Preconditioning SPD Problems

In the previous lecture, we saw that saddle-point systems of the form

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}$$

where A and C are symmetric and positive-definite can be solved using the MINRES Krylov method. The block-diagonal preconditioner

$$\mathcal{B} = \begin{pmatrix} A^{-1} & 0 \\ 0 & -S^{-1} \end{pmatrix}$$

for Schur complement $S = -C - BA^{-1}B^T$ gives a *uniformly well-conditioned system*, in the sense that the eigenvalues λ of $\mathcal{B}\mathcal{A}$ satisfy

$$\lambda \in \left[-1, (1-\sqrt{5})/2\right) \cup \left[1, (1+\sqrt{5})/2\right).$$

This bound is **uniform** meaning that it does not depend on the spectrum or condition number of the blocks of $\mathcal{A}$. In particular, even if A is ill-conditioned (defined on a very fine finite element mesh, for example), then $\mathcal{B}\mathcal{A}$ will be well-conditioned.

The preconditioner $\mathcal{B}$ is not practical, because, in general, none of A^{-1}, S, or S^{-1} can be formed efficiently. We can instead replace A^{-1} with B_A, a preconditioner for A, and replace $-S^{-1}$ with B_S, a preconditioner for $-S$, to obtain the block-diagonal preconditioned

$$\widetilde{\mathcal{B}} = \begin{pmatrix} B_A & 0 \\ 0 & B_S \end{pmatrix}.$$

We have therefore reduced the problem of finding a good preconditioner for an indefinite saddle-point system to the problem of finding good preconditioners for symmetric positive-definite systems.

1 Spectral Equivalence

In the following, A will be a SPD matrix. We say that A is spectrally equivalent to M (another SPD matrix) with constants c and C if

$$c\boldsymbol{x}^T M \boldsymbol{x} \leq \boldsymbol{x}^T A \boldsymbol{x} \leq C \boldsymbol{x}^T M \boldsymbol{x} \tag{1}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$.

Proposition 1. *If A and M are spectrally equivalence with constants c and C, then the spectrum of $M^{-1}A$ is contained in $[c, C]$ (and so $\kappa(M^{-1}A) \leq C/c$).*

Proof. Let $\boldsymbol{v}$ be an eigenvector of $M^{-1}A$, $M^{-1}A\boldsymbol{v} = \lambda\boldsymbol{v}$. Then,

$$A\boldsymbol{v} = \lambda M\boldsymbol{v}$$

and so

$$\boldsymbol{v}^T A\boldsymbol{v} = \lambda\boldsymbol{v}^T M\boldsymbol{v}$$

giving

$$\lambda = \frac{\boldsymbol{v}^T A\boldsymbol{v}}{\boldsymbol{v}^T M\boldsymbol{v}}.$$

from which (together with (1)) it follows that

$$c \leq \lambda \leq C. \qquad \square$$

This means that if M is spectrally equivalent to A (with constants such that C/c is not too large), then $B = M^{-1}$ is a good preconditioner for A. The goal in constructing a preconditioner is, given A, to find a matrix or linear operator M such that:

- A and M are spectrally equivalent
- The action of M^{-1} is "easy" to compute

These two criteria are somewhat in tension. We can take two extremes. If $M = I$, then the action of M^{-1} is trivial to compute (it is a "no-op"), but the constants are $c = \lambda_{\min}$ and $C = \lambda_{\max}$, which result in no improvement at all. On the other hand, if $M = A$, then $c = C = 1$, and the preconditioned system $M^{-1}A = I$ has condition number 1; however, computing the action of $M^{-1} = A^{-1}$ is the same as the original problem. A *good preconditioner* is (much) cheaper to compute than A^{-1}, while simultaneously satisfying $\kappa(M^{-1}A) \ll \kappa(A)$. We want to trade some modest amount of computational effort for a significant improvement in conditioning.

Typically, in finite element problems, the goal is to find a preconditioner such that the constants of equivalence c and C are *independent of discretization parameters*, most importantly, independent of the mesh size h. Such preconditioners are called **uniform**.

