Lecture Note #4 **MTH653: Advanced Numerical Analysis**
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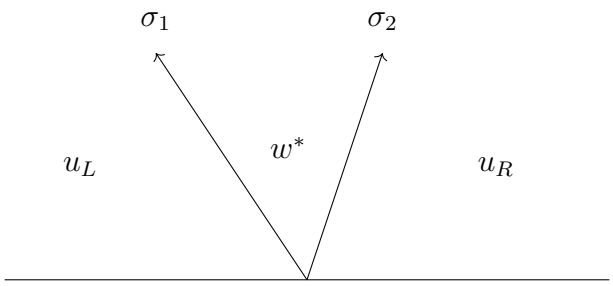
Discontinuous Galerkin methods for hyperbolic conservation laws

1 Approximate Riemann Solvers

As we have seen in the previous lectures, exactly solving the Riemann problem may incur some computational expense. It is not prohibitively expensive to solve a single such problem, but in finite volume methods, such problems need to be solved at every cell interface. As we will see, in discontinuous Galerkin methods, such problems need to be solved at all *quadrature points* on the mesh skeleton. For this reason, it can be computationally advantageous to replace the Godunov numerical flux (i.e. $\widehat{F}(u^-, u^+) = F(w^*)$, where $w*$ solves the Riemann problem with initial states u^- and u^+).

In this section, we will describe the **two-wave approximate Riemann solver** and its very commonly used simplification, the **local Lax–Friedrichs flux**. A two wave solver makes the assumption that there are two waves, traveling at speeds σ_1 and σ_2.



The Rankine–Hugoniot conditions are

$$F_L - F^* = \sigma_1(u_L - w^*),$$
$$F^* - F_R = \sigma_2(w^* - u_R).$$

From these conditions, it is possible to solve for the intermediate state $w*$, giving

$$w^* = \frac{F_L - F_R - \sigma_1 u_L + \sigma_2 u_R}{\sigma_2 - \sigma_1}.$$

Substituting this back into either one of the Rankine–Hugoniot conditions gives

$$F^* = \frac{\sigma_1 F(u_R) - \sigma_2 F(u_L) - \sigma_1\sigma_2(u_R - u_L)}{\sigma_1 - \sigma_2}. \tag{1}$$

Note that this will not generally give the same result as evaluating $F(w^*)$. This can be used as a numerical flux function in the finite volume method. In order for this method to be stable, we need $\sigma_1 \leq c_{\min}$ and $\sigma_2 \geq c_{\max}$, where $c_{\min}$ is minimum of the wave speeds of the Riemann problem, and $c_{\max}$ is the maximum. In the case of a linear hyperbolic system, $c_{\min} = \lambda_{\min}$ and $c_{\max} = \lambda_{\max}$.

1.1 Local Lax–Friedrichs flux

The local Lax–Friedrichs numerical flux is given by the two-wave numerical flux, with the simplification that

$$\sigma_1 = -\alpha, \qquad \sigma_2 = \alpha,$$

with

$$\alpha \geq |c_i|,$$

for all wave speeds c_i. In this case, we use (1) to compute the numerical flux,

$$\begin{aligned}
\widehat{F}(u^-, u^+) := F* &= \frac{\sigma_1 F(u_R) - \sigma_2 F(u_L) - \sigma_1\sigma_2(u_R - u_L)}{\sigma_1 - \sigma_2} \\
&= \frac{-\alpha F(u_R) - \alpha F(u_L) + \alpha^2(u_R - u_L)}{-2\alpha} \\
&= \frac{1}{2}(F(u_L) + F(u_R)) + \frac{\alpha}{2}(u_L - u_R).
\end{aligned}$$

In the context of a finite volume method, we have

$$h_i \frac{\partial u_i}{\partial t} + \widehat{F}(u_i, u_{i+1}) - \widehat{F}(u_{i-1}, u_i) = 0.$$

