Lecture Note #8 **MTH653: Advanced Numerical Analysis**

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$\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ spaces

1 Some Motivating Examples

1.1 Darcy's equations

Recall **Darcy's equations** that model flow through porous media. The unknowns are $\boldsymbol{u}$ (the velocity vector) and p (the pressure). The governing equations are

$$\begin{aligned} \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= g, \end{aligned} \tag{1}$$

with appropriate boundary conditions on $\partial\Omega$. To derive the variational formulation, multiply the first equation by a vector-valued test function and the second equation by a scalar test function, obtaining

$$\begin{aligned} (\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \\ (\nabla \cdot \boldsymbol{u}, q) &= (g, q). \end{aligned}$$

Notice that there are no derivatives acting on p and q, so we can choose $p, q \in L^2(\Omega)$. While we do have derivatives acting on $\boldsymbol{u}$ and $\boldsymbol{v}$, we do not need the whole gradient—we only need the *divergence*. Therefore, it would make sense to choose the minimal regularity space

$$\boldsymbol{u}, \boldsymbol{v} \in [L^2(\Omega)]^d \quad \text{and} \quad \nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v} \in L^2(\Omega).$$

This space is *strictly larger* than $[H^1(\Omega)]^d$.

1.2 Maxwell equations

Maxwell's equations, which govern the physics of electromagnetic fields, are given by

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\ \nabla \times \boldsymbol{H} &= \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}, \\ \nabla \cdot \boldsymbol{D} &= \rho, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \cdot \boldsymbol{J} &= -\frac{\partial \rho}{\partial t}. \end{aligned}$$

In the above, we use the notation:

- $\boldsymbol{E}$ is the electric field intensity

- $\boldsymbol{D}$ is the electric flux density
- $\boldsymbol{H}$ is the magnetic field intensity
- $\boldsymbol{B}$ is the magnetic flux density
- $\boldsymbol{J}$ is the electric current density
- ρ is the electric charge density

Under some assumptions (that the fields are *time harmonic*, i.e. that they oscillate with a single frequency ω) and after introducing constitutive relations, the equations can be simplified, eventually reducing to the so-called "vector wave equation",

$$\nabla \times (\nabla \times \boldsymbol{E}) - \omega^2 \boldsymbol{E} = -j\omega \boldsymbol{J}$$

for the electric field, or

$$\nabla \times (\nabla \times \boldsymbol{H}) - \omega^2 \boldsymbol{H} = \nabla \times \boldsymbol{J}$$

for the magnetic field. These equations motivate considering the model problem

$$\nabla \times (\nabla \times \boldsymbol{E}) + \alpha \boldsymbol{E} = \boldsymbol{f}. \tag{2}$$

We derive the variational formulation for this problem. Recall the vector calculus identity

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}).$$

Divergence theorem and then applying the above identity gives

$$\begin{aligned} \int_{\partial\Omega} (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{n} \, ds &= \int_{\Omega} \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) \, dx \\ &= \int_{\Omega} \left(\boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) \right) \, dx. \end{aligned}$$

Rearranging (and using the vector triple product identity $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = -(\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b}$) we obtain the integration by parts formula

$$\boxed{\int_{\Omega} \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) \, dx = \int_{\Omega} \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) \, dx - \int_{\partial\Omega} (\boldsymbol{u} \times \boldsymbol{n}) \cdot \boldsymbol{v} \, ds. \qquad (3)}$$

Multiply (2) by a test function $\boldsymbol{F}$ and integrate over Ω,

$$\int_{\Omega} (\nabla \times \nabla \times \boldsymbol{E}) \cdot \boldsymbol{F} \, dx + \int_{\Omega} \alpha \boldsymbol{E} \cdot \boldsymbol{F} \, dx = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{F} \, dx.$$

Applying the integration by parts formula to to the first integral on the left-hand side (setting $\boldsymbol{u} = \nabla \times \boldsymbol{E}$, $\boldsymbol{v} = \boldsymbol{F}$), we have

$$\int_{\Omega} (\nabla \times \nabla \times \boldsymbol{E}) \cdot \boldsymbol{F} \, dx = \int_{\Omega} \nabla \times \boldsymbol{E} \cdot \nabla \times \boldsymbol{F} \, dx + \int_{\partial\Omega} ((\nabla \times \boldsymbol{E}) \times \boldsymbol{F}) \cdot \boldsymbol{n} \, ds.$$

