

## Discontinuous Galerkin Method for Advection

The discontinuous Galerkin (DG) method can be used to discretization the advection equation (and other “advection-dominated” equations). The DG method uses the finite element space

$$V_h = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{P}^p(\kappa)\}$$

consisting of all piecewise polynomials of degree  $p$  defined on a mesh  $\mathcal{T} = \{\kappa\}$ . Note that  $V_h$  is a strict superset of the conforming finite space. As opposed to the  $H^1$ -conforming finite element method, elements of  $V_h$  are not generally continuous, and so do not belong to the space  $H^1(\Omega)$ . Instead, they belong to a so-called **broken Sobolev space**. The  $m$ th order **broken Sobolev space**  $H^m(\mathcal{T})$  (which is always defined in terms of a mesh  $\mathcal{T}$ ) is defined by

$$H^m(\mathcal{T}) := \{v \in L^2(\Omega) : v|_\kappa \in H^m(\kappa)\}.$$

Let  $D^\alpha$  denote the order- $\alpha$  weak derivative operator ( $\alpha$  is a multi-index,  $|\alpha| \leq m$ ). Then, the **broken weak derivative** operator  $D_h^\alpha$  is defined on  $H^m(\mathcal{T})$  by

$$(D_h^\alpha u)|_\kappa := D^\alpha(u|_\kappa)$$

for all  $\kappa \in \mathcal{T}$ . In particular, we will often use the “broken gradient operator”  $\nabla_h$  and “broken divergence operator” ( $\nabla_h \cdot$ ).

## 1 Variational Formulation for Linear Advection

As before, we consider the steady linear advection equation

$$(*) \begin{cases} \nabla \cdot (\beta u) + cu = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_{\text{in}}. \end{cases}$$

As before, we assume that  $c + \frac{1}{2} \nabla \cdot \beta \geq \gamma_0 > 0$ . We derive a variational formulation defined on the broken space  $H^1(\mathcal{T})$ . As usual, we multiply by a test function  $v \in H^1(\mathcal{T})$  and integrate over a **single element**  $\kappa$ , obtaining

$$\int_\kappa (\nabla \cdot (\beta u) + cu) v \, dx = \int_\kappa f v \, dx.$$

The terms appearing above are well-defined since  $u$  and  $v$  restricted to  $\kappa$  are in  $H^1(\kappa)$ . We then integrate by parts,

$$-\int_\kappa u \beta \cdot \nabla v \, dx + \int_{\partial\kappa} uv \beta \cdot \mathbf{n} \, ds + \int_\kappa cuv \, dx = \int_\kappa f v \, dx. \quad (1)$$

Since  $u|_\kappa \in H^1(\kappa)$ , the trace of  $u$  on  $\partial\kappa$  from within  $\kappa$  is well-defined, but  $u$  is potentially discontinuous across element boundaries, and so  $u$  does not have a single-valued trace on  $\partial\kappa$ .

Note that for each element  $\kappa$ ,  $\partial\kappa$  can be written as the union of edges  $\partial\kappa = \bigcup e$ . Each edge is incident to two elements, which we will denote  $\kappa^+$  and  $\kappa^-$ . We will use the convention that  $\kappa = \kappa^-$ , so  $\kappa^+$  is used to denote the “neighbor” element. The trace of  $u$  on  $e$  from within  $\kappa^+$  is denoted  $u^+$ , and the trace of  $u$  on  $e$  from within  $\kappa^-$  is denoted  $u^-$ . The outward normal from  $\kappa^+$  is denoted  $\mathbf{n}^+$ , and similarly for  $\mathbf{n}^-$ . We write the boundary integral as the sum over edges,

$$\int_{\partial\kappa} uv\boldsymbol{\beta} \cdot \mathbf{n} \, ds = \sum_{e \subseteq \partial\kappa} \int_e u^- v^- \boldsymbol{\beta} \cdot \mathbf{n}^- \, ds.$$

**We now make the key modification to (1).** We replace the term  $u^- \boldsymbol{\beta}$  in the integral above with a **numerical flux**  $\widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-)$ . We require that the numerical flux function  $\widehat{\mathbf{F}}(\cdot, \cdot, \cdot)$  be **consistent** and **conservative**; these two properties are defined as follows.