Recall that the conditioner number of the stiffness matrix scales as $\kappa(A) = \mathcal{O}(h^{-2})$. If we can find a uniform preconditioner B, then $\kappa(BA) = \mathcal{O}(1)$, and we can solve the finite element problems in a constant number of iterations, *independent of mesh size*. The construction of such preconditioners is usually problem-dependent. We have to leverage some specific structure of the problem giving rise to the matrix A in order to construct a good preconditioner. For elliptic problems, two common (related) approaches are **domain decomposition** and **multigrid**.

2 Domain Decomposition

Domain decomposition methods are sometimes also called **subspace correction** methods. We will describe here the "additive" version of these preconditioners. These sometimes called **additive Schwarz** preconditioners or **parallel subspace correction** preconditioners; both terms refer to the same concept.

Suppose that A is SPD with respect to the inner product $(\cdot\,,\cdot)$ defined on the finite-dimensional space V. (For the stiffness matrix, the space V can be the finite element space V_h, and the inner product can be the L^2 inner product). We conditioner a decomposition of V into $J+1$ subspaces,

$$V = \sum_{j=0}^{J} V_j.$$

This decomposition will define the preconditioner.

For each j, let $Q_j : V \to V_j$ be the **projection** in the inner product $(\cdot\,,\cdot)$ onto V_j,

$$(Q_j u, v_j) = (u, v_j) \qquad \text{for all } v_j \in V_j,$$

and let $P_j : V \to V_j$ be the **elliptic projection** onto V_j,

$$(AP_j u, v_j) = (Au, v_j) \qquad \text{for all } v_j \in V_j.$$

Let $A_j : V_j \to V_j$ be the restriction of A onto V_j defined by

$$(A_j u_j, v_j) = (Au_j, v_j) \qquad \text{for all } v_j \in V_j.$$

We can compute

$$\begin{aligned}(A_j P_j u, v_j) &= (AP_j u, v_j)\\ &= (Au, v_j)\\ &= (Q_j Au, v_j),\end{aligned}$$

which holds for all $u \in V$ and $v_j \in V_j$, from which we derive the identity

$$A_j P_j = Q_j A,$$

or, equivalently,

$$P_j = A_j^{-1} Q_j A$$

The **preconditioned system** $P = BA$ is defined by

$$P = \sum_{j=0}^{J} P_j = \sum_{j=0}^{J} A_j^{-1} Q_j A = \left(\sum_{j=0}^{J} A_j^{-1} Q_j \right) A,$$

and so the "additive Schwarz" or "parallel subspace correction" preconditioner associated with the decomposition $V = \sum V_j$ is

$$B := \sum_{j=0}^{J} A_j^{-1} Q_j.$$

This construction allows us to precondition a large $N \times N$ system by solving a sequence of J smaller systems, each of size $\dim(V_j)$.

2.1 Note on Implementation

Consider $A \in \mathbb{R}^{N \times N}$, and suppose that each subspace V_j is given by

$$V_j = \operatorname{span}\{\boldsymbol{e}_i : i \in \mathcal{I}_j\},$$

in other words, V_j is the subspace spanned by a subset $\mathcal{I}_j$ of the standard basis vectors. Then, the projection Q_j simply extracts the subvector with indices in $\mathcal{I}_j$, and the operator A_j is the submatrix obtained by selecting rows and columns in $\mathcal{I}_j$. The simplest such decomposition is

$$V = \sum_{j=1}^{N} V_j, \qquad V_j := \operatorname{span}\{\boldsymbol{e}_j\}.$$

Here, there are N subspaces, each one-dimensional. In this case, $A_j = a_{jj}$ and $Q_j \boldsymbol{u} = u_j \boldsymbol{e}_j$. The preconditioner B is simply diagonal scaling with the reciprocal of the diagonal of A (i.e. Jacobi preconditioning).

