Lecture Note #7 **MTH653: Advanced Numerical Analysis**
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Preconditioners for discontinuous Galerkin discretizations

1 Condition Number Estimates

In the previous lecture, we analyzed the stability and accuracy of DG methods for the Poisson problem
$$-\Delta u = f.$$
We now turn our attention to the development of preconditioners for the resulting linear system
$$Ax = b.$$
Since the interior penalty bilinear form is symmetric and coercive, the resulting matrix A is symmetric and positive-definite. Recall that the interior penalty bilinear form is defined by
$$a(u, v) := \int_\Omega \nabla_h u \cdot \nabla_h v \, dx - \int_\Gamma [\![u]\!] \cdot \{\nabla_h v\} \, ds - \int_\Gamma \{\nabla_h u\} \cdot [\![v]\!] \, ds + \int_\Gamma \alpha [\![u]\!] \cdot [\![v]\!] \, ds.$$
Since we require $\alpha \gtrsim h^{-1}$, we write
$$\alpha := \eta h^{-1} \tag{1}$$
for some constant η bounded below.

We continue to use the same definition of the DG norm,
$$|\!|\!| u |\!|\!|^2 := \|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1} \|[\![u]\!]\|^2_{L^2(\Gamma)}.$$
We recall the continuity and coercivity properties,
$$\begin{aligned} a(u, v) &\lesssim |\!|\!| u |\!|\!| \, |\!|\!| v |\!|\!|, \\ a(u, u) &\gtrsim |\!|\!| u |\!|\!|^2. \end{aligned}$$

First, we wish to establish some condition number estimates. A useful tool will be the following Poincaré-type inequality.

Lemma 1. *Let $u \in H^1(\mathcal{T})$. Then,*
$$\|u\|^2_{L^2(\Omega)} \lesssim \|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1} \|[\![u]\!]\|^2_{L^2(\Gamma)} = |\!|\!| u |\!|\!|^2. \tag{2}$$

Proof. Let ψ solve the auxiliary problem
$$-\Delta \psi = u.$$
Then, $\|\psi\|_{H^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)}$. Furthermore,
$$\begin{aligned} \|u\|^2_{L^2(\Omega)} &= (u, -\Delta \psi) \\ &= (\nabla_h u, \nabla \psi) - \int_\Gamma \nabla \psi \cdot [\![u]\!] \, ds \\ &\leq \left(\|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1} \|[\![u]\!]\|^2_{L^2(\Gamma)} \right)^{1/2} \left(\|\nabla \psi\|^2_{L^2(\Omega)} + h \|\partial \psi / \partial n\|^2_{L^2(\Gamma)} \right)^{1/2} \end{aligned} \tag{3}$$

A scaling argument shows that

$$\|\partial\psi/\partial n\|^2_{L^2(\Gamma)} \lesssim h^{-1}\|\psi\|^2_{H^2(\Omega)}.$$

Here is a quick recap of this argument; it is very similar to scaling arguments shown before: By the trace theorem and scaling to the reference element, we have for $\theta \in H^1(\kappa)$,

$$\begin{aligned}
\|\theta\|^2_{L^2(\partial\kappa)} &= h^{d-1}\|\hat\theta\|^2_{L^2(\partial\hat\kappa)} \\
&\lesssim h^{d-1}\|\hat\theta\|^2_{H^1(\hat\kappa)} \\
&= h^{d-1}\left(\|\hat\theta\|_{L2(\hat\kappa)} + \|\nabla\hat\theta\|_{L^2(\hat\kappa)}\|^2\right) \\
&= h^{d-1}\left(h^{-d}\|\theta\|_{L2(\kappa)} + h^{-d}h^2\|\nabla\theta\|_{L^2(\kappa)}\|^2\right) \\
&= h^{-1}\|\theta\|_{L2(\kappa)} + h\|\nabla\theta\|_{L^2(\kappa)}\|^2.
\end{aligned}$$

Applying this to $\partial\psi/\partial n$,

$$\begin{aligned}
\|\partial\psi/\partial n\|^2_{L^2(\partial\kappa)} &\lesssim h^{-1}\|\nabla\psi\|^2_{L^2(\kappa)} + h|\psi|^2_{H^2(\kappa)} \\
&\lesssim h^{-1}\|\psi\|^2_{H^2(\kappa)}.
\end{aligned}$$

Summing over all triangles in the mesh, we arrive at the claimed inequality.

