Lecture Note #10 **MTH652: Advanced Numerical Analysis**
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Mixed Finite Elements Continued

1 Well-Posedness of Stokes

Last time, we proved the following theorem

Theorem 1. *Consider continuous bilinear forms*

$$\begin{aligned} a(\cdot\,,\cdot) &: V \times V \to \mathbb{R} \\ b(\cdot\,,\cdot) &: V \times P \to \mathbb{R} \end{aligned}$$

and bounded linear functionals

$$F : V \to \mathbb{R}, \qquad G : P \to \mathbb{R}.$$

Then, the variational problem: find $(u, p) \in V \times P$ such that, for all $(v, q) \in V \times P$

$$\begin{aligned} a(u, v) + b(v, p) &= F(v), \\ b(u, q) &= G(v) \end{aligned}$$

has a unique solution if $a(\cdot\,,\cdot)$ is coercive on Z, i.e.

$$a(v, v) \geq \gamma \|v\|_V \quad \textit{for all } v \in Z,$$

for $\gamma 0$, where Z is defined by

$$Z = \{v \in V : b(v, q) = 0 \textit{ for all } q \in P\}$$

and $b(\cdot\,,\cdot)$ satisfies the inf–sup condition

$$\inf_{0 \neq q \in P} \sup_{0 \neq v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_P} \geq \beta$$

for $\beta 0$. The solution satisfies the stability estimate

$$\|u\|_V + \|p\|_P \leq C \left(\|F\|_{V'} + \|G\|_{P'} \right)$$

for some constant C.

As an example, we apply this theorem to Stokes equations

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0, \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

This corresponds to the variational problem with

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}),$$
$$b(\boldsymbol{v}, p) = -(\nabla \cdot \boldsymbol{v}, p),$$

where $\boldsymbol{u} \in [H_0^1(\Omega)]^d =: \boldsymbol{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$. Recall that $H_0^1(\Omega)$ is the space of zero-trace functions in H^1 (so they satisfy the boundary conditions), and $L_0^2(\Omega)$ is the space of zero-mean functions in L^2 (to ensure that p is unique, since only ∇p appears in the equations). Continuity of these bilinear forms is immediate. We have previously proven coercivity of $a(\boldsymbol{u}, \boldsymbol{v})$ (on all of $\boldsymbol{H}^1(\Omega)$). It remains to show that the inf–sup condition is satisfied.

We will use the following lemma, which we have proven previously (in $\mathbb{R}^2$) in the context of linear elasticity.

Lemma 1. *For any $q \in L_0^2(\Omega)$, there exists $\boldsymbol{v} \in H_0^1(\Omega)$ such that*

$$\nabla \cdot \boldsymbol{v} = q$$

satisfying the bound

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|q\|_{L^2(\Omega)}.$$

Idea of proof. The pure Neumann Poisson problem

$$-\Delta w = -q \quad \text{in } \Omega,$$
$$\frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega.$$

is well-posed since q has zero mean. So, there exists a unique solution w that satisfies the regularity estimate

$$\|w\|_{H^2(\Omega)} \leq C_r \|q\|_{L^2(\Omega)}.$$

Then, $\tilde{\boldsymbol{v}} = \nabla w$ satisfies $\nabla \cdot \boldsymbol{v} = q$; however, it will not be zero-trace, and so a correction is required. For details, see Lemma 1 from Lecture 5.

We will now use this lemma to prove the inf–sup condition. Let $q \in L_0^2(\Omega)$ be arbitrary. Choose $\boldsymbol{v}$ as in the lemma. Then,

$$\begin{aligned}
\frac{b(-\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \|q\|_{L^2(\Omega)}} &= \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \|q\|_{L^2(\Omega)}} \\
&= \frac{(q, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \|q\|_{L^2(\Omega)}} \\
&= \frac{\|q\|_{L^2\Omega}^2}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \|q\|_{L^2(\Omega)}} \\
&\geq \frac{\|q\|_{L^2\Omega}^2}{C \|q\|_{L^2(\Omega)}^2} \\
&= 1/C =: \beta.
\end{aligned}$$

Having verified the conditions of the theorem, we can conclude that the Stokes problem is well-posed, and there exists a unique solution $(\boldsymbol{u}, p)$.