We often use the boundary conditions that $\boldsymbol{E}$ (and the test function $\boldsymbol{F}$) have **zero tangential trace** on $\partial\Omega$, i.e.

$$\boldsymbol{E} \cdot \boldsymbol{t} = 0 \quad \text{and} \quad \boldsymbol{F} \cdot \boldsymbol{t} = 0 \qquad \text{on } \partial\Omega$$

where $\boldsymbol{t}$ tangent to the boundary. Equivalently,

$$\boldsymbol{E} \times \boldsymbol{n} = 0, \qquad \boldsymbol{F} \times \boldsymbol{n} = 0.$$

With these conditions, note that the integrand of the boundary integral is given by

$$((\nabla \times \boldsymbol{E}) \times \boldsymbol{F}) \cdot \boldsymbol{n} = (\nabla \times \boldsymbol{E}) \cdot (\boldsymbol{F} \times \boldsymbol{n}) = 0.$$

Therefore, the variational formulation for (2) (with zero tangential trace conditions) reduces to

$$(\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F}) + (\alpha \boldsymbol{E}, \boldsymbol{F}) = (\boldsymbol{f}, \boldsymbol{F}).$$

Therefore, the conditions on $\boldsymbol{E}$ and $\boldsymbol{F}$ that we require in order for this variational problem to make sense are

$$\boldsymbol{E}, \boldsymbol{F} \in [L^2(\Omega)]^d \quad \text{and} \quad \nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F} \in [L^2(\Omega)]^d.$$

Again, this space is strictly larger than $[H^1(\Omega)]^d$.

2 Sobolev Spaces

So far, we have been working with the $W^{k,p}(\Omega)$ Sobolev spaces (and more specifically, the $H^k(\Omega) := W^{k,2}(\Omega)$ Hilbert spaces). In finite elements, we commonly work in the space $H^1(\Omega)$. However, for the problems described above, it makes sense to work in spaces that do not possess full H^1 regularity. The natural spaces to work in for the above problems are $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, which are defined as

$$\begin{aligned} \boldsymbol{H}(\mathrm{div}, \Omega) &:= \{\boldsymbol{v} \in [L^2(\Omega)]^d : \nabla \cdot \boldsymbol{v} \in L^2(\Omega)\}, \\ \boldsymbol{H}(\mathbf{curl}, \Omega) &:= \{\boldsymbol{v} \in [L^2(\Omega)]^d : \nabla \times \boldsymbol{v} \in [L^2(\Omega)]^d\}. \end{aligned}$$

Each of these spaces is larger than $[H^1(\Omega)]^d$ but smaller than $[L^2(\Omega)]^d$.

2.1 Continuity requirements

Suppose Ω is partitioned into the union of disjoint subdomains Ω_i (to be precise, the interiors of the subdomains are disjoint).

We had previously seen that if $u \in L^2(\Omega)$ and $u|_{\Omega_i} \in C^1(\Omega_i)$, then

$$u \in H^1(\Omega) \Longleftrightarrow u \in C^0(\Omega).$$

We briefly review this argument. Suppose $u \in H^1(\Omega)$. Then, its weak derivative $\nabla_w u \in [L^2(\Omega)]^d$ is well-defined, and $\nabla_w u|_{\Omega_i} = \nabla(u|_{\Omega_i})$. Multiply $\nabla_w u$ by a smooth test function $\boldsymbol{\phi}$ (supported away from $\partial\Omega$) and integrate over Ω. The weak derivative satisfies

$$\int_\Omega \nabla_w u \cdot \boldsymbol{\phi} \, dx = -\int_\Omega u \nabla \cdot \boldsymbol{\phi} \, dx. \tag{4}$$

Take the left-hand side and integrate by parts over each subdomain,

$$\begin{aligned}\int_\Omega \nabla_w u \cdot \boldsymbol{\phi}\, dx &= \sum_i \int_{\Omega_i} \nabla_w u \cdot \boldsymbol{\phi}\, dx \\ &= \sum_i \left(-\int_{\Omega_i} u \nabla \cdot \boldsymbol{\phi}\, dx + \int_{\partial\Omega_i} u \boldsymbol{\phi} \cdot \boldsymbol{n}\, ds \right) \\ &= -\int_\Omega u \nabla \cdot \boldsymbol{\phi}\, dx + \int_\Gamma [\![u]\!] \cdot \boldsymbol{\phi}\, ds\end{aligned}$$

But by (4), this implies that $\int_\Gamma [\![u]\!] \cdot \boldsymbol{\phi}\, ds = 0$, and we can conclude that the function is continuous. The converse direction follows by showing that the function defined by $\nabla(u|_{\Omega_i})$ on each subdomain satisfies the definition of the weak gradient under the condition that $[\![u]\!] = 0$.

