
Gauge Theory --- PMATH 965

Professor Ruxandra Moraru

Contents

1	Fibre Bundles	1
1.1	Bundle Maps	4
1.2	Bundle Atlases	5
1.3	Comparison Between Manifolds and Fibre Bundles	7
1.4	Bundle Maps Revisited	7
1.5	Vector Bundles	8
1.6	Sections	10
1.7	Sections of Vector Bundles	11
1.8	Linear Algebraic Constructions for Vector Bundles	14
2	Connections	15
2.1	Connections on Vector Bundles	15
2.1.1	Definition and Properties	15
2.1.2	Curvature	23

Lecture 1 --- January 7, 2020

1 Fibre Bundles

Definition 1.1. A *fibre bundle* consists of the data (E, B, π, F) where E, B, F are (topological) manifolds and $\pi : E \rightarrow B$ is a continuous surjection that satisfies the *local triviality* condition: For every $p \in B$, there is an open neighbourhood $U \ni p$ such that $\varphi : \pi^{-1}(U) \cong U \times F$ is a homeomorphism such that $\text{pr}_1 \circ \varphi = \pi$, where $\text{pr}_1 : U \times F \rightarrow U$ is the projection. The set of all $\{(U_\alpha, \varphi_\alpha)\}$ is called the *local trivialization* of the bundle.

E is called the *total space*, B is the *base space* and F is the *fibre* and π is the *projection map*.

Note. For all $b \in B$, the set $\pi^{-1}(b) = \{p \in E \mid \pi(p) = b\}$ is called the *fibre at b* , or the *fibre over b* . Since $\text{pr}_1 \circ \varphi = \pi$, we have $\pi^{-1}(b) \cong \{b\} \times F \cong F$. So we can think of E as a family of manifolds homeomorphic to F , parametrized by B .

Note. A fibre bundle (E, B, π, F) is also called an F -bundle.

Example 1.1.

1. $E = B \times F$ with $\pi = \text{pr}_1$ is the *trivial bundle*. Note that taking $\pi = \text{pr}_2$ gives a fibre bundle structure with base F and fibre B .
2. $E = S^1 \times \mathbb{R}$. E is a cylinder. In this case, E has two trivial bundle structures (as above), but with space $B = S^1$ we also have a vector bundle structure, as the fibres are \mathbb{R} .
3. **Möbius strip.** Example of a non-trivial \mathbb{R} -bundle on S^1 . $M = I \times \mathbb{R} / \sim$ where $(0, t) \sim (1, -t)$ for every $t \in \mathbb{R}$.
4. **Hopf fibration.** Example of a non-trivial S^1 -bundle over S^2 . Here,

- $E = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$
- $B = S^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + x^2 = 1\}$
- $F = S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

We take

$$\begin{aligned}\pi : S^3 &\rightarrow S^2 \\ (z_0, z_1) &\mapsto (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)\end{aligned}$$

is called the *Hopf map*. Then $|2z_0z_1|^2 + (|z_0|^2 - |z_1|^2)^2 = 1$, so $\pi(S^3) \subset S^2$, and π is well-defined and continuous. Also, π is surjective with $\pi^{-1}(z, x) \cong S^1$ for every $(z, x) \in S^2$. Indeed, let $(z, x) \in S^2$ so that $|z|^2 + x^2 = 1$ so that $-1 \leq x \leq 1$. Also, if $z = 0$, then $x = \pm 1$. Moreover, one can cover S^2 by the following two open sets:

$$\begin{aligned} U &= \{(z, x) \in S^2 \mid x \neq 1\} \\ &= S^2 \setminus \{(0, 1)\}, \text{ and} \\ V &= \{(z, x) \in S^2 \mid x \neq -1\} \\ &= S^2 \setminus \{(0, -1)\}. \end{aligned}$$

Let us now show that $\pi^{-1}(U) \cong U \times S^1$. let $(z, x) \in U$. So that $x \neq 1$. In particular, $-1 \leq x < 1$. Pick $(z_0, z_1) \in \pi^{-1}(U)$. Then $2z_0\bar{z}_1 = z$ and $|z_0|^2 - |z_1|^2 = x$.

- If $z = 0$, then $(z, x) = (0, -1) \implies z_0 = 0, |z_1|^2 = 1$. Thus $\pi^{-1}(z, x) = \{(0, \lambda) \in \mathbb{C}^2 \mid |\lambda| = 1\} \cong S^1$.
- If $z \neq 0$, then $x \notin \{\pm 1\}$, so $-1 < x < 1$ and $z_0, z_1 \neq 0$ since $2z_0\bar{z}_1 = z$. Then $z_0 = \frac{z}{2\bar{z}_1}$. Replacing z_0 by this in $|z_0|^2 - |z_1|^2 = 1$, one gets $4|z_1|^4 - |z_1|^2 x - |z|^2 = 0$. There is only one positive solution, which is equal to $|z_1|^2 = \frac{1-x}{2}$. So $z_1 = \lambda \sqrt{\frac{1-x}{2}}, \lambda \in S^1$. By the relationship $z_0 = \frac{z}{2\bar{z}_1}$, we have $z_0 = \lambda \frac{z}{\sqrt{2(1-x)}}$. So $\pi^{-1}(z, x) \cong S^1$, as

$$(z_0, z_1) = \lambda \left(\frac{z}{\sqrt{2(1-x)}}, \sqrt{\frac{1-x}{2}} \right)$$

$$\text{And so } \pi^{-1}(z, x) = \left\{ \lambda \left(\frac{z}{\sqrt{2(1-x)}}, \sqrt{\frac{1-x}{2}} \right) \mid \lambda \in S^1 \right\} \cong S^1.$$

This gives the local trivialization

$$\varphi : \pi^{-1}(U) \rightarrow U \times S^1$$

where if $\pi(z, x) = (z_0, z_1)$, $\varphi(z_0, z_1) = \lambda \left(\frac{z}{\sqrt{2(1-x)}}, \sqrt{\frac{1-x}{2}} \right)$. Finally, $\text{pr}_1 \circ \varphi(z_0, z_1) = \pi(z_0, z_1)$. So we have that (E, B, π, F) is a S^1 -bundle. This tells us that S^3 is an S^1 -bundle over S^2 . But, it cannot be a trivial bundle because S^3 is simply connected, but $S^3 \times S^1$ is not.

Lecture 2 --- January 9, 2020

Recall. A *fibre bundle* is a tuple (E, B, π, F) with $\pi : E \rightarrow B$ a continuous surjection that satisfies $\forall b \in B$ there is an open neighbourhood $U \subseteq B$ with $b \in U$ and a homeomorphism $\varphi : \pi^{-1}(U) \rightarrow U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \pi \downarrow & \swarrow \text{proj}_1 & \\ U & & \end{array}$$

Notation.

E = total space
 B = base space
 F = fibre
 π = projection map
 $E_b := \pi^{-1}(b)$ = fibre of E at $b \cong F$
 $E_U = \pi^{-1}(U) \subset E$

A fibre bundle (E, B, π, F) is also called an *F-bundle*.

Definition 1.2. A fibre bundle (E, B, π, F) is called *smooth* if E, B and F are smooth manifolds and $\pi : E \rightarrow B$ is a smooth surjection and for all $b \in B$, there exists an open neighbourhood $U \subset B$ of b and a diffeomorphism $\varphi : \pi^{-1}(U) \rightarrow U \times F$ such that $\text{pr}_1 \circ \varphi = \pi$.

Note. In Definition 1.2, we just replace the continuity/homeomorphism by smooth/diffeomorphism.

Remark. Note that $\pi : E \rightarrow B$ is in fact a smooth submersion (i.e., the differential $\pi_* : TE \rightarrow TB$ is surjective at every point). This follows from the local triviality — not every smooth surjection is a submersion.

Example 1.2. 1. All of the examples from lecture 1 are smooth fibre bundles.

2. **Tangent bundles.** Let M be a smooth manifold of dimension n . Then, TM is a smooth \mathbb{R}^n -bundle. Indeed, let $\{(U_\alpha, \phi_\alpha)\}$ be a smooth atlas for M so that $\phi_\alpha : U_\alpha \subset M \xrightarrow{\text{diffeo}} \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$. Here, of course, ϕ_α are the coordinate charts and $\phi_\alpha \circ \phi_\beta^{-1}$ are the coordinate transformations. In particular, $\phi_\alpha \circ \phi_\beta^{-1}$ is a diffeomorphism whenever $U_\alpha \cap U_\beta \neq \emptyset$ so that, $\forall p \in U_\alpha \cap U_\beta$,

$$(\phi_\alpha \circ \phi_\beta^{-1})_*(\phi_\beta(p)) : T_{\phi_\beta(p)}\mathbb{R}^n \rightarrow T_{\phi_\alpha(p)}\mathbb{R}^n$$

is an isomorphism (of vector spaces).

Recall that the *tangent bundle* TM of M is defined as

$$TM = \coprod_{p \in M} T_p M$$

then, TM has the following smooth manifold structure: Let

$$\begin{aligned} \pi : TM &\rightarrow M \\ X_p \in T_p M &\mapsto p \end{aligned}$$

Suppose that

$$\begin{aligned} \phi_\alpha : U_\alpha &\rightarrow \mathbb{R}^n \\ p &\mapsto (x_1(p), \dots, x_n(p)). \end{aligned}$$

Then, $\forall X \in T_p M$, $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \Big|_p$ for some appropriate scalars a_1, \dots, a_n . Denote by

$$\begin{aligned} \tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) &\rightarrow U_\alpha \times \mathbb{R}^n \\ \left(p, X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \Big|_p \right) &\mapsto (p = \pi(X), (a_1, \dots, a_n)). \end{aligned}$$

Then $\{\pi^{-1}(U_\alpha)\}$ is a basis for a topology on TM with respect to which $\{(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)\}$ is a smooth atlas for TM . Additionally, $\pi : TM \rightarrow M$ is smooth with respect to this smooth structure (see Lee's Introduction to Smooth Manifolds). Note that $\pi \circ \tilde{\phi}_\alpha = \text{pr}_1$ by the definition of $\tilde{\phi}_\alpha$. So $(TM, M, \pi, \mathbb{R}^n)$ is a smooth \mathbb{R}^n -bundle.

Note. Using the notation from above, the coordinate transformations of TM are given by

$$\left(\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}\right)(p, v = (a_1, \dots, a_n)) = (p, (\phi_\alpha \circ \phi_\beta^{-1})_*(p)v)$$

1.1 Bundle Maps

Definition 1.3. Let (E, B, π, F) and (E', B, π', F') be two smooth fibre bundles over the same base space. A *bundle map* or a *bundle morphism* of these bundles is a smooth map $H : E \rightarrow E'$ such that $\pi' \circ H = \pi$ (*). Diagrammatically,

$$\begin{array}{ccc} E & \xrightarrow{H} & E' \\ & \searrow \pi & \swarrow \pi' \\ & B & \end{array}$$

A *bundle isomorphism* is a bundle map which is a diffeomorphism. If such an isomorphism exists, then E and E' are said to be *isomorphic*, denoted $E \cong E'$.

Note. The property (*) tells us that bundle maps are fibre-preserving: $\forall b \in B, H|_{E_b} : E_b \rightarrow E'_b$. Also, if H is an isomorphism, then $H|_b : E_b \rightarrow E'_b$ is an isomorphism.

Definition 1.4. Fibre bundles isomorphic to the trivial bundle are called *trivial*. I.e., if there exists a diffeomorphism $H : E \rightarrow B \times F$ such that $\pi = \text{proj}_1 \circ H$ (with the typical notations).

Note. If E is a trivial bundle, then we have $E = \pi^{-1}(B)$ so that H is a *global* trivialization. All fibre bundles are locally trivial (by definition), but may not be globally trivial (e.g. the Hopf fibration is an S^1 -bundle over S^2 with total space S^3 which is not diffeomorphic (in fact, not even homeomorphic) to $S^1 \times S^2$).

Example 1.3. Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Then, TS^1 is trivial.

Proof. Let us show that $TS^1 \cong S^1 \times \mathbb{R}$. Define the following atlas for S^1 : Let U_1 be the “right half” of the circle with the top and bottom excluded. Then we define the map

$$\begin{aligned} \varphi_1 : U_1 &\rightarrow (-\pi/2, \pi/2) \\ (x, y) &\mapsto \arctan(y/x) =: \theta_1 \end{aligned}$$

We then take the open top U_2 with the map

$$\begin{aligned} \varphi_2 : U_2 &\rightarrow (0, \pi) \\ (x, y) &\mapsto \text{arccot}(x/y) =: \theta_2 \end{aligned}$$

and the bottom half U_3 with

$$\begin{aligned} \varphi_3 : U_3 &\rightarrow (-\pi, 0) \\ (x, y) &\mapsto \text{arccot}(x/y) - \pi =: \theta_3 \end{aligned}$$

and, lastly, the left open semicircle U_4 with

$$\begin{aligned} \varphi_4 : U_4 &\mapsto (\pi/2, 3\pi/2) \\ (x, y) &\mapsto \arctan(y, x) + \pi =: \theta_4 \end{aligned}$$

In all cases, $(\varphi_i \circ \varphi_j^{-1})_* = \text{id}$. Thus, the coordinate transformations for TS^1 are

$$(\tilde{\varphi}_i \circ \tilde{\varphi}_j^{-1})_*(x, v) = ((\varphi_i \circ \varphi_j^{-1})(x), v).$$

We can use the $\tilde{\varphi}_i$'s to construct an isomorphism H between TS^1 and $S^1 \times \mathbb{R}$. Take the usual projection map $\pi : TS^1 \rightarrow S^1$ and set

$$H|_{\pi^{-1}(U_i)} = \tilde{\varphi}_i : TU_i \rightarrow U_i \times \mathbb{R}.$$

Then, the $H|_{\pi^{-1}(U_i)}$ glue together to give a bundle map $H : TS^1 \rightarrow S^1 \times \mathbb{R}$ where we use the atlas $\{(\pi^{-1}(U_i), \tilde{\varphi}_i)\}$ and $((U_i \times \mathbb{R}, \varphi_i \times \text{id}))$, and H is a diffeomorphism, and so $TS^1 \cong S^1 \times \mathbb{R}$. \square

Note. Let $E = B \times F$ be the trivial bundle over B with projection $\pi = \text{proj}_1 : E \rightarrow B$. Then E also admits a projection onto the fibre: proj_2 . For a general fibre bundle, there may only exist a projection onto the fibre locally. We, however, have the following characterisation of trivial bundles:

Proposition 1.1. (E, B, π, F) is trivial if and only if there exists a smooth map $\psi : E \rightarrow F$ such that the restrictions to each fibres $\psi|_{E_b}$ are diffeomorphisms.

