{u}/dt \in \mathbb{R}^N$ denote

$$\frac{d\boldsymbol{u}}{dt} = \begin{pmatrix} du_1/dt \\ du_2/dt \\ \vdots \\ du_N/dt \end{pmatrix}.$$

Therefore (8) can be written as

$$M\frac{d\boldsymbol{u}}{dt} + A\boldsymbol{u} = \boldsymbol{f}, \tag{9}$$

where $\boldsymbol{f} \in \mathbb{R}^N$ is defined by

$$(\boldsymbol{f})_i = \int_\Omega f\phi_i \, dx.$$

This is called the **semi-discrete problem**. We have discretized in space, so that all the spatial derivatives of the unknown have be replaced with algebraic operations. However, we have not discretized in time, and so the temporal derivatives remain in the formulation. The strategy will then be to apply standard methods for ODEs to the problem (9).

Note that we can analyze the semi-discrete problem to prove a similar stability estimates to (4). Let $f = 0$ and take $v = u_h(t)$, obtaining

$$\int_\Omega \frac{\partial u_h}{\partial t} u_h \, dx + \int_\Omega \nabla u_h \cdot \nabla u_h = 0.$$

Note that $\frac{\partial u_h}{\partial t} u_h = \frac{1}{2}\frac{\partial}{\partial t} u_h^2$, and so

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 \, dx + \int_\Omega \nabla u_h \cdot \nabla u_h = 0.$$

Rearranging,

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_{L^2(\Omega)}^2 = -\|\nabla u_h\|_{L^2(\Omega)}^2 \leq 0,$$

which implies

$$\|u_h(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}.$$

We can also prove semi-discrete error estimates. In what follows, we will use the familiar notation

$$\begin{aligned} (u, v) &= \int_\Omega uv \, dx, \\ a(u, v) &= \int_\Omega \nabla u \cdot \nabla v \, dx. \end{aligned}$$

In this notation more compact notation, the semi-discretization is

$$(\partial u_h / \partial_t, v_h) + a(u_h, v_h) = (f, v_h).$$

Theorem 1. *There is a constant C such that, if u solves* (1) *and u_h solves* (8)*, then*

$$\max_{0 \leq t \leq T} \|u(\cdot, t) - u_h(\cdot, t)\|_{L^2(\Omega)} \leq C\left(1 + \left|\log \frac{T}{h^2}\right|\right) \max_{0 \leq t \leq T} h^2 \|u(\cdot, t)\|_{H^2(\Omega)}.$$

Proof. The proof proceeds using a duality argument—this is the same kind of argument that was used to prove L^2 error estimates for the Poisson problem.

We introduce the so-called *dual problem*, and then will use this problem to proceed with the error analysis. Fix some "final time" $t_f \in [0,T]$. Let $w(\boldsymbol{x},t)$, $\boldsymbol{x} \in \Omega$, $t \in [0,t_f]$ satisfy

$$\begin{aligned} -\frac{\partial w}{\partial t} - \Delta w &= 0, \\ w(\boldsymbol{x}, t_f) &= w_f(\boldsymbol{x}), \\ w(\boldsymbol{x}, t) &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Note that instead of an initial condition, there is a "final condition". We can consider the discrete problem associated with the dual problem: find $w_h : [0,t_f] \to V_h$ such that

$$\begin{aligned} -(\partial w_h(t)/\partial t, v_h) + a(w_h(t), v_h) &= 0, \\ w_h(t_f) &= w_f. \end{aligned}$$

for all $v_h \in V_h$. (Here we assume that the final condition w_f is in the space V_h). As before, expanding w_h in a basis for V_h, we obtain the system of ODEs

$$\begin{aligned} -M\frac{d\boldsymbol{w}(t)}{dt} + A\boldsymbol{w} &= 0, \\ \boldsymbol{w}(t_f) &= \boldsymbol{w}_f. \end{aligned}$$