This means that from the point of view of finite elements (where finite element functions u_h restricted to each element $\kappa \in \mathcal{T}$ are smooth), membership in H^1 is determined by continuity at element interfaces. The situation is similar for determining if a finite element function is an element of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$.

2.1.1 Continuity conditions in $\boldsymbol{H}(\mathrm{div})$

A vector-valued function $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ is in $\boldsymbol{H}(\mathrm{div}, \Omega)$ if there exists a function (its **weak divergence**) $\nabla_w \cdot \boldsymbol{v} : \Omega \to \mathbb{R}$, satisfying

$$\int_\Omega (\nabla_w \cdot \boldsymbol{v}) \phi\, dx = -\int_\Omega \boldsymbol{v} \cdot \nabla \phi\, dx,$$

for all smooth test functions ϕ supported away from $\partial\Omega$.

Suppose $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is piecewise-$\boldsymbol{C}^1$ (with respect to the mesh $\mathcal{T}$). Then, $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ if and only if $[\![\boldsymbol{u}]\!] \equiv 0$. (Recall from the lectures on DG diffusion, the jump of a vector is a scalar defined by $[\![\boldsymbol{u}]\!] := \boldsymbol{u}^- \cdot \boldsymbol{n}^- + \boldsymbol{u}^+ \cdot \boldsymbol{n}^+$). First, suppose that $[\![\boldsymbol{u}]\!] = 0$. We claim that $\nabla_h \cdot \boldsymbol{u}$ satisfies the definition of the weak divergence. We compute

$$\begin{aligned}\int_\Omega (\nabla_h \cdot \boldsymbol{u}) \phi\, dx &= \sum_\kappa \int_\kappa (\nabla_h \cdot \boldsymbol{u}) \phi\, dx \\ &= \sum_\kappa \left(\int_{\partial\kappa} \phi \boldsymbol{u} \cdot \boldsymbol{n}\, ds - \int_\kappa \boldsymbol{u} \cdot \nabla \phi\, dx \right) \\ &= \int_\Gamma [\![\boldsymbol{u}]\!] \phi\, ds - \int_\Omega \boldsymbol{u} \cdot \nabla \phi\, dx \\ &= -\int_\Omega \boldsymbol{u} \cdot \nabla \phi\, dx.\end{aligned}$$

Since this holds for all test functions ϕ, we have that $\nabla_w \cdot \boldsymbol{u} = \nabla_h \cdot \boldsymbol{u}$ is well-defined.

On the other hand, if $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}, \Omega)$, then $\nabla_w \cdot \boldsymbol{u}$ is well-defined. We know that, restricted to $\kappa \in \mathcal{T}$,

$$(\nabla_w \cdot \boldsymbol{u})|_\kappa = \nabla \cdot (\boldsymbol{u}|_\kappa) = (\nabla_h \cdot \boldsymbol{u})|_\kappa.$$

By a similar argument to above,

$$\int_\Gamma [\![\boldsymbol{u}]\!]\phi\, ds - \int_\Omega \boldsymbol{u}\cdot\nabla\phi\, dx = \int_\Omega (\nabla_h\cdot\boldsymbol{u})\phi\, dx = \int_\Omega (\nabla_w\cdot\boldsymbol{u})\phi\, dx = \int_\Omega \boldsymbol{u}\cdot\nabla\phi\, dx.$$

From this, we conclude that

$$\int_\Gamma [\![\boldsymbol{u}]\!]\phi\, ds = 0,$$

and since this holds for all test functions, $[\![\boldsymbol{u}]\!] \equiv 0$.

A piecewise-$\boldsymbol{C}^1$ function is in $\boldsymbol{H}(\mathrm{div})$ iff it possesses **normal continuity**.

