

Discontinuous Galerkin methods for elliptic problems

We are now interested in the slightly more general model problem

$$\frac{\partial u}{\partial t} + \nabla \cdot (\beta u) - \epsilon \Delta u = f,$$

which is called an **advection–diffusion** or **convection–diffusion** equation. The vector field β is the **advection velocity** and ϵ is the **diffusion coefficient**. As before, we can also consider the “steady” version of this problem,

$$\nabla \cdot (\beta u) - \epsilon \Delta u + cu = f,$$

which is also called an **advection–diffusion–reaction** equation.

Using the approach developed previously, we know how to construct DG discretizations of the advection and reaction terms; it remains to construct DG discretizations of the diffusion term $-\epsilon \Delta u$. This is now our goal: construct DG discretizations of the Laplace operator. To this end, we return to our original model problem

$$(*) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1 Flux formulation

As before, we consider the degree- p discontinuous finite element space V_h . We also consider the vector-valued version of this space, $\Sigma_h := [V_h]^d$.

We rewrite $(*)$ in first-order form,

$$(**) \quad \begin{cases} \sigma = \nabla u, \\ -\nabla \cdot \sigma = f, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The vector unknown σ is called the **flux** and the scalar unknown u is called the **potential**. To derive a DG discretization for $(*)$, we start by discretizing $(**)$ following the standard methodology.

We multiply the first equation by a test function $\tau \in \Sigma_h$, the second equation by a test function $v \in V_h$, and integrate (by parts) over a triangle $\kappa \in \mathcal{T}$, obtaining

$$\int_{\kappa} \sigma \cdot \tau \, dx = \int_{\kappa} \nabla u \cdot \tau \, dx = - \int_{\kappa} u \nabla \cdot \tau \, dx + \int_{\partial\kappa} u \tau \cdot \mathbf{n} \, ds.$$

and

$$- \int_{\kappa} \nabla \cdot \sigma v \, dx = \int_{\kappa} \sigma \cdot \nabla v \, dx - \int_{\partial\kappa} \sigma \cdot \mathbf{n} v \, ds = \int_{\kappa} f v \, dx.$$

As before, the key step in DG methods is to replace the traces of the unknowns u and $\boldsymbol{\sigma}$ with single-valued **numerical fluxes** on all element interfaces. The numerical fluxes are defined \widehat{u} and $\widehat{\boldsymbol{\sigma}}$. After replacing with numerical fluxes, we obtain

$$\begin{aligned} \int_{\kappa} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx &= - \int_{\kappa} u \nabla \cdot \boldsymbol{\tau} \, dx + \int_{\partial\kappa} \widehat{u} \boldsymbol{\tau} \cdot \mathbf{n} \, ds \\ \int_{\kappa} \boldsymbol{\sigma} \cdot \nabla v \, dx - \int_{\partial\kappa} \widehat{\boldsymbol{\sigma}} \cdot \mathbf{n} v \, ds &= \int_{\kappa} f v \, dx. \end{aligned}$$

Since \widehat{u} and $\widehat{\boldsymbol{\sigma}}$ are single-valued on the interfaces, we can sum over all elements $\kappa \in \mathcal{T}$ to obtain,

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx = - \int_{\Omega} u \nabla_h \cdot \boldsymbol{\tau} \, dx + \int_{\Gamma} \widehat{u} \llbracket \boldsymbol{\tau} \rrbracket \, ds \quad (\text{F1})$$

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \nabla_h v \, dx - \int_{\Gamma} \widehat{\boldsymbol{\sigma}} \cdot \llbracket v \rrbracket \, ds = \int_{\Omega} f v \, dx \quad (\text{F2})$$

where as before $\llbracket \cdot \rrbracket$ denotes the **jump**, which on an edge $e \subseteq \Gamma$

$$\llbracket v \rrbracket = v^- \mathbf{n}^- + v^+ \mathbf{n}^+, \quad \llbracket \boldsymbol{\tau} \rrbracket = \boldsymbol{\tau}^- \cdot \mathbf{n}^- + \boldsymbol{\tau}^+ \cdot \mathbf{n}^+.$$

Note that the jump of a scalar is a vector (in the normal direction), and the jump of a vector is a scalar. We also recall that the average is defined by

$$\{q\} = \frac{1}{2}(q^- + q^+).$$

2 Primal formulation

Rather than working directly with (F1)–(F2), which corresponds to a 2×2 system for the unknowns $\boldsymbol{\sigma}$ and u , we would prefer to **eliminate** the flux $\boldsymbol{\sigma}$, and obtain a system for the potential (primal) unknown u only (this would be analogous to the standard H^1 -conforming finite element discretization for the Poisson problem).

