Lecture Note #3 **MTH653: Advanced Numerical Analysis**
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Hyperbolic Conservation Laws

Previously, we had considered the linear advection equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u) = 0$$

as our model problem. We now consider a more general type of problem (of which the advection equation is one example), called **hyperbolic conservation law**. The general form of such an equation is

$$(*) \quad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{u}) = 0, \\ \qquad\qquad \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}). \end{cases}$$

As before, the spatial domain is $\Omega \subseteq \mathbb{R}^d$ and the temporal domain is $[0, T]$. In the general case, the solution is a **vector** $\boldsymbol{u} \in \mathbb{R}^m$, where m is the **number of solution components**. The function $\boldsymbol{F} : \mathbb{R}^m \to \mathbb{R}^{m \times d}$ is called the **flux function**. In the case of the linear advection equation,

$$m = 1, \qquad \boldsymbol{F}(u) = \boldsymbol{\beta} u.$$

The equation is **linear** because the **flux function is linear**. However, in the general case, $\boldsymbol{F}$ may be nonlinear, in which case the governing PDE is also nonlinear. We haven't yet specified the boundary conditions for $(*)$; this will be discussed later. This type of initial-value problem is called a **Cauchy problem**.

The system $(*)$ is called a conservation law for the following reason. Let $\mathcal{D} \subseteq \Omega$ denote a subdomain of Ω. Integrating $(*)$ over $\mathcal{D}$, using divergence theorem, and switching order of integration and differentiation, results in

$$\frac{\partial}{\partial t} \int_{\mathcal{D}} \boldsymbol{u} \, dx = - \int_{\partial \mathcal{D}} \boldsymbol{F}(\boldsymbol{u}) \cdot \boldsymbol{n} \, ds.$$

Here, $\boldsymbol{u}$ is called a **conserved quantity** and $\boldsymbol{F}$ is the **flux**; the flux on the boundary of $\mathcal{D}$ measures the change in the total conserved quantity in $\mathcal{D}$. In particular, if the flux is zero on $\partial\mathcal{D}$, then the total quantity in $\mathcal{D}$ does not change. The conserved quantities can be used to model quantities for which we have physical conservation laws, such as mass, momentum, energy, etc.

1 Method of Characteristics

Much of the theory will hold in d dimensions, but exposition is simpler for $d = 1$. We will consider for now the case of **scalar equations**, i.e. $m = 1$,

$$(**) \quad \left\{ \frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0. \right.$$

Let u be a smooth solution to this PDE. Consider the ordinary differential equation

$$(***) \begin{cases} \dfrac{\partial \chi_s(t)}{\partial t} = F'(u(\chi_s(t), t)) \\ \chi_s(0) = s. \end{cases}$$

For fixed s, $\chi_s(t)$ defines a curve in $\mathbb{R} \times [0, T]$; these are called **characteristic curves**. Consider the solution to the PDE **along the characteristic curves**, i.e.

$$\psi_s(t) := u(\chi_s(t), t).$$

We can differentiate,

$$\begin{aligned} \frac{\partial \psi_s(t)}{\partial t} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial \chi_s}{\partial t} \\ &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}F'(u(\chi_s(t), t)) \\ &= \frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} \\ &= 0, \end{aligned}$$

since u solves $(**)$. From this, we conclude the the solution u to $(**)$ is **constant along characteristic curves**. Hence, $\psi_s(t) = u_0(s)$ for all t.

From this, we see that $u(\chi_s(t), t) = u_0(s)$, and so the characteristic ODE becomes

$$\begin{aligned} \frac{\partial \chi_s(t)}{\partial t} &= F'(u(\chi_s(t), t)) = F'(u_0(s)) \\ \chi_s(0) &= s. \end{aligned}$$

Since the right-hand side above is constant, we can write the solution to the ODE explicitly,

$$\chi_s(t) = s + F'(u_0(s))t,$$

which defines a **straight line** in (x, t) space.

In the case of the linear advection equation,

$$\frac{\partial u}{\partial t} + \frac{\partial \beta u}{\partial x} = 0,$$

the characteristics are all parallel lines with slope β.

2 Burgers' Equation

Perhaps the simplest nonlinear conservation law is **Burgers' equation**. This is similar to the advection equation, but the **advection velocity** is **proportional to the solution**, $\beta = \frac{1}{2}u$. Hence,

$$F(u) = \frac{1}{2}u^2,$$

giving the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0.$$

In this case, $F'(u) = u$, and so the characteristic curve starting at point s is a **straight line with slope** $u_0(s)$. If the initial condition u_0 is an increasing function, then these curves do not intersect, and we can determine the solution for all (x,t) using the method of characteristics, by tracing back to the initial condition. However, if the initial condition u_0 is **decreasing**, then the characteristics may **intersect**; this means that we cannot trace back to a unique point in the initial state. At the intersection point of multiple characteristics, the solution becomes multi-valued. This is a fundamental property of nonlinear conservation laws; even for smooth initial conditions, the solution may become non-smooth in finite time.

