Gauge Theory --- PMATH 965

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Contents

Lecture 1 --- January 7, 2020

1 Fibre Bundles

Definition 1.1. A *fibre bundle* consists of the data (E, B, π, F) where E, B, F are (topological) manifolds and $\pi : E \to B$ is a continuous surjection that satisfies the *local triviality* condition: For every $p \in B$, there is an open neighbourhood $U \ni p$ such that $\varphi : \pi^{-1}(U) \cong U \times F$ is a homeomorphism such that $\mathrm{pr}_1 \circ \varphi = \pi$, where $\mathrm{pr}_1 : U \times F \to U$ is the projection. The set of all $\{(U_\alpha, \varphi_\alpha)\}$ is called the *local trivialization* of the bundle.

E is called the *total space*, B is the *base space* and F is the *fibre* and π is the *projection map*.

Note. For all $b \in B$, the set $\pi^{-1}(b) = \{p \in E \mid \pi(p) = b\}$ is called the *fibre at* b, or the *fibre over* b. Since $\mathrm{pr}_1 \circ \varphi = \pi$, we have $\pi^{-1}(b) \cong \{b\} \times F \cong F$. So we can think of E as a family of manifolds homeomorphic to F, parametrized by B.

Note. A fibre bundle (E, B, π, F) is also called an F-bundle.

Example 1.1.

1. $E = B \times F$ with $\pi = \mathrm{pr}_1$ is the *trivial bundle*. Note that taking $\pi = \mathrm{pr}_2$ gives a fibre bundle structure with base F and fibre B.
2. $E = S^1 \times \mathbb{R}$. E is a cylinder. In this case, E has two trivial bundle structures (as above), but with space $B = S^1$ we also have a vector bundle structure, as the fibres are $\mathbb{R}$.
3. **Möbius strip.** Example of a non-trivial $\mathbb{R}$-bundle on S^1. $M = I \times \mathbb{R}/_\sim$ where $(0, t) \sim (1, -t)$ for every $t \in \mathbb{R}$.
4. **Hopf fibration.** Example of a non-trivial S^1-bundle over S^2. Here,
   - $E = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$
   - $B = S^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + x^2 = 1\}$

- $F = S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

We take

$$\begin{aligned} \pi : S^3 &\to S^2 \\ (z_0, z_1) &\mapsto (2z_0\overline{z}_1, |z_0|^2 - |z_1|^2) \end{aligned}$$

is called the *Hopf map*. Then $|2z_0z_1|^2 + (|z_0|^2 - |z_1|^2)^2 = 1$, so $\pi(S^3) \subset S^2$, and π is well-defined and continuous. Also, π is surjective with $\pi^{-1}(z, x) \cong S^1$ for every $(z, x) \in S^2$. Indeed, let $(z, x) \in S^2$ so that $|z|^2 + x^2 = 1$ so that $-1 \leqslant x \leqslant 1$. Also, if $z = 0$, then $x = \pm 1$. Moreover, one can cover S^2 by the following two open sets:

$$\begin{aligned} U &= \{(z, x) \in S^2 \mid x \neq 1\} \\ &= S^2 \backslash \{(0, 1)\}, \text{ and} \\ V &= \{(z, x) \in S^2 \mid x \neq -1\} \\ &= S^2 \backslash \{(0, -1)\}. \end{aligned}$$

Let us now show that $\pi^{-1}(U) \cong U \times S^1$. let $(z, x) \in U$. So that $x \neq 1$. In particular, $-1 \leqslant x < 1$. Pick $(z_0 z_1) \in \pi^{-1}(U)$. Then $2z_0\overline{z_1} = z$ and $|z_0|^2 - |z_1|^2 = x$.

- If $z = 0$, then $(z, x) = (0, -1) \implies z_0 = 0, |z_1|^2 = 1$. Thus $\pi^{-1}(z, x) = \{(0, \lambda) \in \mathbb{C}^2 \mid |\lambda| = 1\} \cong S_1$.
- If $z \neq 0$, then $x \notin \{\pm 1\}$, so $-1 < x < 1$ and $z_0, z_1 \neq 0$ since $2z_0\overline{z}_1 = z$. Then $z_0 = \frac{z}{2\overline{z}_1}$. Replacing z_0 by this in $|z_0|^2 - |z_1|^2 = 1$, one gets $4|z_1|^4 - |z_1|^2 x - |z|^2 = 0$. There is only one positive solution, which is equal to $|z_1|^2 = \frac{1-x}{2}$. So $z_1 = \lambda\sqrt{\frac{1-x}{2}}, \lambda \in S^1$. By the relationship $z_0 = \frac{z}{2\overline{z_1}}$, we have $z_0 = \lambda\frac{z}{\sqrt{2(1-x)}}$. So $\pi^{-1}(z, x) \cong S^1$, as

$$(z_0, z_1) = \lambda\left(\frac{z}{\sqrt{2(1-x)}}, \sqrt{\frac{1-x}{2}}\right)$$

  And so $\pi^{-1}(z, x) = \{\lambda\left(\frac{z}{\sqrt{2(1-x)}}, \sqrt{\frac{1-x}{2}})\right) \mid \lambda \in S^1\} \cong S^1$.

This gives the local trivialization

$$\varphi : \pi^{-1}(U) \to U \times S^1$$

where if $\pi(z, x) = (z_0, z_1)$, $\varphi(z_0, z_0) = \lambda\left(\frac{z}{\sqrt{2(1-x)}}, \sqrt{\frac{1-x}{2}}\right)$. Finally, $\mathrm{pr}_1 \circ \varphi(z_0, z_1) = \pi(z_0, z_1)$. So we have that (E, B, π, F) is a S^1-bundle. This tells us that S^3 is an S^1-bundle over S^2. But, it cannot be a trivial bundle because S^3 is simply connected, but $S^3 \times S^1$ is not.

Lecture 2 --- January 9, 2020

Recall. A *fibre bundle* is a tuple (E, B, π, F) with $\pi : E \to B$ a continuous surjection that satisfies $\forall b \in B$ there is an open neighbourhood $U \subseteq B$ with $b \in U$ and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

Notation.

$$\begin{aligned} E &= \text{total space} \\ B &= \text{base space} \\ F &= \text{fibre} \\ \pi &= \text{projection map} \\ E_b := \pi^{-1}(b) &= \text{fibre of } E \text{ at } b \cong F \\ E_U &= \pi^{-1}(U) \subset E \end{aligned}$$

A fibre bundle (E, B, π, F) is also called an *F-bundle*.

Definition 1.2. A fibre bundle (E, B, π, F) is called *smooth* if E, B and F are smooth manifolds and $\pi : E \to B$ is a smooth surjection and for all $b \in B$, there exists and open neighbourhood $U \subset B$ of b and a diffeomorphism $: \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1 \circ \varphi = \pi$.

Note. In Definition 1.2, we just replace the continuity/homeomorphism by smooth/diffeomorphism.

Remark. Note that $\pi : E \to B$ is in fact a smooth submersion (i.e., the differential $\pi_* : TE \to TB$ is surjective at every point). This follows from the local triviality — not every smooth surjection is a submersion.

Example 1.2. 1. All of the examples from lecture 1 are smooth fibre bundles.

2. **Tangent bundles.** Let M be a smooth manifold of dimension n. Then, TM is a smooth $\mathbb{R}^n$-bundle. Indeed, let $\{(U_\alpha, \phi_\alpha)\}$ be a smooth atlas for M so that $\phi_\alpha : U_\alpha \subset M \overset{\text{diffeo}}{\to} \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$. Here, of course, ϕ_α are the coordinate charts and $\phi_\alpha \circ \phi_\beta^{-1}$ are the coordinate transformations. In particular, $\phi_\alpha \circ \phi_\beta^{-1}$ is a diffeomorphism whenever $U_\alpha \cap U_\beta \neq \varnothing$ so that, $\forall p \in U_\alpha \cap U_\beta$,

$$(\phi_\alpha \circ \phi_\beta^{-1})_*(\phi_\beta(p)) : T_{\phi_\beta(p)}\mathbb{R}^n \to T_{\phi_\alpha(p)}\mathbb{R}^n$$

is an isomorphism (of vector spaces).
Recall that the *tangent bundle* TM of M is defined as

$$TM = \coprod_{p \in M} T_p M$$

then, TM has the following smooth manifold structure: Let

$$\begin{aligned} \pi : TM &\to M \\ X_p \in T_p M &\mapsto p \end{aligned}$$

Suppose that

$$\begin{aligned} \phi_\alpha : U_\alpha &\to \mathbb{R}^n \\ p &\mapsto (x_1(p), \ldots, x_n(p)). \end{aligned}$$

Then, $\forall X \in T_p M$, $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\big|_p$ for some appropriate scalars $a_1, \ldots, a_n$. Denote by

$$\begin{aligned} \tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) &\to U_\alpha \times \mathbb{R}^n \\ \left(p, X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\Big|_p\right) &\mapsto (p = \pi(X), (a_1, \ldots, a_n)). \end{aligned}$$

Then $\{\pi^{-1}(U_\alpha)\}$ is a basis for a topology on TM with respect to which $\{(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)\}$ is a smooth atlas for TM. Additionally, $\pi : TM \to M$ is smooth with respect to this smooth structure (see Lee's Introduction to Smooth Manifolds). Note that $\pi \circ \tilde{\phi}_\alpha = \mathrm{pr}_1$ by the definition of $\tilde{\phi}_\alpha$. So $(TM, M, \pi, \mathbb{R}^n)$ is a smooth $\mathbb{R}^n$-bundle.

Note. Using the notation from above, the coordinate transformations of TM are given by

$$\left(\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}\right)(p, v = (a_1, \ldots, a_n)) = (p, (\phi_\alpha \circ \phi_\beta^{-1})_*(p)v)$$

1.1 Bundle Maps

Definition 1.3. Let (E, B, π, F) and (E', B, π', F') be two smooth fibre bundles over the same base space. A *bundle map* or a *bundle morphism* of these bundles is a smooth map $H : E \to E'$ such that $\pi' \circ H = \pi$ (*). Diagrammatically,

$$\begin{tikzcd} E \arrow[rr, "H"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

A *bundle isomorphism* is a bundle map which is a diffeomorphism. If such an isomorphism exists, then E and E' are said to be *isomorphic*, denoted $E \cong E'$.

Note. The property (*) tells us that bundle maps are fibre-preserving: $\forall b \in B$, $H\big|_{E_b} : E_b \to E'_b$. Also, if H is an isomorphism, then $H\big|_b : E_b \to E'_b$ is an isomorphism.

Definition 1.4. Fibre bundles isomorphic to the trivial bundle are called *trivial*. I.e., if there exists a diffeomorphism $H : E \to B \times F$ such that $\pi = \mathrm{proj}_1 \circ H$ (with the typical notations).

Note. If E is a trivial bundle, then we have $E = \pi^{-1}(B)$ so that H is a *global* trivialization. All fibre bundles are locally trivial (by definition), but may not be globally trivial (e.g. the Hopf fibration is an S^1-bundle over S^2 with total space S^3 which is not diffeomorphic (in fact, not even homeomorphic) to $S^1 \times S^2$).

Example 1.3. Let $S^1 = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\right\}$. Then, TS^1 is trivial.

Proof. Let us show that $TS^1 \cong S^1 \times \mathbb{R}$. Define the following atlas for S^1: Let U_1 be the "right half" of the circle with the top and bottom excluded. Then we define the map

$$\begin{aligned} \varphi_1 : U_1 &\to (-\pi/2, \pi/2) \\ (x, y) &\mapsto \arctan(y/x) =: \theta_1 \end{aligned}$$

We then take the open top U_2 with the map

$$\begin{aligned} \varphi_2 : U_2 &\to (0, \pi) \\ (x, y) &\mapsto \operatorname{arccot}(x/y) =: \theta_2 \end{aligned}$$

and the bottom half U_3 with

$$\begin{aligned} \varphi_3 : U_3 &\to (-\pi, 0) \\ (x, y) &\mapsto \operatorname{arccot}(x/y) - \pi =: \theta_3 \end{aligned}$$

and, lastly, the left open semicircle U_4 with

$$\begin{aligned} \varphi_4 : U_4 &\mapsto (\pi/2, 3\pi/2) \\ (x, y) &\mapsto \arctan(y, x) + \pi =: \theta_4 \end{aligned}$$

In all cases, $(\varphi_i \circ \varphi_j^{-1})_* = \mathrm{id}$. Thus, the coordinate transformations for TS^1 are

$$(\tilde{\varphi}_i \circ \tilde{\varphi}_j^{-1})_*(x, v) = \left((\varphi_i \circ \varphi_j^{-1})(x), v\right).$$

We can use the $\tilde{\varphi}_i$'s to construct an isomorphism H between TS^1 and $S^1 \times \mathbb{R}$. Take the usual projection map $\pi : TS^1 \to S^1$ and set

$$H\big|_{\pi^{-1}(U_i)} = \tilde{\varphi}_i : TU_i \to U_i \times \mathbb{R}.$$

Then, the $H\big|_{\pi^{-1}(U_i)}$ glue together to give a bundle map $H : TS^1 \to S^1 \times \mathbb{R}$ where we use the atlas $\left\{\left(\pi^{-1}(U_i), \tilde{\varphi}_i\right)\right\}$ and $((U_i \times \mathbb{R}, \varphi_i \times \mathrm{id}))$, and H is a diffeomorphism, and so $TS^1 \cong S^1 \times \mathbb{R}$. ◻

Note. Let $E = B \times F$ be the trivial bundle over B with projection $\pi = \mathrm{proj}_1 : E \to B$. Then E also admits a projection onto the fibre: proj_2. For a general fibre bundle, there may only exist a projection onto the fibre locally. We, however, have the following characteriszation of trivial bundles:

Proposition 1.1. (E, B, π, F) is trivial if and only if there exists a smooth map $\psi : E \to F$ such that the restrictions to each fibres $\psi\big|_{E_b}$ are diffeomorphisms.