Assuming a constant α and h_i, the Lax–Friedrichs method gives

$$\begin{aligned}
\frac{\partial u_i}{\partial t} &= \frac{1}{h}\left(-\widehat{F}(u_i, u_{i+1}) + \widehat{F}(u_{i-1}, u_i)\right) \\
&= \frac{1}{h}\left(-\frac{1}{2}\left(F(u_{i+1}) + F(u_i)\right) + \frac{\alpha}{2}(u_{i+1} - u_i) + \frac{1}{2}\left(F(u_i) + F(u_{i-1})\right) - \frac{\alpha}{2}(u_i - u_{i-1})\right) \\
&= -\frac{F(u_{i+1}) - F(u_{i-1})}{2h} + \frac{\alpha}{2}\frac{u_{i+1} - 2u_i + u_{i-1}}{h} \\
&= -\frac{F(u_{i+1}) - F(u_{i-1})}{2h} + \frac{h\alpha}{2}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}
\end{aligned}$$

The first term can be interpreted as a centered difference quotient for the flux. The second term can be interpreted as a centered *second-order* difference quotient for the solution, with coefficient $h\alpha/2$. In other words, this can be seen as an approximation of the differential equation

$$\frac{\partial u}{\partial t} + \nabla \cdot F = \epsilon \Delta u,$$

with $\epsilon = h\alpha/2$. The Lax–Friedrichs method can be interpreted as adding **numerical diffusion** to the central difference scheme; the amount of diffusion is proportional to the maximum wave speed, and vanishes as the cell size goes to zero.

2 Discontinuous Galerkin Methods

We now consider discontinuous Galerkin discretizations for systems of hyperbolic conservation laws

$$(*) \quad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{u}) = 0 \end{cases}$$

with m components, and flux function $\boldsymbol{F} : \mathbb{R}^m \to \mathbb{R}^{m \times d}$.

The setup is the same as in the case of linear advection. We are given a mesh $\mathcal{T}$ of elements (e.g. triangles) κ. The mesh skeleton is denoted Γ. Since there are m components, we consider the finite element space $\boldsymbol{V}_h = [V_h]^d$, where V_h is the piecewise polynomial space of degree p,

$$V_h = \{v \in L^2(\Omega) : v_\kappa \in \mathcal{P}^p(\kappa)\}.$$

To derive the DG method for $(*)$, we take $\boldsymbol{u}_h \in \boldsymbol{V}_h$, take the dot product with $\boldsymbol{v}_h \in \boldsymbol{V}_h$, and integrate the flux term by parts over each element, obtaining

$$\sum_{\kappa \in \mathcal{T}} \int_\kappa \frac{\partial \boldsymbol{u}_h}{\partial t} \cdot \boldsymbol{v}_h \, dx - \int_\kappa \boldsymbol{F}(\boldsymbol{u}) : \nabla \boldsymbol{v}_h \, dx + \int_{\partial \kappa} \boldsymbol{F}(\boldsymbol{u}) \boldsymbol{n} \cdot \boldsymbol{v}_h \, ds = 0.$$

As in the finite volume method, the key step is replacing $\boldsymbol{F}(\boldsymbol{u})$ on Γ with a **numerical flux function** $\widehat{\boldsymbol{F}}$. After summing over all $\kappa \in \mathcal{T}$, formulation then becomes

$$\int_\Omega \frac{\partial \boldsymbol{u}_h}{\partial t} \cdot \boldsymbol{v}_h \, dx - \int_\Omega \boldsymbol{F}(\boldsymbol{u}) : \nabla \boldsymbol{v}_h \, dx + \int_\Gamma \widehat{\boldsymbol{F}}(\boldsymbol{u}^-, \boldsymbol{u}^+, \boldsymbol{n}) : [\![\boldsymbol{v}_h]\!] \, ds = 0.$$

We follow the same methodology as in the one-dimensional finite volume method to choose the numerical flux function $\widehat{\boldsymbol{F}}$. Consider an edge e with normal vector $\boldsymbol{n}$. Let $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$ denote the two states $\boldsymbol{u}^\pm$ (they can be assigned arbitrarily). Consider the one-dimensional Riemann problem

$$(**) \quad \begin{cases} \dfrac{\partial \boldsymbol{w}}{\partial t} + \dfrac{\partial \boldsymbol{G}(\boldsymbol{w})}{\partial x} = 0, & \\ \qquad \boldsymbol{w}(x, 0) = \boldsymbol{u}_L, & x \le 0, \\ \qquad \boldsymbol{w}(x, 0) = \boldsymbol{u}_R, & x > 0. \end{cases}$$