Lecture 3 --- January 14, 2020

Definition 1.5. A *smooth fibre bundle* is a tuple (E, B, π, F) such that E, B and F are smooth manifolds and $\pi : E \rightarrow B$ is a smooth surjective map and for all $b \in B$, there is an open $U \ni b$ and a diffeomorphism $\varphi : \pi^{-1}(U) \rightarrow U \times F$ such that $\pi = \text{proj}_1 \circ \varphi$, where $\text{proj}_1 : U \times F \rightarrow U$ is the projection onto the first factor.

Note. From now on we will assume that all manifolds are smooth and all fibre bundles are smooth.

1.2 Bundle Atlases

Definition 1.6. A *bundle atlas* for a fibre bundle (E, B, π, F) is an open covering $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ together with bundle charts $\varphi_\alpha : E_\alpha := \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ of B such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times F$.

Definition 1.7. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a bundle atlas for (E, B, π, F) . If $U_\alpha \cap U_\beta \neq \emptyset$, we define the *transition functions* by

$$g_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} \Big|_{\underbrace{U_\alpha \cap U_\beta}_{\subset U_\beta \times F}} : \underbrace{(U_\alpha \cap U_\beta) \times F}_{\subset U_\alpha \times F} \rightarrow (U_\alpha \cap U_\beta) \times F$$

Note that the $g_{\alpha\beta}$'s are all diffeomorphisms *and* they “preserve the fibres”, i.e., for all $b \in U_\alpha \cap U_\beta$,

$$g_{\alpha\beta} \Big|_{\{b\} \times F} : \{b\} \times F \xrightarrow{\cong} \{b\} \times F$$

(because $\varphi_\alpha \Big|_{\{b\} \times F} : E_b \xrightarrow{\cong} \{b\} \times F$). This implies that for all $b \in U_\alpha \cap U_\beta$,

$$\bar{g}_{\alpha\beta}(b) = g_{\alpha\beta} \Big|_{\{b\} \times F} \in \text{Diff}(\{b\} \times F) \cong \text{Diff}(F)$$

The maps

$$\begin{aligned} \bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta &\rightarrow \text{Diff}(F) \\ b &\mapsto \bar{g}_{\alpha\beta}(b) \end{aligned}$$

are also called the *transition functions* of (E, B, π, F) .

Example 1.4. Hopf fibration. (S^3, S^2, π, S^1) where

- $S^3 = \{(z_0, z_1) \mid |z_0|^2 + |z_1|^2 = 1\} \subset \mathbb{C}^2$
- $S^2 = \{(z, x) \mid |z|^2 + x^2 = 1\} \subset \mathbb{C} \times \mathbb{R}$
- $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$

and

$$\begin{aligned} \pi : S^3 &\rightarrow S^2 \\ (z_0, z_1) &\mapsto (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2) \end{aligned}$$

Set $U = \{(z, x) \in S^2 \mid z \neq 1\} = S^2 \setminus \text{north pole}$ and $V = \{(z, x) \in S^2 \mid x \neq -1\} = S^2 \setminus \text{south pole}$. $\{U, V\}$ is an open cover of S^2 . We have the bundle charts:

$$\begin{aligned} \varphi_U : \underbrace{\pi^{-1}(U)}_{\subset S^3} &\rightarrow \underbrace{U \times S^1}_{\in S^2 \times S^1} \\ (z_0, z_1) &\mapsto ((z, x), \lambda) \end{aligned}$$

where $(z_0, z_1) = \lambda \left(\frac{z}{\sqrt{1(1-x)}}, \sqrt{\frac{1-x}{2}} \right)$, and

$$\begin{aligned} \varphi_V : \pi^{-1}(V) &\rightarrow V \times S^2 \\ (z_0, z_1) &\mapsto ((z, x), \lambda') \end{aligned}$$

where $(z_0, z_1) = \lambda' \left(\sqrt{\frac{x+1}{2}}, \frac{\bar{z}}{\sqrt{2(x+1)}} \right)$. So $\{(U, \varphi_U), (V, \varphi_V)\}$ is a bundle atlas with transition functions

$$g_{UV} = \varphi_U \circ \varphi_V^{-1} : \overbrace{(U \cap V) \times S^1}^{\subset V \times S^1} \rightarrow \overbrace{(U \cap V) \times S^1}^{\subset U \times S^1}$$

$$((z, x), \lambda') \mapsto ((z, x), \lambda)$$

with

$$\lambda' \left(\sqrt{\frac{x+1}{2}}, \frac{\bar{z}}{\sqrt{2(x+1)}} \right) \underset{\varphi_V^{-1}}{=} (z_0, z_1) \underset{\varphi_U}{=} \lambda \left(\frac{z}{\sqrt{2(x+1)}}, \sqrt{\frac{1-x}{2}} \right)$$

This implies that

$$\lambda = \lambda' \left(\frac{\sqrt{1-x^2}}{z} \right) \underset{=}{\overset{\text{since } |z|^2 + |x|^2 = 1}} \lambda' \frac{|z|}{z}.$$

So

$$g_{UV} : (U \cap V) \times S^1 \rightarrow (U \cap V) \times S^1$$

$$((z, x)\lambda') \mapsto ((z, x), \lambda' \left(\frac{|z|}{z} \right))$$

Thus $\bar{g}_{UV}(z, x) = \left(\text{multiplication in } S^1 \text{ by } \frac{|z|}{z} \right) \in \text{Diff}(S^1)$.

It can often be difficult to check that a set we suspect is the total space of a fibre bundle is a manifold. One nonetheless has the following construction:

Definition 1.8. (Formal bundle atlases.) Let B and F be manifolds, E a set and $\pi : E \rightarrow B$ a surjective map.

1. Suppose $U \subset B$ is open and

$$\varphi_U : \pi^{-1}(U) \rightarrow U \times F$$

is a bijection with $\text{proj}_1 \circ \varphi_U = \pi$. Then, we call (U, φ_U) a *formal bundle chart for E* .

2. A family of bundle charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ where $\{U_\alpha\}$ is an open cover of B is called a *formal bundle atlas for E* .
3. The charts in a formal bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ are called *smoothly compatible* iff all transition functions

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \rightarrow (U_\alpha \cap U_\beta) \times F$$

(for $U_\alpha \cap U_\beta \neq \emptyset$) are all diffeomorphisms.

Theorem 1.1. (Formal bundle atlases define fibre bundles.) Let B and F be smooth manifolds, E a set and $\pi : E \rightarrow B$ a surjection. Suppose that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a formal bundle atlas for E of smoothly compatible charts. Then there exists a unique topology and smooth manifold structure on E such that (E, B, π, F) is a smooth fibre bundle with bundle atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$.

Let (E, B, π, F) be a fibre bundle with bundle atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$. Recall that the transition functions

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \rightarrow (U_\alpha \cap U_\beta) \times F.$$

Then they satisfy:

Lemma 1.1. (Cocycle conditions.): If $\bar{g}_{\alpha\beta} = g_{\alpha\beta}|_{\{b\} \times F}$ for all $b \in U_\alpha \cap U_\beta$,

$$\bar{g}_{\alpha\alpha}(b) = \text{id}_F, \forall b \in U_\alpha$$

$$\bar{g}_{\alpha\beta} \circ \bar{g}_{\beta\alpha}(b) = \text{id}_F, \forall b \in U_\alpha \cap U_\beta$$

$$\bar{g}_{\alpha\beta} \circ \bar{g}_{\beta\gamma} \circ \bar{g}_{\gamma\alpha}(b) = \text{id}_F, \forall b \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Remark. A fibre bundle can be (re)-constructed from its transition functions as a quotient using the equivalence relation induced by the cocycle condition:

$$E \cong \left(\coprod_{\alpha \in \mathcal{A}} U_\alpha \times F \right) / \sim$$

where $(b, v) \sim (b', v')$ if and only if $\exists \alpha, \alpha'$ with $b = b' = U_\alpha \cap U_{\alpha'} \neq \emptyset$ and $v = \bar{g}_{\alpha\alpha'}(b')v'$.

1.3 Comparison Between Manifolds and Fibre Bundles

Manifolds	Fibre bundles
coordinate charts $\varphi : U \overset{\text{open}}{\subseteq} M \xrightarrow{\text{diffeo.}} \mathbb{R}^n$	bundle charts / local trivializations $\varphi : \pi^{-1}(U) \rightarrow U \times F$
Coordinate transformations	Transition functions
Atlas	Bundle atlas
Trivial manifold $U \subseteq \mathbb{R}^n$	Trivial bundle $E = B \times F$
Non-trivial manifold	Non-trivial bundle

Notation. (E, B, π, F) is a fibre bundle

- $U \overset{\text{open}}{\subset} B \text{ --- } E_U := \pi^{-1}(U) \subset E$
- $b \in B \text{ --- } E_b := \pi^{-1}(b) \subset E$
- $\{(U_\alpha, \varphi)\}$ a bundle atlas: if $U_\alpha \cap U_\beta \neq \emptyset$, the *transition functions*

$$g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}|_{U_\alpha \cap U_\beta} : (U_\alpha \cap U_\beta) \times F \rightarrow (U_\alpha \cap U_\beta) \times F$$

and for all $b \in U_\alpha \cap U_\beta$,

$$\begin{aligned} g_{\alpha\beta}|_{\{b\} \times F} : \{b\} \times F &\rightarrow \{b\} \times F \\ (b, v) &\mapsto (b, \bar{g}_{\alpha\beta}(b)(v)). \end{aligned}$$

The maps $\bar{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \rightarrow \text{Diff}(F)$ are also called the *transition functions*.

1.4 Bundle Maps Revisited

Let (E, B, π, F) and (E', B, π', F') be two fibre bundles over B . A *bundle map* is a smooth map $H : E \rightarrow E'$ such that $\pi' \circ H = \pi$. Recall that bundle maps are fibre-preserving: For all $b \in B$, $H|_{E_b} : E_b \rightarrow E'_b$. Thus, for all $U \subseteq B$, $H|_{E_U} : E_U \rightarrow E'_U$. Can one obtain a local description of bundle maps? Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of B with respect to which E_{U_α} and E'_{U_α} are trivial for all $\alpha \in \mathcal{A}$. Suppose $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U_\alpha, \varphi'_\alpha)\}$ are bundle atlases for E and E' respectively, and set $H_\alpha = H|_{E_{U_\alpha}} : E_{U_\alpha} \rightarrow E'_{U_\alpha}$.

$$\begin{array}{ccc} E_{U_\alpha} & \xrightarrow{H_\alpha} & E'_{U_\alpha} \\ \downarrow \varphi_\alpha & & \downarrow \varphi'_\alpha \\ U_\alpha \times F & \xrightarrow{\varphi'_\alpha \circ H_\alpha \circ \varphi_\alpha^{-1}} & U_\alpha \times F' \end{array}$$

Where

$$\begin{aligned} \varphi'_\alpha \circ H_\alpha \circ \varphi_\alpha^{-1} : U_\alpha \times F &\rightarrow U_\alpha \times F' \\ (b, v) &\mapsto (b, \bar{H}_\alpha(b)(v)). \end{aligned}$$

Note that $\bar{H}_\alpha(b) : F \rightarrow F'$ are smooth maps, as they are compositions of smooth maps.

Also, if $U_\alpha \cap U_\beta \neq \emptyset$, then $H_\alpha|_{U_\alpha \cap U_\beta} = H|_{U_\alpha \cap U_\beta} = H_\beta|_{U_\alpha \cap U_\beta}$. Thus for any $b \in U_\alpha \cap U_\beta$,

$$\bar{H}_\beta(b) = \bar{g}'_{\beta\alpha}(b) \circ \bar{H}_\alpha(b) \circ \bar{g}_{\alpha\beta}(b)(*)$$

Bundle maps are completely determined by smooth maps

$$\bar{H}_\alpha : U_\alpha \rightarrow C^\infty(F, F')$$

that satisfy (*). Also, if H is a bundle isomorphism, then $\bar{H}_\alpha : U_\alpha \rightarrow \text{Diff}(F, F')$.

Note. When H is a diffeomorphism, (*) can be rewritten as

$$\bar{g}'_{\alpha\beta}(b) = \bar{H}_\alpha(b) \circ \bar{g}_{\alpha\beta}(b) \circ \bar{H}_\beta(b)^{-1}(**).$$

So, (E, B, π, F) is isomorphic to (E', B, π', F') if and only if there is a collection of maps $\{H_\alpha : U_\alpha \rightarrow \text{Diff}(F, F')\}$ which satisfies (**).

Corollary 1.1. (E, B, π, F) is trivial if and only if there is a bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ and smooth maps $\{\overline{H}_\alpha : U_\alpha \rightarrow \text{Diff}(F)\}$ such that $\overline{g}_{\alpha\beta}(b) = \overline{H}_\alpha(b)^{-1} \circ \overline{H}_\beta(b)$ for all $b \in B$. I.e., the cocycle corresponding to the transition functions is a coboundary.

Theorem 1.2. A bundle map $H : E \rightarrow E'$ is an isomorphism if and only if $H|_{E_b} : E_b \rightarrow E'_b$ is a diffeomorphism.

