Lecture Note #7 **MTH653: Advanced Numerical Analysis**
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Preconditioners for discontinuous Galerkin discretizations

In the previous lecture, we analyzed the stability and accuracy of DG methods for the Poisson problem

$$-\Delta u = f.$$

We now turn our attention to the development of preconditioners for the resulting linear system

$$Ax = b.$$

Since the interior penalty bilinear form is symmetric and coercive, the resulting matrix A is symmetric and positive-definite. Recall that the interior penalty bilinear form is defined by

$$a(u,v) := \int_\Omega \nabla_h u \cdot \nabla_h v \, dx - \int_\Gamma [\![u]\!] \cdot \{\nabla_h v\} \, ds - \int_\Gamma \{\nabla_h u\} \cdot [\![v]\!] \, ds + \int_\Gamma \alpha [\![u]\!] \cdot [\![v]\!] \, ds.$$

Since we require $\alpha \gtrsim h^{-1}$, we write

$$\alpha := \eta h^{-1} \tag{1}$$

for some constant η bounded below.

We continue to use the same definition of the DG norm,

$$|\!|\!| u |\!|\!|^2 := \|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1} \|[\![u]\!]\|^2_{L^2(\Gamma)}.$$

We recall the continuity and coercivity properties,

$$\begin{aligned} a(u,v) &\lesssim |\!|\!| u |\!|\!| \, |\!|\!| v |\!|\!|, \\ a(u,u) &\gtrsim |\!|\!| u |\!|\!|^2. \end{aligned}$$

First, we wish to establish some condition number estimates. A useful tool will be the following Poincaré-type inequality.

Lemma 1. *Let $u \in H^1(\mathcal{T})$. Then,*

$$\|u\|^2_{L^2(\Omega)} \lesssim \|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1} \|[\![u]\!]\|^2_{L^2(\Gamma)} = |\!|\!| u |\!|\!|^2. \tag{2}$$

Proof. Let ψ solve the auxiliary problem

$$-\Delta \psi = u.$$

Then, $\|\psi\|_{H^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)}$. Furthermore,

$$\begin{aligned} \|u\|^2_{L^2(\Omega)} &= (u, -\Delta\psi) \\ &= (\nabla_h u, \nabla \psi) - \int_\Gamma \nabla\psi \cdot [\![u]\!] \, ds \\ &\leq \left(\|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1} \|[\![u]\!]\|^2_{L^2(\Gamma)} \right)^{1/2} \left(\|\nabla \psi\|^2_{L^2(\Omega)} + h \|\partial\psi/\partial n\|^2_{L^2(\Gamma)} \right)^{1/2} \end{aligned}$$

A scaling argument shows that

$$\|\partial\psi/\partial n\|_{L^2(\Gamma)}^2 \lesssim h^{-1}\|\psi\|_{H^2(\Omega)}^2,$$

from which we conclude that

$$\|\nabla\psi\|_{L^2(\Omega)}^2 + h\|\partial\psi/\partial n\|_{L^2(\Gamma)}^2 \lesssim \|\psi\|_{H^2(\Omega)}^2 \lesssim \|u\|_{L^2(\Omega)}^2.$$

Dividing through by $\|u\|_{L^2(\Omega)}$, we obtain the desired result. □

We will also make use the following **inverse inequality** and **trace inequality**, which both follow from a scaling argument, and equivalence of finite-dimensional norms. For any $u_h \in V_h$, we have that

$$\begin{aligned} \|u_h\|_{H^1(\Omega)} &\lesssim h^{-1}\|u_h\|_{L^2(\Omega)}, \\ \|u_h\|_{L^2(\partial\kappa)} &\lesssim h^{-1/2}\|u_h\|_{L^2(\kappa)}. \end{aligned}$$

These estimates can be combined to give the following result, which can be used to estimate the eigenvalues of the matrix A.

Proposition 1. *For any $u_h \in V_h$, it holds that*

$$\|u_h\|_{L^2(\Omega)}^2 \lesssim a(u_h, u_h) \lesssim \eta h^{-2}\|u_h\|_{L^2(\Omega)}^2.$$

Proof. The lower bound follows directly from (1) and coercivity of $a(\cdot\cdot)$ in the $|||\cdot|||$ norm. The upper bound can be seen by noting

$$\begin{aligned} a(u_h, u_h) &\lesssim \eta |||u_h|||^2 && \text{(continuity)} \\ &= \eta \left(\|\nabla_h u\|_{L^2(\Omega)}^2 + h^{-1}\|[\![u]\!]\|_{L^2(\Gamma)}^2 \right) && \text{(definition of } |||\cdot|||) \\ &\lesssim \eta \left(h^{-2}\|u\|_{L^2(\Omega)}^2 + h^{-1}\|[\![u]\!]\|_{L^2(\Gamma)}^2 \right) && \text{(inverse inequality).} \end{aligned}$$

On a given edge e adjacent to elements $\kappa^\pm$ with traces $u_h^\pm$,

$$\begin{aligned} \|[\![u_h]\!]\|_{L^2(e)}^2 &\leq 2\left(\|u_h^-\|_{L^2(e)}^2 + \|u_h^+\|_{L^2(e)}^2 \right) && \text{(Young's inequality)} \\ &\lesssim h^{-1}\|u_h\|_{L^2(\kappa^-)}^2 + h^{-1}\|u_h\|_{L^2(\kappa^+)}^2 && \text{(trace inequality)} \end{aligned}$$

and therefore

$$\|[\![u]\!]\|_{L^2(\Gamma)}^2 \lesssim h^{-1}\|u_h\|_{L^2(\Omega)}^2.$$

From this, the upper bound follows. □