Lecture 3 --- January 14, 2020

Definition 1.5. A *smooth fibre bundle* is a tuple (E, B, π, F) such that E, B and F are smooth manifolds and $\pi : E \to B$ is a smooth surjective map and for all $b \in B$, there is an open $U \ni b$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that $\pi = \mathrm{proj}_1 \circ \varphi$, where $\mathrm{proj}_1 : U \times F \to U$ is the projection onto the first factor.

Note. From now on we will assume that all manifolds are smooth and all fibrre bundles are smooth.

1.2 Bundle Atlases

Definition 1.6. A *bundle atlas* for a fibre bundle (E, B, π, F) is an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ together with bundle charts $\varphi_\alpha : E_\alpha =: \pi^{-1}(U_\alpha) \to U_\alpha \times F$ of B such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times F$.

Definition 1.7. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a bundle atlas for (E, B, π, F). If $U_\alpha \cap U_\beta \neq \varnothing$, we define the *transition functions* by

$$g_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}\big|_{U_\alpha\cap U_\beta} : \underbrace{(U_\alpha \cap U_\beta) \times F}_{\subset U_\beta\times F} \to \underbrace{(U_\alpha \cap U_\beta) \times F}_{\subset U_\alpha\times F}$$

Note that the $g_{\alpha\beta}$'s are all diffeomorphisms *and* they "preserve the fibres", i.e., for all $b \in U_\alpha \cap U_\beta$,

$$g_{\alpha\beta}\big|_{\{b\}\times F} : \{b\} \times F \xrightarrow{\cong} \{b\} \times F$$

$\Big($because $\varphi_\alpha\big|_{\{b\}\times F} : E_b \xrightarrow{\cong} \{b\} \times F\Big)$. This implies that for all $b \in U_\alpha \cap U_\beta$,

$$\overline{g}_{\alpha\beta}(b) = g_{\alpha\beta}\big|_{\{b\}\times F} \in \mathrm{Diff}\,(\{b\} \times F) \cong \mathrm{Diff}\,(F)$$

The maps

$$\begin{aligned} \overline{g}_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{Diff}(F) \\ b &\mapsto \overline{g}_{\alpha\beta}(b) \end{aligned}$$

are also called the *transition functions* of (E, B, π, F).

Example 1.4. Hopf fibration. (S^3, S^2, π, S^1) where

- $S^3 = \Big\{(z_0, z_1) \mid |z_0|^2 + |z_1|^2 = 1\Big\} \subset \mathbb{C}^2$
- $S^2 = \Big\{(z, x) \mid |z|^2 + x^2 = 1\Big\} \subset \mathbb{C} \times \mathbb{R}$
- $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$

and

$$\begin{aligned} \pi : S^3 &\to S^2 \\ (z_0, z_1) &\mapsto \left(2z_0\overline{z}_1, |z_0|^2 - |z_1|^2\right) \end{aligned}$$

Set $U = \big\{(z, x) \in S^2 \mid z \neq 1\big\} = S^2\backslash$north pole and $V = \big\{(z, x) \in S^2 \mid x \neq -1\big\} = S^2\backslash$south pole. $\{U, V\}$ is an open cover of S^2. We have the bundle charts:

$$\begin{aligned} \varphi_U : \overbrace{\pi^{-1}(U)}^{\subset S^3} &\to \overbrace{U \times S^1}^{\in S^2\times S^1} \\ (z_0, z_1) &\mapsto ((z, x), \lambda) \end{aligned}$$

where $(z_0, z_1) = \lambda\left(\frac{z}{\sqrt{1(1-x)}}, \sqrt{\frac{1-x}{2}}\right)$, and

$$\begin{aligned} \varphi_V : \pi^{-1}(V) &\to V \times S^2 \\ (z_0, z_1) &\mapsto ((z, x), \lambda') \end{aligned}$$

where $(z_0, z_1) = \lambda'\left(\sqrt{\frac{x+1}{2}}, \frac{\overline{z}}{\sqrt{2(x+1)}}\right)$. So $\{(U, \varphi_U), (V, \varphi_V)\}$ is a bundle atlas with transition functions

$$\begin{aligned} g_{UV} = \varphi_U \circ \varphi_V^{-1} : \overbrace{(U \cap V) \times S^1}^{\subset V \times S^1} &\to \overbrace{(U \cap V) \times S^1}^{\subset U \times S^1} \\ ((z,x), \lambda') &\mapsto ((z,x), \lambda) \end{aligned}$$

with

$$\lambda'\left(\sqrt{\frac{x+1}{2}}, \frac{\overline{z}}{\sqrt{2(x+1)}}\right) \underbrace{=}_{\varphi_V^{-1}} (z_0, z_1) \underbrace{=}_{\varphi_U} \lambda\left(\frac{z}{\sqrt{2(x+1)}}, \sqrt{\frac{1-x}{2}}\right)$$

This implies that

$$\lambda = \lambda'\left(\frac{\sqrt{1-x^2}}{z}\right) \overbrace{=}^{\text{since } |z^2| + |x|^2 = 1} \lambda' \frac{|z|}{z}.$$

So

$$\begin{aligned} g_{UV} : (U \cap V) \times S^1 &\to (U \cap V) \times S^1 \\ ((z,x)\lambda') &\mapsto ((z,x), \lambda'\left(\frac{|z|}{z}\right)) \end{aligned}$$

Thus $\overline{g}_{UV}(z,x) = \left(\text{multiplication in } S^1 \text{ by } \frac{|z|}{z}\right) \in \text{Diff}(S^1)$.

It can often be difficult to check that a set we suspect is the total space of a fibre bundle is a manifold. One nonetheless has the following construction:

Definition 1.8. (**Formal bundle atlases.**) Let B and F be manifolds, E a set and $\pi : E \to B$ a surjective map.

1. Suppose $U \subset B$ is open and

$$\varphi_U : \pi^{-1}(U) \to U \times F$$

   is a bijection with $\text{proj}_1 \circ \varphi_U = \pi$. Then, we call (U, φ_U) a *formal bundle chart for E*.

2. A family of bundle charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ where $\{U_\alpha\}$ is an open cover of B is called a *formal bundle atlas for E*.

3. The charts in a formal bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ are called *smoothly compatible* iff all transition functions

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

   (for $U_\alpha \cap U_\beta \neq \varnothing$) are all diffeomorphisms.

Theorem 1.1. (**Formal bundle atlases define fibre bundles.**) Let B and F be smooth manifolds, E a set and $\pi : E \to F$ a surjection. Suppose that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a fomral bundle atlas for E of smoothly compatible charts. Then there exists a unique topology and smooth manifold structure on E such that (E, B, π, F) is a smooth fibre bundle with bundle atlas $\{(U_{\alpha,\alpha})\}_{\alpha \in \mathcal{A}}$.

Let (E, B, π, F) be a fibre bundle with bundle atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$. Recall that the transition functions

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \longrightarrow (U_\alpha \cap U_\beta) \times F.$$

Then they satisfy:

Lemma 1.1. (**Cocycle conditions.**): If $\overline{g}_{\alpha\beta} = g_{\alpha\beta}\big|_{\{b\} \times F}$ for all $b \in U_\alpha \cap U_\beta$,

$$\begin{aligned} \overline{g}_{\alpha\alpha}(b) &= \text{id}_F, \forall b \in U_\alpha \\ \overline{g}_{\alpha\beta} \circ \overline{g}_{\beta_\alpha}(b) &= \text{id}_F, \forall b \in U_\alpha \cap U_\beta \\ \overline{g}_{\alpha\beta} \circ \overline{g}_{\beta\gamma} \circ \overline{g}_{\gamma\alpha}(b) &= \text{id}_F, \forall b \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}$$

Remark. A fibre bundle can be (re)-constructed from its transition functions as a quotient using the equivalence relation induced by the cocycle condition:

$$E \cong \left(\coprod_{\alpha \in \mathcal{A}} U_\alpha \times F\right) \Big/ {\sim}$$

where $(b, v) \sim (b', v')$ if and only if $\exists \alpha, \alpha'$ with $b = b' = U_\alpha \cap U_{\alpha'} \neq \varnothing$ and $v = \overline{g}_{\alpha\alpha'}(b')v'$.

Lecture 4 --- January 16, 2020

1.3 Comparison Between Manifolds and Fibre Bundles

Manifolds	Fibre bundles
coordinate charts $\varphi : U \overset{\text{open}}{\subseteq} M \xrightarrow{\text{diffeo.}} \mathbb{R}^n$	bundle charts / local trivializations $\varphi : \pi^{-1}(U) \to U \times F$
Coordinate transformations	Transition functions
Atlas	Bundle atlas
Trivial manifold $U \subseteq \mathbb{R}^n$	Trivial bundle $E = B \times F$
Non-trivial manifold	Non-trivial bundle

Notation. (E, B, π, F) is a fibre bundle

- $U \overset{\text{open}}{\subset} B$ — $E_U := \pi^{-1}(U) \subset E$
- $b \in B$ — $E_b := \pi^{-1}(b) \subset E$
- $\{(U_\alpha, \varphi)\}$ a bundle atlas: if $U_\alpha \cap U_\beta \neq \varnothing$, the *transition functions*

$$g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}\big|_{U_\alpha \cap U_\beta} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

and for all $b \in U_\alpha \cap U_\beta$,

$$\begin{aligned} g_{\alpha\beta}\big|_{\{b\}\times F} : \{b\} \times F &\to \{b\} \times F \\ (b, v) &\mapsto (b, \overline{g}_{\alpha\beta}(b)(v)). \end{aligned}$$

The maps $\overline{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to \mathrm{Diff}(F)$ are also called the *transition functions*.

1.4 Bundle Maps Revisited

Let (E, B, π, F) and (E', B, π', F') be two fibre bundles over B. A *bundle map* is a smooth map $H : E \to E'$ such that $\pi' \circ H = \pi$. Recall that bundle maps are fibre-preserving: For all $b \in B$, $H\big|_{E_b} : E_b \to E'_b$. Thus, for all $U \subseteq B$, $H\big|_{E_U} : E_U \to E'_U$. Can one obtain a local description of bundle maps? Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of B with respect to which E_{U_α} and E'_{U_α} are trivial for all $\alpha \in \mathcal{A}$. Suppose $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U_\alpha, \varphi'_\alpha)\}$ are bundle atlases for E and E' respectively, and set $H_\alpha = H\big|_{E_{U_\alpha}} : E_{U_\alpha} \to E'_{U_\alpha}$.

$$\begin{tikzcd} E_{U_\alpha} \arrow[r, "H_\alpha"] \arrow[d, "\varphi_\alpha"] & E'_{U_\alpha} \arrow[d, "\varphi'_\alpha"] \\ U_\alpha \times F \arrow[r, "\varphi'_\alpha \circ H_\alpha \circ \varphi_\alpha^{-1}"] & U_\alpha \times F' \end{tikzcd}$$

Where

$$\begin{aligned} \varphi'_\alpha \circ H_\alpha \circ \varphi_\alpha^{-1} : U_\alpha \times F &\to U_\alpha \times F' \\ (b, v) &\mapsto (b, \overline{H}_\alpha(b)(v)). \end{aligned}$$

Note that $\overline{H}_\alpha(b) : F \to F'$ are smooth maps, as they are compositions of smooth maps.

Also, if $U_\alpha \cap U_\beta \neq \varnothing$, then $H_\alpha\big|_{U_\alpha \cap U_\beta} = H\big|_{U_{\alpha\beta}} = H_\beta\big|_{U_\alpha \cap U_\beta}$. Thus for any $b \in U_\alpha \cap U_\beta$,

$$\overline{H}_\beta(b) = \overline{g}'_{\beta\alpha}(b) \circ \overline{H}_\alpha(b) \circ \overline{g}_{\alpha\beta}(b) (*)$$

.

Bundle maps are completely determined by smooth maps

$$\overline{H}_\alpha : U_\alpha \to C^\infty(F, F')$$

that satisfy (*). Also, if H is a bundle isomorphism, then $\overline{H}_\alpha : U_\alpha \to \mathrm{Diff}(F, F')$.