1.5 Vector Bundles

Definition 1.9. A fibre bundle (E, B, π, F) is called a *vector bundle* (v.b.) if the following are satisfied:

- (i.) F is a finite-dimensional vector space
- (ii.) For all $b \in B$, $\pi^{-1}(b)$ has the structure of an r -dimensional vector space (where $r = \dim F$)
- (iii.) The local trivializations $\varphi_U : E_U \rightarrow U \times F$ restrict to linear maps on the fibres of E . I.e., for all $b \in U$, $\varphi_U|_{E_b} : E_b \rightarrow \{b\} \times F \cong \{b\} \times F$ is a linear isomorphism.

r is called the *rank* of the vector bundle. If $r = 1$, (E, B, π, F) is called a *line bundle*.

Note. Vector bundles are \mathbb{R}^r -bundles, or \mathbb{C}^r -bundles whose bundle charts preserve the linear structure on the fibres.

Example 1.5. 1. $E = B \times \mathbb{R}^r$ or $E = B \times \mathbb{C}^r$ is the trivial bundle of rank r .

- 2. the (infinite) Möbius bundle is a line bundle on S^1 that is non-trivial.
- 3. If M is a manifold of dimension n , then TM is a vector bundle of rank n .
- 4. **Tautological line bundle over \mathbb{P}^n .** Recall that $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ where $x \sim \lambda x$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. I.e., it is the set of all lines in \mathbb{R}^{n+1} through the origin. Set

$$E = \coprod_{[x] \in \mathbb{P}^n} L_{[x]}$$

where $L_{[x]}$ is the line in \mathbb{R}^{n+1} through x and 0. Also,

$$\begin{aligned} \pi : E &\rightarrow \mathbb{P}^n \\ v \in L_{[x]} &\mapsto [x] \end{aligned}$$

note that for every $x \in \mathbb{P}^n$, $\pi^{-1}([x]) = L_{[x]} \cong \mathbb{R}$. Then $(E, \mathbb{P}^n, \pi, \mathbb{R})$ is a line bundle on \mathbb{P}^n .

Lecture 5 --- January 21, 2020

Recall. A *vector bundle* is a fibre bundle (E, B, π, F) such that

- (i) F is a finite-dimensional vector space of dimension r
- (ii) For every $b \in B$, E_b has the structure of a r -dimensional vector space
- (iii) There exist bundle charts $\varphi_U : E_U \rightarrow U \times F$ such that $\varphi_U|_{E_b} : E_b \xrightarrow{\cong} \{b\} \times F$ is a linear isomorphism.

Example 1.6. Tautological line bundle over \mathbb{P}^1 . $\mathbb{P}^1 = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$ where $(x_1, \dots, x_n) \sim (\lambda x_1, \dots, \lambda x_n)$ for all $\lambda \in \mathbb{R}^*$. Let

$$E := \coprod_{[x] \in \mathbb{P}^n} \{[x]\} \times L_{[x]}$$

where $L_{[x]}$ is the line through \mathbb{R}^{n+1} through 0 and x . Then,

$$\begin{aligned} \pi : E &\rightarrow \mathbb{P}^n \\ ([x], v \in L_{[x]}) &\mapsto [x] \end{aligned}$$

is a line bundle over \mathbb{P}^n called the *tautological line bundle over \mathbb{P}^n* , with fibre $E_{[x]} \cong L_{[x]} \cong \mathbb{R}^1$ for every $[x] \in \mathbb{P}^n$.

Proof. let us construct a bundle atlas for E that satisfy condition (iii) of the definition of a vector bundle and whose transition functions are smooth. Cover \mathbb{P}^n by

$$U_i := \{[x] \in \mathbb{P}^n \mid x_i \neq 0\} \underbrace{\subset}_{\text{open}} \mathbb{P}^n.$$

Then, for all $[x] \in U_i$ so that $x_i \neq 0$, and so

$$\begin{aligned} [x] &= [x_1 : \dots, x_i : \dots : x_{n+1}] \\ &= \left[\frac{x_1}{x_i} : \dots : 1 : \dots : \frac{x_{n+1}}{x_i} \right] \end{aligned}$$

Then for all $v \in L_{[x]}$, $v = t \left(\frac{x_1}{x_i}, \dots, 1, \dots, \frac{x_{n+1}}{x_i} \right)$ for some unique $t \in \mathbb{R}$. Set

$$\begin{aligned} \varphi_i : E_{U_i} &= \coprod_{[x] \in U_i} \{[x]\} \times L_{[x]} \longrightarrow U_i \times \mathbb{R}^1 \\ \left([x], t \left(\frac{x_1}{x_i}, \dots, 1, \dots, \frac{x_{n+1}}{x_i} \right) \right) &\mapsto (x, t) \end{aligned}$$

Then φ_i is a bijection. The collection $\{(U_i, \varphi_i)\}_{i=1}^{n+1}$ is a formal atlas for E . Also, if $U_i \cap U_j \neq \emptyset$, $[x] \in U_i \cap U_j$ and $v \in L_{[x]}$,

$$\begin{aligned} s(x_1/x_i, \dots, 1, \dots, x_{n+1}/x_i) &= v = t(x_1/x_j, \dots, 1, \dots, x_{n+1}/x_j) \\ &= t \frac{x_i}{x_j} (x_1/x_i, \dots, 1, \dots, x_{n+1}/x_i) \end{aligned}$$

And thus $s = \left(\frac{x_i}{x_j} \right) t$. Then $\varphi_i([x], v) = ([x], s)$ and $\varphi_j([x], v) = ([x], t)$ and $\varphi_i \circ \varphi_j^{-1}([x], t) = \left([x], \left(\frac{x_i}{x_j} \right) t \right)$, and so $\overline{\varphi}_{ij}([x]) \in \text{Diff}(\mathbb{R}^1)$. So E is a fibre bundle over \mathbb{P}^n with fibre \mathbb{R}^1 . Finally, we need to check that, for $i = 1, \dots, n+1$,

$$\varphi_i|_{E_{[x]}} : E_{[x]} \mapsto \{[x]\} \times \mathbb{R}^1$$

are linear isomorphisms. Here, $E_{[x]} = \{x\} \times L_{[x]}$, with vector space structure: $\forall \alpha \in \mathbb{R}$ and $v, v' \in L_{[x]}$, then $([x], v) + \alpha([x], v') = ([x], v + \alpha v')$. Also, one can write $v = t(x_1/x_i, \dots, x_{n+1}/x_i)$ and $v' = t'(x_1/x_i, \dots, x_{n+1}/x_i)$ for some $t, t' \in \mathbb{R}$. Then $v + \alpha v' = (t + \alpha t')(x_1/x_i, \dots, x_{n+1}/x_i)$. Then

$$\begin{aligned} \varphi_i([x], v + \alpha v') &= \varphi_i([x], v + \alpha v') \\ &= ([x], t + \alpha t') \\ &= ([x], t) + \alpha([x], t') \\ &= \varphi_i([x], v) + \alpha \varphi_i([x], v'). \end{aligned}$$

□

Since $\varphi_i|_{E[x]}$ is also a bijection, it is an isomorphism of vector spaces. This implies that, finally, $(E, \mathbb{P}^n, \pi, \mathbb{R}^1)$ is a vector bundle of rank 1.

Note. In the proof above, the transition functions of the bundle atlas we constructed were the $\overline{\varphi}_{ij} : U_i \cap U_j \rightarrow \text{GL}(1, \mathbb{R}) \subset \text{Diff}(\mathbb{R}^1)$.

Remark. If $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a vector bundle atlas for the vector bundle $(E, B, \pi, \mathbb{R}^r)$ (or $(E, B, \pi, \mathbb{C}^r)$), the transition functions

$$\overline{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{R}) \text{ or } \text{GL}(r, \mathbb{C})$$

In particular, if $r = 1$, then $\text{GL}(1, \mathbb{R}) = \mathbb{R}^\times$ and $\text{GL}(1, \mathbb{C}) = \mathbb{C}^\times$ so that $\overline{g}_{\alpha\beta}$ are just nowhere-vanishing scalar functions.

Definition 1.10. Let $(E, B, \pi, \mathbb{R}^r)$ and $(E', B, \pi', \mathbb{R}^{r'})$ be vector bundles. A map $H : E \rightarrow E'$ is a *(bundle) map of vector bundles* if

$$H|_{E_b} : E_b \rightarrow E'_b$$

is linear for all $b \in B$.

Note. Unless otherwise stated, we will always assume that bundle maps between vector bundles are *actually* bundle maps.

1.6 Sections

Definition 1.11. Let (E, B, π, F) be a fibre bundle. A *section* of (E, B, π, F) is a smooth map $\sigma : B \rightarrow E$ such that $\pi \circ \sigma = \text{id}_B$.

Then for all $b \in B$, $\sigma(b) \in E_b$, since $\pi(\sigma(b)) = b$. Also, $\sigma(B) \subset E$ is a smooth submanifold of E diffeomorphic to B (exercise).

Notation. We write $\Gamma(E) = \{\text{set of all sections of } (E, B, \pi, F)\}$.

Definition 1.12. If $U \subsetneq B$ is open, then a *local section of E over U* is a smooth map $\sigma : U \rightarrow E_U$ such that $\pi \circ \sigma = \text{id}_U$.

Note. Again, $\sigma(b) \in E_b$ for all $b \in U$ if $\sigma : U \rightarrow E$ is a local section over U . We denote

$$\Gamma(U, E) = \{\text{set of local sections of } E \text{ over } U\}.$$

Example 1.7. (i) $E = B \times F$ with $\pi = \text{pr}_1$. Let $\overline{\sigma} : B \rightarrow F$ be any smooth map, and then

$$\begin{aligned} \sigma : B &\rightarrow E \\ b &\mapsto (b, \overline{\sigma}(b)) \end{aligned}$$

Then σ is smooth and $\pi \circ \sigma = \text{id}_B$, so $\sigma \in \Gamma(E)$.

In fact, sections of any fibre bundle look like this locally: Let (U, φ_U) be a bundle chart for (E, B, π, F) and $\sigma \in \Gamma(E)$. Then, $\pi \circ \sigma = \text{id}_B$ and

$$\begin{aligned} \varphi_U \circ \sigma|_U : U &\rightarrow U \times F \\ b &\mapsto (b, \overline{\sigma}_U(b)) \end{aligned}$$

for some $\overline{\sigma}_U : U \rightarrow F$ smooth. [Note: The first component of $\varphi_U \circ \sigma|_U$ is id_U because $\pi \circ \sigma|_U = \text{id}_U$.] Thus, local sections of E over U are completely determined by the smooth functions $\overline{\sigma} : U \rightarrow F$. In particular, local sections *always* exist.

Example 1.8. (i) Vector bundles always admit sections. For example, given any vector bundle $(E, B, \pi, \mathbb{R}^r)$, one can define the *zero section*

$$\begin{aligned} 0 : B &\rightarrow E \\ b &\mapsto 0 \in E_b \end{aligned}$$

(ii) If M is any smooth manifold, then $\Gamma(TM)$ is the collection of smooth vector fields on M , which always exist.

(iii) Consider S^2 and TS^2 . Sections of TS^2 are smooth, tangent vector fields on S^2 . By the Hairy-Ball Theorem, any smooth vector field on S^2 has at least one zero.

(iv) For an example of a fibre bundle that does not admit any global sections, take $E = TS^2 \setminus \{\text{zero section}\}$, which has fibre $\mathbb{R}^2 \setminus \{0\}$ and whose projection is simply $\pi|_E$ where $\pi : TS^2 \rightarrow S^2$ is the standard projection. This fibre bundle does not have a section because any smooth section $\sigma \in \Gamma(E)$ would be a smooth vector field on S^2 and thus must have a zero.

Lecture 6 --- January 23, 2020

Sections. (E, B, π, F) a fibre bundle. A section is a smooth map $\sigma : B \rightarrow E$ such that $\pi \circ \sigma = \text{id}_B$. We denote by $\Gamma(E)$ the set of all sections of (E, B, π, F) .

Given a bundle chart (E_U, φ_U) with $U \subseteq B$ open,

$$\begin{array}{ccc} \varphi_U \circ (\sigma|_U) : U & \xrightarrow{\quad} & U \times F \\ & \searrow \sigma & \nearrow \varphi_U \\ & E_U & \end{array}$$

with $\varphi_U \circ (\sigma|_U)(b) = (b, \bar{\sigma}(b))$ for some smooth $\bar{\sigma} : U \rightarrow F$.

Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of B and $\{(E_{U_\alpha}, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ be a bundle atlas for (E, B, π, F) . Let $\sigma \in \Gamma(E)$. Set

$$\sigma_\alpha := \sigma|_{U_\alpha} : U_\alpha \longrightarrow E_{U_\alpha} = \coprod_{b \in U_\alpha} E_b$$

Then

$$\begin{aligned} \varphi_\alpha \circ \sigma|_{U_\alpha} &\rightarrow U_\alpha \times F \\ b &\mapsto (b, \bar{\sigma}_\alpha(b)) \end{aligned}$$

for some smooth $\bar{\sigma}_\alpha : U_\alpha \rightarrow F$. How are the $\bar{\sigma}_\alpha$'s related? Suppose $U_\alpha \cap U_\beta \neq \emptyset$ and let $b \in U_\alpha \cap U_\beta$. Then

$$\begin{aligned} (b, \bar{\sigma}_\alpha(b)) &= \varphi_\alpha \circ \sigma_\alpha(b) \\ &= \varphi_\alpha \circ \sigma_\beta(b) \\ &= \underbrace{\varphi_\alpha \circ \varphi_\beta^{-1}}_{g_{\alpha\beta}} \circ \varphi_\beta \circ \sigma_\beta(b) \\ &= (b, \bar{g}_{\alpha\beta}(b) (\bar{\sigma}_\beta(b))) \end{aligned}$$

which implies that

$$\bar{\sigma}_\beta(b) = \bar{g}_{\alpha\beta}(b) (\bar{\sigma}_\alpha(b)) (***)$$

for all $b \in U_\alpha \cap U_\beta$.

So, given a bundle atlas $\{(E_{U_\alpha}, \varphi_\alpha)\}$ of (E, B, π, α) , we can think of sections of the bundle as families of smooth maps $\{\sigma_\alpha : U_\alpha \rightarrow F\}$ that satisfy (***) .

