Lecture Note #9 **MTH653: Advanced Numerical Analysis**
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$\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ finite element spaces

1 Finite Element Spaces

In order to construct finite element discretizations of problems like Darcy's equations and the curl-curl (Maxwell-type) problem, we must construct finite element spaces that are **subspaces** of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$. Note that since $\boldsymbol{H}^1(\Omega)$ is a subspace of both $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, the standard vector-valued finite element spaces are also subspaces of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, but, for reasons that are perhaps not immediately obvious, these spaces are often not suitable for discretization of problems that are posed naturally in $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$. For example, in Darcy's equations, taking the vector flux $\boldsymbol{u} \in \boldsymbol{V}_h \subseteq \boldsymbol{H}^1(\Omega)$ will lead to discretizations that are not inf-sup stable.

Finite element spaces are constructed by gluing together piecewise polynomial functions; these functions are piecewise-C^∞, and so membership in $\boldsymbol{H}(\mathrm{div})$ or $\boldsymbol{H}(\mathbf{curl})$ is completely characterized by the continuity conditions listed above. For example, we would like to construct a finite-dimensional piecewise-polynomial subspace of $\boldsymbol{H}(\mathrm{div})$, which requires *normal continuity* but permits *tangential discontinuities*. This space can be thought of us somehow "in between" H^1 finite element spaces (which have full C^0 continuity) and L^2 (DG) finite element spaces (which have no continuity conditions).

1.1 Raviart–Thomas Finite Elements

Consider a triangle κ. Define the local lowest-order **Raviart–Thomas** space $\boldsymbol{RT}_0(\kappa)$ by

$$\boldsymbol{RT}_0(\kappa) := \left\{\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x} : \boldsymbol{a} \in \mathbb{R}^2, b \in \mathbb{R}\right\}.$$

In other words, every element of $\boldsymbol{RT}_0(\kappa)$ has the form

$$\boldsymbol{v}(\boldsymbol{x}) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + b \begin{pmatrix} x \\ y \end{pmatrix}.$$

A simple computation shows that $\nabla \cdot \boldsymbol{v} = 2b$. From this, it is simple to see that $\nabla \cdot \boldsymbol{v} = 0$ implies that $\boldsymbol{v}$ is constant. Furthermore, the **normal component** of $\boldsymbol{v} \in \boldsymbol{RT}_0$ is **constant** along each face (edge) of κ. This can be seen as follows. The line containing the edge e is defined by the equation $\boldsymbol{n} \cdot \boldsymbol{x} = c$, for some constant c, where $\boldsymbol{n}$ is the normal vector to e. Then, the normal component of $\boldsymbol{v}$ is $(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$, and

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{n} &= \boldsymbol{a} \cdot \boldsymbol{n} + b\boldsymbol{x} \cdot \boldsymbol{n} \\ &= \boldsymbol{a} \cdot \boldsymbol{n} + bc, \end{aligned}$$

which is a constant.

In order to create a finite element space on all of Ω, we need to **"glue"** the local spaces $\boldsymbol{RT}_0(\kappa)$ together in such a way that the normal components match for every $e \in \Gamma$. The

way that we do this is by assigning **degrees of freedom** to each face $e \in \Gamma$, such that the degrees of freedom on a single triangle uniquely determine $\boldsymbol{v}|_\kappa$. In the case of H^1 elements, the natural degrees of freedom were point values. In the case of $\boldsymbol{H}(\mathrm{div})$ elements, it is natural to work in terms of "moments", i.e. integrals of the normal components. Let e_1, e_2, e_3 denote the three edges of a triangle κ, and let c_i denote the integrated normal component,

$$c_i := \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_{e_i} (\boldsymbol{a} \cdot \boldsymbol{n} + bc) \, ds.$$

We claim that these degrees of freedom are **unisolvent**, i.e. the values c_i uniquely determine every element of $\boldsymbol{RT}_0(\kappa)$.

First, we show uniqueness. Suppose that $c_i = 0$ for all i. Then,

$$\begin{aligned} 0 &= \sum_i c_i \\ &= \sum_i \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds \\ &= \int_{\partial\kappa} \boldsymbol{v} \cdot \boldsymbol{n} \, ds \\ &= \int_\kappa \nabla \cdot \boldsymbol{v} \, dx \\ &= \int_\kappa 2b \, dx \\ &= 2|\kappa| b. \end{aligned}$$

So, $b = 0$, and $\boldsymbol{v} = \boldsymbol{a}$, a constant. Then,

$$c_i = \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_{e_i} \boldsymbol{a} \cdot \boldsymbol{n} = |e_i| \boldsymbol{a} \cdot \boldsymbol{n},$$

so $\boldsymbol{a}$ is normal to three linearly independent vectors in $\mathbb{R}^2$, and so $\boldsymbol{a} = 0$.