Note. When H is a diffeomorphism, (*) can be rewritten as

$$\overline{g}'_{\alpha\beta}(b) = \overline{H}_\alpha(b) \circ \overline{g}_{\alpha\beta}(b) \circ \overline{H}_\beta(b)^{-1} (**).$$

So, (E, B, π, F) is isomorphic to (E', B, π', F') if and only if there is a collection of maps $\{H_\alpha : U_\alpha \to \text{Diff}(F, F')\}$ which satisfies $(**)$.

Corollary 1.1. (E, B, π, F) is trivial if and only if there is a bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ and smooth maps $\{\overline{H}_\alpha : U_\alpha \to \text{Diff}(F)\}$ such that $\overline{g}_{\alpha\beta}(b) = \overline{H}_\alpha(b)^{-1} \circ \overline{H}_\beta(b)$ for all $b \in B$. I.e., the cocycle corresponding to the transition functions is a coboundary.

Theorem 1.2. A bundle map $H : E \to E'$ is an isomorphism if and only if $H\big|_{E_b} : E_b \to E'_b$ is a diffeomorphism.

1.5 Vector Bundles

Definition 1.9. A fibre bundle (E, B, π, F) is called a *vector bundle* (v.b.) if the following are satisfied:

(i.) F is a finite-dimensional vector space

(ii.) For all $b \in B$, $\pi^{-1}(b)$ has the structure of an r-dimensional vector space (where $r = \dim F$)

(iii.) The local trivializations $\varphi_U : E_U \to U \times F$ restrict to linear maps on the fibres of E. I.e., for all $b \in U$, $\varphi_U\big|_{E_b} : E_b \to \{b\} \times F \cong \{b\} \times F$ is a linear isomorphism.

r is called the *rank* of the vector bundle. If $r = 1$, (E, B, π, F) is called a *line bundle*.

Note. Vector bundles are $\mathbb{R}^r$-bundles, or $\mathbb{C}^r$-bundles whose bundle charts preserve the linear structure on the fibres.

Example 1.5.

1. $E = B \times \mathbb{R}^r$ or $E = B \times \mathbb{C}^r$ is the trivial bundle of rank r.
2. the (infinite) Möbius bundle is a line bundle on S^1 that is non-trivial.
3. If M is a manifold of dimension n, then TM is a vector bundle of rank n.
4. **Tautological line bundle over $\mathbb{P}^n$.** Recall that $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{0\} \big/ \sim$ where $x \sim \lambda x$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. I.e., it is the set of all lines in $\mathbb{R}^{n+1}$ through the orgin. Set
$$E = \coprod_{[x] \in \mathbb{P}^n} L_{[x]}$$
where $L_[x]$ is the line in $\mathbb{R}^{n+1}$ through x and 0. Also,
$$\begin{aligned} \pi : E &\to \mathbb{P}^n \\ v \in L_{[x]} &\mapsto [x] \end{aligned}$$
note that for every $x \in \mathbb{P}^n$, $\pi^{-1}([x]) = L_{[x]} \cong \mathbb{R}$. Then $(E, \mathbb{P}^n, \pi, \mathbb{R})$ is a line bundle on $\mathbb{P}^n$.

Lecture 5 --- January 21, 2020

Recall. A *vector bundle* is a fibre bundle (E, B, π, F) such that

(i) F is a finite-dimensional vector space of dimension r

(ii) For every $b \in B$, E_b has the structure of a r-dimensional vector space

(iii) There exist bundle charts $\varphi_U : E_U \to U \times F$ such that $\varphi_U\big|_{E_b} : E_b \xrightarrow{\cong} \{b\} \times F$ is a linear isomorphism.

Example 1.6. Tautological line bundle over $\mathbb{P}^1$. $\mathbb{P}^1 = \left(\mathbb{R}^{n+1}\{0\}\right)\Big/\sim$ where $(x_1, \dots, x_n) \sim (\lambda x_1, \dots, \lambda x_n)$ for all $\lambda \in \mathbb{R}^*$. Let

$$E := \coprod_{[x]\in\mathbb{P}^n} \{[x]\} \times L_{[x]}$$

where $L_[x]$ is the line through $\mathbb{R}^{n+!}$ through 0 and x. Then,

$$\begin{aligned} \pi : E &\to \mathbb{P}^n \\ \left([x], v \in L_{[x]}\right) &\mapsto [x] \end{aligned}$$

is a line bundle over $\mathbb{P}^n$ called the *tautological line bundle over $\mathbb{P}^n$*, with fibre $E_{[x]} \cong L_{[x]} \cong \mathbb{R}^1$ for every $[x] \in \mathbb{P}^n$.

Proof. let us construct a bundle atlas for E that satisfy condition (iii) of the definition of a vector bundle and whose transition functions are smooth. Cover $\mathbb{P}^n$ by

$$U_i := \{[x] \in \mathbb{P}^n \mid x_i \neq 0\} \underbrace{\subset}_{\text{open}} \mathbb{P}^n.$$

Then, for all $[x] \in U_i$ so that $x_i \neq 0$, and so

$$\begin{aligned} [x] &= [x_1 : \dots, x_i : \dots : x_{n+1}] \\ &= \left[\frac{x_1}{x_i} : \dots : 1 : \dots \frac{x_{n+1}}{x_i}\right] \end{aligned}$$

Then for all $v \in L_{[x]}$, $v = t\left(\frac{x_1}{x_i}, \dots, 1, \dots, \frac{x_{n+1}}{x_i}\right)$ for some unique $t \in \mathbb{R}$. Set

$$\begin{aligned} \varphi_i : E_{U_i} = \coprod_{[x]\in U_i} \{[x]\} \times L_{[x]} &\longrightarrow U_i \times \mathbb{R}^1 \\ \left([x], t\left(\frac{x_1}{x_i}, \dots, 1, \dots, \frac{x_{n+1}}{x_i}\right)\right) &\mapsto (x, t) \end{aligned}$$

Then φ_i is a bijection. The collection $\{(U_i, \varphi_i)\}_{i=1}^{n+1}$ is a formal atlas for E. Also, if $U_i \cap U_j \neq \varnothing$, $[x] \in U_i \cap U_j$ and $v \in L_[x]$,

$$\begin{aligned} s\left(x_1/x_i, \dots, 1, \dots, x_{n+1}/x_i\right) = v &= t\left(x_1/x_j, \dots, 1, \dots, x_{n+1}/x_j\right) \\ &= t\frac{x_i}{x_j}\left(x_1/x_i, \dots, 1, \dots, x_{n+1}/x_i\right) \end{aligned}$$

And thus $s = \left(\frac{x_i}{x_j}\right)t$. Then $\varphi_i([x], v) = ([x], s)$ and $\varphi_j([x], v) = ([x], t)$ and $\varphi_i \circ \varphi_j^{-1}([x], t) = \left([x], \left(\frac{x_i}{x_j}\right)t\right)$, and so $\overline{\varphi}_{ij}([x]) \in \mathrm{Diff}(\mathbb{R}^1)$. So E is a fibre bundle over $\mathbb{P}^n$ with fibre $\mathbb{R}^1$. Finally, we need to check that, for $i = 1, \dots, n+1$,

$$\varphi_i\big|_{E_{[x]}} : E_{[x]} \mapsto \{[x]\} \times \mathbb{R}^1$$

are linear isomorphisms. Here, $E_{[x]} = \{x\} \times L_{[x]}$, with vector space structure: $\forall \alpha \in \mathbb{R}$ and $v, v' \in L_{[x]}$, then $([x], v) + \alpha([x], v') = ([x], v + \alpha v')$. Also, one can write $v = t\left(x_1/x_i, \dots, x_{n+1}/x_i\right)$ and $v' = t'\left(x_1/x_i, \dots, x_{n+1}/x_i\right)$ for some $t, t' \in \mathbb{R}$. Then $v + \alpha v' = (t +')\left(x_1/x_i, \dots, x_{n+1}/x_i\right)$ Then

$$\begin{aligned} \varphi_i\left(([x], v) + \alpha([x], v')\right) &= \varphi_i\left([x], v + \alpha v'\right) \\ &= \left([x], t + \alpha t'\right) \\ &= ([x], t) + \alpha\left([x], t'\right) \\ &= \varphi_i([x], v) + \alpha\varphi_i([x], v'). \end{aligned}$$

□

Since $\varphi_i\big|_{E_{[x]}}$ is also a bijection, it is an isomorphism of vector spaces. This implies that, finally, $(E, \mathbb{P}^n, \pi, \mathbb{R}^1)$ is a vector bundle of rank 1.

Note. In the proof above, the transition functions of the bundle atlas we constructed were the $\overline{\varphi}ij : U_i \cap U_j \longrightarrow \mathrm{GL}\,(1, \mathbb{R}) \subset \mathrm{Diff}(\mathbb{R}^1)$.

Remark. If $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a vector bundle atlas for the vector bundle $(E, B, \pi, \mathbb{R}^r)$ (or $(E, B, \pi, \mathbb{C}^r)$), the transition functions

$$\overline{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}\,(r, \mathbb{R}) \text{ or } \mathrm{GL}\,(r, \mathbb{C})$$

In particular, if $r = 1$, then $\mathrm{GL}\,(1, \mathbb{R}) = \mathbb{R}^\times$ and $\mathrm{GL}\,(1, \mathbb{C}) = \mathbb{C}^\times$ so that $\overline{g}_{\alpha\beta}$ are just nowhere-vanishing scalar functions.

Definition 1.10. Let $(E, B, \pi, \mathbb{R}^r)$ and $(E', B, \pi'.\mathbb{R}^{r'})$ be vector bundles. A map $H : E \to E'$ is a *(bundle) map of vector bundles* if

$$H\big|_{E_b} : E_b \to E'_b$$

is linear for all $b \in B$.

Note. Unless otherwise stated, we will always assume that bundle maps between vector bundles are *actually* bundle maps.

1.6 Sections

Definition 1.11. Let (E, B, π, F) be a fibre bundle. A *section* of (E, B, π, F) is a smooth map $\sigma : B \to E$ such that $\pi \circ \sigma = \mathrm{id}_B$.

Then for all $b \in B$, $\sigma(b) \in E_b$, since $\pi(\sigma(b)) = b$. Also, $\sigma(B) \subset E$ is a smooth submanifold of E diffeomorphic to B (exercise).

Notation. We write $\Gamma\,(E) = \{$set of *all* sections of $(E, B, \pi, F)\}$.

Definition 1.12. If $U \subsetneq B$ is open, then a *local section of E over U* is a smooth map $\sigma : U \to E_U$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Note. Again, $\sigma(b) \in E_b$ for all $b \in U$ if $\sigma : U \to E$ is a local section over U. We denote

$$\Gamma\,(U, E) = \{\text{ set of local sections of } E \text{ over } U\}\,.$$

Example 1.7. (i) $E = B \times F$ with $\pi = \mathrm{pr}_1$. Let $\overline{\sigma} : B \to F$ be any smooth map, and then

$$\begin{aligned} \sigma : B &\to E \\ b &\mapsto (b, \overline{\sigma}(b)) \end{aligned}$$

Then σ is smooth and $\pi \circ \sigma = \mathrm{id}_B$, so $\sigma \in \Gamma\,(E)$.

In fact, sections of any fibre bundle look like this locally: Let (U, φ_U) be a bundle chart for (E, B, π, F) and $\sigma \in \Gamma\,(E)$. Then, $\pi \circ \sigma = \mathrm{id}_B$ and

$$\begin{aligned} \varphi_U \circ \sigma\big|_U : U &\to U \times F. \\ b &\mapsto (b, \overline{\sigma}_U(b)) \end{aligned}$$

for some $\overline{\sigma}_U : U \to F$ smooth. [Note: The first component of $\varphi_U \circ \sigma\big|_U$ is id_U because $\pi \circ \sigma\big|_U = \mathrm{id}_U$.] Thus, local sections of E over U are completely determined by the smooth functions $\overline{\sigma} : U$ In particular, local sections *always* exist.

Example 1.8. (i) Vector bundles always admit sections. For example, given any vector bundle $(E, B, \pi, \mathbb{R}^r)$, one can define the *zero section*

$$\begin{aligned} 0 : B &\to E \\ b &\mapsto 0 \in E_b \end{aligned}$$

(ii) If M is any smooth manifold, them $\Gamma\,(TM)$ is the collection of smooth vector fields on M, which always exist.

(iii) Consider S^2 and TS^2. Sections of TS^2 are smooth, tangent vector fields on S^2. By the Hairy-Ball Theorem, any smooth vector field on S^2 has at least one zero.

(iv) For an example of a fibre bundle that does not admit any global sections, take $E = TS^2 \backslash \{\text{zero section}\}$, which has fibre $\mathbb{R}^2 \backslash \{0\}$ and whose projection is simply $\pi\big|_E$ where $\pi : TS^2 \to S^2$ is the standard projection. This fibre bundle does not have a section because any smooth section $\sigma \in \Gamma\,(E)$ would be a smooth vector field on S^2 and thus must have a zero.

Lecture 6 --- January 23, 2020

Sections. (E, B, π, F) a fibre bundle. A section is a smooth map $\sigma : B \to E$ such that $\pi \circ \sigma = \mathrm{id}_B$. We denote by $\Gamma(E)$ the set of all sections of (E, B, π, F).