- **Consistent:**  $\widehat{\mathbf{F}}(u, u, \mathbf{n}) = u\boldsymbol{\beta}$  for all  $u, \mathbf{n}$ .
- **Conservative:**  $\widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-) = \widehat{\mathbf{F}}(u^+, u^-, \mathbf{n}^+)$  (recalling  $\mathbf{n}^+ = -\mathbf{n}^-$ ).

Having made this replacement, (1) becomes

$$-\int_{\kappa} u\boldsymbol{\beta} \cdot \nabla v \, dx + \int_{\partial\kappa} \widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-) \cdot \mathbf{n}^- v^- \, ds + \int_{\kappa} cuv \, dx = \int_{\kappa} fv \, dx.$$

Summing over all elements  $\kappa \in \mathcal{T}$ , we obtain

$$-\int_{\Omega} u\boldsymbol{\beta} \cdot \nabla_h v \, dx + \sum_{\kappa} \int_{\partial\kappa} \widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-) \cdot \mathbf{n}^- v^- \, ds + \int_{\Omega} cuv \, dx = \int_{\Omega} fv \, dx.$$

Each boundary edge will be included once in the above sum, and each interior edge (i.e. each edge incident to two mesh elements) will be included twice, giving rise to sums

$$\int_e \widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-) \cdot \mathbf{n}^- v^- \, ds + \int_e \widehat{\mathbf{F}}(u^+, u^-, \mathbf{n}^+) \cdot \mathbf{n}^+ v^+ \, ds.$$

Since the numerical flux is conservative, we can combine these terms to obtain the single integral

$$\int_e \widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-) \cdot (\mathbf{n}^- v^- + \mathbf{n}^+ v^+) \, ds.$$

The term involving the test functions in the above integral is known as the **jump** of  $v$ . This is denoted using the special notation

$$[[v]] := \mathbf{n}^- v^- + \mathbf{n}^+ v^+.$$

On boundary edges, there is a single well-defined trace and one outward-pointing normal, and so we can set

$$[[v]] := \mathbf{n}v \quad (\text{on } \partial\Omega).$$

Let  $\Gamma$  denote the **mesh skeleton**; that is, the union of all the edges in the mesh  $\mathcal{T}$ . We can write  $\Gamma = \Gamma_0 \cup \Gamma_{\partial}$ , where  $\Gamma_0$  consists of all the interior edges, and  $\Gamma_{\partial}$  consists of all the

boundary edges. On the boundary edges, it doesn't make sense to use  $\widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-)$  since there is only one trace; we will use  $\widehat{\mathbf{F}}_\partial(u, \mathbf{n})$ , which will be defined as in the weak enforcement of boundary conditions for the conforming finite element method. Then, the DG method becomes

$$-\int_{\Omega} u \boldsymbol{\beta} \cdot \nabla_h v \, dx + \int_{\Gamma_0} \widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-) \cdot \llbracket v \rrbracket \, ds + \int_{\Gamma_\partial} \widehat{\mathbf{F}}_\partial(u, \mathbf{n}) \cdot \llbracket v \rrbracket \, ds + \int_{\Omega} cuv \, dx = \int_{\Omega} fv \, dx.$$

To unify notation, we will use the shorthand  $\widehat{\mathbf{F}}$  to mean either  $\widehat{\mathbf{F}}(u^-, u^+, \mathbf{n}^-)$  or  $\widehat{\mathbf{F}}_\partial(u, \mathbf{n})$ , depending on whether it is evaluated on an interior or boundary edge. With this simplification, we obtain the following.