1.7 Sections of Vector Bundles

Let $(E, B, \pi, \mathbb{R}^r)$ be a vector bundle, which we will denote by E . Let $\{U_\alpha\}$ be an open cover of B and $\{(E_{U_\alpha}, \varphi_\alpha)\}$ be a vector bundle atlas of E . Then, the transition functions of the atlas are

$$\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{R})$$

So, for all $b \in U_\alpha \cap U_\beta$, $\bar{g}_{\alpha\beta}(b)$ = (invertible matrix), and, for all $v \in \mathbb{R}^r$,

$$\bar{g}_{\alpha\beta}(b)(v) = \underbrace{\bar{g}_{\alpha\beta}(b) \cdot v}_{\text{matrix multiplication}}$$

For this reason, $\bar{g}_{\alpha\beta}(b)$ are sometimes called *transition matrices*.

Also, any section of E is determined by a family

$$\{\bar{\sigma}_\alpha : U_\alpha \rightarrow \mathbb{R}^r\}$$

of smooth vector-valued functions such that

$$\bar{\sigma}_\alpha(b) = \underbrace{\bar{g}_{\alpha\beta}(b) \cdot \bar{\sigma}_\beta(b)}_{\text{matrix multiplication}}$$

by (***) .

Note. On a vector bundle, any local section can be extended globally (possibly by zero outside of the open set on which it is defined) by using bump functions (exercise).

Definition 1.13. Let $\sigma_1, \dots, \sigma_l \in \Gamma(E)$. We say that the set $\{\sigma_1, \dots, \sigma_l\}$ is *linearly independent* if

$$\{\sigma_1(b), \dots, \sigma_l(b)\} \subseteq E_b$$

is linearly independent for every $b \in B$. If $l = r$ (the rank of E), then $\{\sigma_1, \dots, \sigma_l\}$ is called a *frame for E* .

Note. (i) If $\{\sigma_1, \dots, \sigma_r\}$ is a frame of E so that $\{\sigma_1(b), \dots, \sigma_l(b)\}$ is linearly independent in E_b for all $b \in B$, then $\{\sigma_1(b), \dots, \sigma_l(b)\}$ is a basis for E_b for all $b \in B$. Then $\sigma_i(b) \neq 0$ for all $i = 1, \dots, l$. So, the σ_i 's are nowhere-vanishing.

(ii) If $r = 1$, then any frame of E consists solely of a nowhere-vanishing section.

Example 1.9. 1) Let S^{2n} be an even-dimensional sphere. Then, by the Hairy Ball theorem, any tangent vector field of S^{2n} has at least one zero. Thus, TS^{2n} does not admit nowhere-vanishing sections. So, TS^{2n} does not admit any (global) frames.

2) $S^{2n+1} \subset \mathbb{R}^{2n+2} = \{(x_1, \dots, x_{2n+2})\}$.

- $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$. Then $X_{(x_1, x_2)} = (-x_2, x_1)$ is a nowhere-vanishing, tangent vector field of S^1 .
- On $S^{2n+1} \subset \mathbb{R}^{2n+2}$, We define

$$X_{(x_1, \dots, x_{2n+2})} = (-x_2, x_1, \dots, -x_{2k}, x_{2k+1}, \dots, -x_{2n-1}, x_{2n+2}).$$

- On S^3 , we have that

$$\begin{aligned} X_1(x_1, \dots, x_4) &= (-x_2, x_1, -x_3, x_4) \\ X_2(x_1, \dots, x_4) &= (-x_3, -x_4, x_1, x_2) \\ X_3(x_1, \dots, x_4) &= (x_4, -x_3, -x_2, x_1) \end{aligned}$$

comprise a frame for TS^3 .

- On S^7 , one can use the octonions to construct a frame for TS^7
 - On S^{2n+1} for $n \geq 4$, TS^{2n+1} does not admit a global frame.
- 3) Let $E = B \times \mathbb{R}^r$ be the trivial vector bundle with $\pi(b, v) = b$. Then suppose that $\{e_1, \dots, e_r\}$ is the standard basis for \mathbb{R}^r . Then a global frame is given by $\{\sigma_1, \dots, \sigma_r\}$ where

$$\begin{aligned} \sigma_i : B &\rightarrow E \\ b &\mapsto (b, e_i). \end{aligned}$$

We will refer to this as the *standard frame on the trivial bundle*. So, the trivial bundle admits at least one frame (in fact... many).

In general, we have:

Proposition 1.2. A vector bundle E is trivial if and only if it admits a frame.

Proof. (\implies) If E is trivial, then it is isomorphic to $B \times \mathbb{R}^r$. Thus, there exists a vector bundle isomorphism $H : B \times \mathbb{R}^r \rightarrow E$. In particular, $H|_{\{b\} \times \mathbb{R}^r} \rightarrow E_b$ is a linear isomorphism. Let $\{\sigma_1, \dots, \sigma_r\}$ be the standard frame on $B \times \mathbb{R}^r$, and define

$$\begin{aligned} \tilde{\sigma}_i : B &\rightarrow E \\ b &\mapsto H \circ \sigma_i(b). \end{aligned}$$

Then each $\tilde{\sigma}_i$ is a section of E , because $\pi \circ \tilde{\sigma}_i = \pi \circ H \circ \sigma_i = \text{proj}_1 \circ \sigma_i = \text{id}_B$. Also, for all $b \in B$,

$$\{\tilde{\sigma}_1(b), \dots, \tilde{\sigma}_r(b)\} = \underbrace{H|_b(\{\sigma_1(b), \dots, \sigma_r(b)\})}_{\text{linearly independent}}.$$

So $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_r(b)\}$ is a frame for E . (\impliedby) Assume that E admits the frame $\{\sigma_1, \dots, \sigma_r\}$ and use it to construct an isomorphism given by

$$\begin{aligned} H : B \times \mathbb{R}^r &\rightarrow E \\ (b, (a_1, \dots, a_r)) &\mapsto \sum_{i=1}^r a_i \sigma_i(b) \in E_b, \end{aligned}$$

which is an isomorphism because $\{\sigma_1, \dots, \sigma_r\}$ is a frame. So, H is a vector bundle isomorphism. □

Corollary 1.2. A line bundle is trivial if and only if it admits a nowhere-vanishing section.

Corollary 1.3. TS^k is trivial if and only if $k \in \{1, 3, 7\}$.

Definition 1.14. A manifold M is called *parallelizable* if its tangent bundle is trivial.

Example 1.10. 1. S^1, S^3, S^7 are parallelizable.

2. Any Lie group G is parallelizable.

Proposition 1.3. The tautological line bundle on \mathbb{P}^n is not trivial.

Proof. The tautological line bundle on \mathbb{P}^n does not admit any nowhere-vanishing sections. □

Lecture 7 --- January 28, 2020

Let (E, B, π, F) be a vector bundle. A **frame** is a set $\{\sigma_1, \dots, \sigma_l\}$ of linearly independent sections $\sigma_i \in \Gamma(E)$.

Proposition 1.4. E is trivial if and only if E admits a frame.

Corollary 1.4. A line bundle is trivial if and only if it admits a nowhere-vanishing section.

Proposition 1.5. The tautological line bundle over \mathbb{P}^n is *not* trivial.

Proof. It is enough to show that the tautological line bundle E over \mathbb{P}^n does not admit any nowhere-vanishing sections. We do it by contradiction: Suppose instead that E admits a nowhere-vanishing section $\sigma : \mathbb{P}^n \rightarrow E$ so that $\sigma([x]) \neq 0$ for every $[x] \in \mathbb{P}^n$. Recall that we constructed a vector bundle atlas for E with open cover $\{U_i\}_{i=1}^{n+1}$ where

$$U_i := \{[x_1 : \dots : x_{n+1}] \mid x_i \neq 0\}$$

and transition functions

$$\begin{aligned} g_{ij} : U_i \cap U_j &\rightarrow \mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^\times \\ [x] &\mapsto \frac{x_i}{x_j}. \end{aligned}$$

Then σ is given by scalar functions

$$\bar{\sigma}_i : U_i \rightarrow \mathbb{R}$$

such that (without loss of generality)

$$\begin{aligned} \underbrace{\bar{\sigma}_i([x])}_{>0} &= \bar{g}_{ij}([x]) \bar{\sigma}_j([x]) \\ &= \left(\frac{x_i}{x_j}\right) \underbrace{\bar{\sigma}_j([x])}_{>0}. \end{aligned}$$

but

$$\begin{aligned} U_i \cap U_j &\rightarrow \mathbb{R}^\times \\ [x] &\mapsto \frac{x_i}{x_j} \end{aligned}$$

is surjective. □

Thus, not all vector bundles admit frames, but they ALL admit “local frames”:

Definition 1.15. Let $U \subseteq B$ be open and $e_1, \dots, e_r \in \Gamma(U, E)$. Then $\{e_1, \dots, e_r\}$ is a **local frame of E over U** if, for all $b \in U$, $\{e_1(b), \dots, e_r(b)\}$ is linearly independent.

So, for all $U \subseteq B$ open over which E admits a vector bundle chart $\varphi_U : E_U \rightarrow U \times \mathbb{R}^r$, one has the local frame $\{e_1, \dots, e_r\}$ given by

$$\begin{aligned} e_i : U &\rightarrow E_U \\ b &\mapsto \varphi_U^{-1}(b, \vec{e}_i) \end{aligned}$$

where $\{\vec{e}_1, \dots, \vec{e}_r\}$ is the standard basis in \mathbb{R}^r .

Local frames are useful for describing frames locally. Given a local frame $\{e_1, \dots, e_r\}$ of E over U and a section $\sigma \in \Gamma(E)$,

$$\sigma|_U = \bar{\sigma}_1 e_1 + \dots + \bar{\sigma}_r e_r$$

for some $\bar{\sigma}_1, \dots, \bar{\sigma}_r \in C^\infty(U)$. Also, if $\{e'_1, \dots, e'_r\}$ is another local frame of E over U' with $U \cap U' \neq \emptyset$, for all $b \in U \cap U'$, we have

$$e'_j(b) = \sum_{i=1}^r h_{ij}(b) e_i(b)$$

for some smooth $h_{ij} \in C^\infty(U)$. Thus, we get a map

$$\begin{aligned} h : U \cap U' &\rightarrow \mathrm{GL}(r, \mathbb{R}) \\ b &\mapsto [h_{ij}(b)]_{i,j=1}^r \end{aligned}$$

where $h(b)$ is the “change of basis matrix” from $\{e_i(b)\}$ to $\{e'_i(b)\}$.

Note. $\Gamma(U, E)$ has the following $C^\infty(U)$ —module structure: For all $\sigma, \tau \in \Gamma(U, E)$ and $f \in C^\infty(U)$, set

$$(f\sigma + \tau) : U \mapsto E_U \\ b \mapsto f(b)\sigma(b) + \tau(b).$$

Then, since $f(b) \in \mathbb{R}$ and $\sigma(b), \tau(b) \in E_b$, so $f(b)\sigma(b) + \tau(b) \in E_b$. Thus $f\sigma + \tau \in \Gamma(U, E)$. In terms of a local frame $\{e_1, \dots, e_r\}$ of E over U , we have $\sigma = \sum_{j=1}^r \bar{\sigma}_j e_j$, $\tau = \sum_{j=1}^r \bar{\tau}_j e_j$ and

$$f\sigma + \tau = \sum_{j=1}^r (f\bar{\sigma}_j + \bar{\tau}_j) e_j.$$

1.8 Linear Algebraic Constructions for Vector Bundles

Let $(E, B, \pi, \mathbb{R}^r)$ and $(E', B, \pi', \mathbb{R}^{r'})$ be vector bundles. One can construct new vector bundles by applying linear algebra constructions fibrewise:

$$E \oplus E', \quad E \otimes E', \quad E^*, \quad \bigwedge^k E, \quad \text{Hom}(E, E').$$

(i) To construct the direct sum of E and E' , we take the underlying set

$$E \oplus E' = \bigsqcup_{b \in B} \underbrace{E_b \oplus E'_b}_{\text{rank } r+r'}.$$

Given an open cover $\{U_\alpha\}$ of B and vector bundle atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\alpha, \varphi'_\alpha)\}$ for E and E' , respectively, we define

$$\varphi_\alpha \oplus \varphi'_\alpha : \bigsqcup_{b \in B} E_b \oplus E'_b \rightarrow U_\alpha \times (\mathbb{R}^r \oplus \mathbb{R}^{r'}) \\ E_b \oplus E'_b \ni (e, e') \mapsto (b, (\varphi_\alpha(e), \varphi'_\alpha(e'))).$$

These are bundle charts for $E \oplus E'$, for all α . Then we get transition functions

$$\bar{g}_{\alpha\beta} \oplus \bar{g}'_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \text{GL}(r + r', \mathbb{R}).$$

(ii) The tensor product is given (as a set) by

$$E \otimes E' = \bigsqcup_{b \in B} \underbrace{E_b \otimes E'_b}_{\text{rank } rr'}.$$

(iii) The dual bundle is given (as a set) by

$$E^* = \bigsqcup_{b \in B} \underbrace{E_b^*}_{\text{rank } r}.$$

(iv) The exterior power bundles are given (as sets) by

$$\bigwedge^k E = \bigsqcup_{b \in B} \underbrace{\bigwedge^k E_b}_{\text{rank } \binom{n}{r}}$$

(v) The hom bundles are given (as sets) by

$$\text{Hom}_E(E') = \bigsqcup_{b \in B} \underbrace{\text{Hom}(E_b, E'_b)}_{\text{rank } rr'}$$

Example 1.11. 1. • Let M be a smooth manifold and TM its tangent bundle. Then $(TM)^* = T^*M$ is the cotangent bundle. Smooth sections of this bundle are the smooth 1-forms: $\Gamma(T^*M) = \Omega^1(M)$.

• $\bigwedge^k T^*M =: \bigwedge^k M$ have the k -forms as sections: $\Gamma(\bigwedge^k T^*M) = \Omega^k(M)$.