Gien a bundle chart (E_U, φ_U) with $U \subseteq B$ open,

$$\begin{tikzcd} \varphi_U \circ (\sigma\big|_U) : U \arrow[rr] \arrow[dr, "\sigma"'] & & U \times F \\ & E_U \arrow[ur, "\varphi_U"'] & \end{tikzcd}$$

with $\varphi_U \circ (\sigma\big|_U)(b) = (b, \overline{\sigma}(b))$ for some smooth $\overline{\sigma} : U \to F$.

Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open conver of B and $\{(E_{U_\alpha}, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ be a bundle atlas for (E, B, π, F). Let $\sigma \in \Gamma(E)$. Set

$$\sigma_\alpha := \sigma\big|_{U_\alpha} : U_\alpha \longrightarrow E_{U_\alpha} = \coprod_{b \in U_\alpha} E_b$$

Then

$$\begin{aligned} \varphi_\alpha \circ \sigma \colon U_\alpha &\to U_\alpha \times F \\ b &\mapsto (b, \overline{\sigma}_\alpha(b)) \end{aligned}$$

for some smooth $\overline{\sigma}_\alpha : U_\alpha \to F$. How are the $\overline{\sigma}_\alpha$'s related? Suppose $U_\alpha \cap U_\beta \neq \varnothing$ and let $b \in U_\alpha \cap U_\beta$. Then

$$\begin{aligned} (b, \overline{\sigma}_\alpha(b)) &= \varphi_\alpha \circ \sigma_\alpha(b) \\ &= \varphi_\alpha \circ \sigma_\beta(b) \\ &= \underbrace{\varphi_\alpha \circ \varphi_\beta^{-1}}_{g_{\alpha\beta}} \circ \varphi_\beta \circ \sigma_\beta(b) \\ &= \left(b, \overline{g}_{\alpha\beta}(b)\left(\overline{\sigma}_\beta(b)\right)\right) \end{aligned}$$

which implies that

$$\overline{\sigma}_\beta(b) = \overline{g}_{\alpha\beta}(b)\left(\overline{\sigma}_\beta(b)\right) \; (***)$$

for all $b \in U_\alpha \cap U_\beta$.

So, given a bundle atlas $\{(E_{U_\alpha}, \varphi_\alpha)\}$ of (E, B, π, α), we can think of sections of the bundle as families of smooth maps $\{\sigma_\alpha : U_\alpha \to F\}$ that satisfy $(***)$.

1.7 Sections of Vector Bundles

Let $(E, B, \pi, \mathbb{R}^r)$ be a vector bundle, which we will denote by E. Let $\{U_\alpha\}$ be an open cover of B and $\{(E_{U_\alpha}, \varphi_\alpha)\}$ be a vector bundle atlas of E. Then, the transition functions of the atlas are

$$\overline{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}\,(r, \mathbb{R})$$

So, for all $b \in U_\alpha \cap U_\beta$, $\overline{g}_{\alpha\beta}(b) =$ (invertible matrix), and, for all $v \in \mathbb{R}^r$,

$$\overline{g}_{\alpha\beta}(b)(v) = \underbrace{\overline{g}_{\alpha\beta}(b) \cdot v}_{\text{matrix multiplication}}$$

For this reason, $\overline{g}_{\alpha\beta}(b)$ are sometimes called *transition matrices*.

Also, any section of E is determined by a family

$$\{\overline{\sigma}_\alpha : U_\alpha \to \mathbb{R}^r\}$$

of smooth vector-valued functions such that

$$\overline{\sigma}_\alpha(b) = \underbrace{\overline{g}_{\alpha\beta}(b) \cdot \sigma_\beta(b)}_{\text{matrix multiplication}}$$

by $(***)$.

Note. On a vector bundle, any local section can be extended globally (possibly by zero outside of the open set on which it is defined) by using bump functions (exercise).

Definition 1.13. Let $\sigma_1, \dots, \sigma_l \in \Gamma(E)$. We say that the set $\{\sigma_1, \dots, \sigma_l\}$ is *linearly independent* if

$$\{\sigma_1(b), \dots, \sigma_l(b)\} \subseteq E_b$$

is linearly independent for every $b \in B$. If $l = r$ (the rank of E), then $\{\sigma_1, \dots, \sigma_l\}$ is called a *frame for E*.

Note. (i) If $\{\sigma_1, \dots, \sigma_r\}$ is a frame of E so that $\{\sigma_1(b), \dots, \sigma_l(b)\}$ is linearly independnet in E_b for all $b \in B$, then $\{\sigma_1(b), \dots, \sigma_l(b)\}$ is a basis for E_b for all $b \in B$. Then $\sigma_i(b) \neq 0$ for all $i = 1, \dots, l$. So, the σ_i's are nowhere-vanishing.

(ii) If $r = 1$, then any frame of E consists solely of a nowhere-vanishing section.

Example 1.9. 1) Let S^{2n} be an even-dimensional sphere. Then, by the Hairy Ball theorem, any tangent vector field of S^{2n} has at least one zero. Thus, TS^{2n} does not admit nowhere-vanishing sections. So, TS^{2n} does not admit any (global) frames.

2) $S^{2n+1} \subset \mathbb{R}^{2n+2} = \{(x_1, \dots, x_{2n+2})\}$.

- $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$. Then $X_{(x_1,x_2)} = (-x_2, x_1)$ is a nowhere-vanishing, tangent vector field of S^1.
- On $S^{2n+1} \subset \mathbb{R}^{2n+2}$, We define

$$X_{(x_1,\dots,x_{2n+2})} = (-x_2, x_1, \dots, -x_{2k}, x_{2k+1}, \dots, -x_{2n-1}, x_{2n+2}).$$

- On S^3, we have that

$$\begin{aligned} X_1(x_1, \dots x_4) &= (-x_2, x_1, -x_3, x_4) \\ X_2(x_1, \dots x_4) &= (-x_3, -x_4, x_1, x_2) \\ X_3(x_1, \dots x_4) &= (x_4, -x_3, -x_2, x_1) \end{aligned}$$

comprise a frame for TS^3.
- On S^7, one can use the octonions to construct a frame for TS^7
- On S^{2n+1} for $n \geqslant 4$, TS^{2n+1} does not admit a global frame.

3) Let $E = B \times \mathbb{R}^r$ be the trivial vector bundle with $\pi(b, v) = b$. Then suppose that $\{e_1, \dots, e_r\}$ is the standard basis for $\mathbb{R}^r$. Then a global frame is given by $\{\sigma_1, \dots, \sigma_r\}$ where

$$\begin{aligned} \sigma_i : B &\to E \\ b &\mapsto (b, e_i). \end{aligned}$$

We will refer to this as the *standard frame on the trivial bundle*. So, the trivial bundle admits at least one frame (in fact... many).

In general, we have:

Proposition 1.2. A vector bundle E is trivial if and only if it admits a frame.

Proof. ($\implies$) If E is trivial, then it is isomorphic to $B \times \mathbb{R}^r$. Thus, there exists a vector bundle isomorphism $H : B \times \mathbb{R}^r \to E$. In particular, $H\big|_{\{b\}\times\mathbb{R}^r} \to E_b$ is a linear isomorphism. Let $\{\sigma_1, \dots, \sigma_r\}$ be the standard frame on $B \times \mathbb{R}^r$, and define

$$\begin{aligned} \tilde{\sigma}_i : B &\to E \\ b &\mapsto H \circ \sigma_i(b). \end{aligned}$$

Then each $\tilde{\sigma}_i$ is a section of E, because $\pi \circ \tilde{\sigma}_i = \pi \circ H \circ \sigma_i = \mathrm{proj}_1 \circ \sigma_i = \mathrm{id}_B$. Also, for all $b \in B$,

$$\{\tilde{\sigma}_1(b), \dots, \tilde{\sigma}_r(b)\} = \underbrace{H\big|_b (\{\sigma_1(b), \dots, \sigma_r(b)\})}_{\text{linearly independent}}.$$

So $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_r(b)\}$ is a frame for E. ($\impliedby$) Assume that E admits the frame $\{\sigma_1, \dots, \sigma_r\}$ and use it to construct an isomorphism given by

$$\begin{aligned} H : B \times \mathbb{R}^r &\to E \\ (b, (a_1, \dots, a_r)) &\mapsto \sum_{i=1}^r a_i \sigma_i(b) \in E_b, \end{aligned}$$

which is an isomorphism because $\{\sigma_1, \dots, \sigma_r\}$ is a frame. So, H is a vector bundle isomorphism. □

Corollary 1.2. A line bundle is trivial if and only if it admits a nowhere-vanishing section.

Corollary 1.3. TS^k is trivial if and only if $k \in \{1, 3, 7\}$.

Definition 1.14. A manifold M is called *parallelizable* if its tangent bundle is trivial.

Example 1.10.
1. S^1, S^3, S^7 are parallelizable.
2. Any Lie group G is parallelizable.

Proposition 1.3. The tautological line bundle on $\mathbb{P}^n$ is not trivial.

Proof. The tautological line bundle on $\mathbb{P}^n$ does not admit any nowhere-vanishing sections. $\square$

Lecture 7 --- January 28, 2020

Let (E, B, π, F) be a vector bundle. A *frame* is a set $\{\sigma_1, \ldots, \sigma_l\}$ of linearly independent sections $\sigma_i \in \Gamma(E)$.

Proposition 1.4. *E is trivial if and only if E admits a frame.*

Corollary 1.4. *A line bundle is trivial if and only if it admits a nowhere-vanishing section.*

Proposition 1.5. *The tautological line bundle over $\mathbb{P}^n$ is not trivial.*

Proof. It is enough to show that the tautological line bundle E over $\mathbb{P}^n$ does not admit any nowhere-vanishing sections. We do it by contradiction: Suppose instead that E admits a nowhere-vanishing section $\sigma : \mathbb{P}^n \to E$ so that $\sigma([x]) \neq 0$ for every $[x] \in \mathbb{P}^n$. Recall that we constructed a vector bundle atlas for E with open cover $\{U_i\}_{i=1}^{n+1}$ where

$$U_i := \{[x_1 : \cdots : x_{n+1}] \mid x_i \neq 0\}$$

and transition functions

$$\begin{aligned} g_{ij} : U_i \cap U_j &\to \mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^\times \\ [x] &\mapsto \frac{x_i}{x_j}. \end{aligned}$$

Then σ is given by scalar functions

$$\overline{\sigma}_i : U_i \to \mathbb{R}$$

such that (without loss of generality)

$$\begin{aligned} \underbrace{\overline{\sigma}_i([x])}_{>0} &= \overline{g}_{ij}([x])\overline{\sigma}_j([x]) \\ &= \left(\frac{x_i}{x_j}\right) \underbrace{\overline{\sigma}_j([x])}_{>0}. \end{aligned}$$

but

$$\begin{aligned} U_i \cap U_j &\to \mathbb{R}^\times \\ [x] &\mapsto \frac{x_i}{x_j} \end{aligned}$$

is surjective. $\square$

Thus, not all vector bundles admit frames, but they ALL admit "local frames":

Definition 1.15. Let $U \subseteq B$ be open and $e_1, \ldots, e_r \in \Gamma(U, E)$. Then $\{e_1, \ldots, e_r\}$ is a *local frame of E over U* if, for all $b \in U$, $\{e_1(b), \ldots, e_r(b)\}$ is linearly independent.

So, for all $U \subseteq B$ open over which E adits a vector bundle chart $\varphi_U : E_U \to U \times \mathbb{R}^r$, one has the local frame $\{e_1, \ldots, e_r\}$ given by

$$\begin{aligned} e_i : U &\to E_U \\ b &\mapsto \varphi_U^{-1}(b, \vec{e}_i) \end{aligned}$$

where $\{\vec{e}_1, \ldots, \vec{e}_r\}$ is the standard basis in $\mathbb{R}^r$.

Local frames are useful for describing frames locally. Given a local fram $\{e_1, \ldots, e_r\}$ of E over U and a section $\sigma \in \Gamma(E)$,

$$\sigma\big|_U = \overline{\sigma}_1 e_1 + \cdots + \overline{\sigma}_r e_r$$

for some $\overline{\sigma}_1, \ldots, \overline{\sigma}_r \in C^\infty(U)$. Also, if $\{e'_1, \ldots, e'_r\}$ is another local frame of E over U' with $U \cap U' \neq \varnothing$, for all $b \in U \cap U'$, we have

$$e'_j(b) = \sum_{i=1}^r h_{ij}(b) e_j(b)$$

for some smooth $h_{ij} \in C^\infty(U)$. Thus, we get a map

$$\begin{aligned} h : U \cap U' &\to \mathrm{GL}(r, \mathbb{R}) \\ b &\mapsto [h_{ij}(b)]_{i,j=1}^r \end{aligned}$$

where $h(b)$ is the "change of basis matrix" from $\{e_i(b)\}$ to $\{e'_1(b)\}$.