Recall that the mass matrix M is symmetric and positive-definite, and so it can be written as $M = E^TE$ for some invertible matrix E. Introduce the change of variables $\boldsymbol{\eta} = E\boldsymbol{w}$ (and so $w = E^{-1}\boldsymbol{w}$) to obtain

$$\begin{aligned} 0 &= -M\frac{d\boldsymbol{w}(t)}{dt} + A\boldsymbol{w} \\ &= -E^TE\frac{d\boldsymbol{w}(t)}{dt} + AE^{-1}\boldsymbol{\eta} && = 0, \\ &= -E^T\frac{d\boldsymbol{\eta}(t)}{dt} + AE^{-1}\boldsymbol{\eta} && = 0. \end{aligned}$$

Multiplying both sides by E^{-T} and setting $\bar{A} = E^{-T}AE^{-1}$, we have

$$-\frac{d\boldsymbol{\eta}(t)}{dt} + \bar{A}\boldsymbol{\eta} = 0$$

or

$$\begin{aligned} \frac{d\boldsymbol{\eta}(t)}{dt} &= \bar{A}\boldsymbol{\eta}, \\ \boldsymbol{\eta}(t_f) &= \boldsymbol{\eta}_f. \end{aligned}$$

The solution to this system of ODEs is given by

$$\boldsymbol{\eta}(t) = e^{\bar{A}(t-t_f)}\boldsymbol{\eta}_f,$$

where we recall that the matrix exponential is defined by

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

(Note: this series always converges). Since $\bar{A}$ is SPD, we have that

$$\|e^{\bar{A}(t-t_f)}\| = e^{\|\bar{A}\|(t-t_f)},$$

and so for any $t \in [0, t_f]$ it holds that

$$\|e^{\bar{A}(t-t_f)}\| \leq 1.$$

This means that for $t \in [0, t_f]$

$$\|\boldsymbol{\eta}(t)\| \leq \|\boldsymbol{\eta}_f\|.$$

We now estimate the quantity

$$\int_0^{t_f} \|\partial \boldsymbol{\eta}(t)/\partial t\| \, dt.$$

First, we note that $\|\bar{A}\boldsymbol{\eta}\| \leq Ch^{-2}\|\boldsymbol{\eta}\|$. (Why?) Furthermore, using the above results,

$$\|\partial \boldsymbol{\eta}(t)/\partial t\| \leq \|\bar{A}\| e^{\|\bar{A}\|(t-t_f)} \|\boldsymbol{\eta}_f\| \leq \frac{C}{t-t_f} \|\boldsymbol{\eta}_f\|.$$

Splitting the above integral as

$$\int_0^{t_f} \|\partial \boldsymbol{\eta}(t)/\partial t\| \, dt = \int_0^{t_f-h^2} \|\partial \boldsymbol{\eta}(t)/\partial t\| \, dt + \int_{t_f-h^2}^{t_f} \|\partial \boldsymbol{\eta}(t)/\partial t\| \, dt$$

and using the above two estimates gives the result

$$\int_0^{t_f} \|\partial \boldsymbol{\eta}(t)/\partial t\| \, dt \leq C\left(1 + \left|\log \frac{C}{h^2}\right|\right) \|\boldsymbol{\eta}_f\|$$

(see problem 8.2 from the textbook).

Let $s \in [0, T]$. Define $\widetilde{u_h} \in V_h$ by

$$a(\widetilde{u_h}(s), v_h) = a(u(s), v_h) \qquad \text{for all } v_h \in V_h.$$

$\widetilde{u_h}(s)$ is called the *elliptic projection* of $u(s)$; it is the projection of $u(s)$ onto the subspace V_h in the inner product defined by the bilinear form $a(\cdot\,,\cdot)$. Let $e_h(s) \in V_h$ be given by the difference between the semi-discrete solution $u_h(s)$ and the elliptic projection $\widetilde{u_h}(s)$,

$$e_h(s) := u_h(s) - \widetilde{u_h}(s).$$

Consider the dual problem where the final condition is given by $e_h(t_f)$. We look for $w_h : [0, T] \to V_h$ such that

$$\begin{aligned} -(\partial w_h(t)/\partial t, v_h) + a(w_h(t), v_h) &= 0, \\ w_h(t_f) &= e_h(t_f). \end{aligned}$$