2 Application to Mixed Finite Elements

To apply the finite element method to problems of the form (1), we need to choose finite-dimensional subspaces

$$V_h \subseteq V, \qquad P_h \subseteq P.$$

such that the hypotheses of Theorem 1 are satisfied. The bilinear forms $a(\cdot\,,\cdot)$, $b(\cdot\,,\cdot)$ are automatically continuous. Specifically for Stokes, since $a(\cdot\,,\cdot)$ is coercive on the entire space, it is automatically coercive on a subspace. So, the nontrivial condition to prove is the inf–sup condition.

To see how this can fail, consider the so-called $\mathcal{P}^1$–$\mathcal{P}^0$ discretization. This uses the space of continuous piecewise linear functions for the velocity and (discontinuous) piecewise constants for the pressure. Let n_V be the number of interior (non-boundary) vertices, and let n_T be the number of triangles in the mesh. Then, the dimension (in 2D) of $\boldsymbol{V}_h$ is $2n_V$, and the dimension of P_h is n_T. So, the matrix B associated with the divergence bilinear form $b(\cdot\,,\cdot)$ is of size $n_T \times 2n_V$. We count the angles in the mesh. Each triangle has π radians, and each vertex has 2π radians. Since n_V is the number of interior vertices, we have that

$$2\pi n_V < \pi n_T$$

which implies $2n_V < n_T$. This means that B is taller than it is wide (it has more rows than columns). Consequently, B **cannot be surjective**. However, the inf–sup condition implies (as we proved last time) surjectivity of B. Therefore, this discretization cannot be used for the Stokes problem. In fact, using this discretization, the resulting matrix

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

will be singular.

It will be useful to establish to sufficient conditions that we can check to ensure inf–sup stability of pairs of spaces. One way of doing this is using the so-called **Fortin operator**.

Definition 1. *Let $V_h \subseteq V$ and $P_h \subseteq P$. A linear operator $\Pi_h : V \to V_h$ is called a **Fortin operator** if*

(i) $b(\Pi_h v, q_h) = b(v, q_h)$ *for all* $v \in V$ *and* $q_h \in P_h$

(ii) $\|\Pi_h v\|_V \le C_f \|v\|_V$ *for all* $v \in V$

Note that in the case of Stokes, condition (i) is equivalent to commutativity of the following diagram

$$\begin{array}{ccc} \boldsymbol{V} & \xrightarrow{\nabla\cdot} & P \\ {\scriptstyle \Pi_h}\downarrow & & \downarrow{\scriptstyle Q_h} \\ \boldsymbol{V}_h & \xrightarrow{\nabla\cdot} & P_h \end{array}$$

where $Q_h : P \to P_h$ is L^2-projection, i.e.

$$(Q_h p, q_h) = (p, q_h) \qquad \text{for all } q_h \in P_h.$$

Theorem 2. *Suppose the hypotheses of Theorem 1 hold for the continuous problem. Let $V_h \subseteq V$ and $P_h \subset P$ be subspaces that admit a Fortin operator Π_h satisfying conditions (i) and (ii). Then, the discrete inf–sup condition*

$$\inf_{0\neq q\in P_h} \sup_{0\neq v\in V_h} \frac{b(v,q)}{\|v\|_V \|q\|_P} \geq \beta/C_f$$

holds.

Proof. Let $q_h \in P_h$ be given. The continuous inf–sup condition implies existence of $v \in V$ such that

$$b(v, q_h) \geq \beta \|v\|_V \|q_h\|_P.$$

Set $v_h = \Pi_h v \in V_h$. Then,

$$b(v_h, q_h) = b(v, q_h) \geq \beta\|v\|_V \|q_h\|_P \geq \frac{\beta}{C_f}\|v_h\|_V \|q_h\|_P. \qquad \square$$

Demonstrating existence of a Fortin operator is one of the primary ways that a discretization is shown to be inf–sup stable. Returning to the example of the $\mathcal{P}^1$–$\mathcal{P}^0$ discretization, what went wrong was that $\dim(\boldsymbol{V}_h) < \dim(P_h)$, ruling out surjectivity of B. This means that we need a richer (higher dimensional) velocity space. We can try to use the $\mathcal{P}^2$–$\mathcal{P}^0$ discretization, with continuous piecewise quadratics for the velocity space.