We first derive an identity that will be useful for element-by-element integration by parts. Let q be scalar-valued and $\boldsymbol{\varphi}$ be vector-valued. Then,

$$\begin{aligned} \llbracket q \rrbracket \cdot \{\boldsymbol{\varphi}\} \, ds + \{q\} \llbracket \boldsymbol{\varphi} \rrbracket &= \frac{1}{2} ((q^- \mathbf{n}^- + q^+ \mathbf{n}^+) \cdot (\boldsymbol{\varphi}^- + \boldsymbol{\varphi}^+) + (q^- + q^+) (\boldsymbol{\varphi}^- \cdot \mathbf{n}^- + \boldsymbol{\varphi}^+ \cdot \mathbf{n}^+)) \\ &= \frac{1}{2} (q^- \boldsymbol{\varphi}^- \cdot \mathbf{n}^- + q^- \boldsymbol{\varphi}^+ \cdot \mathbf{n}^- + q^+ \boldsymbol{\varphi}^+ \cdot \mathbf{n}^+ + q^+ \boldsymbol{\varphi}^- \cdot \mathbf{n}^+ \\ &\quad q^- \boldsymbol{\varphi}^- \cdot \mathbf{n}^- + q^- \boldsymbol{\varphi}^+ \cdot \mathbf{n}^+ + q^+ \boldsymbol{\varphi}^- \cdot \mathbf{n}^- + q^+ \boldsymbol{\varphi}^+ \cdot \mathbf{n}^+) \\ &= q^- \boldsymbol{\varphi}^- \cdot \mathbf{n}^- + q^- \boldsymbol{\varphi}^+ \cdot \mathbf{n}^+. \end{aligned}$$

Therefore,

$$\sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa} q \boldsymbol{\varphi} \cdot \mathbf{n} \, ds = \int_{\Gamma} (\llbracket q \rrbracket \cdot \{\boldsymbol{\varphi}\} \, ds + \{q\} \llbracket \boldsymbol{\varphi} \rrbracket) \, ds.$$

Integrating the first term on the right-hand side of (F1) by parts element-by-element, we obtain

$$-\int_{\Omega} u \nabla_h \cdot \boldsymbol{\tau} dx = \int_{\Omega} \nabla_h u \cdot \boldsymbol{\tau} dx - \int_{\Gamma} (\llbracket u \rrbracket \cdot \{\boldsymbol{\tau}\} + \{u\} \llbracket \boldsymbol{\tau} \rrbracket) ds$$

Inserting this expression into (F1) yields

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} dx = \int_{\Omega} \nabla_h u \cdot \boldsymbol{\tau} dx - \int_{\Gamma} (\llbracket u \rrbracket \cdot \{\boldsymbol{\tau}\} + \{u\} \llbracket \boldsymbol{\tau} \rrbracket) ds + \int_{\Gamma} \hat{u} \llbracket \boldsymbol{\tau} \rrbracket ds \quad (1)$$

Note that by definition of the spaces V_h and $\boldsymbol{\Sigma}_h$, we have that $\nabla_h v \in \boldsymbol{\Sigma}_h$, and so, choosing the test function $\boldsymbol{\tau} = \nabla_h v$, (1) gives

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \nabla_h v dx = \int_{\Omega} \nabla_h u \cdot \nabla_h v dx - \int_{\Gamma} (\llbracket u \rrbracket \cdot \{\nabla_h v\} + \{u\} \llbracket \nabla_h v \rrbracket) ds + \int_{\Gamma} \hat{u} \llbracket \nabla_h u \rrbracket ds.$$