We have defined a map (the "degree of freedom" mapping)

$$C : \boldsymbol{RT}_0(\kappa) \to \mathbb{R}^3$$

defined by

$$C\boldsymbol{v} = (c_1, c_2, c_3).$$

This is a linear map between 3-dimensional spaces. The above argument shows that C is injective, therefore it is a bijection, and every element of $\boldsymbol{RT}_0(\kappa)$ is uniquely determined by its degrees of freedom.

Following the standard construction, each local degree of freedom c_i corresponds with a local basis function $\boldsymbol{\phi}_i$, satisfying $c_i(\boldsymbol{\phi}_j) = \delta_{ij}$ (denoting here c_i as a linear functional). The standard procedure in finite elements is to transform the "physical element" κ to the **reference element** $\hat{\kappa}$ through an affine transformation $T : \hat{\kappa} \mapsto \kappa$. For H^1 finite elements, we simply defined

$$\psi(\boldsymbol{x}) = \hat{\psi}(T^{-1}(\boldsymbol{x})).$$

This **does not work** for the Raviart–Thomas basis functions, because in general

$$c_i(\hat{\boldsymbol{\phi}}(T^{-1}(\boldsymbol{x}))) \neq \delta_{ij}.$$

The reason for this is that the normal vectors on $\hat{\kappa}$ are not mapped to normal vectors on κ through the transformation T. To appropriately map functions in $\boldsymbol{H}(\mathrm{div})$, we need a transformation that preserves normal vectors. Given $\boldsymbol{v} : \kappa \to \mathbb{R}^d$, we want to define $\hat{\boldsymbol{v}} : \hat{\kappa} \to \mathbb{R}^d$ such that

$$\int_{\partial\kappa} w\boldsymbol{v} \cdot \boldsymbol{n}\, ds = \int_{\partial\hat{\kappa}} \hat{w}\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}\, ds$$

for all scalar test functions w (where $\hat{w} := w \circ T$). This can be achieved by ensuring that

$$\int_\kappa (\nabla \cdot \boldsymbol{v}) w\, dx = \int_{\hat{\kappa}} (\nabla \cdot \hat{\boldsymbol{v}})\hat{w}\, dx \qquad \text{and} \qquad \int_\kappa \boldsymbol{v} \cdot \nabla w\, dx = \int_{\hat{\kappa}} \hat{\boldsymbol{v}} \cdot \nabla \hat{w}\, dx$$

since, if this holds, integrating by parts,

$$\begin{aligned}
\int_{\partial\kappa} w\boldsymbol{v} \cdot \boldsymbol{n}\, ds &= \int_\kappa (\nabla \cdot \boldsymbol{v}) w\, dx + \int_\kappa \boldsymbol{v} \cdot \nabla w\, dx \\
&= \int_{\hat{\kappa}} (\nabla \cdot \hat{\boldsymbol{v}})\hat{w}\, dx + \int_{\hat{\kappa}} \hat{\boldsymbol{v}} \cdot \nabla \hat{w}\, dx \\
&= \int_{\partial\hat{\kappa}} \hat{w}\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}\, ds.
\end{aligned}$$

Therefore, we find $\hat{\boldsymbol{v}}$ such that

$$\int_\kappa \boldsymbol{v} \cdot \nabla w\, dx = \int_{\hat{\kappa}} \hat{\boldsymbol{v}} \cdot \nabla \hat{w}\, dx.$$

