Lecture Note #8 **MTH653: Advanced Numerical Analysis**
Spring 2025 Instructor: Will Pazner

$\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ spaces

1 Some Motivating Examples

1.1 Darcy's equations

Recall **Darcy's equations** that model flow through porous media. The unknowns are $\boldsymbol{u}$ (the velocity vector) and p (the pressure). The governing equations are

$$\begin{aligned} \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= g, \end{aligned} \tag{1}$$

with appropriate boundary conditions on $\partial\Omega$. To derive the variational formulation, multiply the first equation by a vector-valued test function and the second equation by a scalar test function, obtaining

$$\begin{aligned} (\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \\ (\nabla \cdot \boldsymbol{u}, q) &= (g, q). \end{aligned}$$

Notice that there are no derivatives acting on p and q, so we can choose $p, q \in L^2(\Omega)$. While we do have derivatives acting on $\boldsymbol{u}$ and $\boldsymbol{v}$, we do not need the whole gradient—we only need the *divergence*. Therefore, it would make sense to choose the minimal regularity space

$$\boldsymbol{u}, \boldsymbol{v} \in [L^2(\Omega)]^d \quad \text{and} \quad \nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v} \in L^2(\Omega).$$

This space is *strictly larger* than $[H^1(\Omega)]^d$.

1.2 Maxwell equations

Maxwell's equations, which govern the physics of electromagnetic fields, are given by

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\ \nabla \times \boldsymbol{H} &= \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}, \\ \nabla \cdot \boldsymbol{D} &= \rho, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \cdot \boldsymbol{J} &= -\frac{\partial \rho}{\partial t}. \end{aligned}$$

In the above, we use the notation:

- $\boldsymbol{E}$ is the electric field intensity

- $\boldsymbol{D}$ is the electric flux density
- $\boldsymbol{H}$ is the magnetic field intensity
- $\boldsymbol{B}$ is the magnetic flux density
- $\boldsymbol{J}$ is the electric current density
- ρ is the electric charge density

Under some assumptions (that the fields are *time harmonic*, i.e. that they oscillate with a single frequency ω) and after introducing constitutive relations, the equations can be simplified, eventually reducing to the so-called "vector wave equation",

$$\nabla \times (\nabla \times \boldsymbol{E}) - \omega^2 \boldsymbol{E} = -j\omega \boldsymbol{J}$$

for the electric field, or

$$\nabla \times (\nabla \times \boldsymbol{H}) - \omega^2 \boldsymbol{H} = \nabla \times \boldsymbol{J}$$

for the magnetic field. These equations motivate considering the model problem

$$\nabla \times (\nabla \times \boldsymbol{E}) + \alpha \boldsymbol{E} = \boldsymbol{f}. \tag{2}$$

We derive the variational formulation for this problem. Recall the vector calculus identity

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}).$$

Divergence theorem and then applying the above identity gives

$$\begin{aligned} \int_{\partial\Omega} (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{n} \, ds &= \int_{\Omega} \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) \, dx \\ &= \int_{\Omega} \left(\boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) \right) \, dx. \end{aligned}$$

Rearranging (and using the vector triple product identity $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = -(\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b}$) we obtain the integration by parts formula

$$\boxed{\int_{\Omega} \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) \, dx = \int_{\Omega} \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) \, dx - \int_{\partial\Omega} (\boldsymbol{u} \times \boldsymbol{n}) \cdot \boldsymbol{v} \, ds. \qquad (3)}$$

Multiply (2) by a test function $\boldsymbol{F}$ and integrate over Ω,

$$\int_{\Omega} (\nabla \times \nabla \times \boldsymbol{E}) \cdot \boldsymbol{F} \, dx + \int_{\Omega} \alpha \boldsymbol{E} \cdot \boldsymbol{F} \, dx = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{F} \, dx.$$