We can replace the first term on the left-hand side of (F2) with this expression, giving

$$\begin{aligned} \int_{\Omega} \nabla_h u \cdot \nabla_h v dx - \int_{\Gamma} (\llbracket u \rrbracket \cdot \{\nabla_h v\} + \{u\} \llbracket \nabla_h v \rrbracket) ds \\ + \int_{\Gamma} \hat{u} \llbracket \nabla_h u \rrbracket ds - \int_{\Gamma} \hat{\boldsymbol{\sigma}} \cdot \llbracket v \rrbracket ds = \int_{\Omega} f v dx, \end{aligned} \quad (2)$$

which is called the DG **primal formulation**. Once the numerical fluxes \hat{u} and $\hat{\boldsymbol{\sigma}}$ are specified, the description of the DG method is complete. The choice of numerical flux can be somewhat subtle. Different choices have different implications for stability, accuracy, and computational efficiency.

Note that (since $\nabla_h u \in \boldsymbol{\Sigma}_h$), (1) can be used to obtain an explicit expression for $\boldsymbol{\sigma}$. We first define the **lifting operators** $\ell : L^2(\Gamma) \rightarrow \boldsymbol{\Sigma}_h$ and $r : [L^2(\Gamma)]^d \rightarrow \boldsymbol{\Sigma}_h$ by

$$\begin{aligned} \int_{\Omega} \ell(q) \cdot \boldsymbol{\tau} dx &= - \int_{\Gamma} q \llbracket \boldsymbol{\tau} \rrbracket ds \\ \int_{\Omega} r(\boldsymbol{\varphi}) \cdot \boldsymbol{\tau} dx &= - \int_{\Gamma} \boldsymbol{\varphi} \cdot \{\boldsymbol{\tau}\} ds \end{aligned}$$

for all test functions $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$. Then, (1) implies that

$$\boldsymbol{\sigma} = \nabla_h u + r(\llbracket u \rrbracket) + \ell(\{u\} - \hat{u}). \quad (3)$$

This gives an explicit expression for the discrete flux in terms of the scalar variable (and numerical flux \hat{u}).

3 Examples of DG methods

Different choices of numerical fluxes give rise to different DG discretizations. The following table (from the paper D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM Journal on Numerical Analysis*, 39(5):1749–1779, 2002) enumerates some commonly used DG methods.

TABLE 3.1
Some DG methods and their numerical fluxes.

Method	\hat{u}_K	$\hat{\sigma}_K$
Bassi–Rebay [10]	$\{u_h\}$	$\{\sigma_h\}$
Brezzi et al. [22]	$\{u_h\}$	$\{\sigma_h\} - \alpha_r(\llbracket u_h \rrbracket)$
LDG [41]	$\{u_h\} - \beta \cdot \llbracket u_h \rrbracket$	$\{\sigma_h\} + \beta \llbracket \sigma_h \rrbracket - \alpha_j(\llbracket u_h \rrbracket)$
IP [50]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_j(\llbracket u_h \rrbracket)$
Bassi et al. [13]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_r(\llbracket u_h \rrbracket)$
Baumann–Oden [15]	$\{u_h\} + n_K \cdot \llbracket u_h \rrbracket$	$\{\nabla_h u_h\}$
NIPG [64]	$\{u_h\} + n_K \cdot \llbracket u_h \rrbracket$	$\{\nabla_h u_h\} - \alpha_j(\llbracket u_h \rrbracket)$
Babuška–Zlámal [7]	$(u_h _K) _{\partial K}$	$-\alpha_j(\llbracket u_h \rrbracket)$
Brezzi et al. [23]	$(u_h _K) _{\partial K}$	$-\alpha_r(\llbracket u_h \rrbracket)$

Perhaps the simplest choice is that of Bassi and Rebay (commonly called the “BR1” method). This simply uses the average of u and σ as the numerical fluxes. However, **this choice is not stable**. The following diagram (from the same paper) illustrates the consistency and stability of these methods.