Note. $\Gamma(U,E)$ has the following $C^\infty(U)$—module structure: For all $\sigma,\tau\in\Gamma(U,E)$ and $f\in C^\infty(U)$, set

$$\begin{aligned}(f\sigma+\tau):U&\mapsto E_U\\ b&\mapsto f(b)\sigma(b)+\tau(b).\end{aligned}$$

Then, since $f(b)\in\mathbb{R}$ and $\sigma(b),\ \tau(b)\in E_b$, so $f(b)\sigma(b)+\tau(b)\in E_b$. Thus $f\sigma+\tau\in\Gamma(U,E)$. In terms of a local frame $\{e_1,\ldots,e_r\}$ of E over U, we have $\sigma=\sum_{j=1}^r\overline{\sigma}_je_j$, $\tau=\sum_{j=1}^r\overline{\tau}e_i$ and

$$f\sigma+\tau=\sum_{j=1}^r(f\overline{\sigma}_j+\overline{\tau})e_j.$$

1.8 Linear Algebraic Constructions for Vector Bundles

Let $(E,B,\pi,\mathbb{R}^r)$ and $(E',B,\pi',\mathbb{R}^{r'})$ be vector bundles. One can construct new vector bundles by applying linear algebra constructions fibrewise:

$$E\oplus E',\ E\otimes E',\ E^*,\bigwedge^k E,\ \mathrm{Hom}\left(E,E'\right).$$

(i) To construct the direct sum of E and E', we take the underlying set

$$E\oplus E'=\bigsqcup_{b\in B}\underbrace{E_b\oplus E_b'}_{\text{rank } r+r'}.$$

Gien an open cover $\{U_\alpha\}$ of B and vector bundle atlases $\{(U_\alpha,\varphi_\alpha)\}$ and $\{(U_\alpha',\varphi_\alpha')\}$ for E and E', respectively, we define

$$\begin{aligned}\varphi_\alpha\oplus\varphi_\alpha':\bigsqcup_{b\in B}E_b\oplus E_b'&\to U_\alpha\times(\mathbb{R}^r\oplus\mathbb{R}^{r'})\\ E_b\oplus E_b'\ni(e,e')&\mapsto\left(b,\left(\varphi_\alpha(e),\varphi_\alpha'(e')\right)\right).\end{aligned}$$

These are bundle charts for $E\oplus E'$, for all α. Then we get transition functions

$$\overline{g}_{\alpha\beta}\oplus\overline{g}_{\alpha\beta}':U_\alpha\cap U_\beta\longrightarrow\mathrm{GL}\left(r+r',\mathbb{R}\right).$$

(ii) The tensor product is given (as a set) by

$$E\otimes E'=\bigsqcup_{b\in B}\underbrace{E_b\otimes E_b'}_{\text{rank } rr'}$$

.

(iii) The dual bundle is given (as a set) by

$$E^*=\bigsqcup_{b\in B}\underbrace{E_b^*}_{\text{rank } r}$$

.

(iv) The exterior power bundles are given (as sets) by

$$\bigwedge^k E=\bigsqcup_{b\in B}\underbrace{\bigwedge^k E_b}_{\text{rank }\binom{n}{r}}$$

(v) The hom bundles are given (as sets) by

$$\mathrm{Hom}_E\left(E'\right)=\bigsqcup_{b\in B}\underbrace{\mathrm{Hom}\left(E_b,E_b'\right)}_{\text{rank } rr'}$$

Example 1.11. 1. • Let M be a smooth manifold and TM its tangent bundle. Then $(TM)^*=T^*M$ is the cotangent bundle. Smooth sections of this bundle are the smooth 1-forms: $\Gamma(T^*M)=\Omega^1(M)$.

- $\bigwedge^k T^*M=:\bigwedge^k M$ have the k-forms as sections: $\Gamma\left(\bigwedge^k T^*M\right)=\Omega^k(M)$.

2. We will be interested in $\left(\bigwedge^k M\right)\otimes E$ with E a vector bundle on M. Locally, sections of $\left(\bigwedge^k M\right)\otimes E$ look like: Given a local frame $\{e_1,\ldots,e_r\}$ of E over U, for all $s\in\Gamma\left(\left\{\bigwedge^k M\right\}\otimes E\right)$,

$$s\big|_U=\sum_{i=1}^r\omega_i\otimes e_i$$

for some $\omega_1, \dots, \omega_r \in \Omega^k(U)$.

2 Connections

2.1 Connections on Vector Bundles

2.1.1 Definition and Properties

Fix $(E, B, \pi, \mathbb{R}^r)$ be a vector bundle of rank r. Our goal is to find a way of differentiating sections of E. Let us first assume that $E = B \times \mathbb{R}^r$. In this case, a seciton $\sigma \in \Gamma(E)$ is just

$$\begin{aligned} \sigma : B &\to B \times \mathbb{R}^r \\ b &\mapsto (b, \overline{\sigma}(b)) \end{aligned}$$

for some smooth map $\overline{\sigma} : B \to \mathbb{R}^r$. In particular,

$$\begin{aligned} \overline{\sigma} : B &\to \mathbb{R}^r \\ b &\mapsto (\overline{\sigma}_1(b), \dots \overline{\sigma}_r(b)) \end{aligned}$$

for some $\overline{\sigma}_i \in C^\infty(B)$. Also, if $\{e_1, \dots, e_r\}$ is the standard frame for $B \times \mathbb{R}^r$ (so that $e_i(b) = (b, \vec{e}_i)$), then

$$\sigma = \sum_{i=1}^{r} \overline{\sigma}_i e_i.$$

So, one possible way of differentiating σ is to differentiate $\overline{\sigma}$ component-wise:

$$d\sigma(b) = (b, d\overline{\sigma}(b))$$

where $d\overline{\sigma}(b)L = (d\overline{\sigma}_1(b), \dots, d\overline{\sigma}_r(b)) = \sum_{i=1}^r d\overline{\sigma}(b) \otimes \vec{e}_i$. In terms of the local frame $\{e_1, \dots, e_r\}$,

$$d\sigma = \sum_{i=1}^{r} \underbrace{(d\overline{\sigma}_i)}_{\text{form}} \otimes \underbrace{e_i}_{\in \Gamma(E)} \in \Gamma(T^*M \otimes E).$$

Then:

$$\begin{aligned} d : \Gamma(E) &\to \Gamma(T^*M \otimes E) \\ \sigma = \sum_{i=1} \overline{\sigma}_i e_i &\mapsto \sum_{i=1}^{r} (d\overline{\sigma}_i) \otimes e_i \end{aligned}$$

which satisfies

- $\mathbb{R}$-linearity.
- (Leibniz rule): $d(f\sigma) = df \otimes \sigma + f d\sigma \in \Gamma(T^*M \otimes E)$.

Lecture 8 --- January 30, 2020

Recall. $(E, B, \pi, \mathbb{R}^r)$ the trivial bundle with $E = B \times \mathbb{R}^r$. Pick a frame $\{e_1, \ldots, e_r\}$ with $e_i(b) = (b, \vec{e}_i)$. Then any section looks like $\sigma = \sum_{i=1}^r \overline{\sigma}_i e_i$. One possible way of differentiating σ is to set

$$d\sigma(b) := (b, d\overline{\sigma}(b))$$

where $d\overline{\sigma}(b) = (d\overline{\sigma}_1(b), \ldots, d\overline{\sigma}_r(b))$. So we get

$$d\sigma := \sum_{i=1}^r \overbrace{d\overline{\sigma}_i}^{\in\Omega^1(B)} \otimes \overbrace{e_i}^{\in\Gamma(E)}$$

Note.

- d is $\mathbb{R}$-linear: for $\sigma, \tau \in \Gamma(E)$ so that $\sigma = \sum +i = 1^r \overline{\sigma}_i e_i$ and $\tau = \sum_{i=1}^r \overline{\tau}_i e_i$. Then for any $c \in \mathbb{R}$,

$$d(c\sigma + \tau) := \sum_{i=1}^r d(c\overline{\sigma}_i + \overline{\tau}_i) \otimes e_i = cd\sigma + d\tau.$$

- d satisfies the *Leibniz rule*: For any $\sigma = \sum_{i=1}^r \overline{\sigma}_i e_i$ and $f \in C^\infty(B)$,

$$d(f\sigma) = df \otimes \sigma + f d\sigma.$$

  Indeed,

$$\begin{aligned} d(f\sigma) &= d\left(\sum_{i=1}^r (f\overline{\sigma}_i) \otimes e_i\right) \\ &= \sum_{i=1}^r d(f\overline{\sigma}_i) \otimes e_i \\ &= \sum_{i=1}^r (\overline{\sigma}_i df + f d\overline{\sigma}_i) \otimes e_i \\ &= df \otimes \left(\sum_{i=1}^r \overline{\sigma}_i e_i\right) + f\left(\sum_{i=1}^r d\overline{\sigma}_i \otimes e_i\right) \\ &= df \otimes \sigma + f d\sigma. \end{aligned}$$

Definition 2.1. A *connection on E* is an $\mathbb{R}$-linear map

$$D : \Gamma(E) to \Gamma(T^*B \otimes E)$$

that safisfies the *Leibniz rule*: For all $f \in C^\infty(B)$ and $\sigma \in \Gamma(E)$, we have

$$D(f\sigma) = df \otimes \sigma + f D(\sigma).$$

Note. Connections generalize the notion of exterior derivative "d" to sections of *any vector bundle.*

Example 2.1. 1. Take $E = B \times \mathbb{R}^r$.

- $D = d$ is called the *trivial connection.*
- What do the others look like? Let $D : \Gamma(E) \to \Gamma(T^*B \otimes E)$ be a connection on $E = B \times \mathbb{R}^r$. Consider the frame $\{e_1, \ldots, e_r\}$ with $e_i(b) = (b, \vec{e}_i)$. Then, for all $j = 1, \ldots, r$, $D(e_j) \in \Gamma(T^*B \otimes E)$. Then

$$D(e_j) = \sum_{i=1}^r a_{ij} \otimes e_i$$

for some $a_{ij} \in \Gamma(T^*B)$. If we pick $\sigma \in \Gamma(E)$, then $\sigma = \sum_{j=1}^r \overline{\sigma}_j e_j$ for $\overline{\sigma}_j \in C^\infty(B)$. Then

$$\begin{aligned}
D(\sigma) &= \sum_{j=1}^r D(\overline{\sigma}_j e_j) \\
&= \sum_{j=1}^r (d\overline{\sigma}_j \otimes e_j + \overline{\sigma}_j D(e_j)) \\
&= \sum_{j=1}^r d\overline{\sigma}_j \otimes e_j + \sum_{i,j=1}^r \overline{\sigma}_j (a_{ij} \otimes e_i) \\
&= \sum_{j=1}^r d\overline{\sigma}_j \otimes e_j + \sum_{i=1}^r \left(\sum_{j=1}^r a_{ij} \overline{\sigma}_j \right) \otimes e_i \\
&=: d\sigma + A\sigma =: (d + A)\sigma
\end{aligned}$$

where we set $A = [a_{ij}]_{i,j=1}^r$ is a $r \times r$ matrix of 1-forms, called the *connection matrix of D* and $\overline{\sigma} = [\overline{\sigma}_i]_{i=1}^r$. Here, we mean

$$A\sigma = \sum_i \left(\sum_j a_{ij} \overline{\sigma}_j \right) \otimes e_i.$$

Note. The connection matrix depends on the frame $\{e_1, \ldots, e_r\}$: To br precise, if $\{e_1, \ldots, e_r\}$ and $\{e'_1, \ldots, e'_r\}$ are frames of $E = B \times \mathbb{R}^r$ and

$$e'_i = \sum_k h_{ki} e_k$$

so that $h = (h_{ij})_{i,j=1}^r$ is the change of basis matrix. Then:

$$D(e_j) = \sum_i a_{ij} \otimes e_i \qquad\qquad D(e'_j) = \sum_i a'_{ij} \otimes e'_i$$

Then $A' = (a'_{ij})_{i,j=1}^r$ satisfies

$$A' = h^{-1} dh + h^{-1} A h \quad \text{(exercise.)}$$

2. E is any vector bundle and $\{(U_\alpha, \varphi_\alpha)\}$ is a vector bundle atlas for E with $\{U_\alpha\}$ an open cover of B. Then, for all α, $E_{U_\alpha} \cong U_\alpha \times \mathbb{R}^r$ and hence admits a local frame $\{e_1^\alpha, \ldots, e_r^\alpha\}$ with

$$e_1^\alpha(b) = \varphi_\alpha^{-1}(b, \vec{e}_i).$$

Let D be a connection on E. Then on E_{U_α}, $D = d + A_\alpha$ where A_α is the connection matrix of $D\big|_{E_{U_\alpha}}$ in terms of the frame $\{e_i^\alpha\}$. Note that on $U_\cap U_\beta$, the change of basis matrix from $\left\{e_1^\beta, \ldots, e_r^\beta\right\}$ to $\{e_1^\alpha, \ldots, e_r^\alpha\}$ is $\overline{g}_{\alpha\beta}$ so that

$$A_\alpha = \overline{g}_{\alpha\beta}^{-1} d\overline{g}_{\alpha\beta} + \overline{g}_{\alpha\beta}^{-1} A_\beta \overline{g}_{\alpha\beta}.$$

Proposition 2.1. Connections *always exist.*

Proof. Let $(E, B\pi, \mathbb{R}^r)$ be a vector bundle with the vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ and corresponding local frames $\{e_1^\alpha, \ldots, e_r^\alpha\}$. Then, on every E_{U_α}, we can pick the trivial connection $d_\alpha = d\big|_{E_{U_\alpha}}$ (i.e., $A_\alpha \equiv 0$). Let $\{\psi_\alpha : B \to \mathbb{R}\}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}$. Then for every $b \in B$,

- $\operatorname{supp}(\psi_\alpha) \subset U_\alpha$,
- only a finite number of ψ_α's are nonzero at b, and
- $\sum_\alpha \psi_\alpha(b) = 1$.

