Lecture Note #4 **MTH653: Advanced Numerical Analysis**
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Finite Volume Method for Hyperbolic Conservation Laws

Consider the one-dimensional scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \qquad (*)$$

with flux function F. For simplicity, we assume periodic boundary conditions. We are interested in designing a numerical method for this equation analogous to the first-order finite volume method for the advection equation.

As before, we represent u using its cell averages,

$$u_i(t) := \frac{1}{h_i}\int_{x_i}^{x_{i+1}} u(x,t)\,dx.$$

Integrating $(*)$ and using divergence theorem on the second term, we obtain

$$h_i\frac{\partial u_i}{\partial t} + F(u(x_{i+1})) - F(u(x_i)) = 0.$$

In order to make this a well-defined numerical method, we need some way of approximating $F(u(x_i))$ and $F(u(x_{i+1}))$ in terms of the cell averages of u. In the case of the advection equation, we used the idea of **upwinding** to approximate $\beta u(x_i)$, i.e.

$$u(x_i) \approx \begin{cases} u_{i-1}, & \beta \geq 0 \\ u_i, & \beta < 0 \end{cases}$$

1 Godunov Method

For general nonlinear conservation laws, we generalize this idea using the concept of a **Riemann problem**. Different approximations of $F(u(x_i))$ give rise to different finite volume methods. The **Godunov method** approximates $u(x_i)$ by the **solution to the Riemann problem** with initial data u_{i-1} and u_i. Consider the Riemann problem

$$\frac{\partial w}{\partial t} + \frac{\partial F(w)}{\partial x} = 0,$$
$$w(x,0) = \begin{cases} u_{i-1}, & x \leq x_i \\ u_i, & x > x_i \end{cases}$$

This is the Riemann problem that one obtains if we assume that the initial conditions are given by the piecewise constants corresponding to the cell averages. There is a unique physically relevant self-similar solution w, i.e. $w^* := w(x_i, t)$ is uniquely defined for all $t > 0$. The Godunov method uses the approximation

$$F(u_{x_i}) \approx F(w^*).$$

Note that in the case of linear advection, the solution w^* to the Riemann problem is just the upwind value; therefore, for linear advection, the Godunov method corresponds exactly to the first-order upwind method.

1.1 Application to Burgers' Equation

Recall the solution to the Riemann problem for a 1D scalar conservation law (with convex flux F and initial data u_L and u_R) is given by

$$\text{if } u_L \geq u_R \text{, shock wave:} \qquad u(x,t) = \begin{cases} u_L, & x/t \leq \sigma \\ u_R, & x/t > \sigma \end{cases}$$

$$\text{if } u_L < u_R \text{, rarefaction wave:} \qquad u(x,t) = \begin{cases} u_L, & x/t \leq F'(u_L) \\ (F')^{-1}(x/t), & F'(u_L) < x/t < F'(u_R) \\ u_R, & x/t \geq F'(u_R) \end{cases}$$

Applying this to Burgers equation, we have

$$F(u) = \frac{1}{2}u^2 \quad \Longrightarrow \quad F'(u) = u \quad \Longrightarrow \quad (F')^{-1}(u) = u.$$

and

$$\sigma = \frac{F(u_L) - F(u_R)}{u_L - u_R} = \frac{1}{2}\frac{u_L^2 - u_R^2}{u_L - u_R} = \frac{1}{2}(u_L + u_R).$$

So, the solution w^* to the Riemann problem at $x/t = 0$ for $t > 0$ is given by

$$\text{if } u_L \geq u_R \text{, shock wave:} \qquad w^* = \begin{cases} u_L, & u_L + u_R \geq 0 \\ u_R, & u_L + u_R < 0 \end{cases}$$

$$\text{if } u_L < u_R \text{, rarefaction wave:} \qquad w^* = \begin{cases} u_L, & 0 \leq u_L \\ 0, & u_L < 0 < u_R \\ u_R, & 0 \geq u_R \end{cases}$$

from which the Godunov approximation to $F(u(x_i))$ can be evaluated simply.