From this scaling argument we conclude that

$$\|\nabla\psi\|^2_{L^2(\Omega)} + h\|\partial\psi/\partial n\|^2_{L^2(\Gamma)} \lesssim \|\psi\|^2_{H^2(\Omega)} \lesssim \|u\|^2_{L^2(\Omega)}. \tag{4}$$

Combining (3) with (4), we see that

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)} &\leq \left(\|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1}\|[\![u]\!]\|^2_{L^2(\Gamma)}\right)^{1/2}\left(\|\nabla\psi\|^2_{L^2(\Omega)} + h\|\partial\psi/\partial n\|^2_{L^2(\Gamma)}\right)^{1/2} \\
&\leq \left(\|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1}\|[\![u]\!]\|^2_{L^2(\Gamma)}\right)^{1/2}\|u\|_{L^2(\Omega)}
\end{aligned}$$

Dividing through by $\|u\|_{L^2(\Omega)}$, we obtain the desired result. □

We will also make use the following **inverse inequality** and **trace inequality**. These both follow from a scaling argument combined with equivalence of finite-dimensional norms. The proofs are given here for completeness.

Proposition 1. *For any $u_h \in V_h$, we have that*

$$\|u_h\|_{H^1(\mathcal{T})} \lesssim h^{-1}\|u_h\|_{L^2(\Omega)}, \tag{5}$$

$$\|u_h\|_{L^2(\partial\kappa)} \lesssim h^{-1/2}\|u_h\|_{L^2(\kappa)}. \tag{6}$$

Proof. On a triangle $\kappa = h\hat\kappa$,

$$\begin{aligned}
\|u_h\|^2_{H^1(\kappa)} &= \|u_h\|^2_{L^2(\kappa)} + \|\nabla u_h\|^2_{L^2(\kappa)} \\
&= h^d\|\hat u_h\|^2_{L^2(\hat\kappa)} + h^{d-2}\|\nabla\hat u_h\|^2_{L^2(\hat\kappa)}.
\end{aligned}$$

Recall that all norms defined on a finite-dimensional space are equivalent. Consider the space $\mathcal{P}^p(\hat{\kappa})$; for any $q \in \mathcal{P}^p(\hat{\kappa})$,

$$\|q\|^2_{L^2(\hat{\kappa})} \approx \|q\|^2_{H^1(\hat{\kappa})}$$

and so

$$\|\nabla q\|_{L^2(\hat{\kappa})} \leq \|q\|^2_{H^1(\hat{\kappa})} \approx \|q\|^2_{L^2(\hat{\kappa})},$$

and therefore

$$\|\nabla \hat{u}_h\|^2_{L^2(\hat{\kappa})} \lesssim \|u_h\|_{L^2(\hat{\kappa})}.$$

This implies that

$$\begin{aligned}
\|u_h\|^2_{H^1(\kappa)} &= h^d \|\hat{u}_h\|^2_{L^2(\hat{\kappa})} + h^{d-2}\|\nabla \hat{u}_h\|^2_{L^2(\hat{\kappa})} \\
&\lesssim h^d \|\hat{u}_h\|^2_{L^2(\hat{\kappa})} + h^{d-2}\|\hat{u}_h\|^2_{L^2(\hat{\kappa})} \\
&= (h^d + h^{d-2})\|\hat{u}_h\|^2_{L^2(\hat{\kappa})} \\
&= (1 + h^{-2})\|u_h\|^2_{L^2(\kappa)} \\
&\lesssim h^{-2}\|u_h\|^2_{L^2(\kappa)}.
\end{aligned}$$

Summing over all triangles yields the inverse inequality (5).