The discontinuous Galerkin variational formulation for the steady advection–reaction equation is: find  $u \in H^1(\mathcal{T})$  such that

$$-\int_{\Omega} u \boldsymbol{\beta} \cdot \nabla_h v \, dx + \int_{\Gamma} \widehat{\mathbf{F}} \cdot \llbracket v \rrbracket \, ds + \int_{\Omega} cuv \, dx = \int_{\Omega} fv \, dx \quad (2)$$

for all  $v \in H^1(\mathcal{T})$ , where  $\widehat{\mathbf{F}}$  is a numerical flux function. Let  $a(\cdot, \cdot)$  denote the bilinear form appearing on the left-hand side of the above equality.

**Proposition 1.** *The variational formulation (2) is **consistent** (meaning that if  $u$  is solution to  $(*)$ , then  $a(u, v) = F(v)$  for all  $v \in H^1(\mathcal{T})$ ) if and only if  $\widehat{\mathbf{F}}$  is a consistent numerical flux.*

### 1.1 Mean-value and upwind fluxes

We now give some examples of consistent and conservative fluxes. The **mean-value flux** (also called the **central flux**) is defined by

$$\widehat{\mathbf{F}}_{\text{mv}}(u^-, u^+, \mathbf{n}^-) = \frac{u^- + u^+}{2} \boldsymbol{\beta}.$$

The **average** of  $u$  also has a special notation,

$$\{u\} := \frac{u^- + u^+}{2}$$

(and  $\{u\} := u$  on  $\partial\Omega$ ). While the mean-value flux may seem to be a very natural choice, it is not a good choice for DG discretizations.

The **upwind flux** is defined by

$$\widehat{\mathbf{F}}_{\text{uw}}(u^-, u^+, \mathbf{n}^-) = \begin{cases} u^- \boldsymbol{\beta}, & \boldsymbol{\beta} \cdot \mathbf{n}^- \geq 0, \\ u^+ \boldsymbol{\beta}, & \boldsymbol{\beta} \cdot \mathbf{n}^- < 0. \end{cases}$$

The motivation behind the upwind flux is the same as in finite volume methods. We note that both the mean-value flux and the upwind flux can be written in the form of a “**general flux**”. The general flux is

$$\widehat{\mathbf{F}}_{b_0}(u^-, u^+, \mathbf{n}^-) = (\{u\} + \mathbf{b}_0 \cdot \llbracket u \rrbracket) \boldsymbol{\beta}.$$

Clearly, for  $\mathbf{b}_0 = 0$ , the mean-value flux is recovered. When  $\mathbf{b}_0$  is defined by

$$\mathbf{b}_0 = \begin{cases} \frac{1}{2}\mathbf{n}^-, & \boldsymbol{\beta} \cdot \mathbf{n}^- \geq 0, \\ \frac{1}{2}\mathbf{n}^+, & \boldsymbol{\beta} \cdot \mathbf{n}^- < 0, \end{cases}$$

the upwind flux is recovered. This can also be written as  $\mathbf{b}_0 = \frac{1}{2}\mathbf{n}^- \operatorname{sgn}(\boldsymbol{\beta} \cdot \mathbf{n}^-)$ . We generally write  $\mathbf{b}_0 = b_0 \mathbf{n}^-$ .

Our goal is to show some coercivity and continuity results that will let us analyze this method. For simplicity, we will work with periodic boundary conditions, so that  $\Gamma_\partial = \emptyset$ .

**Theorem 1.** *With  $\widehat{\mathbf{F}} = \widehat{\mathbf{F}}_{b_0}$ , we have that*

$$a(u, u) \geq \gamma_0 \|u\|^2 + \int_{\Gamma} \mathbf{b}_0 \cdot \boldsymbol{\beta} \llbracket u \rrbracket^2 ds.$$

*Proof.* First, note that

$$u \boldsymbol{\beta} \cdot \nabla u = \frac{1}{2} \boldsymbol{\beta} \cdot \nabla u^2 = \frac{1}{2} \nabla \cdot (\boldsymbol{\beta} u^2) - \frac{1}{2} u^2 \nabla \cdot \boldsymbol{\beta}.$$