Set $D = \sum_\alpha \psi_\alpha d_\alpha$ so that $D(\sigma) = \sum_\alpha \psi_\alpha d_\alpha \sigma$ for all $\sigma \in \Gamma(E)$. D is a connection because it is $\mathbb{R}$ linear, and the Leibniz rule hols:

$$\begin{aligned}
D(f\sigma) &= \sum_\alpha \psi_\alpha d_\alpha (f\sigma) \\
&= \sum_\alpha \psi_\alpha (df \otimes \sigma + f d_\alpha \sigma) \\
&= \left(\sum_\alpha \psi_\alpha \right) df \otimes \sigma + f \left(\sum_\alpha \psi_\alpha d_\alpha \sigma \right) \\
&= df \otimes \sigma + f d\sigma.
\end{aligned}$$

□

Let $\mathcal{A}(E)$ be the set of all connections on E. This set is not closed under addition! Let $D, D' \in \mathcal{A}(E)$ and define

$$\begin{aligned} D + D' : \Gamma(E) &\to \Gamma(T^*B \otimes E) \\ \sigma &\mapsto D(\sigma) + D'(\sigma). \end{aligned}$$

Although $D + D'$ is a well-defined map, it does not satisfy Leibniz: Let $\sigma \in \Gamma(E)$ and $f \in C^\infty(B)$. Then

$$\begin{aligned} (D + D')(f\sigma) &= D(f\sigma) + D'(f\sigma) \\ & df \otimes \sigma + fD(\sigma) + df \otimes \sigma + fD'(\sigma) \\ &= 2df \otimes \sigma + f(D + D')(\sigma) \\ &\neq df \otimes \sigma + f(D + D')(\sigma). \end{aligned}$$

However, if we had considered $a_1 D + a_2 D'$ such that $a_1 + a_2 = 1$, then we would have a connection. So $\mathcal{A}(E)$ is convex: For all $D_1, \ldots, D_l \in \mathcal{A}(E)$ and $a_1, \ldots, a_l \in \mathbb{R}$ such that $\sum_{i=1}^{l} a_i = 1$, then $a_1 D_1 + \cdots + a_l D_l \in \mathcal{A}(E)$.

$\mathcal{A}(E)$ is an affine space. To see this, we need to following notation:

Notation. Let $(V, B, \tilde{\pi}, \mathbb{R}^m)$ be a vector bundle. We set

$$\Omega^k(B) := \Gamma\left(\bigwedge^k T^*B \otimes V\right).$$

In particular,

$$\Omega^1(V) = \Gamma(T^*B \otimes V).$$

Proposition 2.2. $\mathcal{A}(E)$ is an affine space modelled on $\Omega^1(\mathrm{End}\ E)$. To be more precise, if D_0 is any connection on E, then

$$\mathcal{A}(E) = \left\{D_0 + a \mid a \in \Omega^1(\mathrm{End}\ E)\right\}$$

Lecture 9 --- February 4, 2020

Recall.

- A *connection* on a vector bundle $(E, B, \pi, \mathbb{R}^r)$ is a map $D : \Gamma(E) \to \Gamma(T^*B \otimes E)$ that is $\mathbb{R}$-linear and satisfies $D(f\sigma) = df \otimes \sigma + fD(\sigma)$ for any $f \in C^\infty(B)$ and $\sigma \in \Gamma(E)$.
- Given an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of E and local frames $e_i^\alpha = \varphi_\alpha^{-1}(-, \vec{e}_i)$,

$$D(e_j^\alpha) = \sum_i \alpha_{ij}^\alpha \otimes e_i$$

where $a_{ij}^\alpha \in \Omega^1(U_\alpha)$, so that $A_\alpha = (a_{ij}^\alpha)$ is a matrix of 1-forms, called the connection matrix of D over U_α.

Claim. For all $b \in U_\alpha \cap U_\beta \neq \varnothing$,

$$e_j^\beta(b) = \sum_i \left(\overline{g}_{\alpha\beta}(b)\right)_{ij} e_i^\alpha(b).$$

Proof.

$$\begin{aligned} e_j^\beta(b) &= \varphi_\beta^{-1}(b, \vec{e}_j) \\ &= \varphi_\alpha^{-1} \circ g_\alpha \beta(b, \vec{e}_j) \\ &= \varphi_\alpha^{-1}(b, \overline{g}_{\alpha\beta}(b)\vec{e}_j) \\ &= \sum_i \left(\overline{g}_{\alpha\beta}(b)\right)_{ij} \varphi_\alpha^{-1}(b, \vec{e}_i) \\ &= \sum_i \left(\overline{g}_{\alpha\beta}(b)\right)_{ij} e_i^\alpha. \end{aligned}$$

□

So the change of basis matrix from $\{e_1^\alpha, \dots, e_r^\alpha\}$ to $\left\{e_1^\beta, \dots, e_r^\beta\right\}$ is $\overline{g}_{\alpha\beta}$, so

$$A_\beta = \overline{g}_{\alpha\beta}^{-1} d\overline{g}_{\alpha\beta} + \overline{g}_{\alpha\beta} A_\alpha \overline{g}_{\alpha\beta}.$$

- $\mathcal{A}(E) = \{$all connections on E $\}$ is not closed under addition. Nonetheless, it is convex: For all $D_1, \dots, D_l \in \mathcal{A}(E)$ and $a_1, \dots, a_l \in \mathbb{R}$ such that $\sum_{j=1}^l a_j = 1$, we have that

$$a_1 D_1 + \cdots + a_l D_l \in \mathcal{A}(E).$$

Proposition 2.3. A(E) is an affine space modeled on $\Omega^1(\mathrm{End}(E)) := \Gamma(T^*M \otimes \mathrm{End}(E))$.

Note. Let $(V, B, \pi, \mathbb{R}^r)$ be a vector bundle and set $\Omega^k(V) := \Gamma\left(\bigwedge^k B \otimes V\right)$. Locally, $\tau \in \Omega^k(V)$ looks like $\tau = \sum_{i=1}^m \omega_i \otimes e_i$ where $\{e_1, \dots, e_m\}$ is a local frame of V and $\omega_1, \dots, \omega_m \in \bigwedge^k U$ with $U \subseteq B$ open. For any $X_1, \dots, X_k \in \Gamma(TB)$, we define

$$\begin{aligned} \tau(X_1, \dots, X_k) &:= \sum_{i=1}^m \omega_i(X_1, \dots, X_k) \otimes e_i \\ &= \sum_{i=1}^m \omega_i(X_1, \dots, X_k) e_i \in \Gamma(V). \end{aligned}$$

Note that the definition of $\tau(X_1, \dots, X_k)$ is independent of the local description of τ.

Proof. Let $D_0 \in \mathcal{A}(E)$. It is enough to show that

$$\mathcal{A}(E) = \left\{D_0 + a \mid a \in \Omega^1(\mathrm{End}(E))\right\}$$

What do elements of $\Omega^1(\mathrm{End}(E))$ look like? Locally, $a = \sum_i a_i \otimes \psi_i$ where the a_i are 1-forms and $\psi_i \in \mathrm{End}(E|_U)$ where $U \subset B$ is open. Then for all $\sigma \in \Gamma\left(E|_U\right)$,

$$a(\sigma) = \sum_i a_i \otimes \psi_i(\sigma)$$

so

$$
\begin{aligned}
a : \Gamma(E) &\to \Gamma(T^*B \otimes E) \\
\sigma &\mapsto a(\sigma).
\end{aligned}
$$

So a is $C^\infty(B)$-linear because, for any $f \in C^\infty(B)$,

$$
\begin{aligned}
a(f\sigma) &= \sum_i a_\otimes \psi_i(f\sigma) \\
&= \sum_i a_i \otimes f\psi_i(\sigma) \\
&= f\sum_i a_i \otimes \psi_i(\sigma) \\
&= fa(\sigma).
\end{aligned}
$$

So any $a \in \Omega^1(\mathrm{End}(E))$ induces a $C^\infty(B)$-linear map $a : \Gamma(E) \to \Gamma(T^*B \otimes E)$. Conversely, any $C^\infty(B)$-linear map $a : \Gamma(E) \to \Gamma(T^*B \otimes E)$ induces an element of $\Omega^1(\mathrm{End}(E))$.

Let $D, D' \in \mathcal{A}(E)$. Let us check that

$$
D - D' \in \Omega^1(\mathrm{End}(E)).
$$

It is enough to check that the induced map

$$
\begin{aligned}
D - D' : \Gamma(E) &\to \Gamma(T^*B \otimes E) \\
\sigma &\mapsto D(\sigma) - D'(\sigma)
\end{aligned}
$$

is $C^\infty(B)$-linear. let $\sigma, \sigma' \in \Gamma(E)$ and $f \in C^\infty(B)$. Then

$$
\begin{aligned}
(D - D')(f\sigma + \sigma') &= \big(D(f\sigma) + D(\sigma')\big) - \big(D'(f\sigma) + D'(\sigma')\big) \\
&= \big(df \otimes \sigma + fD(\sigma) + D(\sigma')\big) - \big(df \otimes \sigma + fD'(\sigma) - D'(\sigma')\big) \\
&= f(D - D')(\sigma) + (D - D')(\sigma').
\end{aligned}
$$

and so $D - D' \in \Omega^1(\mathrm{End}(E))$.

$\square$

We have seen that connections generalize the exterior derivative.

Recall. Let $U \subset B$ be open with coordinates $(x_1, \ldots, x_n)$. Then for any $f \in C^\infty(U)$, then

$$
df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.
$$

In particular, if for any $i \in \{1, \ldots, n\}$, we geta

$$
df\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}.
$$

In general, for any $X = \sum_i a_i \frac{\partial}{\partial x_i}$, then

$$
df(X) = \sum_i a_i \frac{\partial f}{\partial x_i} = \nabla f \cdot (a_1, \ldots, a_n).
$$

Also, for all $\omega \in \Omega^1(U)$,

$$
\omega = \sum_i \omega\left(\frac{\partial}{\partial x_i}\right) dx_i.
$$

Lets go back to a connection $D \in \mathcal{A}(E)$. Let $U \subset B$ be an open set over which B has coordinates $x_1, \ldots, x_n$ and E is trivial with local frame $\{e_1, \ldots, e_r\}$. Then for all $\sigma \in \Gamma\left(E\big|_U\right)$,

$$
D(\sigma) = \sum_{i=1}^r \omega_i \otimes e_i
$$

with $\omega_i \in \Omega^1(U)$. And, for all $X \in \Gamma(TU)$,

$$
D(\sigma)(X) := \sum_{i=1}^r \omega_i(X) e_i \in \Gamma\left(E\big|_U\right).
$$

So for fixed $X \in \Gamma(TB)$, we get a map

$$\begin{aligned} D_X : \Gamma(E) &\to \Gamma(E) \\ \sigma &\mapsto D(\sigma)(X) \end{aligned}$$

Note that D_X is $\mathbb{R}$-linear and satisfies Leibniz in σ. We say that $D_X(\sigma)$ is the *covariant derivative of σ in the direction X*. Also note that for any $f \in C^\infty(B)$,

$$D(\sigma)(fX) = f\left(D(\sigma)(X)\right), \text{ or } D_{fX}(\sigma) = fD_X(\sigma).$$

We then get a map

$$\begin{aligned} \nabla : \Gamma(TB) \times \Gamma(E) &\to \Gamma(E) \\ (X, \sigma) &\mapsto D_X(\sigma) \end{aligned}$$

such that it is

- $C^\infty(B)$-linear in X
- $\mathbb{R}$ linear in σ
- Satisfies Leibniz in σ:

$$\begin{aligned} D_X(f\sigma) &= D(f\sigma)(X) \\ &= \left(df \otimes \sigma + fD(\sigma)\right)(X) \\ &= df(X)\sigma + fD(\sigma)(X) \\ &= X(f)\sigma + fD_X(\sigma). \end{aligned}$$

Definition 2.2. A map $\nabla : \Gamma(TB) \times \Gamma(E) \to \Gamma(E)$ such that

- $C^\infty(B)$-linear in X,
- $\mathbb{R}$-linear in σ, and
- $\nabla(X, f\sigma) = X(f)\sigma + f\nabla(X, \sigma)$

is called a *linear connection on E*, or a *covariant derivative on E*.