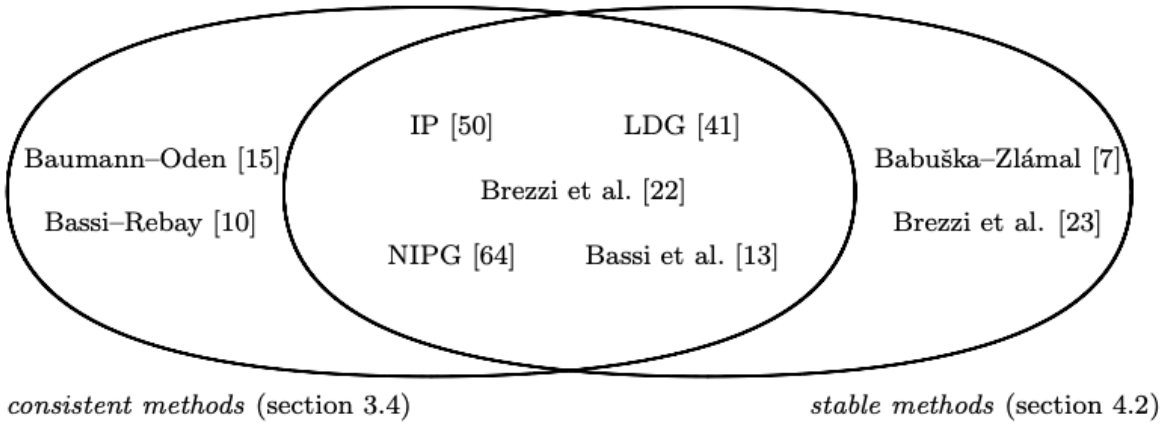


FIG. 3.1. Consistency and stability of some DG methods.

We will focus on two methods:

- The (symmetric) **interior penalty** (IP) method
- The **local discontinuous Galerkin** (LDG) method

The interior penalty method defines the fluxes

$$\hat{u} := \{u\}, \quad \hat{\sigma} := \{\nabla_h u\} - \alpha \llbracket u \rrbracket. \quad (4)$$

This can be thought of as an improvement on the unstable BR1 method by adding a **penalization term** $\alpha \llbracket u \rrbracket$ to the definition of the flux $\hat{\sigma}$ (and using directly $\{\nabla_h u\}$ rather than $\{\sigma\}$). This term ensures that the method is stable, but the coefficient α has to be chosen carefully.

The **local discontinuous Galerkin** method defines the fluxes

$$\hat{u} := \{u\} - \beta \cdot \llbracket u \rrbracket, \quad \hat{\sigma} := \{\sigma\} + \beta \llbracket \sigma \rrbracket - \alpha \llbracket u \rrbracket \quad (5)$$

where β is a prescribed vector field. Note that if $|\beta \cdot \mathbf{n}| = 1/2$, then the LDG method corresponds to choosing \hat{u} to be an “upwind” value of u (according to a potentially arbitrarily chosen direction), and $\hat{\sigma}$ will be the “downwind value” (with the addition of a penalty term). The LDG method can be thought of as a natural DG discretization of the first-order system (**), where the upwind direction is prescribed, rather than being part of the problem definition.

3.1 Symmetric interior penalty method

Substituting (4) into the left-hand side of (2), we obtain

$$\begin{aligned} & \int_{\Omega} \nabla_h u \cdot \nabla_h v \, dx - \int_{\Gamma} (\llbracket u \rrbracket \cdot \{\nabla_h v\} + \{u\} \llbracket \nabla_h v \rrbracket) \, ds \\ & \quad + \int_{\Gamma} \{u\} \llbracket \nabla_h u \rrbracket \, ds - \int_{\Gamma} (\{\nabla_h u\} - \alpha \llbracket u \rrbracket) \cdot \llbracket v \rrbracket \, ds \\ &= \int_{\Omega} \nabla_h u \cdot \nabla_h v \, dx - \int_{\Gamma} (\llbracket u \rrbracket \cdot \{\nabla_h v\} + \{u\} \llbracket \nabla_h v \rrbracket) \, ds \\ & \quad + \int_{\Gamma} \{u\} \llbracket \nabla_h u \rrbracket \, ds - \int_{\Gamma} (\{\nabla_h u\} - \alpha \llbracket u \rrbracket) \cdot \llbracket v \rrbracket \, ds \\ &= \int_{\Omega} \nabla_h u \cdot \nabla_h v \, dx - \int_{\Gamma} \llbracket u \rrbracket \cdot \{\nabla_h v\} \, ds - \int_{\Gamma} \{\nabla_h u\} \cdot \llbracket v \rrbracket \, ds + \int_{\Gamma} \alpha \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds. \end{aligned}$$