Applying the integration by parts formula to to the first integral on the left-hand side (setting $\boldsymbol{u} = \nabla \times \boldsymbol{E}$, $\boldsymbol{v} = \boldsymbol{F}$), we have

$$\int_{\Omega} (\nabla \times \nabla \times \boldsymbol{E}) \cdot \boldsymbol{F} \, dx = \int_{\Omega} \nabla \times \boldsymbol{E} \cdot \nabla \times \boldsymbol{F} \, dx + \int_{\partial\Omega} ((\nabla \times \boldsymbol{E}) \times \boldsymbol{F}) \cdot \boldsymbol{n} \, ds.$$

We often use the boundary conditions that $\boldsymbol{E}$ (and the test function $\boldsymbol{F}$) have **zero tangential trace** on $\partial\Omega$, i.e.

$$\boldsymbol{E} \cdot \boldsymbol{t} = 0 \quad \text{and} \quad \boldsymbol{F} \cdot \boldsymbol{t} = 0 \qquad \text{on } \partial\Omega$$

where $\boldsymbol{t}$ tangent to the boundary. Equivalently,

$$\boldsymbol{E} \times \boldsymbol{n} = 0, \qquad \boldsymbol{F} \times \boldsymbol{n} = 0.$$

With these conditions, note that the integrand of the boundary integral is given by

$$((\nabla \times \boldsymbol{E}) \times \boldsymbol{F}) \cdot \boldsymbol{n} = (\nabla \times \boldsymbol{E}) \cdot (\boldsymbol{F} \times \boldsymbol{n}) = 0.$$

Therefore, the variational formulation for (2) (with zero tangential trace conditions) reduces to

$$(\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F}) + (\alpha \boldsymbol{E}, \boldsymbol{F}) = (\boldsymbol{f}, \boldsymbol{F}).$$

Therefore, the conditions on $\boldsymbol{E}$ and $\boldsymbol{F}$ that we require in order for this variational problem to make sense are

$$\boldsymbol{E}, \boldsymbol{F} \in [L^2(\Omega)]^d \quad \text{and} \quad \nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F} \in [L^2(\Omega)]^d.$$

Again, this space is strictly larger than $[H^1(\Omega)]^d$.

2 Sobolev Spaces

So far, we have been working with the $W^{k,p}(\Omega)$ Sobolev spaces (and more specifically, the $H^k(\Omega) := W^{k,2}(\Omega)$ Hilbert spaces). In finite elements, we commonly work in the space $H^1(\Omega)$. However, for the problems described above, it makes sense to work in spaces that do not possess full H^1 regularity. The natural spaces to work in for the above problems are $\boldsymbol{H}(\text{div})$ and $\boldsymbol{H}(\textbf{curl})$, which are defined as

$$\begin{aligned} \boldsymbol{H}(\text{div}, \Omega) &:= \{\boldsymbol{v} \in [L^2(\Omega)]^d : \nabla \cdot \boldsymbol{v} \in L^2(\Omega)\}, \\ \boldsymbol{H}(\textbf{curl}, \Omega) &:= \{\boldsymbol{v} \in [L^2(\Omega)]^d : \nabla \times \boldsymbol{v} \in [L^2(\Omega)]^d\}. \end{aligned}$$

Each of these spaces is larger than $[H^1(\Omega)]^d$ but smaller than $[L^2(\Omega)]^d$.

2.1 Continuity requirements

Suppose Ω is partitioned into the union of disjoint subdomains Ω_i (to be precise, the interiors of the subdomains are disjoint).