We wish to construct a Fortin operator $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$. This operator must satisfy

$$b(\Pi_h \boldsymbol{v}, q_h) = b(\boldsymbol{v}, q_h),$$

i.e.

$$0 = \int_\kappa \nabla \cdot (\boldsymbol{v} - \Pi_h \boldsymbol{v}) q_h \, dx = \int_{\partial\kappa} (\boldsymbol{v} - \Pi_h \boldsymbol{v}) \cdot \boldsymbol{n} q_h \, ds$$

for all q_h, for all triangles κ in the mesh. Since q_h is piecewise constant, it suffices to find (stable) $\Pi_h \boldsymbol{v}$ such that

$$\int_e \boldsymbol{v} \, ds = \int_e \Pi_h \boldsymbol{v} \, ds$$

for every edge e in the mesh. Suppose e has vertices i and j. Define $b_e = 6\phi_i\phi_j/|e|$. This is a quadratic "bubble function". It is nonzero on the edge e, and zero on all other edges in the mesh. Clearly, $\boldsymbol{c} b_e \in \boldsymbol{V}_h$ for any $\boldsymbol{c} \in \mathbb{R}^2$. It is possible to verify that

$$\int_e b_e \, ds = 1,$$

and so, if we set $\boldsymbol{c}_e = \int_e \boldsymbol{v} \, ds$, then

$$\int_e \boldsymbol{v} \, ds = \int_e \boldsymbol{c}_e b_e \, ds.$$

Therefore, we can define

$$\Pi_h \boldsymbol{v} = \sum_e \boldsymbol{c}_e b_e.$$

This will satisfy the condition $b(\Pi_h \boldsymbol{v}, q_h) = b(\boldsymbol{v}, q_h)$. Stability of Π_h can be proved using Cauchy–Schwarz, inverse inequality, and trace inequality. This construction shows that $\mathcal{P}^2$–$\mathcal{P}^1$ is a stable pair for Stokes. There are "better" choices of spaces that have more advantageous properties (better convergence rates, exactly divergence free velocities, etc).

The following theorems give stability and error estimates for the finite element discretization.

Theorem 3. *Suppose the variational problem satisfies the conditions for well-posedness (let C_a and C_b denote the continuity constants for the bilinear forms, α the coercivity constant for $a(\cdot,\cdot)$ on the kernel Z, and β the inf-sup constant for b). Let (u,p) denote the unique solution. Then,*

$$\begin{aligned} \|u\|_V &\leq \frac{1}{\alpha}\|\ell\|_{V'} + \frac{2C_a}{\alpha\beta}\|\xi\|_{P'} \\ \|p\|_P &\leq \frac{2C_a}{\alpha\beta}\|\ell\|_{V'} + \frac{2C_a^2}{\alpha\beta^2}\|\xi\|_{P'} \end{aligned}$$

Proof. Let $B : V \to P'$ denote the mapping $v \mapsto b(v,\cdot)$. The inf-sup condition implies that B' (the adjoint of B) is injective. From some functional analysis (Closed Range Theorem), this implies that B is surjective. This means that there is some u_ξ such that $Bu_\xi = \xi \in P'$, i.e.

$$b(u_\xi, q) = \xi(q)$$

for all $q \in P$. Then,

$$\begin{aligned} \|\xi\|_{Q'} &= \sup_{q\in Q} \frac{|\xi(q)|}{\|q\|_P} \\ &= \sup_{q\in Q} \frac{|b(u_\xi, q)|}{\|q\|_P} \\ &= \|u_\xi\|_V \sup_{q\in Q} \frac{|b(u_\xi, q)|}{\|u_\xi\|_V \|q\|_P} \\ &\geq \beta \|u_\xi\|_V \end{aligned}$$

and so

$$\|u_\xi\|_V \leq \frac{1}{\beta}\|\xi\|_{Q'}$$

Then, the problem

$$\begin{aligned} a(u,v) + b(v,p) &= \ell(v) \\ b(u,q) &= \xi(q) \end{aligned}$$

is equivalent to finding u_z with $u = u_z + u_\xi$ such that

$$\begin{aligned} a(u_z, v) + b(v,p) &= \ell(v) - a(u_\xi, v) =: \ell'(v) \\ b(u_z, q) &= 0 \end{aligned}$$

The Lax-Milgram theorem implies the existence of such a $u_z \in Z$ satisfying

$$\begin{aligned}
\|u_z\|_V &\leq \frac{1}{\alpha}\|\ell'\|_{V'} \\
&\leq \frac{1}{\alpha}\left(\|\ell\|_{V'} + \sup_{v\in Z}\frac{a(u_\xi, v)}{\|v\|_v}\right) \\
&\leq \frac{1}{\alpha}\|\ell\|_{V'} + \frac{C_a}{\alpha}\|u_\xi\|_V \\
&\leq \frac{1}{\alpha}\|\ell\|_{V'} + \frac{C_a}{\alpha\beta}\|\xi\|_{Q'}
\end{aligned}$$

Then, given $u = u_z + u_\xi$, we have

$$\|u\|_V \leq \|u_z\|_V + \|u_\xi\|_V \leq \frac{1}{\alpha}\|\ell\|_{V'} + \frac{2C_a}{\alpha\beta}\|\xi\|_{Q'}$$

proving the first stability result.