So, the **IP-DG** variational formulation is: find $u \in V_h$ such that, for all $v \in V_h$,

$$\int_{\Omega} \nabla_h u \cdot \nabla_h v \, dx - \int_{\Gamma} \llbracket u \rrbracket \cdot \{\nabla_h v\} \, ds - \int_{\Gamma} \{\nabla_h u\} \cdot \llbracket v \rrbracket \, ds + \int_{\Gamma} \alpha \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds = \int_{\Omega} f v \, dx.$$

The IP-DG bilinear form is

$$a(u, v) := \int_{\Omega} \nabla_h u \cdot \nabla_h v \, dx - \int_{\Gamma} \llbracket u \rrbracket \cdot \{\nabla_h v\} \, ds - \int_{\Gamma} \{\nabla_h u\} \cdot \llbracket v \rrbracket \, ds + \int_{\Gamma} \alpha \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds.$$

4 Analysis

We want to show boundedness (continuity), stability (coercivity), and accuracy of the DG variational formulation

$$a(u, v) = (f, v).$$

As in the case of DG methods for the advection equation, it is convenient to work with a special mesh-dependent norm denotes $\llbracket \cdot \rrbracket$ defined by

$$\llbracket v \rrbracket^2 := \|\nabla_h v\|_{L^2(\Omega)}^2 + h^{-1} \|\llbracket v \rrbracket\|_{L^2(\Gamma)}^2.$$

4.1 Boundedness

We wish to show that

$$a(u, v) \lesssim \llbracket u \rrbracket \llbracket v \rrbracket$$

for all $u, v \in V_h$. Clearly, the first first defining $a(\cdot, \cdot)$ is bounded by the first term in the definition of $\llbracket \cdot \rrbracket$, and similarly for the last term. It remains now to consider terms of the form

$$\int_{\Gamma} \llbracket u \rrbracket \cdot \{\nabla_h v\} ds \quad \text{and} \quad \int_{\Gamma} \{\nabla_h u\} \cdot \llbracket v \rrbracket ds.$$

By symmetry, we consider only the first of these two terms. Note that, by definition of the lifting operator $r(\cdot)$,

$$\int_{\Gamma} \llbracket u \rrbracket \cdot \{\nabla_h v\} ds = - \int_{\Omega} \nabla_h v \cdot r(\llbracket u \rrbracket) dx.$$

So, by Cauchy–Schwarz,

$$\int_{\Gamma} \llbracket u \rrbracket \cdot \{\nabla_h v\} ds \leq \|\nabla_h v\|_{L^2(\Omega)} \|r(\llbracket u \rrbracket)\|_{L^2(\Omega)}$$

Considering the term

$$\begin{aligned} \|r(\llbracket u \rrbracket)\|_{L^2(\Omega)}^2 &= \int_{\Omega} r(\llbracket u \rrbracket)^2 dx \\ &= - \int_{\Gamma} \llbracket u \rrbracket \cdot \{r(\llbracket u \rrbracket)\} ds \\ &\leq \|\llbracket u \rrbracket\|_{L^2(\Gamma)} \|\{r(\llbracket u \rrbracket)\}\|_{L^2(\Gamma)} \\ &\lesssim h^{-1/2} \|\llbracket u \rrbracket\|_{L^2(\Gamma)} \|r(\llbracket u \rrbracket)\|_{L^2(\Omega)} \end{aligned}$$

where the last step follows by a trace inequality and scaling argument. This implies that

$$\|r(\llbracket u \rrbracket)\|_{L^2(\Omega)} \lesssim h^{-1/2} \|\llbracket u \rrbracket\|_{L^2(\Gamma)}$$

from which we obtain

$$\begin{aligned} \int_{\Gamma} \llbracket u \rrbracket \cdot \{\nabla_h v\} ds &\leq \|\nabla_h v\|_{L^2(\Omega)} \|r(\llbracket u \rrbracket)\|_{L^2(\Omega)} \\ &\lesssim \|\nabla_h v\|_{L^2(\Omega)} h^{-1/2} \|\llbracket u \rrbracket\|_{L^2(\Gamma)}. \end{aligned}$$

Combining these estimate, we conclude that $a(u, v) \lesssim \llbracket u \rrbracket \llbracket v \rrbracket$.