We had previously seen that if $u \in L^2(\Omega)$ and $u|_{\Omega_i} \in C^1(\Omega_i)$, then

$$u \in H^1(\Omega) \iff u \in C^0(\Omega).$$

We briefly review this argument. Suppose $u \in H^1(\Omega)$. Then, its weak derivative $\nabla_w u \in [L^2(\Omega)]^d$ is well-defined, and $\nabla_w u|_{\Omega_i} = \nabla(u|_{\Omega_i})$. Multiply $\nabla_w u$ by a smooth test function $\boldsymbol{\phi}$ (supported away from $\partial\Omega$) and integrate over Ω. The weak derivative satisfies

$$\int_\Omega \nabla_w u \cdot \boldsymbol{\phi} \, dx = -\int_\Omega u \nabla \cdot \boldsymbol{\phi} \, dx. \tag{4}$$

Take the left-hand side and integrate by parts over each subdomain,

$$\begin{aligned}
\int_\Omega \nabla_w u \cdot \boldsymbol{\phi}\, dx &= \sum_i \int_{\Omega_i} \nabla_w u \cdot \boldsymbol{\phi}\, dx \\
&= \sum_i \left(-\int_{\Omega_i} u \nabla \cdot \boldsymbol{\phi}\, dx + \int_{\partial\Omega_i} u \boldsymbol{\phi} \cdot \boldsymbol{n}\, ds \right) \\
&= -\int_\Omega u \nabla \cdot \boldsymbol{\phi}\, dx + \int_\Gamma [\![u]\!] \cdot \boldsymbol{\phi}\, ds
\end{aligned}$$

But by (4), this implies that $\int_\Gamma [\![u]\!] \cdot \boldsymbol{\phi}\, ds = 0$, and we can conclude that the function is continuous. The converse direction follows by showing that the function defined by $\nabla(u|_{\Omega_i})$ on each subdomain satisfies the definition of the weak gradient under the condition that $[\![u]\!] = 0$.

This means that from the point of view of finite elements (where finite element functions u_h restricted to each element $\kappa \in \mathcal{T}$ are smooth), membership in H^1 is determined by continuity at element interfaces. The situation is similar for determining if a finite element function is an element of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$.

2.1.1 Continuity conditions in $\boldsymbol{H}(\mathrm{div})$

A vector-valued function $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ is in $\boldsymbol{H}(\mathrm{div}, \Omega)$ if there exists a function (its **weak divergence**) $\nabla_w \cdot \boldsymbol{v} : \Omega \to \mathbb{R}$, satisfying

$$\int_\Omega (\nabla_w \cdot \boldsymbol{v}) \phi\, dx = -\int_\Omega \boldsymbol{v} \cdot \nabla \phi\, dx,$$

for all smooth test functions ϕ supported away from $\partial\Omega$.

Suppose $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is piecewise-$\boldsymbol{C}^1$ (with respect to the mesh $\mathcal{T}$). Then, $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ if and only if $[\![\boldsymbol{u}]\!] \equiv 0$. (Recall from the lectures on DG diffusion, the jump of a vector is a scalar defined by $[\![\boldsymbol{u}]\!] := \boldsymbol{u}^- \cdot \boldsymbol{n}^- + \boldsymbol{u}^+ \cdot \boldsymbol{n}^+$). First, suppose that $[\![\boldsymbol{u}]\!] = 0$. We claim that $\nabla_h \cdot \boldsymbol{u}$ satisfies the definition of the weak divergence. We compute

$$\begin{aligned}
\int_\Omega (\nabla_h \cdot \boldsymbol{u}) \phi\, dx &= \sum_\kappa \int_\kappa (\nabla_h \cdot \boldsymbol{u}) \phi\, dx \\
&= \sum_\kappa \left(\int_{\partial\kappa} \phi \boldsymbol{u} \cdot \boldsymbol{n}\, ds - \int_\kappa \boldsymbol{u} \cdot \nabla \phi\, dx \right) \\
&= \int_\Gamma [\![\boldsymbol{u}]\!] \phi\, ds - \int_\Omega \boldsymbol{u} \cdot \nabla \phi\, dx \\
&= -\int_\Omega \boldsymbol{u} \cdot \nabla \phi\, dx.
\end{aligned}$$

Since this holds for all test functions ϕ, we have that $\nabla_w \cdot \boldsymbol{u} = \nabla_h \cdot \boldsymbol{u}$ is well-defined.