More generally, for each subspace V_j, we have the inclusion $I_j : V_j \hookrightarrow V$. The projection Q_j with respect to the ℓ^2 inner product is simply I_j^T. If $A_j : V_j \to V_j$ is the restriction of A to V_j, then the preconditioner B is given by

$$B = \sum_{j=0}^{J} I_j A_j^{-1} I_j^T.$$

2.2 Conditioning

Note that if the subspaces V_j are themselves large, we might want to replace A_j^{-1} with a preconditioner B_j, to obtain

$$\widetilde{B} := \sum_{j=0}^{J} B_j Q_j.$$

This can be viewed as approximating the projections $P_j = A_j^{-1} Q_j A$ with approximations $T_j = B_j Q_j A$, to obtain

$$T := \sum_{j=0}^{J} T_j = \widetilde{B} A.$$

We wish to estimate the conditioner number of T (which is the preconditioned system $\widetilde{B}A$).

Lemma 1. *If the approximate projections T_j are symmetric and positive-definite on V_j, then*

$$(T^{-1} v, v) = \inf_{\sum v_j = 0} \sum_{j=0}^{J} (T_j^{-1} v_j, v_j). \tag{2}$$

The infimum is taken over all decompositions $v = \sum_{j=0}^{J} v_j$ with $v_j \in V_j$.

Proof. Let $v \in V$ and define $v_j = T_j T^{-1} v$, so that

$$\sum_{j=0}^{J} v_j = \sum_{j=0}^{J} T_j T^{-1} v = \left(\sum_{j=0}^{J} T_j \right) T^{-1} v = v.$$

Then,

$$\begin{aligned}
\inf_{\sum v_j = 0} \sum_{j=0}^{J} (T_j^{-1} v_j, v_j) &= \inf_{\sum w_j = 0} \sum_{j=0}^{J} (T_j^{-1}(v_j + w_j), v_j + w_j) \\
&= \inf_{\sum w_j = 0} \sum_{j=0}^{J} (T^{-1} v + T_j^{-1} w_j, v_j + w_j) \\
&= \inf_{\sum w_j = 0} \sum_{j=0}^{J} \big((T^{-1} v, v_j) + (T^{-1} v, w_j) + (T_j^{-1} w_j, v_j) + (T_j^{-1} w_j, w_j) \big) \\
&= (T^{-1} v, v) + \inf_{\sum w_j = 0} \sum_{j=0}^{J} \big((T_j^{-1} w_j, v_j) + (T_j^{-1} w_j, w_j) \big) \\
&= (T^{-1} v, v) + \inf_{\sum w_j = 0} \sum_{j=0}^{J} \big((w_j, T v) + (T_j^{-1} w_j, w_j) \big) \\
&= (T^{-1} v, v) + \inf_{\sum w_j = 0} \sum_{j=0}^{J} (T_j^{-1} w_j, w_j) \\
&= (T^{-1} v, v).
\end{aligned}$$

□

Note the special case when the projections are exact, i.e. $T_j = P_j$. Then, $P_j|_{V_j} = I$, and

$$(T^{-1} v, v) = \inf_{\sum v_j = 0} \sum_{j=0}^{J} (v_j, v_j).$$

Recall that $T_j = B_j Q_j A$. If B_j is spectrally equivalent to A_j^{-1}, then

$$(T_j^{-1} v_j, v_j) \approx (v_j, v_j),$$

and

$$(T^{-1} v, v) \approx \inf_{\sum v_j = 0} \sum_{j=0}^{J} (v_j, v_j).$$

It will usually be convenient to work in the inner product induced by the bilinear form $a(\cdot\,, \cdot)$, giving

$$a(T^{-1} v, v) \approx \inf_{\sum v_j = 0} \sum_{j=0}^{J} a(v_j, v_j).$$

The above arguments show that estimating the condition number of the preconditioned system is reduced to bounding above and below the quantity

$$\inf_{\sum v_j = 0} \sum_{j=0}^{J} a(v_j, v_j).$$

These will give bounds on the extremal eigenvalues of T^{-1} (and hence on the preconditioned operator T).