2.1.2 Continuity conditions in $\boldsymbol{H}(\mathbf{curl})$

A vector-valued function $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ ($d = 2$ or $d = 3$) is in $\boldsymbol{H}(\mathbf{curl})$ if it has a well-defined **weak curl** $\nabla_w \times \boldsymbol{v}$. Note that while the curl in $\mathbb{R}^3$ is vector-valued, in $\mathbb{R}^2$ the curl is the scalar-valued quantity

$$\nabla\times\boldsymbol{v} := \frac{\partial v_w}{\partial x} - \frac{\partial v_1}{\partial y}$$

(which can be interpreted as the z-component of a vector field in the xy-plane). The weak curl is characterized by the integration by parts formula

$$\int_\Omega (\nabla_w\times\boldsymbol{v})\cdot\boldsymbol{\phi}\, dx = \int_\Omega \boldsymbol{v}\cdot(\nabla\times\boldsymbol{\phi})\, dx$$

for all smooth vector-valued test functions $\boldsymbol{\phi}$ supported away from $\partial\Omega$.

If $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is piecewise-$\boldsymbol{C}^1$, then $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ if and only if $[\![\boldsymbol{u}\times\boldsymbol{n}]\!] \equiv 0$ (where $[\![\boldsymbol{u}\times\boldsymbol{n}]\!] = \boldsymbol{u}^-\times\boldsymbol{n}^- + \boldsymbol{u}^+\times\boldsymbol{n}^+$). The argument follows the same structure as in the case of $\boldsymbol{H}(\mathrm{div})$. It can be seen that $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ if and only if

$$\int_\Gamma [\![\boldsymbol{u}\times\boldsymbol{n}]\!]\cdot\boldsymbol{\phi}\, ds = 0$$

for all smooth test functions $\boldsymbol{\phi}$.

As discussed previously, a vector field on a face e can be decomposed into its normal and tangential components,

$$\begin{aligned}\boldsymbol{\phi} &= \boldsymbol{\phi}_\parallel + \boldsymbol{\phi}_\perp \\ &= \boldsymbol{\phi}_\parallel + (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}.\end{aligned}$$

Then,

$$\begin{aligned}\boldsymbol{\phi}\times\boldsymbol{n} &= \boldsymbol{\phi}_\parallel\times\boldsymbol{n} + (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\times\boldsymbol{n} \\ &= \boldsymbol{\phi}_\parallel\times\boldsymbol{n},\end{aligned}$$

and so $[\![\boldsymbol{u}\times\boldsymbol{n}]\!] = 0$ is equivalent to **tangential continuity** of the vector field $\boldsymbol{u}$.

A piecewise-$\boldsymbol{C}^1$ function is in $\boldsymbol{H}(\mathrm{div})$ iff it possesses **tangential continuity**.

3 Finite Element Spaces

In order to construct finite element discretizations of problems like (1) and (2), we must construct finite element spaces that are **subspaces** of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$. Note that since $\boldsymbol{H}^1(\Omega)$ is a subspace of both $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, the standard vector-valued finite element spaces are also subspaces of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, but, for reasons that are perhaps not immediately obvious, these spaces are often not suitable for discretization of problems that are posed naturally in $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$. For example, in Darcy's equations, taking the vector flux $\boldsymbol{u} \in \boldsymbol{V}_h \subseteq \boldsymbol{H}^1(\Omega)$ will lead to discretizations that are not inf-sup stable.

Finite element spaces are constructed by gluing together piecewise polynomial functions; these functions are piecewise-C^∞, and so membership in $\boldsymbol{H}(\mathrm{div})$ or $\boldsymbol{H}(\mathbf{curl})$ is completely characterized by the continuity conditions listed above. For example, we would like to construct a finite-dimensional piecewise-polynomial subspace of $\boldsymbol{H}(\mathrm{div})$, which requires *normal continuity* but permits *tangential discontinuities*. This space can be thought of us somehow "in between" H^1 finite element spaces (which have full C^0 continuity) and L^2 (DG) finite element spaces (which have no continuity conditions).