On the other hand, if $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}, \Omega)$, then $\nabla_w \cdot \boldsymbol{u}$ is well-defined. We know that, restricted to $\kappa \in \mathcal{T}$,

$$(\nabla_w \cdot \boldsymbol{u})|_\kappa = \nabla \cdot (\boldsymbol{u}|_\kappa) = (\nabla_h \cdot \boldsymbol{u})|_\kappa.$$

By a similar argument to above,

$$\int_\Gamma [\![\boldsymbol{u}]\!]\phi\, ds - \int_\Omega \boldsymbol{u}\cdot\nabla\phi\, dx = \int_\Omega (\nabla_h\cdot\boldsymbol{u})\phi\, dx = \int_\Omega (\nabla_w\cdot\boldsymbol{u})\phi\, dx = \int_\Omega \boldsymbol{u}\cdot\nabla\phi\, dx.$$

From this, we conclude that

$$\int_\Gamma [\![\boldsymbol{u}]\!]\phi\, ds = 0,$$

and since this holds for all test functions, $[\![\boldsymbol{u}]\!] \equiv 0$.

A piecewise-$\boldsymbol{C}^1$ function is in $\boldsymbol{H}(\text{div})$ iff it possesses **normal continuity**.

2.1.2 Continuity conditions in $\boldsymbol{H}(\mathbf{curl})$

A vector-valued function $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ ($d = 2$ or $d = 3$) is in $\boldsymbol{H}(\mathbf{curl})$ if it has a well-defined **weak curl** $\nabla_w \times \boldsymbol{v}$. Note that while the curl in $\mathbb{R}^3$ is vector-valued, in $\mathbb{R}^2$ the curl is the scalar-valued quantity

$$\nabla\times\boldsymbol{v} := \frac{\partial v_w}{\partial x} - \frac{\partial v_1}{\partial y}$$

(which can be interpreted as the z-component of a vector field in the xy-plane). The weak curl is characterized by the integration by parts formula

$$\int_\Omega (\nabla_w\times\boldsymbol{v})\cdot\boldsymbol{\phi}\, dx = \int_\Omega \boldsymbol{v}\cdot(\nabla\times\boldsymbol{\phi})\, dx$$

for all smooth vector-valued test functions $\boldsymbol{\phi}$ supported away from $\partial\Omega$.

If $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is piecewise-$\boldsymbol{C}^1$, then $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ if and only if $[\![\boldsymbol{u}\times\boldsymbol{n}]\!] \equiv 0$ (where $[\![\boldsymbol{u}\times\boldsymbol{n}]\!] = \boldsymbol{u}^-\times\boldsymbol{n}^- + \boldsymbol{u}^+\times\boldsymbol{n}^+$). The argument follows the same structure as in the case of $\boldsymbol{H}(\text{div})$. It can be seen that $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ if and only if

$$\int_\Gamma [\![\boldsymbol{u}\times\boldsymbol{n}]\!]\cdot\boldsymbol{\phi}\, ds = 0$$

for all smooth test functions $\boldsymbol{\phi}$.

As discussed previously, a vector field on a face e can be decomposed into its normal and tangential components,

$$\begin{aligned}\boldsymbol{\phi} &= \boldsymbol{\phi}_\parallel + \boldsymbol{\phi}_\perp \\ &= \boldsymbol{\phi}_\parallel + (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}.\end{aligned}$$

Then,

$$\begin{aligned}\boldsymbol{\phi}\times\boldsymbol{n} &= \boldsymbol{\phi}_\parallel\times\boldsymbol{n} + (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\times\boldsymbol{n} \\ &= \boldsymbol{\phi}_\parallel\times\boldsymbol{n},\end{aligned}$$

and so $[\![\boldsymbol{u}\times\boldsymbol{n}]\!] = 0$ is equivalent to **tangential continuity** of the vector field $\boldsymbol{u}$.

A piecewise-$\boldsymbol{C}^1$ function is in $\boldsymbol{H}(\text{div})$ iff it possesses **tangential continuity**.