The trace inequality (6) follows from a similar argument. □

These estimates can be combined to give the following result, which can be used to estimate the eigenvalues of the matrix A.

Proposition 2. *For any $u_h \in V_h$, it holds that*

$$\|u_h\|^2_{L^2(\Omega)} \lesssim a(u_h, u_h) \lesssim \eta h^{-2}\|u_h\|^2_{L^2(\Omega)}.$$

Proof. The lower bound follows directly from (1) and coercivity of $a(\cdot\cdot)$ in the $|\!|\!| \cdot |\!|\!|$ norm. The upper bound can be seen by noting

$$\begin{aligned}
a(u_h, u_h) &\lesssim \eta |\!|\!| u_h |\!|\!|^2 && \text{(continuity)} \\
&= \eta \left(\|\nabla_h u\|^2_{L^2(\Omega)} + h^{-1}\|[\![u]\!]\|^2_{L^2(\Gamma)} \right) && \text{(definition of } |\!|\!| \cdot |\!|\!|\text{)} \\
&\lesssim \eta \left(h^{-2}\|u\|^2_{L^2(\Omega)} + h^{-1}\|[\![u]\!]\|^2_{L^2(\Gamma)} \right) && \text{(inverse inequality).}
\end{aligned}$$

On a given edge e adjacent to elements $\kappa^\pm$ with traces $u_h^\pm$,

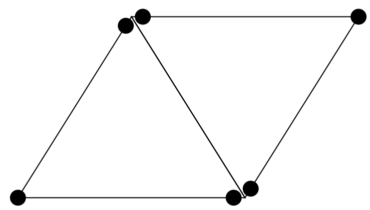
$$\begin{aligned}
\|[\![u_h]\!]\|^2_{L^2(e)} &\leq \frac{3}{2}\left(\|u_h^-\|^2_{L^2(e)} + \|u_h^+\|^2_{L^2(e)} \right) && \text{(Young's inequality)} \\
&\lesssim h^{-1}\|u_h\|^2_{L^2(\kappa^-)} + h^{-1}\|u_h\|^2_{L^2(\kappa^+)} && \text{(trace inequality)}
\end{aligned}$$

and therefore

$$\|[\![u]\!]\|^2_{L^2(\Gamma)} \lesssim h^{-1}\|u_h\|^2_{L^2(\Omega)}.$$

From this, the upper bound follows. □

In order to discuss the condition number of the matrix A, it is necessary to describe the basis $\{\varphi_i\}$ such that $A_{ij} = a(\varphi_i, \varphi_j)$. The standard Lagrange basis for the H^1-conforming space can be modified slightly to obtain a basis for the DG space. Instead of "gluing" basis functions that are associated with the same nodal point, we simply do not glue them, and obtain distinct basis functions for each element. To make this precise: for each triangle $\kappa \in \mathcal{T}$, there are nodal points $\{\boldsymbol{x}_i\} \subseteq \kappa$, and associated Lagrange polynomials ℓ_i satisfying $\ell_i(\boldsymbol{x}_j) = \delta_{ij}$. Each function ℓ_i is extended to $L^2(\Omega)$ by setting $\ell_i(\boldsymbol{x}) = 0$ for $x \notin \kappa$.



In this basis, the nodal points at vertices are **duplicated** across all elements containing that vertex. Each basis function φ_i is associated with a unique element κ, and satisfies the property that $\varphi_i(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \notin \kappa$. This construction immediately generalizes to higher-order elements. It is also worth noting that the DG space affords more freedom in the selection of nodal points: unlike the H^1 space, which requires that enough nodal points lie on all element vertices/edges/faces to ensure continuity, in the DG method, no continuity is required, and all nodal points can lie in the interior of the element. This has some algorithmic implications. In what follows, we will consider the low-order case ($p = 1$), with nodal points at mesh vertices.