2. We will be interested in $(\bigwedge^k M) \otimes E$ with E a vector bundle on M . Locally, sections of $(\bigwedge^k M) \otimes E$ look like: Given a local frame $\{e_1, \dots, e_r\}$ of E over U , for all $s \in \Gamma(\bigwedge^k M \otimes E)$,

$$s|_U = \sum_{i=1}^r \omega_i \otimes e_i$$

for some $\omega_1, \dots, \omega_r \in \Omega^k(U)$.

2 Connections

2.1 Connections on Vector Bundles

2.1.1 Definition and Properties

Fix $(E, B, \pi, \mathbb{R}^r)$ be a vector bundle of rank r . Our goal is to find a way of differentiating sections of E . Let us first assume that $E = B \times \mathbb{R}^r$. In this case, a section $\sigma \in \Gamma(E)$ is just

$$\begin{aligned}\sigma : B &\rightarrow B \times \mathbb{R}^r \\ b &\mapsto (b, \bar{\sigma}(b))\end{aligned}$$

for some smooth map $\bar{\sigma} : B \rightarrow \mathbb{R}^r$. In particular,

$$\begin{aligned}\bar{\sigma} : B &\rightarrow \mathbb{R}^r \\ b &\mapsto (\bar{\sigma}_1(b), \dots, \bar{\sigma}_r(b))\end{aligned}$$

for some $\bar{\sigma}_i \in C^\infty(B)$. Also, if $\{e_1, \dots, e_r\}$ is the standard frame for $B \times \mathbb{R}^r$ (so that $e_i(b) = (b, \vec{e}_i)$), then

$$\sigma = \sum_{i=1}^r \bar{\sigma}_i e_i.$$

So, one possible way of differentiating σ is to differentiate $\bar{\sigma}$ component-wise:

$$d\sigma(b) = (b, d\bar{\sigma}(b))$$

where $d\bar{\sigma}(b)L = (d\bar{\sigma}_1(b), \dots, d\bar{\sigma}_r(b)) = \sum_{i=1}^r d\bar{\sigma}_i(b) \otimes \vec{e}_i$. In terms of the local frame $\{e_1, \dots, e_r\}$,

$$d\sigma = \sum_{i=1}^r \underbrace{(d\bar{\sigma}_i)}_{\text{form}} \otimes \underbrace{e_i}_{\in \Gamma(E)} \in \Gamma(T^*M \otimes E).$$

Then:

$$d : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$$

$$\sigma = \sum_{i=1}^r \bar{\sigma}_i e_i \mapsto \sum_{i=1}^r (d\bar{\sigma}_i) \otimes e_i$$

which satisfies

- \mathbb{R} -linearity.
- (Leibniz rule): $d(f\sigma) = df \otimes \sigma + f d\sigma \in \Gamma(T^*M \otimes E)$.

Lecture 8 --- January 30, 2020

Recall. $(E, B, \pi, \mathbb{R}^r)$ the trivial bundle with $E = B \times \mathbb{R}^r$. Pick a frame $\{e_1, \dots, e_r\}$ with $e_i(b) = (b, \vec{e}_i)$. Then any section looks like $\sigma = \sum_{i=1}^r \bar{\sigma}_i e_i$. One possible way of differentiating σ is to set

$$d\sigma(b) := (b, d\bar{\sigma}(b))$$

where $d\bar{\sigma}(b) = (d\bar{\sigma}_1(b), \dots, d\bar{\sigma}_r(b))$. So we get

$$d\sigma := \sum_{i=1}^r \underbrace{\in \Omega^1(B)}_{d\bar{\sigma}_i} \otimes \underbrace{\in \Gamma(E)}_{e_i}$$

Note. • d is \mathbb{R} -linear: for $\sigma, \tau \in \Gamma(E)$ so that $\sigma = \sum_{i=1}^r \bar{\sigma}_i e_i$ and $\tau = \sum_{i=1}^r \bar{\tau}_i e_i$. Then for any $c \in \mathbb{R}$,

$$d(c\sigma + \tau) := \sum_{i=1}^r d(c\bar{\sigma}_i + \bar{\tau}_i) \otimes e_i = cd\sigma + d\tau.$$

• d satisfies the *Leibniz rule*: For any $\sigma = \sum_{i=1}^r \bar{\sigma}_i e_i$ and $f \in C^\infty(B)$,

$$d(f\sigma) = df \otimes \sigma + f d\sigma.$$

Indeed,

$$\begin{aligned} d(f\sigma) &= d\left(\sum_{i=1}^r (f\bar{\sigma}_i) \otimes e_i\right) \\ &= \sum_{i=1}^r d(f\bar{\sigma}_i) \otimes e_i \\ &= \sum_{i=1}^r (\bar{\sigma}_i df + f d\bar{\sigma}_i) \otimes e_i \\ &= df \otimes \left(\sum_{i=1}^r \bar{\sigma}_i e_i\right) + f \left(\sum_{i=1}^r d\bar{\sigma}_i \otimes e_i\right) \\ &= df \otimes \sigma + f d\sigma. \end{aligned}$$

Definition 2.1. A *connection on E* is an \mathbb{R} -linear map

$$D : \Gamma(E) \rightarrow \Gamma(T^*B \otimes E)$$

that satisfies the *Leibniz rule*: For all $f \in C^\infty(B)$ and $\sigma \in \Gamma(E)$, we have

$$D(f\sigma) = df \otimes \sigma + f D(\sigma).$$

Note. Connections generalize the notion of exterior derivative “ d ” to sections of *any vector bundle*.

Example 2.1. 1. Take $E = B \times \mathbb{R}^r$.

- $D = d$ is called the *trivial connection*.
- What do the others look like? Let $D : \Gamma(E) \rightarrow \Gamma(T^*B \otimes E)$ be a connection on $E = B \times \mathbb{R}^r$. Consider the frame $\{e_1, \dots, e_r\}$ with $e_i(b) = (b, \vec{e}_i)$. Then, for all $j = 1, \dots, r$, $D(e_j) \in \Gamma(T^*B \otimes E)$. Then

$$D(e_j) = \sum_{i=1}^r a_{ij} \otimes e_i$$

for some $a_{ij} \in \Gamma(T^*B)$. If we pick $\sigma \in \Gamma(E)$, then $\sigma = \sum_{j=1}^r \bar{\sigma}_j e_j$ for $\bar{\sigma}_j \in C^\infty(B)$. Then

$$\begin{aligned} D(\sigma) &= \sum_{j=1}^r D(\bar{\sigma}_j e_j) \\ &= \sum_{j=1}^r (d\bar{\sigma}_j \otimes e_j + \bar{\sigma}_j D(e_j)) \\ &= \sum_{j=1}^r d\bar{\sigma}_j \otimes e_j + \sum_{i,j=1}^r \bar{\sigma}_j (a_{ij} \otimes e_i) \\ &= \sum_{j=1}^r d\bar{\sigma}_j \otimes e_j + \sum_{i=1}^r \left(\sum_{j=1}^r a_{ij} \bar{\sigma}_j \right) \otimes e_i \\ &=: d\sigma + A\sigma =: (d + A)\sigma \end{aligned}$$

where we set $A = [a_{ij}]_{i,j=1}^r$ is a $r \times r$ matrix of 1-forms, called the *connection matrix of D* and $\bar{\sigma} = [\bar{\sigma}_i]_{i=1}^r$. Here, we mean

$$A\sigma = \sum_i \left(\sum_j a_{ij} \bar{\sigma}_j \right) \otimes e_i.$$

Note. The connection matrix depends on the frame $\{e_1, \dots, e_r\}$: To be precise, if $\{e_1, \dots, e_r\}$ and $\{e'_1, \dots, e'_r\}$ are frames of $E = B \times \mathbb{R}^r$ and

$$e'_i = \sum_k h_{ki} e_k$$

so that $h = (h_{ij})_{i,j=1}^r$ is the change of basis matrix. Then:

$$D(e_j) = \sum_i a_{ij} \otimes e_i \qquad D(e'_j) = \sum_i a'_{ij} \otimes e'_i$$

Then $A' = (a'_{ij})_{i,j=1}^r$ satisfies

$$A' = h^{-1}dh + h^{-1}Ah \quad (\text{exercise.})$$

2. E is any vector bundle and $\{(U_\alpha, \varphi_\alpha)\}$ is a vector bundle atlas for E with $\{U_\alpha\}$ an open cover of B . Then, for all α , $E_{U_\alpha} \cong U_\alpha \times \mathbb{R}^r$ and hence admits a local frame $\{e_1^\alpha, \dots, e_r^\alpha\}$ with

$$e_1^\alpha(b) = \varphi_\alpha^{-1}(b, \vec{e}_1).$$

Let D be a connection on E . Then on E_{U_α} , $D = d + A_\alpha$ where A_α is the connection matrix of $D|_{E_{U_\alpha}}$ in terms of the frame $\{e_i^\alpha\}$. Note that on $U_\alpha \cap U_\beta$, the change of basis matrix from $\{e_1^\beta, \dots, e_r^\beta\}$ to $\{e_1^\alpha, \dots, e_r^\alpha\}$ is $\bar{g}_{\alpha\beta}$ so that

$$A_\alpha = \bar{g}_{\alpha\beta}^{-1} d\bar{g}_{\alpha\beta} + \bar{g}_{\alpha\beta}^{-1} A_\beta \bar{g}_{\alpha\beta}.$$

Proposition 2.1. Connections *always exist*.

Proof. Let $(E, B\pi, \mathbb{R}^r)$ be a vector bundle with the vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ and corresponding local frames $\{e_1^\alpha, \dots, e_r^\alpha\}$. Then, on every E_{U_α} , we can pick the trivial connection $d_\alpha = d|_{E_{U_\alpha}}$ (i.e., $A_\alpha \equiv 0$). Let $\{\psi_\alpha : B \rightarrow \mathbb{R}\}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}$. Then for every $b \in B$,

- $\text{supp}(\psi_\alpha) \subset U_\alpha$,
- only a finite number of ψ_α 's are nonzero at b , and
- $\sum_\alpha \psi_\alpha(b) = 1$.

Set $D = \sum_\alpha \psi_\alpha d_\alpha$ so that $D(\sigma) = \sum_\alpha \psi_\alpha d_\alpha \sigma$ for all $\sigma \in \Gamma(E)$. D is a connection because it is \mathbb{R} linear, and the Leibniz rule holds:

$$\begin{aligned} D(f\sigma) &= \sum_\alpha \psi_\alpha d_\alpha(f\sigma) \\ &= \sum_\alpha \psi_\alpha (df \otimes \sigma + f d_\alpha \sigma) \\ &= \left(\sum_\alpha \psi_\alpha \right) df \otimes \sigma + f \left(\sum_\alpha \psi_\alpha d_\alpha \sigma \right) \\ &= df \otimes \sigma + f D\sigma. \end{aligned}$$

□

Let $\mathcal{A}(E)$ be the set of all connections on E . This set is not closed under addition! Let $D, D' \in \mathcal{A}(E)$ and define

$$\begin{aligned} D + D' : \Gamma(E) &\rightarrow \Gamma(T^*B \otimes E) \\ \sigma &\mapsto D(\sigma) + D'(\sigma). \end{aligned}$$

Although $D + D'$ is a well-defined map, it does not satisfy Leibniz: Let $\sigma \in \Gamma(E)$ and $f \in C^\infty(B)$. Then

$$\begin{aligned} (D + D')(f\sigma) &= D(f\sigma) + D'(f\sigma) \\ &= df \otimes \sigma + fD(\sigma) + df \otimes \sigma + fD'(\sigma) \\ &= 2df \otimes \sigma + f(D + D')(\sigma) \\ &\neq df \otimes \sigma + f(D + D')(\sigma). \end{aligned}$$

However, if we had considered $a_1D + a_2D'$ such that $a_1 + a_2 = 1$, then we would have a connection. So $\mathcal{A}(E)$ is convex: For all $D_1, \dots, D_l \in \mathcal{A}(E)$ and $a_1, \dots, a_l \in \mathbb{R}$ such that $\sum_{i=1}^l a_i = 1$, then $a_1D_1 + \dots + a_lD_l \in \mathcal{A}(E)$.

$\mathcal{A}(E)$ is an affine space. To see this, we need the following notation:

Notation. Let $(V, B, \tilde{\pi}, \mathbb{R}^m)$ be a vector bundle. We set

$$\Omega^k(B) := \Gamma\left(\bigwedge^k T^*B \otimes V\right).$$

In particular,

$$\Omega^1(V) = \Gamma(T^*B \otimes V).$$

Proposition 2.2. $\mathcal{A}(E)$ is an affine space modelled on $\Omega^1(\text{End } E)$. To be more precise, if D_0 is any connection on E , then

$$\mathcal{A}(E) = \{D_0 + a \mid a \in \Omega^1(\text{End } E)\}$$

Lecture 9 --- February 4, 2020

Recall.

- A **connection** on a vector bundle $(E, B, \pi, \mathbb{R}^r)$ is a map $D : \Gamma(E) \rightarrow \Gamma(T^*B \otimes E)$ that is \mathbb{R} -linear and satisfies $D(f\sigma) = df \otimes \sigma + fD(\sigma)$ for any $f \in C^\infty(B)$ and $\sigma \in \Gamma(E)$.
- Given an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of E and local frames $e_i^\alpha = \varphi_\alpha^{-1}(-, \vec{e}_i)$,

$$D(e_j^\alpha) = \sum_i \alpha_{ij}^\alpha \otimes e_i$$

where $a_{ij}^\alpha \in \Omega^1(U_\alpha)$, so that $A_\alpha = (a_{ij}^\alpha)$ is a matrix of 1-forms, called the connection matrix of D over U_α .