3 Weak Solutions

To make sense of potentially non-smooth solutions to the governing equation $(*)$, we consider a weak formulation, by multiplying by a smooth test function ϕ and integrating by parts. For any $\mathcal{D} \subseteq \Omega$ with $\phi \in C_0^1(\mathcal{D} \times [0,T])$, we have

$$\int_0^T \int_{\mathcal{D}} \left(u\frac{\partial \phi}{\partial t} + \boldsymbol{F}(u) \cdot \nabla\phi \right) dx\, dt + \int_{\mathcal{D}} \phi(\boldsymbol{x}, 0) u_0(\boldsymbol{x})\, dx = 0. \tag{1}$$

The function u is a **weak solution** to $(*)$ if it satisfies (1) for all test functions.

However, (1) is not sufficient to determine a **unique solution**. For example, consider Burgers' equation with

$$u_0(x) = H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Consider

$$u_1(x,t) = H(x - t/2), \qquad u_2(x,t) = \begin{cases} 0, & x < 0 \\ x/t, & 0 < x < t \\ 1, & x > t. \end{cases}$$

It is possible to check (exercise) that **both** u_1 and u_2 are weak solutions to Burgers' equations with initial conditions u_0.

Therefore, in order to specify a well-defined **unique** solution, we need some additional criteria. Among all the (possibly infinite) weak solutions to the PDE, we select one unique solution that we say is **physically relevant**. The physically relevant solution is the unique limit of the solution to the perturbed problem

$$\frac{\partial u_\epsilon}{\partial t} + \nabla \cdot \boldsymbol{F}(u_\epsilon) - \epsilon \Delta u_\epsilon = 0. \tag{2}$$

Here, we have added a **viscous term** $\epsilon \Delta u_\epsilon$. Formally, as $\epsilon \to 0$, the perturbed problem (2) approaches the original problem $(*)$. For any $\epsilon > 0$, there exists a unique solution u_ϵ to (1). It can be shown that among all weak solutions, there is one unique solution u such that

$$\lim_{\epsilon \to 0} \| u - u_\epsilon \|_{L^1(\Omega \times [0,T])} = 0.$$

This is chosen as the unique **physically relevant** solution to $(*)$. We would like numerical methods that, when applied to $(*)$, converge to the physically relevant weak solution.

4 Riemann Problems

A **Riemann problem** is a particular instance of the **Cauchy problem**. Consider the Cauchy problem associated with the one-dimensional scalar equation $(**)$. The **Riemann problem** is

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \tag{3}$$

with **discontinuous initial conditions**

$$u(x,0) = \begin{cases} u_L, & x \leq 0 \\ u_R, & x > 0 \end{cases}$$

Note that if $u_L = u_R$ the problem becomes trivial, and so we consider $u_L \neq u_R$. The solution to the Riemann problem is **self-similar**, meaning that it depends only on the ratio x/t and not on x and t separately. To see this, suppose $u(x,t)$ is a solution to (3). Let $u_\lambda(x,t) := u(\lambda x, \lambda t)$. Then, it is immediate that for $\lambda > 0$, u_λ is also a solution to (3). So, let $\xi := x/t$, and we look for a solution $u(x,t) = w(x/t) = w(\xi)$. Substitute this form of $u(x,t)$ into (3) to obtain

$$-\frac{xw'(\xi)}{t^2} + \frac{1}{t}F'(w(\xi))w'(\xi) = 0$$

From this, we see

$$\begin{aligned} tF'(w(\xi))w'(\xi) &= xw'(\xi) \\ tF'(w(\xi)) &= x \\ F'(w(\xi)) &= \xi \end{aligned}$$

which must hold in order for $u(x,t)$ to be a solution (where $w'(\xi) \neq 0$).

We assume that the flux F is convex. Then, if $u_L > u_R$, the unique physically relevant solution is

$$u(x,t) = \begin{cases} u_L, & x/t \leq \sigma \\ u_R, & x/t > \sigma \end{cases}$$

where

$$\sigma := \frac{F(u_L) - F(u_R)}{u_L - u_R}.$$

This is called a **shock wave**, and σ is the **shock speed**. The shock speed can be derived as follows. Consider the interval $[-a, a]$. Then,

$$\frac{\partial}{\partial t}\int_{-a}^{a} u(x,t)\,dx = \int_{-a}^{a} \frac{\partial}{\partial x}F(u)\,dx = F(a) - F(-a).$$

On the other hand,

$$\int_{-a}^{a} u(x,t)\,dx = (M+\sigma t)u_L + (M-\sigma t)u_R.$$

From this,

$$\frac{\partial}{\partial t}\int_{-a}^{a} u(x,t)\,dx = \sigma(u_L - u_R).$$

Combining the above, we obtain

$$\sigma = \frac{F(u_L)-F(u_R)}{u_L-u_R}.$$

If $u_L < u_R$, the unique physically relevant solution is

$$u(x,t) = \begin{cases} u_L, & x/t \le F'(u_L) \\ (F')^{-1}(x/t), & F'(u_L) < x/t < F'(u_R) \\ u_R, & x/t \ge F'(u_R) \end{cases}$$

This is called a **rarefaction wave** or **expansion**.