Note that the Godunov approximation

$$F(u(x_i)) \approx F(w^*)$$

can be interpreted as a **numerical flux function**. We write

$$F(u(x_i)) \approx \widehat{F}(u_L, u_R) := F(w^*(u_L, u_R)),$$

since the Riemann solution w^* can be evaluated as a function of u_L and u_R. We say that the Godunov method uses an (exact) **Riemann solver** for the numerical flux function, since the numerical flux is given by the flux function evaluated at the (exact) solution to the associated Riemann problem. For complicated problems (especially for multi-component systems, which we have not yet covered), it can be complicated or computationally expensive to find the exact solution to the Riemann problem (even for scalar problems, it requires evaluating $(F')^{-1}$, which could require solving a nonlinear root finding problem e.g. using Newton's method or another iterative method). This will motivate using **approximate Riemann solvers** to compute the numerical flux; this topic will be covered later.

2 Hyperbolic Systems

2.1 Linear Hyperbolic Systems

We now consider one-dimensional multi-component linear hyperbolic systems. Let $\boldsymbol{u} \in \mathbb{R}^m$, with $m \geq 1$. Define the linear flux function $F : \mathbb{R}^m \to \mathbb{R}^m$,

$$F(\boldsymbol{u}) = A\boldsymbol{u},$$

where $A \in \mathbb{R}^{m \times m}$. The system

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial A\boldsymbol{u}}{\partial x} = 0 \qquad (**)$$

is **hyperbolic** if the matrix A is diagonalizable and has real eigenvalues. Let R denote the matrix of eigenvectors, so that

$$AR = R\Lambda,$$

where Λ is the diagonal matrix of eigenvectors. So,

$$R^{-1}AR = \Lambda, \qquad A = R\Lambda R^{-1}.$$

We perform a change of variables, writing

$$\boldsymbol{w} = R^{-1}\boldsymbol{u}, \qquad \boldsymbol{u} = R\boldsymbol{w}.$$

Then,

$$\frac{\partial R\boldsymbol{w}}{\partial t} + \frac{\partial AR\boldsymbol{w}}{\partial x} = 0$$

Left-multiplying by R^{-1}, we obtain

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial R^{-1}AR\boldsymbol{w}}{\partial x} = 0,$$

equivalently

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial \Lambda \boldsymbol{w}}{\partial x} = 0. \qquad (***)$$

Therefore, the linear hyperbolic system $(**)$ is **equivalent** to a **decoupled system** of m **linear advection equations**, where the ith equation has advection velocity λ_i.

In order to solve $(**)$, we can change variables from $\boldsymbol{u}$ to $\boldsymbol{w}$ and solve $(***)$, or, we can implement a numerical flux function based on solving the Riemann problem.

2.2 Nonlinear Hyperbolic Systems

A system written in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + A(\boldsymbol{u}, x, t)\frac{\partial \boldsymbol{u}}{\partial x} = 0$$

is called **quasilinear**. Such a system is **hyperbolic at** $(\boldsymbol{u}, x, t)$ if the matrix $A(\boldsymbol{u}, x, t)$ is diagonalizable with real eigenvalues.

The fully nonlinear system

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{u})}{\partial x} = 0$$

can be written in quasilinear form as

$$\frac{\partial \boldsymbol{u}}{\partial t} + D_F(\boldsymbol{u}, x, t)\frac{\partial \boldsymbol{u}}{\partial x} = 0,$$

where D_F is the Jacobian matrix

$$D_F := \left(\frac{\partial F_i}{\partial u_j}\right).$$

The nonlinear system is **hyperbolic** iff its quasilinear form is hyperbolic.

2.3 Example: Simple Fluid Equations

Consider a gas with density ρ and velocity u. The law of conservation of mass takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0.$$

This is often called the *continuity equation*. In order to close the system, we also need an equation the velocity, which will come from conservation of momentum. Note that the integral of ρu over a region $\mathcal{D}$ gives the **total momentum** in $\mathcal{D}$; the quantity ρu is the density of momentum, and it is a conserved variable. Since the fluid moves with its velocity, the momentum is advected with velocity u, leading to a contribution to the momentum flux of the form $(\rho u)u$. The momentum is also acted on by the **pressure** p of the fluid, leading to the equation

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial t} = 0.$$

Here we have introduced a third variable (pressure); pressure is not a conserved variable. Instead, we typically use the law of conservation of energy to introduce a third equation that closes the system (using an **equation of state** that determines the pressure given density, momentum, and energy). For the sake of simplicity we make the assumption that the fluid is **isentropic**. In this simplified case, the pressure can be determined solely from the density,

$$p := P(\rho),$$

for a given function P. In our example, we will take

$$P(\rho) = \hat{\kappa}\rho^{\gamma},$$

where $\hat{\kappa} > 0$ and $\gamma > 1$ are constants.