3.1 Raviart–Thomas Finite Elements

Consider a triangle κ. Define the local lowest-order **Raviart–Thomas** space $\boldsymbol{RT}_0(\kappa)$ by

$$\boldsymbol{RT}_0(\kappa) := \left\{ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x} : \boldsymbol{a} \in \mathbb{R}^2, b \in \mathbb{R} \right\}.$$

In other words, every element of $\boldsymbol{RT}_0(\kappa)$ has the form

$$\boldsymbol{v}(\boldsymbol{x}) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + b \begin{pmatrix} x \\ y \end{pmatrix}.$$

A simple computation shows that $\nabla \cdot \boldsymbol{v} = 2b$. From this, it is simple to see that $\nabla \cdot \boldsymbol{v} = 0$ implies that $\boldsymbol{v}$ is constant. Furthermore, the **normal component** of $\boldsymbol{v} \in \boldsymbol{RT}_0$ is **constant** along each face (edge) of κ. This can be seen as follows. The line containing the edge e is defined by the equation $\boldsymbol{n} \cdot \boldsymbol{x} = c$, for some constant c, where $\boldsymbol{n}$ is the normal vector to e. Then, the normal component of $\boldsymbol{v}$ is $(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$, and

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{n} &= \boldsymbol{a} \cdot \boldsymbol{n} + b\boldsymbol{x} \cdot \boldsymbol{n} \\ &= \boldsymbol{a} \cdot \boldsymbol{n} + bc, \end{aligned}$$

which is a constant.

In order to create a finite element space on all of Ω, we need to **"glue"** the local spaces $\boldsymbol{RT}_0(\kappa)$ together in such a way that the normal components match for every $e \in \Gamma$. The way that we do this is by assigning **degrees of freedom** to each face $e \in \Gamma$, such that the degrees of freedom on a single triangle uniquely determine $\boldsymbol{v}|_\kappa$. In the case of H^1 elements, the natural degrees of freedom were point values. In the case of $\boldsymbol{H}(\mathrm{div})$ elements, it is natural to work in terms of "moments", i.e. integrals of the normal components. Let e_1, e_2, e_3 denote the three edges of a triangle κ, and let c_i denote the integrated normal component,

$$c_i := \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_{e_i} (\boldsymbol{a} \cdot \boldsymbol{n} + bc) \, ds.$$

We claim that these degrees of freedom are **unisolvent**, i.e. the values c_i uniquely determine every element of $\boldsymbol{RT}_0(\kappa)$.

First, we show uniqueness. Suppose that $c_i = 0$ for all i. Then,

$$\begin{aligned}
0 &= \sum_i c_i \\
&= \sum_i \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds \\
&= \int_{\partial\kappa} \boldsymbol{v} \cdot \boldsymbol{n} \, ds \\
&= \int_{\kappa} \nabla \cdot \boldsymbol{v} \, dx \\
&= \int_{\kappa} 2b \, dx \\
&= 2|\kappa| b.
\end{aligned}$$

So, $b = 0$, and $\boldsymbol{v} = \boldsymbol{a}$, a constant. Then,

$$c_i = \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_{e_i} \boldsymbol{a} \cdot \boldsymbol{n} = |e_i| \boldsymbol{a} \cdot \boldsymbol{n},$$

so $\boldsymbol{a}$ is normal to three linearly independent vectors in $\mathbb{R}^2$, and so $\boldsymbol{a} = 0$.

We have defined a map (the "degree of freedom" mapping)

$$C : \boldsymbol{RT}_0(\kappa) \to \mathbb{R}^3$$

defined by

$$C\boldsymbol{v} = (c_1, c_2, c_3).$$

This is a linear map between 3-dimensional spaces. The above argument shows that C is injective, therefore it is a bijection, and every element of $\boldsymbol{RT}_0(\kappa)$ is uniquely determined by its degrees of freedom.

The lowest-order Raviart–Thomas finite element space $\boldsymbol{V}_h$ is defined by

$$\boldsymbol{V}_h = \{ \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) : \boldsymbol{v}|_\kappa \in \boldsymbol{RT}_0(\kappa) \}.$$

This space is constructed by assigning to each edge $e \in \Gamma$ a degree of freedom c. The global basis functions for this space are associated with each edge of the mesh, and are supported in the two triangles containing that edge.

However, note that, in contrast to H^1 finite elements, the local basis functions on a given element κ are **not** given by simply evaluating the basis functions of the reference element with transformed coordinates. In order to transform from reference to physical coordinates, we must introduce the **normal preserving Piola transformation**.