Therefore,

$$\begin{aligned} - \int_{\kappa} u \boldsymbol{\beta} \cdot \nabla u dx &= -\frac{1}{2} \int_{\kappa} \nabla \cdot (\boldsymbol{\beta} u^2) dx + \frac{1}{2} \int_{\kappa} u^2 \nabla \cdot \boldsymbol{\beta} dx \\ &= -\frac{1}{2} \int_{\partial \kappa} \boldsymbol{\beta} \cdot \mathbf{n} u^2 ds + \frac{1}{2} \int_{\kappa} u^2 \nabla \cdot \boldsymbol{\beta} dx. \end{aligned}$$

Summing over all  $\kappa \in \mathcal{T}$ ,

$$- \int_{\Omega} u \boldsymbol{\beta} \cdot \nabla u dx = -\frac{1}{2} \int_{\Gamma} \boldsymbol{\beta} \cdot \llbracket u^2 \rrbracket ds + \frac{1}{2} \int_{\Omega} u^2 \nabla \cdot \boldsymbol{\beta} dx.$$

We compute

$$\begin{aligned} a(u, u) &= - \int_{\Omega} u \boldsymbol{\beta} \cdot \nabla_h u dx + \int_{\Gamma} \widehat{\mathbf{F}} \cdot \llbracket u \rrbracket ds + \int_{\Omega} c u^2 dx \\ &= - \int_{\Omega} u \boldsymbol{\beta} \cdot \nabla_h u dx + \int_{\Gamma} (\{u\} + \mathbf{b}_0 \cdot \llbracket u \rrbracket) \boldsymbol{\beta} \cdot \llbracket u \rrbracket ds + \int_{\Omega} c u^2 dx \\ &= -\frac{1}{2} \int_{\Gamma} \boldsymbol{\beta} \cdot \llbracket u^2 \rrbracket ds + \int_{\Gamma} (\{u\} + \mathbf{b}_0 \cdot \llbracket u \rrbracket) \boldsymbol{\beta} \cdot \llbracket u \rrbracket ds + \int_{\Omega} (c + \frac{1}{2} \nabla \cdot \boldsymbol{\beta}) u^2 dx. \end{aligned}$$

Now notice that

$$\begin{aligned} \{u\} \llbracket u \rrbracket &= \frac{u^- + u^+}{2} (u^- \mathbf{n}^- + u^+ \mathbf{n}^+) \\ &= \frac{u^- + u^+}{2} (u^- - u^+) \mathbf{n}^- \\ &= \frac{1}{2} ((u^-)^2 - (u^+)^2) \mathbf{n}^- \\ &= \frac{1}{2} \llbracket u^2 \rrbracket, \end{aligned}$$

(this is analogous to a “product rule” for jumps), so

$$-\frac{1}{2} \int_{\Gamma} \boldsymbol{\beta} \cdot \llbracket u^2 \rrbracket dx + \int_{\Gamma} \{u\} \boldsymbol{\beta} \cdot \llbracket u \rrbracket ds = 0.$$

Furthermore,

$$\begin{aligned} \mathbf{b}_0 \cdot \llbracket u \rrbracket \boldsymbol{\beta} \cdot \llbracket u \rrbracket &= b_0(u^- - u^+) \boldsymbol{\beta} \cdot \mathbf{n}^- (u^- - u^+) \\ &= b_0 \boldsymbol{\beta} \cdot \mathbf{n}^- (u^- - u^+)^2 \\ &= b_0 \boldsymbol{\beta} \cdot \mathbf{n}^- \llbracket u \rrbracket^2, \end{aligned}$$

slightly abusing notation to write  $\llbracket u \rrbracket^2 = \llbracket u \rrbracket \cdot \llbracket u \rrbracket = (u^- - u^+)^2$ . Therefore,

$$a(u, u) = \int_{\Gamma} \mathbf{b}_0 \cdot \boldsymbol{\beta} \llbracket u \rrbracket^2 ds + \int_{\Omega} (c + \tfrac{1}{2} \nabla \cdot \boldsymbol{\beta}) u^2 dx.$$

□