Theorem 1. *Let A be the matrix associated with the DG bilinear form $a(\cdot\,\cdot)$ in the standard (Lagrange) basis. Then, the extremal eigenvalues of A satisfy*

$$\lambda_{\min}(A) \approx h^2, \qquad \lambda_{\max}(A) \approx \eta.$$

Consequently,

$$\kappa(A) \approx \eta h^{-2}.$$

Proof. Let $u_h \in V_h$, and let $\boldsymbol{u}$ denote the associated vector of degrees of freedom. From the textbook (Lemma 7.3), we know

$$\|u_h\|^2_{L^2(\Omega)} \approx h^2 \|\boldsymbol{u}\|^2_{\ell^2}.$$

Then, the Rayleigh quotient satisfies

$$\frac{\boldsymbol{u}^T A \boldsymbol{u}}{\|\boldsymbol{u}\|^2_{\ell^2}} \approx h^2 \frac{a(u_h, u_h)}{\|u_h\|^2_{L^2(\Omega)}}.$$

By Proposition 2,

$$\frac{\|u_h\|^2_{L^2(\Omega)}}{\|u_h\|^2_{L^2(\Omega)}} \lesssim \frac{a(u_h, u_h)}{\|u_h\|^2_{L^2(\Omega)}} \lesssim \frac{\eta h^{-2} \|u_h\|^2_{L^2(\Omega)}}{\|u_h\|^2_{L^2(\Omega)}}$$

and so

$$h^2 \lesssim \frac{\boldsymbol{u}^T A \boldsymbol{u}}{\|\boldsymbol{u}\|^2_{\ell^2}} \lesssim \eta. \qquad \square$$

This theorem tells us that the DG method has essentially the same conditioning as the standard H^1-conforming method, but the system becomes worse-conditioned as the size of the penalty parameter increases.

2 Domain Decomposition Preconditioning

We now develop a preconditioner for the DG method using the ideas of **domain decomposition/subspace correction**. We will decompose the DG finite element space V_h into the sum

$$V_h = V_0 + V_1 + \cdots + V_N.$$

The **coarse space** V_0 will be the H^1-conforming finite element space on the same mesh,

$$V_0 := V_h \cap H^1(\Omega).$$

The **local spaces** V_i, $i \geq 1$ will be one-dimensional spaces spanned by the basis functions,

$$V_i := \operatorname{span}\{\varphi_i\}.$$

We recall briefly the additive subspace correction preconditioner. Let A_i be the restriction of A to the subspace V_i, and let $I_i : V_i \hookrightarrow V_h$ denote the inclusion. Define the preconditioner B by

$$B := \sum_{i=0}^{N} I_i A_i^{-1} I_i^T.$$

Since the coarse space V_0 is quite large, we can replace A_0^{-1} with a preconditioner B_0. Any good preconditioner B_0 for A_0 will work. For example, we can use the two-level domain decomposition preconditioner developed in the previous term, or we can use a multigrid method, etc. The requirement is simply that $\kappa(B_0 A_0) \lesssim 1$. Our goal is now to estimate the condition number of the preconditioned system $T = BA$.

Recall from MTH 652 Lecture 14, it suffices to check two conditions on the preconditioner in order to bound the maximum and minimum eigenvalues of T. These two conditions are

(C1) **Finite overlap**

(C2) **Stable decomposition**

The first condition is the **finite overlap property**. This is the property that

$$a(v_i, v_j) = 0 \quad \text{for all but } \textbf{finitely many} \text{ (bounded) } i, j > 0$$

where $v_i \in V_i$, $v_j \in V_j$. This condition is easy to check. For $v_i \in V_i$,

$$a(v_i, v_j) = \int_\Omega \nabla_h v_i \cdot \nabla_h v_j \, dx - \int_\Gamma [\![v_i]\!] \cdot \{\nabla_h v_j\} \, ds - \int_\Gamma \{\nabla_h v_i\} \cdot [\![v_j]\!] \, ds + \int_\Gamma \alpha [\![v_i]\!] \cdot [\![v_j]\!] \, ds.$$

Recall that the local spaces are the spans of basis functions. The first term on the right-hand side is nonzero only when the basis functions φ_i and φ_j belong to the **same element**. The remaining terms are nonzero only when the basis functions lie on the **same edge**. The total number of such basis functions is clearly bounded. As a result, the maximum eigenvalue of T is bounded (independent of discretization parameters h and η).