These equations can be written in the form of a conservation law with

$$\boldsymbol{u} = \begin{pmatrix} \rho \\ \rho u \end{pmatrix} =: \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad \boldsymbol{F}(\boldsymbol{u}) = \begin{pmatrix} \rho u \\ \rho u^2 + P(\rho) \end{pmatrix} = \begin{pmatrix} q_2 \\ q_2^2/q_1 + P(q_1) \end{pmatrix}.$$

The associated system of conservation laws

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{u})}{\partial x} = 0$$

is hyperbolic if the Jacobian of $\boldsymbol{F}$ is diagonalizable with real eigenvalues. We compute

$$D_F = \left(\frac{\partial F_i}{\partial u_j}\right) = \begin{pmatrix} 0 & 1 \\ -(q_2/q_1)^2 + P'(q_1) & 2q_2/q_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u^2 + P'(\rho) & 2u \end{pmatrix}.$$

The eigenvalues of this matrix can be computed,

$$\lambda_1 = u - \sqrt{P'(\rho)}, \quad \lambda_2 = u + \sqrt{P'(\rho)}.$$

The eigenvalues are real iff $P'(\rho) \geq 0$. For the equation of state $P(\rho) = \hat{\kappa}\rho^\gamma$, we have

$$P'(\rho) = \hat{\kappa}\gamma\rho^{\gamma-1}$$

Since $\hat{\kappa} > 0$, and $\gamma > 1$, we have that $P'(\rho) > 0$ when $\rho > 0$. Therefore, these equations are hyperbolic for all **physically relevant states** (i.e. those with positive density).

Example: Linear Acoustics

Using the same simple fluid model as in the preceding section, and solving for the **perturbation** of density and velocity $\tilde{\rho}$ and $\tilde{u}$ around an initial state ρ_0 and u_0, one can obtain the system (dropping the tildes)

$$\boldsymbol{u} = (\rho, u), \qquad \boldsymbol{F}(\boldsymbol{u}) = A\boldsymbol{u} = \begin{pmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{pmatrix} \boldsymbol{u},$$

where $K_0 = \rho_0 P'(\rho_0)$.

As an example, we will take $u_0 = 0$, $\rho_0 = 1/4$, $K = 2$. Then,

$$A = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$$

which has eigenvalues $\lambda_i = \pm 2\sqrt{2}$ and eigenvectors $(\pm 1/\sqrt{2}, 1)$. So this system has **one left-moving wave** and **one right-moving wave**.

3 Local Lax–Friedrichs Flux

Since solving a Riemann problem exactly may be challenging, it can be useful to approximate the solution to the Riemann problem using an approximate Riemann solver. Perhaps the simplest of these is the **local Lax–Friedrichs**, which is defined as follows

$$\widehat{\boldsymbol{F}}(\boldsymbol{u}_L, \boldsymbol{u}_R) := \frac{1}{2}(\boldsymbol{F}(\boldsymbol{u}_L) + \boldsymbol{F}(\boldsymbol{u}_R)) + \alpha(\boldsymbol{u}_L - \boldsymbol{u}_R),$$

where α is the **maximum wave speed**, i.e. a bound on the maximum speed of any wave of the true solution to the associated Riemann problem. The local Lax–Friedrichs method

does not require solving the Riemann problem, only bounding the speed of the waves of the solution. For example, for a linear system, the maximum wave speed is the maximum (absolute) eigenvalue. For Burgers' equation, the maximum wave speed is $\max\{|u_L|, |u_R|\}$.

In the first-order finite volume method, the local Lax–Friedrichs method is much more **dissipative** than the Godunov method; it adds much more **numerical diffusion** (and smears the solution out). So this method trades simplicity for accuracy. However, for DG methods, the effect is greatly reduced, and local Lax–Friedrichs can be used as a numerical flux without overly sacrificing solution quality.