Claim. For all $b \in U_\alpha \cap U_\beta \neq \emptyset$,

$$e_j^\beta(b) = \sum_i (\bar{g}_{\alpha\beta}(b))_{ij} e_i^\alpha(b).$$

Proof.

$$\begin{aligned} e_j^\beta(b) &= \varphi_\beta^{-1}(b, \vec{e}_j) \\ &= \varphi_\alpha^{-1} \circ g_{\alpha\beta}(b, \vec{e}_j) \\ &= \varphi_\alpha^{-1}(b, \bar{g}_{\alpha\beta}(b) \vec{e}_j) \\ &= \sum_i (\bar{g}_{\alpha\beta}(b))_{ij} \varphi_\alpha^{-1}(b, \vec{e}_i) \\ &= \sum_i (\bar{g}_{\alpha\beta}(b))_{ij} e_i^\alpha. \end{aligned}$$

□

So the change of basis matrix from $\{e_1^\alpha, \dots, e_r^\alpha\}$ to $\{e_1^\beta, \dots, e_r^\beta\}$ is $\bar{g}_{\alpha\beta}$, so

$$A_\beta = \bar{g}_{\alpha\beta}^{-1} d\bar{g}_{\alpha\beta} + \bar{g}_{\alpha\beta} A_\alpha \bar{g}_{\alpha\beta}.$$

- $\mathcal{A}(E) = \{\text{all connections on } E\}$ is not closed under addition. Nonetheless, it is convex: For all $D_1, \dots, D_l \in \mathcal{A}(E)$ and $a_1, \dots, a_l \in \mathbb{R}$ such that $\sum_{j=1}^l a_j = 1$, we have that

$$a_1 D_1 + \dots + a_l D_l \in \mathcal{A}(E).$$

Proposition 2.3. $\mathcal{A}(E)$ is an affine space modeled on $\Omega^1(\text{End}(E)) := \Gamma(T^*M \otimes \text{End}(E))$.

Note. Let $(V, B, \pi, \mathbb{R}^r)$ be a vector bundle and set $\Omega^k(V) := \Gamma(\bigwedge^k B \otimes V)$. Locally, $\tau \in \Omega^k(V)$ looks like $\tau = \sum_{i=1}^m \omega_i \otimes e_i$ where $\{e_1, \dots, e_m\}$ is a local frame of V and $\omega_1, \dots, \omega_m \in \bigwedge^k U$ with $U \subseteq B$ open. For any $X_1, \dots, X_k \in \Gamma(TB)$, we define

$$\begin{aligned} \tau(X_1, \dots, X_k) &:= \sum_{i=1}^m \omega_i(X_1, \dots, X_k) \otimes e_i \\ &= \sum_{i=1}^m \omega_i(X_1, \dots, X_k) e_i \in \Gamma(V). \end{aligned}$$

Note that the definition of $\tau(X_1, \dots, X_k)$ is independent of the local description of τ .

Proof. Let $D_0 \in \mathcal{A}(E)$. It is enough to show that

$$\mathcal{A}(E) = \{D_0 + a \mid a \in \Omega^1(\text{End}(E))\}$$

What do elements of $\Omega^1(\text{End}(E))$ look like? Locally, $a = \sum_i a_i \otimes \psi_i$ where the a_i are 1-forms and $\psi_i \in \text{End}(E|_U)$ where $U \subset B$ is open. Then for all $\sigma \in \Gamma(E|_U)$,

$$a(\sigma) = \sum_i a_i \otimes \psi_i(\sigma)$$

so

$$\begin{aligned} a : \Gamma(E) &\rightarrow \Gamma(T^*B \otimes E) \\ \sigma &\mapsto a(\sigma). \end{aligned}$$

So a is $C^\infty(B)$ -linear because, for any $f \in C^\infty(B)$,

$$\begin{aligned} a(f\sigma) &= \sum_i a_i \psi_i(f\sigma) \\ &= \sum_i a_i \otimes f\psi_i(\sigma) \\ &= f \sum_i a_i \otimes \psi_i(\sigma) \\ &= fa(\sigma). \end{aligned}$$

So any $a \in \Omega^1(\text{End}(E))$ induces a $C^\infty(B)$ -linear map $a : \Gamma(E) \rightarrow \Gamma(T^*B \otimes E)$. Conversely, any $C^\infty(B)$ -linear map $a : \Gamma(E) \rightarrow \Gamma(T^*B \otimes E)$ induces an element of $\Omega^1(\text{End}(E))$.

Let $D, D' \in \mathcal{A}(E)$. Let us check that

$$D - D' \in \Omega^1(\text{End}(E)).$$

It is enough to check that the induced map

$$\begin{aligned} D - D' : \Gamma(E) &\rightarrow \Gamma(T^*B \otimes E) \\ \sigma &\mapsto D(\sigma) - D'(\sigma) \end{aligned}$$

is $C^\infty(B)$ -linear. let $\sigma, \sigma' \in \Gamma(E)$ and $f \in C^\infty(B)$. Then

$$\begin{aligned} (D - D')(f\sigma + \sigma') &= (D(f\sigma) + D(\sigma')) - (D'(f\sigma) + D'(\sigma')) \\ &= (df \otimes \sigma + fD(\sigma) + D(\sigma')) - (df \otimes \sigma + fD'(\sigma) - D'(\sigma')) \\ &= f(D - D')(\sigma) + (D - D')(\sigma'). \end{aligned}$$

and so $D - D' \in \Omega^1(\text{End}(E))$. □

We have seen that connections generalize the exterior derivative.

Recall. Let $U \subset B$ be open with coordinates (x_1, \dots, x_n) . Then for any $f \in C^\infty(U)$, then

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

In particular, if for any $i \in \{1, \dots, n\}$, we geta

$$df \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial f}{\partial x_i}.$$

In general, for any $X = \sum_i a_i \frac{\partial}{\partial x_i}$, then

$$df(X) = \sum_i a_i \frac{\partial f}{\partial x_i} = \nabla f \cdot (a_1, \dots, a_n).$$

Also, for all $\omega \in \Omega^1(U)$,

$$\omega = \sum_i \omega_i \left(\frac{\partial}{\partial x_i} \right) dx_i.$$

Lets go back to a connection $D \in \mathcal{A}(E)$. Let $U \subset B$ be an open set over which B has coordinates x_1, \dots, x_n and E is trivial with local frame $\{e_1, \dots, e_r\}$. Then for all $\sigma \in \Gamma(E|_U)$,

$$D(\sigma) = \sum_{i=1}^r \omega_i \otimes e_i$$

with $\omega_i \in \Omega^1(U)$. And, for all $X \in \Gamma(TU)$,

$$D(\sigma)(X) := \sum_{i=1}^r \omega_i(X) e_i \in \Gamma(E|_U).$$

So for fixed $X \in \Gamma(TB)$, we get a map

$$\begin{aligned} D_X : \Gamma(E) &\rightarrow \Gamma(E) \\ \sigma &\mapsto D(\sigma)(X) \end{aligned}$$

Note that D_X is \mathbb{R} -linear and satisfies Leibniz in σ . We say that $D_X(\sigma)$ is the *covariant derivative of σ in the direction X* . Also note that for any $f \in C^\infty(B)$,

$$D(\sigma)(fX) = f(D(\sigma)(X)), \text{ or } D_{fX}(\sigma) = fD_X(\sigma).$$

We then get a map

$$\begin{aligned} \nabla : \Gamma(TB) \times \Gamma(E) &\rightarrow \Gamma(E) \\ (X, \sigma) &\mapsto D_X(\sigma) \end{aligned}$$

such that it is

- $C^\infty(B)$ -linear in X
- \mathbb{R} linear in σ
- Satisfies Leibniz in σ :

$$\begin{aligned} D_X(f\sigma) &= D(f\sigma)(X) \\ &= (df \otimes \sigma + fD(\sigma))(X) \\ &= df(X)\sigma + fD(\sigma)(X) \\ &= X(f)\sigma + fD_X(\sigma). \end{aligned}$$

Definition 2.2. A map $\nabla : \Gamma(TB) \times \Gamma(E) \rightarrow \Gamma(E)$ such that

- $C^\infty(B)$ -linear in X ,
- \mathbb{R} -linear in σ , and
- $\nabla(X, f\sigma) = X(f)\sigma + f\nabla(X, \sigma)$

is called a *linear connection on E* , or a *covariant derivative on E* .

Note. 1. Tu defines connections this way.

2. There is a one-to-one correspondence between elements of $\mathcal{A}(E)$ and linear connections $\nabla : \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E)$. We saw that any $D \in \mathcal{A}(E)$ induces a ∇ . Conversely, given a linear connection ∇ , we can define $D \in \mathcal{A}(E)$ by

$$\begin{aligned} D : \Gamma(E) &\rightarrow \Gamma(T^*B \otimes E) \\ \sigma &\mapsto \nabla(-, \sigma) \end{aligned}$$

3. When $E = TB$, linear connections

$$\nabla : \Gamma(TB) \times \Gamma(TB) \rightarrow \Gamma(TB)$$

are called *affine connections*. In local coordinates (x_1, \dots, x_n) on B and a local frame $\{e_1, \dots, e_r\}$ on E :

$$\begin{aligned} D(\sigma) &= \sum_i \omega_i \otimes e_i \quad (\text{with } \omega_i \in \bigwedge^1(U)) \\ &= \sum_{i,j} \omega_i \left(\frac{\partial}{\partial x_j} dx_j \otimes e_i \right) \\ &= \sum_j dx_j \otimes \left(\sum_i \omega_i \left(\frac{\partial}{\partial x_j} e_i \right) \right) \\ &= \sum_j dx_j \otimes D_{\frac{\partial}{\partial x_j}}(\sigma) \end{aligned}$$

Lecture 10 --- February 6, 2020

Recall. $(E, B, \pi, \mathbb{R}^r)$ a vector bundle and $D : \Gamma(E) \rightarrow \Gamma(T^*B \otimes E)$ a connection on E . For any $X \in \Gamma(TB)$ and $\sigma \in \Gamma(E)$, we can define

$$D_X \sigma = (\text{covariant derivative on } \sigma \text{ in the direction of } X)$$

where if, locally, $D(\sigma) = \sum_i \omega_i \otimes e_i$ where $\{e_1, \dots, e_r\}$ is a local frame of E and ω_i are local 1-forms, then

$$D_X \sigma := \sum_i \omega_i(X) e_i.$$

Note. If $f \in C^\infty(B)$, $D_{fX} \sigma = f D_X \sigma$. So $D_X : \Gamma(E) \rightarrow \Gamma(T^*B \otimes E)$ is such that $D_{fX} \sigma = f D_X \sigma$ (hence too \mathbb{R} -linear) and D_X satisfies a Leibniz rule:

$$D_X(f\sigma) = X(f)\sigma + f D_X(\sigma).$$

$$\begin{aligned} \nabla : \Gamma(TB) \times \Gamma(E) &\rightarrow \Gamma(E) \\ (X, \sigma) &\mapsto D_X \sigma \end{aligned}$$

is called a linear connection.

Note. The connection D is completely determined by the D_X 's, for all $X \in \Gamma(TB)$. In particular, if $\{e_1, \dots, e_r\}$ is a local frame of E and $D = d + A$ with A the connection matrix in this frame and $\{x_1, \dots, x_n\}$ are local coordinates for B , then

$$a_{ij} = \sum_k a_{ij} \left(\frac{\partial}{\partial x_k} \right) dx_k$$

and

$$\begin{aligned} D_{\frac{\partial}{\partial x_k}}(e_j) &= D(e_j) \left(\frac{\partial}{\partial x_j} \right) \\ &= \sum_i a_{ij} \left(\frac{\partial}{\partial x_k} \right) e_i. \end{aligned}$$

So, the connection D is completely determined (locally) by $D_{\frac{\partial}{\partial x_k}}(e_j)$ for $j = 1, \dots, r$ and $k = 1, \dots, n$.

Example 2.2. 1. $M \subseteq \mathbb{R}^n$ a submanifold so that $TM \subset T\mathbb{R}^n|_M \cong M \times \mathbb{R}^n$. Let $\sigma \in \Gamma(TM)$. Then we can think of it as

$$\begin{aligned} \sigma : M &\rightarrow TM \subseteq M \times \mathbb{R}^n \\ x &\mapsto (x, \bar{\sigma}(x)) \end{aligned}$$

for some smooth $\bar{\sigma} : M \rightarrow \mathbb{R}^n$ such that $\sigma(x) \in T_x M$ for each $x \in M$. Since $\bar{\sigma} : M \rightarrow \mathbb{R}^n$ is smooth with $M \subset \mathbb{R}^n$, there is an open $U \subset \mathbb{R}^n$ with $M \subset U$ and $\bar{\sigma} : U \rightarrow \mathbb{R}^n$ (i.e., $\bar{\sigma}$ extends to a smooth function on a neighbourhood of M). So, we can think of σ as $\sigma : U \rightarrow T\mathbb{R}^n|_U$, and we can apply the trivial connection d on $T\mathbb{R}^n|_U$ to it:

$$d\sigma \in \Gamma(T^*U \otimes TU).$$

But, $d\sigma(X) \in \Gamma(TU)$ for any $X \in \Gamma(TU)$. So, we may not have that $d\sigma(X) \in \Gamma(TM)$. So, we just take $\text{pr}_{TM}(d\sigma)$. Thus, we get the connection D on TM : For every $\sigma \in \Gamma(TM)$ and every $X \in TM$,

$$D_X(\sigma) := \text{pr}_{TM}(d\sigma(X)),$$

where $\text{pr}_{TM} : TU|_{TM} \rightarrow TM$.

2. Let $(E, B, \pi, \mathbb{R}^r)$ and $(E', B, \pi', \mathbb{R}^{r'})$ be two vector bundles on B with two connections D, D' , respectively. Then there exist natural induced connections on $E \oplus E', E \otimes E', E^*, \text{Hom}(E, E')$ and f^*E for all $f : N \rightarrow B$ smooth.