Showing the **stable decomposition property** is (as usual) more involved. Recall the **key identity**

$$(T^{-1}v, v)_A = \inf_{\sum v_j = v} \sum_{j=0}^{J} (T_j^{-1} v_j, v_j)_A. \tag{7}$$

This means that to bound the **minimal eigenvalue** of T, it suffices that show that for all $u_h \in V_h$, there exists a decomposition

$$u_h = u_0 + u_1 + \cdots + u_N$$

such that

$$\sum_{i=0}^{N} a(u_i, u_i) \lesssim a(u_h, u_h).$$

This situation is similar to the two-level case studied previously: the local spaces V_i for $i > 0$ give a **direct sum** of the space V_h,

$$V_h = V_1 \oplus V_2 \oplus \cdots \oplus V_N,$$

and so, once u_0 is determined, the other u_i are defined uniquely. We define

$$u_0 = Q_0 u_h,$$

where Q_0 is the **vertex averaging operator**: $Q_0 u_h$ is the unique H^1-conforming function whose value et each vertex $\boldsymbol{x}_i$ is given by the average of the trace of u_h at $\boldsymbol{x}_i$ from within each triangle incident to that vertex. We now bound $\|u_h - u_0\|_{L^2(\Omega)}$. It is clear that if u_h happens to be continuous, then $Q_0 u_h = u_h$, and $\|u_h - u_0\|_{L^2(\Omega)} = 0$. Therefore, it is logical to posit that $\|u_h - u_0\|_{L^2(\Omega)}$ can be bounded in terms of the jumps of u_h. In fact, it is possible to prove the following result

$$\|u_h - u_0\|_{L^2(\Omega)}^2 \lesssim h \|[\![u]\!]\|_{L^2(\Gamma)}^2. \tag{8}$$

Then,

$$\begin{aligned} |\!|\!| u_h - u_0 |\!|\!|^2 &\lesssim \eta h^{-2} \|u_h - u_0\|_{L^2(\Omega)}^2 \\ &\lesssim \eta h^{-1} \|u_h\|_{L^2(\Gamma)}^2 \\ &\lesssim |\!|\!| u_h |\!|\!|^2. \end{aligned}$$

Triangle inequality then implies that

$$|\!|\!| u_0 |\!|\!| \lesssim |\!|\!| u_h |\!|\!|.$$

It remains to show that

$$\sum_{i=1}^{N} |\!|\!| u_i |\!|\!|^2 \lesssim |\!|\!| u_h |\!|\!|^2,$$

where $\sum_{i=1}^N u_i$ is the unique decomposition of $u_h - u_0$. We compute

$$\begin{aligned}
\sum_{i=1}^N |||u_i|||^2 &\lesssim \eta h^{-2} \sum_{i=1}^N \|u_i\|_{L^2(\Omega)}^2 \\
&\approx \eta h^{-2} \|u_h - u_0\|_{L^2(\Omega}^2 \\
&\lesssim \eta h^{-1} \|[\![u_h]\!]\|_{L^2(\Gamma)}^2 \\
&\lesssim |||u_h|||
\end{aligned}$$

where we used that, for $\boldsymbol{v} = (V_i)$ vector representation of $v_h \in V_h$, and $v_i = V_i \varphi_i$,

$$\sum_{i=1}^N \|v_i\|_{L^2(\Omega)}^2 \approx h^2 \sum_{i=1}^N V_i^2 \approx \|v_h\|_{L^2(\Omega)}.$$

This establishes the stable decomposition property (independent of h and η). Therefore, the additive subspace preconditioner is a robust preconditioner for the DG discretization, and the problem is reduced to the well-studied problem of finding a good preconditioner for the H^1-conforming method.