The 1D flux function $\boldsymbol{G} : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $\boldsymbol{G}(\boldsymbol{w}) := \boldsymbol{F}(\boldsymbol{w})\boldsymbol{n}$. Generally, any numerical flux function $\widehat{\boldsymbol{G}}$ that can be used in the 1D finite volume method for $(**)$ can also be used in the discontinuous Galerkin method, by taking $\widehat{\boldsymbol{F}} = \widehat{\boldsymbol{G}}\boldsymbol{n}$. For example:

- **Godunov flux:** Let $\boldsymbol{w}^*$ be the solution to the 1D Riemann problem $(**)$. Let $\widehat{\boldsymbol{F}} = \boldsymbol{F}(\boldsymbol{w}^*)$.
- **Local Lax–Friedrichs flux:** Let α denote the maximum wave speed of Riemann problem $(**)$. Define

$$\widehat{\boldsymbol{F}}(\boldsymbol{u}^-, \boldsymbol{u}^+, \boldsymbol{n}) := \frac{1}{2} \left(\boldsymbol{F}(\boldsymbol{u}^-) + \boldsymbol{F}(\boldsymbol{u}^+) \right) + \alpha(\boldsymbol{u}^- - \boldsymbol{u}^+)\boldsymbol{n}.$$

- **Roe flux:** Let $\widehat{\boldsymbol{F}} = \boldsymbol{F}(\tilde{\boldsymbol{w}})$, where $\tilde{\boldsymbol{w}}$ is the solution to an appropriately chosen **linearized** Riemann problem.

These are only three possible choices of numerical flux function; there is a huge literature on (approximate) Riemann solvers and their impact on the associated numerical method.

It is worth mentioning that the choice of numerical flux function has a much greater impact on the solution quality of finite volume methods than on the discontinuous Galerkin method (especially for higher polynomial degrees p). This is because (for smooth solutions), the jump $[\![\boldsymbol{u}]\!]$ will be small ($[\![\boldsymbol{u}]\!] = \mathcal{O}(h^{p+1/2})$), and for continuous solutions, all consistent numerical flux functions agree.

High-order methods and Godunov's theorem

Note that for $p = 0$ (so the space V_h consists of piecewise constants), the discontinuous Galerkin method is exactly the first-order finite volume method. In this way, the DG method can be thought of as a:

- generalization of the finite volume method, allowing for higher-order convergence
- generalization of the finite element method, suitable for hyperbolic conservation laws

The DG method in some ways blends elements of both methods.

Of particular interest in hyperbolic conservation laws are the properties of **positivity preservation** and **monotonicity**. Scalar hyperbolic conservation laws satisfy the **maximum principle** that if $u_0(x) \in [a, b]$ for all x, then $u(t, x) \in [a, b]$ for all x and $t > 0$. It makes sense to look for numerical methods such that the approximate solution also satisfies the maximum principle. A method is **monotonicity-preserving** if it does not introduce new extrema. A famous theorem of Godunov from the 1950s is the following:

> **Godunov's Theorem:** Linear numerical methods having the property of not generating new extrema (monotonicity-preserving) can be at most first-order accurate.

The finite volume method is monotonicity-preserving under the condition that the numerical flux function $\widehat{F}(u_L, u_R)$ is monotone, meaning that it is non-decreasing in u_L and non-increasing in u_R. This is true for e.g. the Lax–Friedrichs flux. However, for $p > 0$, since the method is high-order accurate for smooth solutions (and the DG method is **linear**, meaning that the discretization of a linear equation is itself linear), we cannot expect the DG method to be monotonicity-preserving. This means that if we want to "fix" the higher-order DG method to make it monotonicity preserving, then the procedure necessarily has to be nonlinear (i.e. solution-dependent).