Let $\sigma \in \Gamma(E|_U)$ and $\sigma' \in \Gamma(E'|_U)$ and suppose that on U , Let $D(\sigma) = \sum_i \omega_i \otimes \sigma_i$ for $\omega_i \in \Omega^1(U)$ and $\sigma_i \in \Gamma(E|_U)$ and $D'(\sigma') = \sum_j \omega'_j \otimes \sigma'_j$ for $\omega'_j \in \Omega^1(U)$ and $\sigma'_j \in \Gamma(E'|_U)$. Then

(i) $E \oplus E'$. Define a connection ∇ by

$$\begin{aligned} \nabla(\sigma \oplus \sigma') &= D(\sigma) \oplus D'(\sigma') \\ &= \sum_i \omega_i \otimes (\sigma_i \oplus 0) + \sum_j \omega'_j \otimes (0 \oplus \sigma'_j). \end{aligned}$$

(ii) $E \otimes E'$.

$$\begin{aligned}\nabla(\sigma \otimes \sigma') &= D(\sigma) \otimes \sigma' + \sigma \otimes D'(\sigma') \\ &= \sum_i \omega_i \otimes (\sigma_i \otimes \sigma') + \sum_j \omega'_j \otimes (\sigma \otimes \sigma'_j)\end{aligned}$$

(iii) E^* . We have a natural connection on E^* defined by:

$$D^* : \Gamma(E^*) \rightarrow \Gamma(T^*B \otimes E^*)$$

where for all $\psi \in \Gamma(E^*)$, $D^*(\psi) \in \Gamma(T^*B \otimes E^*)$ is completely determined by $D^*(\psi)(\sigma) \in \Gamma(T^*B)$ for all $\sigma \in \Gamma(E)$. So, we set

$$D^*(\psi)(\sigma) := d(\psi(\sigma)) - \psi(D(\sigma))$$

where

$$\psi(D(\sigma)) = \underbrace{\sum_i \psi(\sigma_i) \omega_i}_{\in \Gamma(T^*B)}$$

(iv) $\mathbf{Hom}(E, E')$. We have a natural connection ∇ given by, for all $\psi \in \Gamma(\mathbf{Hom}(E, E'))$ and for all $\sigma \in \Gamma(E)$ we set

$$\nabla(\psi)(\sigma) := D'(\psi(\sigma)) - \psi(D(\sigma)).$$

(v) If $f : N \rightarrow B$ is smooth and we have a local frame $\{e_1, \dots, e_r\}$ of E on U , and $D = d + A$, then on $f^{-1}(U)$,

$$f^*D := d + f^*A$$

is a connection matrix, where $f^*A = (f^*a_{ij})$ where $A = (a_{ij})$

2.1.2 Curvature

Recall. Suppose M is a smooth manifold with local coordinates (x_1, \dots, x_n) .

$$\Omega^0(M) := C^\infty(M)$$

$$\Omega^k(M) = (\text{smooth } k\text{-forms on } M) = \Gamma\left(\bigwedge^k T^*M\right), 1 \leq k \leq n$$

$$\Omega^k(M) = 0, k > n.$$

Note. • For all $f \in C^\infty(M)$, $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.

• For all $\omega = \sum_I a_I dx_I \in \Omega^k(M)$, $d\omega = \sum_I da_I \wedge dx_I$.

• **Leibniz.** For all $\eta \in \Omega^p(M)$ and $\omega \in \Omega^q(M)$,

$$d(\eta \wedge \omega) = d\eta \wedge \omega + (-1)^p \eta \wedge d\omega.$$

• **de Rham Complex.**

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \rightarrow \dots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

this is a complex because $d \circ d = 0$.

Now, fix a vector bundle $(E, B, \pi, \mathbb{R}^r)$ with $n = \dim B$. Set

$$\Omega^0(E) := \Gamma(E)$$

$$\Omega^k(E) := \Gamma\left(\bigwedge^k B \otimes E\right) = (\text{bundle-valued } k\text{-forms}), 1 \leq k \leq n$$

$$\Omega^k(E) := 0, k > n.$$

If $\omega \in \Omega^p(B)$ and $\tau \in \Omega^q(E)$ so that locally

$$\tau = \sum_i \eta_i \otimes \sigma_i$$

where η_i are k -forms and $\sigma_i \in \Gamma(E)$. We define

$$\omega \wedge \tau := \sum_i (\omega \wedge \eta_i) \otimes \sigma_i \in \Omega^{p+q}(E|_U).$$

Let D be a connection on E so that

$$D : \Omega^0(E) \rightarrow \Omega^1(E)$$

is \mathbb{R} -linear and satisfies Leibniz. How can we extend this to a map

$$D : \Omega^p(E) \rightarrow \Omega^{p+1}(E)?$$

If ω is a local p -form on B and σ is a local section of E so that $\omega \otimes \sigma \in \Omega^p(E|_U)$. We set

$$D(\omega \otimes \sigma) := d\omega \otimes \sigma + (-1)^p \omega \wedge D(\sigma) \in \Omega^{p+1}(E|_U),$$

and extend this definition \mathbb{R} -linearly.

- If $k = 0$: $D(f\sigma) = df \otimes \sigma + fD(\sigma)$. This is just the usual Leibniz.
- If $k > 0$, then for all $f \in C^\infty(B)$, $(f\omega) \otimes \sigma = \omega \otimes (f\sigma)$.

$$\begin{aligned} D(f\omega \otimes \sigma) &= d(f\omega) \otimes \sigma + f\omega \wedge D(\sigma) \\ &= df \wedge \omega \otimes \sigma + f d\omega \otimes \sigma + (-1)^p f\omega \wedge D(\sigma) \end{aligned}$$

and

$$\begin{aligned} D(\omega \otimes (f\sigma)) &= d\omega \otimes (f\sigma) + (-1)^p \omega \wedge D(f\sigma) \\ &= f d\omega \otimes \sigma + (-1)^p \omega \wedge df \otimes \sigma + (-1)^p f\omega \wedge D(\sigma) \end{aligned}$$

We get

$$0 \xrightarrow{d} \Omega^0(E) \xrightarrow{d} \Omega^1(E) \rightarrow \dots \xrightarrow{d} \Omega^{n-1}(E) \xrightarrow{d} \Omega^n(E) \xrightarrow{d} 0$$

but we may not have $D \circ D = 0$.

Definition 2.3. $F_D := D \circ D$ is the *curvature of D* . We say that D is *flat* if and only if $F_D = 0$.

Lecture 11 --- February 11, 2020

Recall. Fix a vector bundle $(E, B, \pi, \mathbb{R}^r)$. We define

$$\Omega^k(E) = \Gamma \left(\bigwedge^k B \otimes E \right)$$

$$\Omega^k(\text{End}(E)) = \Gamma \left(\bigwedge^k B \otimes \text{End}(E) \right)$$

and if we have a connection $D : \Omega^0(E) \rightarrow \Omega^1(E)$, we extend D to $\Omega^p(E)$ as follows:

$$D : \Omega^p(E) \rightarrow \Omega^{p+1}(E)$$

is defined on elements of $\Omega^p(B)$ of the form $\omega \otimes \sigma$, $\omega \in \Omega^p(E)$ and $\sigma \in \Gamma(E)$, then we take

$$D(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge D(\sigma) \quad (*).$$

(where $(-1)^p$ is necessary to ensure that $D(f\omega \otimes \sigma) = D(\omega \otimes f\sigma)$ for all $f \in C^\infty(B)$). We extend $(*)$ \mathbb{R} -linearly.

Then D satisfies a generalized Leibniz rule: For all $\tau \in \Omega^q(E)$ and $\alpha \in \Omega^p(B)$, then we have $\alpha \wedge \tau \in \Omega^{p+q}(E)$ and

$$D(\alpha \wedge \tau) = \underbrace{(d\alpha)}_{\in \Omega^{p+1}(B)} \wedge \tau + (-1)^p \alpha \wedge D(\tau).$$

Proof. Indeed, suppose that $\tau = \omega \wedge \sigma$ with $\omega \in \Omega^q(B)$ and $\sigma \in \Gamma(E)$. Then,

$$\begin{aligned} \alpha \wedge \tau &= \alpha \wedge (\omega \otimes \sigma) \\ &= (\alpha \wedge \omega) \otimes \sigma, \end{aligned}$$

so that by $(*)$, we have

$$\begin{aligned} D(\alpha \wedge \tau) &= D((\alpha \wedge \omega) \otimes \sigma) \\ &= d(\alpha \wedge \omega) \otimes \sigma + (-1)^{p+q} (\alpha \wedge \omega) \wedge D(\sigma) \\ &= (d\alpha \wedge \omega + (-1)^p \alpha \wedge d\omega) \otimes \sigma + (-1)^{p+q} (\alpha \wedge \omega) \wedge D(\sigma) \\ &= (d\alpha \wedge \omega) \otimes \sigma + (-1)^p (\alpha \wedge d\omega) \otimes \sigma + (-1)^{p+q} \alpha \wedge \omega \wedge D(\sigma) \\ &= d\alpha \wedge \tau + (-1)^p \alpha \wedge (d\omega \otimes \sigma + (-1)^q \omega \wedge D(\sigma)) \\ &= d\alpha \wedge \tau + (-1)^p \alpha \wedge D(\tau). \end{aligned}$$

By \mathbb{R} -linearity, we get the formula for all elements in $\Omega^q(E)$. □

By extending D to $\Omega^p(E)$, we get a chain

$$0 \xrightarrow{D} \Omega^0(E) \xrightarrow{D} \Omega^1(E) \rightarrow \dots \xrightarrow{D} \Omega^{n-1}(E) \xrightarrow{D} \Omega^n(E) \xrightarrow{D} 0$$

where $n = \dim B$. In general, $D \circ D \neq 0$ so that this is not a complex.

Definition 2.4. Given a connection D on E , we define $F_D = D \circ D$, which is called the *curvature of D* . Furthermore, D is called *flat* if $F_D = 0$.

Example 2.3. If $E = B \times \mathbb{R}^r$ is the trivial bundle and $D = d$ is the trivial connection on E , then $F_D = d \circ d = 0$, so the trivial connection is flat. We will see that, locally, any flat connection can be given by d in an appropriate local frame.

What are some of the properties of

$$F_D : \Omega^0(E) \rightarrow \Omega^2(E)?$$

1) F_D is $C^\infty(B)$ -linear: For all $\sigma \in \Gamma(E)$ and $f \in C^\infty(B)$, we have

$$F_D(f\sigma) := fF_D(\sigma).$$

Proof.

$$\begin{aligned}
F_D(f\sigma) &= D(D(f\sigma)) \\
&= D(df \otimes \sigma + fD(\sigma)) \\
&\stackrel{\text{defn}}{=} (d(df) \otimes \sigma + (-1)^1 df \wedge D(\sigma)) + (df \wedge D(\sigma) + fD^2(\sigma)) \\
&= fD(\sigma).
\end{aligned}$$

□

In general,

$$D \circ D : \Omega^p(E) \rightarrow \Omega^{p+1}(E)$$

is $C^\infty(B)$ -linear.

2) Locally, in terms of local coordinates (x_1, \dots, x_n) on B , we have seen that, for any local section σ of E ,

$$D(\sigma) = \sum_{i=1}^n dx_i \otimes D_{\frac{\partial}{\partial x_i}}(\sigma)$$

(where $D_{\frac{\partial}{\partial x_i}} : \Gamma(E) \rightarrow \Gamma(E)$ is so that $D_{\frac{\partial}{\partial x_i}}$ are local sections of E). Given this, we also have

$$\begin{aligned}
F_D(\sigma) &= \sum_{i,j} (dx_i \wedge dx_j) \otimes \left(D_{\frac{\partial}{\partial x_i}} \left(D_{\frac{\partial}{\partial x_j}}(\sigma) \right) \right) \\
\Rightarrow F_D \left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right) &= \sum_{i,j} (dx_i \wedge dx_j) \left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right) \otimes D_{\frac{\partial}{\partial x_i}} \left(D_{\frac{\partial}{\partial x_j}}(\sigma) \right) \\
&= D_{\frac{\partial}{\partial x_k}} \left(D_{\frac{\partial}{\partial x_l}}(\sigma) \right) - D_{\frac{\partial}{\partial x_l}} \left(D_{\frac{\partial}{\partial x_k}}(\sigma) \right).
\end{aligned}$$

We then see that $F_D = 0$ if and only if $D_{\frac{\partial}{\partial x_l}} \left(D_{\frac{\partial}{\partial x_k}}(\sigma) \right) = D_{\frac{\partial}{\partial x_k}} \left(D_{\frac{\partial}{\partial x_l}}(\sigma) \right)$ for all $k, l = 1, \dots, n$. So the connection is flat if and only if the covariant derivatives commute (with respect to the coordinate directions).

As with connections, the curvature can be described as a matrix of 2-forms in terms of a local frame as follows:

Example 2.4. $E = B \times \mathbb{R}^r$ and frame $\{e_1, \dots, e_r\}$ where $e_i(b) = (b, \vec{e}_i)$. Suppose that D is a connection on E with connection matrix $A = (a_{ij})$, where $D(e_j) = \sum_i a_{ij} \otimes e_i$. Then

$$\begin{aligned}
F_D(e_j) &= D(D(e_j)) \\
&= D \left(\sum_i a_{ij} \otimes e_j \right) \\
&= \sum_i D(a_{ij} \otimes e_j) \\
&= \sum_i (da_{ij} \otimes e_i + (-1)^1 a_{ij} \wedge D(e_i)) \\
&= \sum_i da_{ij} \otimes e_i - \sum_i a_{ij} \wedge D(e_i) \\
&= \sum_i da_{ij} \otimes e_i - \sum_i a_{ij} \left(\sum_k a_{ki} e_k \right) \\
&= \sum_i da_{ij} \otimes e_i - \sum_{i,k} (a_{ij} \wedge a_{ki}) \otimes e_k \\
&= \sum_i da_{ij} \otimes e_i + \underbrace{\sum_k \left(\sum_i a_{ki} \wedge a_{ij} \right)}_{(A \wedge A)_{kj}} \otimes e_k \\
&= \sum_i (dA)_{ij} \otimes e_i + \sum_k (A \wedge A)_{kj} \otimes e_k \\
&= \sum_i (dA + A \wedge A)_{ij} \otimes e_i \\
\Rightarrow F_D(e_j) &= \sum_i (dA + A \wedge A)_{ij} \otimes e_i.
\end{aligned}$$

In general, any local section σ of E can be written as $\sigma = \sum_{i=1}^r \bar{\sigma}_i e_i$ for some smooth functions $\bar{\sigma}_1, \dots, \bar{\sigma}_r$. By $C^\infty(B)$ -linearity of F_D , we get:

$$\begin{aligned} F_D(\sigma) &= \sum_{j=1}^r \bar{\sigma}_j F_D(e_j) \\ &= \sum_{j=1}^r \bar{\sigma}_j \left(\sum_i (dA + A \wedge A)_{ij} \right) \otimes e_i. \\ \implies F_D(\sigma) &= \sum_{i=1}^r \left(\sum_j (dA + A \wedge A)_{ij} \bar{\sigma}_j \right) \otimes e_i \\ &=: (dA + A \wedge A) \cdot \sigma. \end{aligned}$$

Here, $F_A := dA + A \wedge A$ is the *curvature matrix of D* with respect to $\{e_1, \dots, e_r\}$.