Note.

1. Tu defines connections this way.
2. There is a one-to-one correspondence between elements of $\mathcal{A}(E)$ and linear connections $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$. We saw that any $D \in \mathcal{A}(E)$ induces a ∇. Conversely, given a linear connection ∇, we can define $D \in \mathcal{A}(E)$ by

$$\begin{aligned} D : \Gamma(E) &\to \Gamma(T^*B \otimes E) \\ \sigma &\mapsto \nabla(-, \sigma) \end{aligned}$$

3. When $E = TB$, linear connections

$$\nabla : \Gamma(TB) \times \Gamma(TB) \to \Gamma(TB)$$

are called *affine connections*. In local coordinates $(x_1, \ldots, x_n)$ on B and a local frame $\{e_1, \ldots, e_r\}$ on E:

$$\begin{aligned} D(\sigma) &= \sum_i \omega_i \otimes e_i \quad (\text{with } \omega_i \in \bigwedge^1(U)) \\ &= \sum_{i,j} \omega_i \left(\frac{\partial}{\partial x_j} dx_j \otimes e_i\right) \\ &= \sum_j dx_j \otimes \left(\sum_i \omega_i \left(\frac{\partial}{\partial x_j} e_j\right)\right) \\ &= \sum_j dx_j \otimes D_{\frac{\partial}{\partial x_j}}(\sigma) \end{aligned}$$

Lecture 10 --- February 6, 2020

Recall. $(E, B, \pi, \mathbb{R}^r)$ a vector bundle and $D : \Gamma(E) \to \Gamma(T^*B \otimes E)$ a connection on E. For any $X \in \Gamma(TB)$ and $\sigma \in \Gamma(E)$, we can define

$$D_X\sigma = (\text{covariant derivative on } \sigma \text{ in the direciton of } X)$$

where if, locally, $D(\sigma) = \sum_i \omega_i \otimes e_i$ where $\{e_1, \ldots, e_r\}$ is a local frame of E and ω_i are local 1-forms, then

$$D_X\sigma := \sum_i \omega_i(X)e_i.$$

Note. If $f \in C^\infty(B)$, $D_{fX}\sigma = fD_X\sigma$. So $D_X : \Gamma(E) \to \Gamma(T^*B \otimes E)$ is such that $D_{fX}\sigma = fD_X\sigma$ (hence too $\mathbb{R}$-linear) and D_X satisfies a Leibniz rule:

$$D_X(f\sigma) = X(f)\sigma + fD_X(\sigma).$$

$$\begin{aligned} \nabla : \Gamma(TB) \times \Gamma(E) &\to \Gamma(E) \\ (X, \sigma) &\mapsto D_X\sigma \end{aligned}$$

is called a linear connection.

Note. The connection D is completely determined by the D_X's, for all $X \in \Gamma(TB)$. In particular, if $\{e_1, \ldots, e_r\}$ is a local frame of E and $D = d + A$ with A the connection matrix in this frame and $\{x_1, \ldots, x_n\}$ are local coordinates for B, then

$$a_{ij} = \sum_k a_{ij}\left(\frac{\partial}{\partial x_k}\right)dx_k$$

and

$$\begin{aligned} D_{\frac{\partial}{\partial x_k}}(e_j) &= D(e_j)\left(\frac{\partial}{\partial x_j}\right) \\ &= \sum_i a_{ij}\left(\frac{\partial}{\partial x_k}\right)e_i. \end{aligned}$$

So, the connection D is completely determined (locally) by $D_{\frac{\partial}{\partial x_k}}(e_j)$ for $j = 1, \ldots r$ and $k = 1, \ldots, n$.

Example 2.2.

1. $M \subseteq \mathbb{R}^n$ a submanifold so that $TM \subset T\mathbb{R}^n\big|_M \cong M \times \mathbb{R}^n$. Let $\sigma \in \Gamma(TM)$. Then we can think of it as

$$\begin{aligned} \sigma : M &\to TM \subseteq M \times \mathbb{R}^n \\ x &\mapsto (x, \overline{\sigma}(x)) \end{aligned}$$

for some smooth $\overline{\sigma} : M \to \mathbb{R}^n$ such that $\sigma(x) \in T_xM$ for each $x \in M$. Since $\overline{\sigma} : M \to \mathbb{R}^n$ is smooth with $M \subset \mathbb{R}^n$, there is an open $U \subset \mathbb{R}^n$ with $M \subset U$ and $\overline{\sigma} : U \to \mathbb{R}^n$ (i.e., $\overline{\sigma}$ extends to a smooth function on a neigbourhood of M). So, we can think of σ as $\sigma : U \to T\mathbb{R}^n\big|_U$, and we can apply the trivial connection d on $T\mathbb{R}^n\big|_U$ to it:

$$d\sigma \in \Gamma(T^*U \otimes TU).$$

But, $d\sigma(X) \in \Gamma(TU)$ for any $X \in \Gamma(TU)$. So, we may not have that $d\sigma(X) \in \Gamma(TM)$. So, we just take $\mathrm{pr}_{TM}(d\sigma)$. Thus, we get the connection D on TM: For every $\sigma \in \Gamma(TM)$ and every $X \in TM$,

$$D_X(\sigma) := \mathrm{pr}_{TM}(d\sigma(X)),$$

where $\mathrm{pr}_{TM} : TU\big|_{TM} \to TM$.

2. Let $(E, B, \pi, \mathbb{R}^r)$ and $(E', B, \pi', \mathbb{R}^{r'})$ be two vector bundles on B with two connecitons D, D', respectively. Then there exist natural induced connections on $E \oplus E', E \otimes E', E^*, \mathrm{Hom}(E, E')$ and f^*E for all $f : N \to B$ smooth.

Let $\sigma \in \Gamma\left(E\big|_U\right)$ and $\sigma' \in \Gamma\left(E'\big|_U\right)$ and suppose that on U, Let $D(\sigma) = \sum_i \omega_i \otimes \sigma_i$ for $\omega_i \in \Omega^1(U)$ and $\sigma_i \in \Gamma\left(E\big|_U\right)$ and $D'(\sigma') = \sum_j \omega'_j \otimes \sigma'_j$ for $\omega'_j \in \Omega^1(U)$ and $\sigma_j \in \Gamma\left(E'\big|_U\right)$. Then

   (i) $E \oplus E'$**.** Define a connection ∇ by

$$\begin{aligned} \nabla(\sigma \oplus \sigma') &= D(\sigma) \oplus D'(\sigma') \\ &= \sum_i \omega_i \otimes (\sigma_i \oplus 0) + \sum_j \omega'_j \otimes (0 \oplus \sigma'_j). \end{aligned}$$

(ii) $E \otimes E'$.

$$\begin{aligned}\nabla(\sigma\otimes\sigma') &= D(\sigma)\otimes\sigma' + \sigma\otimes D'(\sigma') \\ &= \sum_i \omega_i \otimes \left(\sigma_i\otimes\sigma'\right) + \sum_j \omega_j' \otimes \left(\sigma\otimes\sigma_j'\right)\end{aligned}$$

(iii) E^*. We have a natural connection on E^* defined by:

$$D^* : \Gamma\left(E^*\right) \to \Gamma\left(T^*B\otimes E^*\right)$$

where for all $\psi \in \Gamma\left(E^*\right)$, $D^*(\psi) \in \Gamma\left(T^*B\otimes E^*\right)$ is completely determined by $D^*(\psi)(\sigma) \in \Gamma\left(T^*B\right)$ for all $\sigma \in \Gamma\left(E\right)$. So, we set

$$D^*(\psi)(\sigma) := d\left(\psi(\sigma)\right) - \psi\left(D\left(\sigma\right)\right)$$

where

$$\psi(D(\sigma)) = \underbrace{\sum_i \psi(\sigma_i)\omega_i}_{\in\Gamma(T^*B)}$$

(iv) **Hom** (E, E'). We have a natural connection ∇ given by, for all $\psi \in \Gamma\left(\mathrm{Hom}\left(E,E'\right)\right)$ and for all $\sigma \in \Gamma\left(E\right)$ we set

$$\nabla(\psi)(\sigma) := D'\left(\psi(\sigma)\right) - \psi\left(D(\sigma)\right).$$

(v) If $f : N \to B$ is smooth and we have a local frame $\{e_1, \ldots, e_r\}$ of E on U, and $D = d + A$, then on $f^{-1}(U)$,

$$f^*D := d + f^*A$$

is a connection matrix, where $f^*A = (f^*a_{ij})$ where $A = (a_{ij})$

2.1.2 Curvature

Recall. Suppose M is a smooth manifold with local coordinates $(x_1 \ldots, x_n)$.

$$\begin{aligned}&\Omega^0(M) := C^\infty(M) \\ &\Omega^k(M) = (\text{smooth } k\text{-forms on } M) = \Gamma\left(\bigwedge^k T^*M\right), 1 \leqslant k \leqslant n \\ &\Omega^k(M) = 0, k > n.\end{aligned}$$

Note.

- For all $f \in C^\infty(M)$, $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.
- For all $\omega = \sum_I a_I dx_I \in \Omega^k(M)$, $d\omega = \sum_I da_I \wedge dx_I$.
- **Leibniz.** For all $\eta \in \Omega^p(M)$ and $\omega \in \Omega^q(M)$,

$$d\left(\eta\wedge\omega\right) = d\eta\wedge\omega + (-1)^p \eta\wedge d\omega.$$

- **de Rham Complex.**

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \ldots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

  this is a complex *because* $d \circ d = 0$.

Now, fix a vector bundle $(E, B, \pi, \mathbb{R}^r)$ with $n = \dim B$. Set

$$\begin{aligned}&\Omega^0(E) := \Gamma\left(E\right) \\ &\Omega^k(E) := \Gamma\left(\bigwedge^k B\otimes E\right) = (\text{bundle-valued } k\text{-forms}), 1 \leqslant k \leqslant n \\ &\Omega^k(E) := 0, k > n.\end{aligned}$$

If $\omega \in \Omega^p(B)$ and $\tau \in \Omega^q(E)$ so that locally

$$\tau = \sum_i \eta_i \otimes \sigma_i$$

where η_i are $k-$forms and $\sigma_i \in \Gamma\left(E\right)$. We define

$$\omega\wedge\tau := \sum_i (\omega\wedge\eta_i)\otimes\sigma_i \in \Omega^{p+q}(E\big|_U).$$

Let D be a connection on E so that

$$D : \Omega^0(E) \to \Omega^1(E)$$

is $\mathbb{R}$-linear and satisfies Leibniz. How can we extend this to a map

$$D : \Omega^p(E) \to \Omega^{p+1}(E)?$$

If ω is a local p-form on B and σ is a local section of E so that $\omega \otimes \sigma \in \Omega^p(E\big|_U)$. We set

$$D(\omega \otimes \sigma) := d\omega \otimes \sigma + (-1)^p \omega \wedge D(\sigma) \in \Omega^{p+1}(E\big|_U),$$

and extend this definition $\mathbb{R}$-linearly.

- If $k = 0$: $D(f\sigma) = df \otimes \sigma + fD(\sigma)$. This is just the usual Leibniz.
- If $k > 0$, then for all $f \in C^\infty(B)$, $(f\omega) \otimes \sigma = \omega \otimes (f\sigma)$.

$$\begin{aligned} D(f\omega \otimes \sigma) &= d(f\omega) \otimes \sigma + f\omega \wedge D(\sigma) \\ &= df \wedge \omega \otimes \sigma + fd\omega \otimes \sigma + (-1)^p f\omega \wedge D(\sigma) \end{aligned}$$

  and

$$\begin{aligned} D(\omega \otimes (f\sigma)) &= d\omega \otimes (f\sigma) + (-1)^p \omega D(f\sigma) \\ &= fd\omega \otimes \sigma + (-1)^p \omega \wedge df \otimes \sigma + (-1)^p f\omega \wedge D(\sigma) \end{aligned}$$

We get

$$0 \xrightarrow{d} \Omega^0(E) \xrightarrow{d} \Omega^1(E) \to \dots \xrightarrow{d} \Omega^{n-1}(E) \xrightarrow{d} \Omega^n(E) \xrightarrow{d} 0$$

but we may not have $D \circ D = 0$.

Definition 2.3. $F_D := D \circ D$ is the *curvature of D*. We say that D is *flat* if and only if $F_D = 0$.

Lecture 11 --- February 11, 2020

Recall. Fix a vector bundle $(E, B, \pi, \mathbb{R}^r)$. We define

$$\Omega^k(E) = \Gamma\left(\bigwedge^k B \otimes E\right)$$
$$\Omega^k(\mathrm{End}(E)) = \Gamma\left(\bigwedge^k B \otimes \mathrm{End}(E)\right)$$

and if we have a connection $D : \Omega^0(E) \to \Omega^1(E)$, we extend D to $\Omega^p(E)$ as follows:

$$D : \Omega^p(E) \to \Omega^{p+1}(E)$$

is defined on elements of $\Omega^p(B)$ of the form $\omega \otimes \sigma, \omega \ \in \Omega^p(E)$ and $\sigma \in \Gamma(E)$, then we take

$$D(\omega \otimes \sigma) = da \otimes \sigma + (-1)^p \omega \wedge D(\sigma) \quad (*).$$

(where $(-1)^p$ s neceesary do ensure that $D(f\omega \otimes \sigma) = D(\omega \otimes f\sigma)$ for all $f \in C^\infty(B)$. We extend $(*)$ $\mathbb{R}$-linearly.