Now, choose the test function $v_h = e_h(t)$. Integrating by parts in time, we have

$$\begin{aligned} 0 &= \int_0^{t_f} \Big[-\left(\partial w_h(t)/\partial t, e_h(t)\right) + a(w_h(t), e_h(t)) \Big]\, dt \\ &= \int_0^{t_f} \Big[(\partial e_h(t)/\partial t, w_h(t)) + a(w_h(t), e_h(t)) \Big]\, dt - (w_h(t_f), e_h(t_f)) + (w_h(0), e_h(0)) \end{aligned}$$

Rearranging and using that $w_h(t_f) = e_h(t_f)$,

$$\|e_h(t_f)\|^2_{L^2(\Omega)} = \int_0^{t_f} \Big[(\partial e_h(t)/\partial t, w_h(t)) + a(w_h(t), e_h(t)) \Big]\, dt + (w_h(0), e_h(0)).$$

Define $\theta(t) = u(t) - \widetilde{u_h(t)}$. Note that

$$(u_h(t), v_h) + a(u_h(t), v_h) = (u(t), v_h) + a(u(t), v_h)$$

for all $v_h \in V_h$, and so

$$(e_h(t), v_h) + a(e_h(t), v_h) = (\theta(t), v_h) + a(\theta(t), v_h).$$

Also, $(u_h(0), v_h) = (u(0), v_h)$. This means that

$$\|e_h(t_f)\|^2_{L^2(\Omega)} = \int_0^{t_f} \Big[(\partial \theta(t)/\partial t, w_h(t)) + a(w_h(t), \theta(t)) \Big]\, dt + (w_h(0), \theta(0)).$$

Since $\theta(t)$ is the error associated with the elliptic projection, the term involving $a(\cdot\,,\cdot)$ vanishes (Galerkin orthogonality), leaving

$$\|e_h(t_f)\|^2_{L^2(\Omega)} = \int_0^{t_f} (\partial \theta(t)/\partial t, w_h(t))\, dt + (w_h(0), \theta(0)).$$

Integrating by parts again, we obtain

$$\begin{aligned} \|e_h(t_f)\|^2_{L^2(\Omega)} &= -\int_0^{t_f} (\partial w_h(t)/\partial t, \theta(t))\, dt + (w_h(t_f), \theta(t_f)) \\ &= -\int_0^{t_f} (\partial w_h(t)/\partial t, \theta(t))\, dt + (e_h(t_f), \theta(t_f)). \end{aligned}$$

Then, by Cauchy-Schwarz,

$$\begin{aligned} -\int_0^{t_f} (\partial w_h(t)/\partial t, \theta(t))\, dt + (w_h(t_f), \theta(t_f)) &\le \int_0^{t_f} \|\partial w_h(t)/\partial t\| \|\theta(t)\|\, dt + \|w_h(t_f)\| \|\theta(t_f)\| \\ &\le \max_{t \in [0, t_f]} \|\theta(t)\| \left[\int_0^{t_f} \|\partial w_h(t)/\partial t\|\, dt + \|e_h(t_f)\| \right] \\ &\le \max_{t \in [0, t_f]} \|\theta(t)\| \left[C \left(1 + \left| \log \frac{t_f}{h^2} \right| \right) \|e_h(t_f)\| \right] \end{aligned}$$

Dividing both sides by $\|e_h(t_f)\|$,

$$\|e_h(t_f)\|_{L^2(\Omega)} \leq C\left(1 + \left|\log \frac{t_f}{h^2}\right|\right) \max_{t\in[0,t_f]} \|\theta(t)\|.$$

Since $\theta(t)$ is the error associated with elliptic projection of $u(t)$ onto V_h, we have that

$$\|\theta(t)\|_{L^2(\Omega)} \leq Ch^2\|u\|_{H^2(\Omega)},$$

and the conclusion follows. $\square$