Let $J = D_T$ denote the Jacobian matrix of the element transformation T. Then, since $\hat{w} = w \circ T$, we have $\nabla \hat{w}(\hat{\boldsymbol{x}}) = (J^T \nabla w)(T(\hat{\boldsymbol{x}}))$, from which it follows that $\nabla w(T(\hat{\boldsymbol{x}})) = J^{-T} \nabla \hat{w}(\hat{\boldsymbol{x}})$. Therefore,

$$\begin{aligned}
\int_\kappa \boldsymbol{v} \cdot \nabla w\, dx &= \int_{\hat{\kappa}} \det(J) \boldsymbol{v}(T(\hat{\boldsymbol{x}})) \cdot \nabla w(T(\hat{\boldsymbol{x}}))\, d\hat{x} \\
&= \int_{\hat{\kappa}} \det(J) \boldsymbol{v}(T(\hat{\boldsymbol{x}})) \cdot J^{-T} \nabla \hat{w}(\hat{\boldsymbol{x}})\, d\hat{x} \\
&= \int_{\hat{\kappa}} \underbrace{\det(J) J^{-1} \boldsymbol{v}(T(\hat{\boldsymbol{x}}))}_{\hat{\boldsymbol{v}}} \cdot \nabla \hat{w}(\hat{\boldsymbol{x}})\, d\hat{x},
\end{aligned}$$

so $\hat{\boldsymbol{v}}$ is defined by

$$\hat{\boldsymbol{v}} := \det(J) J^{-1} \boldsymbol{v} \circ T. \tag{1}$$

Given this definition of $\hat{\boldsymbol{v}}$, we can show (omitting some tedious chain rule calculations) that

$$\nabla \cdot \hat{\boldsymbol{v}} = \det(J)(\nabla \cdot \boldsymbol{v}) \circ T.$$

From this, it follows that

$$\begin{aligned}\int_\kappa (\nabla \cdot \boldsymbol{v}) w \, dx &= \int_{\hat{\kappa}} \det(J)(\nabla \cdot \boldsymbol{v} \circ T) w \circ T \, dx \\ &= \int_{\hat{\kappa}} ()\nabla \cdot \hat{\boldsymbol{v}}) \hat{w} \, dx,\end{aligned}$$

as desired, and so we obtain the desired relation

$$\int_{\partial\kappa} w \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_{\partial\hat{\kappa}} \hat{w} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} \, ds.$$

The transformation from $\boldsymbol{v}$ to $\hat{\boldsymbol{v}}$ defined by (1) can easily be inverted to yield

$$\boldsymbol{v} := \frac{1}{\det(J)} J\hat{\boldsymbol{v}} \circ T^{-1} \tag{2}$$

This is called the (normal-preserving) **Piola transformation**. Using this transformation, the basis functions $\boldsymbol{\phi}$ on each triangle $\kappa \in \mathcal{T}$ can be defined in terms of the basis functions on the reference triangle $\hat{\kappa}$.

The lowest-order Raviart–Thomas finite element space $\boldsymbol{V}_h$ is defined by

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) : \boldsymbol{v}|_\kappa \in \boldsymbol{RT}_0(\kappa)\}.$$

This space is constructed by assigning to each edge $e \in \Gamma$ a degree of freedom c. The global basis functions for this space are associated with each edge of the mesh, and are supported in the two triangles containing that edge. Note that although this space contains some piecewise linear functions, the local space is strictly smaller than the space of all linear functions.

1.1.1 The Raviart–Thomas Interpolant

We wish to define an **interpolation operator** $\mathcal{I}_h : [\boldsymbol{C}^1(\Omega)] \to \boldsymbol{V}_h$. In other words, given $\boldsymbol{v} \in [C^1(\Omega)]^d$, we want to define an "interpolant" $\boldsymbol{v}_h = \mathcal{I}_h(\boldsymbol{v}) \in \boldsymbol{V}_h$. The standard finite element interpolant uses the **degrees of freedom** of the space (considered as functionals) to define the interpolant. In other words, for any such $\boldsymbol{v}$, there is a unique element $\boldsymbol{v}_h \in \boldsymbol{V}_h$ such that

$$\int_e \boldsymbol{v}_h \cdot \boldsymbol{n} \, ds = \int_e \boldsymbol{v} \cdot \boldsymbol{n} \, ds$$

for all edges (faces) e in the mesh. By the properties of the induced basis functions, $\boldsymbol{v}_h$ can be written explicitly as

$$\boldsymbol{v}_h = \sum_{e \in \Gamma} \left(\int_e \boldsymbol{v} \cdot \boldsymbol{n} \, ds \right) \boldsymbol{\phi}_e,$$

where $\boldsymbol{\phi}_e$ is the basis function associated with edge e. From this expression (and using the trace theorem), it is possible to show that

$$\|\mathcal{I}_h \boldsymbol{v}\|_{L^2(\Omega)} \lesssim \|\boldsymbol{v}\|_{H^1(\Omega)}$$