Also, if $\{e'_1, \dots, e'_r\}$ is another form where

$$e'_j = \sum_i h_{ij} e_i$$

where $h = (h_{ij}) : B \rightarrow \text{GL}(r, \mathbb{R})$ is the change of basis matrix, and A' is the connection matrix of D with respect to $\{e'_1, \dots, e'_r\}$ then:

$$A' = h^{-1} A h + h^{-1} dh$$

and

$$F_{A'} = h^{-1} F_A h \quad (\text{exercise.})$$

Note. If $F_D = 0$, then $F_A = 0$ with respect to *any* local frame on E .

In general, for any vector bundle E with vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ and corresponding local frames $\{e_1^\alpha, \dots, e_r^\alpha\}$ where $e_i^\alpha = \varphi_\alpha^{-1}(-, \vec{e}_i)$. Suppose that the connection D on E is given by the connection matrices A_α . Then $U_\alpha \cap U_\beta \neq \emptyset$,

$$A_\beta = \bar{g}_{\alpha\beta}^{-1} A_\alpha \bar{g}_{\alpha\beta} + \bar{g}_{\alpha\beta}^{-1} d\bar{g}_{\alpha\beta}$$

and

$$F_{A_\beta} = \bar{g}_{\alpha\beta}^{-1} F_{A_\alpha} \bar{g}_{\alpha\beta}$$

where $\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{R})$.

Theorem 2.1. A connection D on E is flat if and only if there exists a vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ such that $A_\alpha = 0$ for all $\alpha \in \mathcal{A}$.

Remark. If D is flat, then the vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ for which $A_\alpha = 0$ is such that $\bar{g}_{\alpha\beta} \equiv \text{constant}$, because $d\bar{g}_{\alpha\beta} = 0$ for all α, β .

Definition 2.5. A vector bundle E is called *flat* if and only if there exists a vector bundle atlas on E whose transition functions are constant.

Corollary 2.1. A vector bundle is flat if and only if it admits a flat connection.

Lecture 12 --- February 13, 2020

Let $(E, B, \pi, \mathbb{R}^r)$ be a vector bundle and D a connection on E . If $\{e_1, \dots, e_r\}$ is a local frame of E , then locally, if σ is a local section of E given by $\sigma = \sum_i \bar{\sigma}_i e_i$, then

$$F_D(\sigma) = \sum_i (F_A)_{ij} \bar{\sigma}_j \otimes e_i$$

where $F_A := dA + A \wedge A$ is the curvature matrix (with respect to this local frame).

Also, if $\{e'_1, \dots, e'_r\}$ is another local frame with $e'_j = \sum_i h_{ij} e_i$ (where $h : U \rightarrow \text{GL}(r, \mathbb{R})$ is a smooth map with $h = (h_{ij})$), then if A' is the connection matrix of D with respect to $\{e'_1, \dots, e'_r\}$, then:

$$A' = h^{-1} A h + {}^{-1} dh$$

and

$$F_{A'} = h^{-1} F_A h.$$

So, we have that $F_D = 0 \iff F_A = 0$ for every connection matrix A .

Proposition 2.4. D is flat if and only if there exists a vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ on E with respect to which every $A_\alpha = 0$ for all $\alpha \in \mathcal{A}$, where A_α is the connection matrix of D with respect to the frame $\{e_1^\alpha, \dots, e_r^\alpha\}$.

Before proving the proposition, we need some notation. Let $U \subset B$ be an open set with local coordinates (x_1, \dots, x_n) and assume that E admits a vector bundle chart for U with associated local frame $\{e_i\} = \{\varphi^{-1}(-, \bar{e}_i)\}$. Let A be the corresponding connection matrix of D . So

$$A = \sum_{k=1}^n A_k dx_k$$

where $A_k : U \rightarrow \mathfrak{gl}(r, \mathbb{R})$ is a smooth map, and so

$$F_A = dA + A \wedge A = \sum_{k < l} \left(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} = [A_k, A_l] \right) dx_k \wedge dx_l.$$

Proof.

$$\begin{aligned} dA &= \sum_{k=1}^n dA_k \wedge dx_k \\ &= \sum_{k=1}^n \left(\sum_{l=1}^n \frac{\partial A_k}{\partial x_l} dx_l \right) \wedge dx_k \\ &= \sum_{k < l} \left(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} \right) dx_k \wedge dx_l \end{aligned}$$

and

$$\begin{aligned} A \wedge A &= \left(\sum_{k=1}^n A_k dx_k \right) \wedge \left(\sum_{l=1}^n A_l dx_l \right) \\ &= \sum_{k, l=1}^n A_k A_l dx_k \wedge dx_l \\ &= \sum_{k < l} (A_k A_l - A_l A_k) dx_k \wedge dx_l \\ &= \sum_{k < l} [A_k, A_l] dx_k \wedge dx_l. \end{aligned}$$

□

So, $F_D = 0$ iff $F_A = 0$ for all A iff $\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} + [A_k, A_l] = 0$ for all $k < l$.

Suppose that $\{e'_1, \dots, e'_r\}$ is related to $\{e_1, \dots, e_r\}$ by $h : U \rightarrow \text{GL}(r, \mathbb{R})$ so that its connection matrix is

$$A' = h^{-1} A h + h^{-1} dh$$

If $A = \sum_{k=1}^n A_k dx_k$ and $A' = \sum_{k=1}^n A'_k dx_k$, then:

$$A'_k = h^{-1} A_k h + h^{-1} \frac{\partial h}{\partial x_k}.$$

Therefore, if there exists a local frame $\{e'_1, \dots, e'_r\}$ with respect to which $A' = 0$ then there exists $h : U \rightarrow \text{GL}(r, \mathbb{R})$ such that

$$h^{-1} A_k h + h^{-1} \frac{\partial h}{\partial x_k}.$$

Proof. (\Leftarrow) If there is a vector bundle atlas such that $A_\alpha = 0$ for all α , then $F_{A_\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha = 0$. (\Rightarrow) Suppose that $F_D = 0$, so that $F_A = 0$ for any connection matrix A . Let us first assume that B is a hypercube: $B = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_i| \leq 1\}$. Then E is trivial on B , so there exists a global vector bundle chart $\varphi : E \rightarrow B \times \mathbb{R}^r$ and a corresponding global frame $\{e_i = \varphi^{-1}(-, \vec{e}_i)\}_{i=1}^r$. Let A be the connection matrix of D with respect to this frame and let us write it:

$$A = \sum_{k=1}^n A_k dx_k$$

with each $A_k : U \rightarrow \mathfrak{gl}(r, \mathbb{R})$ smooth for all $k = 1, \dots, n$. Then $F_A = 0$, which implies

$$\frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k} + [A_k, A_l] = 0 \quad (*).$$

We want to find $h : B \rightarrow \text{GL}(r, \mathbb{R})$ smooth such that

$$h^{-1} A_k h + h^{-1} \frac{\partial h}{\partial x_k}.$$

We do this in several steps by finding smooth maps $B \rightarrow \text{GL}(r, \mathbb{R})$ that take A to a connection matrix \tilde{A} with $\tilde{A}_1 = 0$, then $\tilde{A}_2 = 0$, etc.

- Can we find $h : B \rightarrow \text{GL}(r, \mathbb{R})$ smooth such Mathematics

$$\tilde{A}_1 = h^{-1} A_1 h + h^{-1} \frac{\partial h}{\partial x_1} \iff A_1 h + \frac{\partial h}{\partial x_1} = 0.$$

This is a linear ODE for h in the variable x_1 with x_2, \dots, x_n fixed (but also with the equation varying smoothly in x_2, \dots, x_n). So there exists a smooth solution by the ODE theorem (exercise)

- Suppose that there is $h : B \rightarrow \text{GL}(r, \mathbb{R})$ smooth taking A to a connection matrix \tilde{A} with $\tilde{A}_1, \dots, \tilde{A}_p = 0$. Let us show that there is a new $\tilde{h} : B \rightarrow \text{GL}(r, \mathbb{R})$ taking \tilde{A} to $\tilde{\tilde{A}}$ with

$$\tilde{\tilde{A}}_1, \dots, \tilde{\tilde{A}}_p = 0.$$

Then \tilde{h} much satisfy

$$\begin{aligned} \tilde{\tilde{A}}_k &= \tilde{h}^{-1} \tilde{A}_k \tilde{h} + \tilde{h}^{-1} \frac{\partial \tilde{h}}{\partial x_k} = 0, \forall k = 1, \dots, p+1 \\ \iff \begin{cases} \frac{\partial \tilde{h}}{\partial x_k} = 0 & \forall k = 1, \dots, p \quad (**) \\ \tilde{A}_{p+1} \tilde{h} + \frac{\partial \tilde{h}}{\partial x_{p+1}} = 0 & (***) \end{cases} \end{aligned}$$

As before, by the ODE theorem, there exists a solution \tilde{h} to $(***)$. Also, since $F_{\tilde{A}} = 0$ by $(*)$, for all $k < p+1$, since D is flat we have

$$\begin{aligned} \frac{\partial \tilde{\tilde{A}}_{p+1}}{\partial x_k} - \underbrace{\frac{\partial \tilde{\tilde{A}}_k}{\partial x_{p+1}}}_{=0} + [\tilde{\tilde{A}}_k, \underbrace{\tilde{\tilde{A}}_{p+1}}_{=0}] &= 0 \\ \iff \frac{\partial \tilde{\tilde{A}}_{p+1}}{\partial x_k} &= 0 \quad \forall k = 1, \dots, p. \end{aligned}$$

So $\tilde{\tilde{A}}_{p+1}$ does not depend on x_1, \dots, x_p . So \tilde{h} satisfies $(**)$.

- Now for a general vector bundle, start with a vector bundle atlas whose open cover of B consists of open sets diffeomorphic to hypercubes, and replace every vector bundle chart by a chart with respect to which the connection matrix is 0, as above.

□

We will end with a few more facts about curvature:

- We have seen that if D_0 is a fixed connection on E , then the set of all connections on E is

$$\mathcal{A}(E) = \{D_0 + a \mid a \in \Omega^1(\text{End}(E))\}.$$

One can show that

$$F_{D_0+a} = F_{D_0} + D_0(a) + a \wedge a$$

for every $a \in \Omega^1(\text{End}(E))$, where D_0 also denotes the induced connection on $\text{End}(E)$.

- **Bianchi identity.** Let D be a connection on E . Then,

$$F_D : \Gamma(E) \rightarrow \Omega^2(E)$$

and is $C^\infty(B)$ -linear. We can therefore think of F_D as an element of $\Omega^2(\text{End}(E))$.

As an aside: In general, if E_1 and E_2 are vector bundles on B , then $\Gamma(\text{Hom}(E_1, E_2))$ is identified with the set

$$\{C^\infty(B) - \text{linear maps } \Gamma(E_1) \rightarrow \Gamma(E_2)\}$$

Indeed, given $\psi \in \Gamma(\text{Hom}(E_1, E_2))$ so that

$$\psi : B \rightarrow \text{Hom}(E_1, E_2) = \bigsqcup_{b \in B} \text{Hom}((E_1)_b, (E_2)_b)$$

so that $\psi(b) : (E_1)_b \rightarrow (E_2)_b$ is \mathbb{R} -linear. Then ψ induces

$$\begin{aligned} \tilde{\psi} : \Gamma(E_1) &\rightarrow \Gamma(E_2) \\ \sigma &\mapsto \tilde{\psi}(\sigma) \end{aligned}$$

where

$$\begin{aligned} \tilde{\psi}(\sigma) : B &\rightarrow E_2 \\ b &\mapsto \psi(b)(\sigma(b)) \in (E_2)_b. \end{aligned}$$

Conversely, let $\tilde{\psi} : \Gamma(E_1) \rightarrow \Gamma(E_2)$ be $C^\infty(B)$ -linear. Set

$$\begin{aligned} \psi : B &\rightarrow \text{Hom}(E_1, E_2) \\ b &\mapsto \psi(b) \in \text{Hom}((E_1)_b, (E_2)_b) \end{aligned}$$

where, for all $b \in B$,

$$\begin{aligned} \psi(b) : (E_1)_b &\rightarrow (E_2)_b \\ e = \sigma(b) &\mapsto \tilde{\psi}(\sigma)(b) \end{aligned}$$

for some local section σ . One can show that this definition of $\psi(b)$ is independent of the choice of σ by the $C^\infty(B)$ -linearity of $\tilde{\psi}$ and $\psi(b)$ is \mathbb{R} -linear.

Proposition 2.5. For any connection D on E ,

$$D(F_D) = 0$$

where D also denotes the induced connection on $\text{End}(E)$.

Proof. $F_D \in \Omega^2(\text{End}(E))$ and for all $\psi \in \Gamma(\text{End}(E))$, then induced connection on $\text{End}(E)$ is such that for all $\sigma \in \Gamma(E)$,

$$D(\psi)(\sigma) := D(\psi(\sigma)) - \psi(D(\sigma)).$$

In general if $\tau \in \Omega^k(\text{End}(E))$, for all $\sigma \in \Gamma(E)$,

$$D(\tau)(\sigma) = D(\tau(\sigma)) - \tau(D(\sigma)).$$

So we have

$$\begin{aligned} D(F_D)(\sigma) &= D(F_D(\sigma)) - F_D(D(\sigma)) \\ &= D \circ D \circ D(\sigma) - D \circ D \circ D(\sigma) \\ &= 0. \end{aligned}$$

□