(Why? In the above, we used that $1/\beta \leq C_a/(\alpha\beta)$ which follows from $\alpha < C_a$. This can be seen from $\alpha\|v\|_V^2 \leq a(v,v) \leq C_a\|v\|_V^2$). ☐

Theorem 4. *Let $V_h \subseteq V$ and $P_h \subseteq P$. Suppose the conditions hold as above. Then,*

$$\begin{aligned}
\|u - u_h\|_V &\leq \frac{4C_aC_b}{\alpha\beta}\inf_v \|u - v\|_V + \frac{C_b}{\alpha}\inf_q \|p - q\|_P \\
\|p - p_h\|_P &\leq \frac{4C_a^2C_b}{\alpha\beta^2}\inf_v \|u - v\|_V + \frac{3C_aC_b}{\alpha\beta}\inf_q \|p - q\|_P
\end{aligned}$$

Proof. We first note the following version of Galerkin orthogonality for the mixed problem

$$\begin{aligned}
a(u - u_h, v_h) + b(v_h, p - p_h) &= 0 \\
b(u - u_h, q_h) &= 0
\end{aligned}$$

for all $v_h \in V_h, q_h \in P_h$.

Let $(w, r) \in V \times P$ be arbitrary. Setting $u = u + w - w$ and $p = p + r - r$, we obtain

$$\begin{aligned}
a(w - u_h, v_h) + b(v_h, r - p_h) &= a(w - u, v_h) + b(v_h, r - p) =: f_{w,r}w, r(v_h) \\
b(w - u_h, q_h) &= b(w - u, q_h) =: g_w(q_h)
\end{aligned}$$

The above stability result implies that (dropping subscripts)

$$\begin{aligned}
\|u_h - w\|_V &\leq \frac{1}{\alpha}\|f\|_{V'} + \frac{2C_a}{\alpha\beta}\|g\|_{P'} \\
\|p_h - r\|_P &\leq \frac{2C_a}{\alpha\beta}\|f\|_{V'} + \frac{2C_a^2}{\alpha\beta^2}\|g\|_{P'}
\end{aligned}$$

Continuity of a and b implies that

$$\begin{aligned}
\|f_{w,r}\|_{V'} &= \sup_v \frac{a(w-u, v)}{\|v\|_V} + \sup_v \frac{b(v, r-p)}{\|v\|_V} \leq C_a\|u - w\|_V + C_b\|p - r\|_P \\
\|g_w\|_{P'} &= \sup_q \frac{b(w-u, q)}{\|q\|_P} \leq C_b\|u - w\|_V
\end{aligned}$$

Putting these together

$$\begin{aligned}
\|u_h - w\|_V &\le \frac{1}{\alpha}(C_a\|u-w\|_V + C_b\|p-r\|_P) + \frac{2C_a}{\alpha\beta}C_b\|u-w\|_V \\
&\le 3\frac{2C_aC_b}{\alpha\beta}\|u-w\|_V + \frac{C_b}{\alpha}\|p-r\|_P
\end{aligned}$$

and

$$\begin{aligned}
\|p_h - r\|_P &\le \frac{2C_a}{\alpha\beta}(C_a\|u-w\|_V + C_b\|p-r\|_P) + \frac{2C_a^2}{\alpha\beta^2}C_b\|u-w\|_V \\
&\le 4\frac{2C_a^2C_b}{\alpha\beta^2}\|u-w\|_V + \frac{2C_aC_b}{\alpha\beta}\|p-r\|_P
\end{aligned}$$

Finally,

$$\begin{aligned}
\|u - u_h\|_V &\le \|u-w\|_V + \|u_h - w\|_V \\
\|p - p_h\|_P &\le \|p-r\|_P + \|p_h - r\|_P
\end{aligned}$$

and (since this holds for **any** $w \in V$, and $r \in P$), the result follows by taking the infimum. $\square$