Let $\boldsymbol{v} \in [\boldsymbol{C}^1(\Omega)]^d$, and let $\boldsymbol{v}_h$ be its RT interpolant. Consider the space of piecewise constants,

$$P_h = \{p \in L^2(\Omega) : p|_\kappa \text{ is constant}\}.$$

Then, for any $w_h \in P_h$,

$$\begin{aligned}
\int_\Omega (\nabla \cdot \boldsymbol{v}_h)\, w_h \, dx &= \sum_\kappa \int_\kappa (\nabla \cdot \boldsymbol{v}_h)\, w_h \, dx \\
&= \sum_\kappa \left(-\int_\kappa \boldsymbol{v}_h \cdot \nabla w_h \, dx + \int_{\partial\kappa} \boldsymbol{v}_h \cdot \boldsymbol{n} w_h \, ds \right) && \text{(integration by parts)} \\
&= \sum_\kappa \int_{\partial\kappa} \boldsymbol{v}_h \cdot \boldsymbol{n} w_h \, ds && (w_h \text{ piecewise constant}) \\
&= \sum_\kappa \int_{\partial\kappa} \boldsymbol{v} \cdot \boldsymbol{n} w_h \, ds && \text{(interpolant)} \\
&= \int_\Omega (\nabla \cdot \boldsymbol{v})\, w_h \, dx
\end{aligned}$$

From this, we conclude that $\nabla \cdot \boldsymbol{v}_h$ is the L^2 **projection** of $\nabla \cdot \boldsymbol{v}$ onto the space of piecewise constants. The definition of the interpolant and subsequent arguments can be extended from $[\boldsymbol{C}^1(\Omega)]^d$ to all of $[H^1(\Omega)]^d$, resulting in the following commutative diagram

$$\begin{array}{ccc}
[H^1(\Omega)]^d & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\
\Big\downarrow{\scriptstyle \mathcal{I}_h} & & \Big\downarrow{\scriptstyle \Pi_0} \\
\boldsymbol{V}_h & \xrightarrow{\nabla\cdot} & P_h
\end{array}$$

1.1.2 Application to Darcy's equations

Recall Darcy's equations

$$\begin{aligned}
\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\
\nabla \cdot \boldsymbol{u} &= g,
\end{aligned} \tag{3}$$

leading to the variational problem

$$\begin{aligned}
(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \\
(\nabla \cdot \boldsymbol{u}, q) &= (g, q).
\end{aligned}$$

Here $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p, q \in L^2(\Omega)$.

We consider the finite element approximation obtained by taking $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{RT}_0$, and $p, q \in P_h$, where

$$P_h = \{p \in L^2(\Omega) : p|_\kappa \text{ is constant}\}.$$

(We can identify P_h as the lowest-order DG space). In order for this discretization to be convergent, we need the spaces to satisfy the **inf-sup condition**, i.e.

$$\inf_{0 \neq q \in P_h} \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}_h} \|q\|_{P_h}} \geq \beta > 0,$$

where $b(\cdot,\cdot)$ is the bilinear form

$$b(\boldsymbol{v}, q) := (\nabla \cdot \boldsymbol{v}, q).$$

Previously, we have seen that, given $q \in L^2(\Omega)$, there is $\boldsymbol{v} \in [H^1(\Omega)]^d$ with $\nabla \cdot \boldsymbol{v} = q$, and $\|\boldsymbol{v}\|_{H^1(\Omega)} \lesssim \|q\|_{L^2(\Omega)}$. For any $q \in P_h$, let $\boldsymbol{v} \in [H^1(\Omega)]^d$ satisfy these conditions, and define $\boldsymbol{v}_h = \mathcal{I}_h(\boldsymbol{v}) \in \boldsymbol{V}_h$. By commutativity of the diagram above, we have that $\nabla \cdot \boldsymbol{v}_h = q$. Then, for any such $q \in P_h$,

$$\begin{aligned}
\frac{b(\boldsymbol{v}_h, q)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h}\|q\|_{P_h}} &= \frac{(\nabla \cdot \boldsymbol{v}_h, q)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h}\|q\|_{P_h}} \\
&= \frac{(q, q)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h}\|q\|_{P_h}} \\
&= \frac{\|q\|_{P_h}^2}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h}\|q\|_{P_h}} \\
&\gtrsim \frac{\|q\|_{P_h}^2}{\|q\|_{P_h}\|q\|_{P_h}} \\
&\approx 1.
\end{aligned}$$