Then D satisfies a generalized Leibniz rule: For all $\tau \in \Omega^q(E)$ and $\alpha \in \Omega^p(B)$, then we have $\alpha \wedge \tau \in \Omega^{p+q}(E)$ and

$$D(\alpha \wedge \tau) = \underbrace{(d\alpha)}_{\in \Omega^{p+1}(B)} \wedge \tau + (-1)^p \alpha \wedge D(\tau).$$

Proof. Indeed, suppose that $\tau = \omega \wedge \sigma$ with $\omega \in \Omega^q(B)$ and $\sigma \in \Gamma(E)$. Then,

$$\begin{aligned} \alpha \wedge \tau &= \alpha \wedge (\omega \otimes \sigma) \\ &= (\alpha \wedge \omega) \otimes \sigma, \end{aligned}$$

so that by $(*)$, we have

$$\begin{aligned} D(\alpha \wedge \tau) &= D((\alpha \wedge \omega) \otimes \sigma) \\ &= d(a \wedge \omega) \otimes \sigma + (-1)^{p+q}(\alpha \wedge \omega) \wedge D(\sigma) \\ &= (d\alpha \wedge \omega + (-1)^p \alpha \wedge d\omega) \otimes \sigma + (-1)^{p+q}(\alpha \wedge \omega) \wedge D(\sigma) \\ &= (d\alpha \wedge \omega) \otimes \sigma + (-1)^p(\alpha \wedge d\omega) \otimes \sigma + (-1)^{p+q}\alpha \wedge \omega \wedge D(\sigma) \\ &= d\alpha \wedge \tau + (-1)^p \alpha \wedge (d\omega \otimes \sigma + (-1)^q \omega \wedge D(\sigma)) \\ &= d\alpha \wedge \tau + (-1)^p \alpha \wedge D(\tau). \end{aligned}$$

By $\mathbb{R}$-linearity, we get the formula for all elements in $\Omega^q(E)$. □

By extending D to $\Omega^p(E)$, we get a chain

$$0 \xrightarrow{D} \Omega^0(E) \xrightarrow{D} \Omega^1(E) \to \ldots \xrightarrow{D} \Omega^{n-1}(E) \xrightarrow{D} \Omega^n(E) \xrightarrow{D} 0$$

where $n = \dim B$. In genereal, $D \circ D$ so that this is not a complex.

Definition 2.4. Given a connection D on E, we define $F_D = D \circ D$, which is called the *curvature of D*. Furthermore, D is called *flat* if $F_D = 0$.

Example 2.3. If $E = B \times \mathbb{R}^r$ is the trivial bundle and $D = d$ is the trivial connection on E, then $F_D = d \circ d = 0$, so the trivial connection is flat. We will see that, locally, any flat connection can be given by d in an appropriate local frame.

What are some of the properties of

$$F_D : \Omega^0(E) \to \Omega^2(E)?$$

1) F_D is $C^\infty(B)$-linear: For all $\sigma \in \Gamma(E)$ and $f \in C^\infty(B)$, we have

$$F_D(f\sigma) := fF_D(\sigma).$$

Proof.

$$\begin{aligned} F_D(f\sigma) &= D(D(f\sigma)) \\ &= D(df \otimes \sigma + fD(\sigma)) \\ &\overset{\text{defn}}{=} \big(d(df)\otimes\sigma + (-1)^1 df \wedge D(\sigma)\big) + \big(df \wedge D(\sigma) + fD^2(\sigma)\big) \\ &= fD(\sigma). \end{aligned}$$

□

In genereal,

$$D \circ D : \Omega^p(E) \to \Omega^{p+1}(E)$$

is $C^\infty(B)$-linear.

2) Locally, in terms of local coordinates $(x_1, \ldots, x_n)$ on B, we have seen that, for any local section σ of E,

$$D(\sigma) = \sum_{i=1}^n dx_i \otimes D_{\frac{\partial}{\partial x_i}}(\sigma)$$

(where $D_{\frac{\partial}{\partial x_i}} : \Gamma(E) \to \Gamma(E)$ is so that $D_{\frac{\partial}{\partial x_i}}$ are local sections of E). Given this, we also have

$$\begin{aligned} F_D(\sigma) &= \sum_{i,j}(dx_i \wedge dx_j) \otimes \left(D_{\frac{\partial}{\partial x_i}}\left(D_{\frac{\partial}{\partial x_j}}(\sigma)\right)\right) \\ \implies F_D\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) &= \sum_{i,j}(dx_i \wedge dx_j)\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) \otimes D_{\frac{\partial}{\partial x_i}}\left(D_{\frac{\partial}{\partial x_j}}(\sigma)\right) \\ &= D_{\frac{\partial}{\partial x_k}}\left(D_{\frac{\partial}{\partial x_l}}(\sigma)\right) - D_{\frac{\partial}{\partial x_l}}\left(D_{\frac{\partial}{\partial x_k}}(\sigma)\right). \end{aligned}$$

We then see that $F_D = 0$ if and only if $D_{\frac{\partial}{\partial x_l}}\left(D_{\frac{\partial}{\partial x_k}}\right)(\sigma) = D_{\frac{\partial}{\partial x_k}}\left(D_{\frac{\partial}{\partial x_l}}\right)(\sigma)$ for all $k, l = 1, \ldots, n$. So the connection is flat if and only if the covariant derivatives commute (with respect to the coordinate directions).

As with connections, the curvature can be described as a matrix of 2-forms in terms of a local frame as follows:

Example 2.4. $E = B \times \mathbb{R}^r$ and frame $\{e_1, \ldots, e_r\}$ where $e_i(b) = (b, \vec{e}_i)$. Suppose that D is a connection on E with connection matrix $A = (a_{ij})$, where $D(e_j) = \sum_i a_{ij} \otimes e_i$. Then

$$\begin{aligned} F_D(e_j) &= D(D(e_j)) \\ &= D\left(\sum_i a_{ij} \otimes e_j\right) \\ &= \sum_i D(a_{ij} \otimes e_j) \\ &= \sum_i \big(da_{ij} \otimes e_i + (-1)^1 a_{ij} \wedge D(e_i)\big) \\ &= \sum_i da_{ij} \otimes e_i - \sum_i a_{ij} \wedge D(e_i) \\ &= \sum_i da_{ij} \otimes e_i - \sum_i a_{ij}\left(\sum_k a_{ki} e_k\right) \\ &= \sum_i da_{ij} \otimes e_i - \sum_{i,k}(a_{ij} \wedge a_{ki}) \otimes e_k \\ &= \sum_i da_{ij} \otimes e_i + \underbrace{\sum_k \left(\sum_i a_{ki} \wedge a_{ij}\right)}_{(A\wedge A)_{kj}} \otimes e_k \\ &= \sum_i (dA)_{ij} \otimes e_i + \sum_k (A \wedge A)_{kj} \otimes e_k \\ &= \sum_i (dA + A \wedge A)_{ij} \otimes e_i \\ \implies F_D(e_j) &= \sum_i (dA + A \wedge A)_{ij} \otimes e_i. \end{aligned}$$

In general, any local section σ of E can be written as $\sigma = \sum_{i=1}^{r} \overline{\sigma}_j e_j$ for some smooth functions $\overline{\sigma}_1, \ldots, \overline{\sigma}_r$. By $C^\infty(B)$-linearity of F_D, we get:

$$\begin{aligned} F_D(\sigma) &= \sum_{j=1}^{r} \overline{\sigma}_j F_D(e_j) \\ &= \sum_{j=1}^{r} \overline{\sigma}_j \left(\sum_i (dA + A \wedge A)_{ij} \right) \otimes e_i. \\ \implies F_D(\sigma) &= \sum_{i=1}^{r} \left(\sum_j (dA + A \wedge A)_{ij} \, \overline{\sigma}_j \right) \otimes e_i \\ &=: (dA + A \wedge A) \cdot \sigma. \end{aligned}$$

Here, $F_A := dA + A \wedge A$ is the *curvature matrix of D* with respect to $\{e_1, \ldots e_r\}$.

Also, if $\{e'_1, \ldots, e'_r\}$ is another form where

$$e'_j = \sum_i h_{ij} e_j$$

where $h = (h_{ij}) : B \to \mathrm{GL}\,(r, \mathbb{R})$ is the change of basis matrix, and A' is the connection matrix of D with respect to $\{e'_1, \ldots, e'_r\}$ then:

$$A' = h^{-1} A h + h^{-1} dh$$

and

$$F_{A'} = h^{-1} F_A h \quad \text{(exercise.)}$$

Note. If $F_D = 0$, then $F_A = 0$ with respect to *any* local frame on E.

In general, for any vector bundle E with vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ and corresponding local frames $\{e_1^\alpha, \ldots, e_r^\alpha\}$ where $e_i^\alpha = \varphi_\alpha^{-1}(-, \vec{e}_i)$. Suppose that the connection D on E is given by the connection matrices A_α. Then $U_\alpha \cap U_\beta \neq \varnothing$,

$$A_\beta = \overline{g}_{\alpha\beta}^{-1} A_\alpha \overline{g}_{\alpha\beta} + \overline{g}_{\alpha\beta}^{-1} d\overline{g}_{\alpha\beta}$$

and

$$F_{A_\beta} = \overline{g}_{\alpha\beta}^{-1} F_{A_\alpha} \overline{g}_{\alpha\beta}$$

where $\overline{g}_{\alpha\beta} : U_\alpha \cap \beta \to \mathrm{GL}\,(r, \mathbb{R})$.

Theorem 2.1. A connection D on E is flat if and only if there exists a vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ such that $A_\alpha = 0$ for all $\alpha \in \mathcal{A}$.

Remark. If D is flat, then the vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ for which $A_\alpha = 0$ is such that $\overline{g}_{\alpha\beta} \equiv$ constant, because $d\overline{g}_{\alpha\beta} = 0$ for all α, β.

Definition 2.5. A vector bundle E is called *flat* if and only if there exists a vector bundle atlas on E whose transition functions are constant.

Corollary 2.1. A vector bundle is flat if and only if it admits a flat connection.

Lecture 12 --- February 13, 2020

Let $(E, B, \pi, \mathbb{R}^r)$ be a vector bundle and D a connection on E. If $\{e_1, \dots, e_r\}$ is a local frame of E, then locally, if σ is a local section of E given by $\sigma = \sum_i \overline{\sigma}_i e_i$, then

$$F_D(\sigma) = \sum_i (F_A)_{ij} \overline{\sigma}_j \otimes e_i$$

where $F_A := dA + A \wedge A$ is the curvature matrix (with respect to this local frame).

Also, if $\{e_1', \dots, e_r'\}$ is another local frame with $e_j' = \sum_i h_{ij} e_i$ (where $h : U \to \mathrm{GL}\,(r, \mathbb{R})$ is a smooth map with $h = (h_{ij})$), then if A' is the connection matrix of D with respect ot $\{e_1', \dots, e_r'\}$, then:

$$A' = h^{-1} A h +^{-1} dh$$

and

$$F_{A'} = h^{-1} F_A h.$$

So, we have that $F_D = 0 \iff F_A = 0$ for every connection matrix A.

Proposition 2.4. D is flat if and only if there exists a vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ on E with respect to which every $A_\alpha = 0$ for all $\alpha \in \mathcal{A}$, where A_α is the connection matrix of D with respect to the frame $\{e_1^\alpha, \dots, e_r^\alpha\}$.

Before proving the proposition, we need some notation. Let $U \subset B$ be an open set with local coordinates $(x_1, \dots, x_n)$ and assume that E admits a vector bundle chart for U with associated local frame $\{e_i\} = \left\{\varphi^{-1}(-, \vec{e}_i)\right\}$. Let A be the corresponding connection matrix of D. So

$$A = \sum_{k=1}^n A_k dx_k$$

where $A_k : U \to \mathfrak{gl}(r, \mathbb{R})$ is a smooth map, and so

$$F_A = dA + A \wedge A = \sum_{k<l} \left(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} = [A_k, A_l] \right) dx_k \wedge dx_l.$$

Proof.

$$\begin{aligned}
dA &= \sum_{k=1}^n dA_k \wedge dx_k \\
&= \sum_{k=1}^n \left(\sum_{l=1}^n \frac{\partial A_k}{\partial x_l} dx_l \right) \wedge dx_k \\
&= \sum_{k<l} \left(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} \right) dx_k \wedge dx_l
\end{aligned}$$

and

$$\begin{aligned}
A \wedge A &= \left(\sum_{k=1}^n A_k dx_k \right) \wedge \left(\sum_{l=1}^n A_l dx_l \right) \\
&= \sum_{k,l=1}^n A_k A_l dx_k \wedge dx_l \\
&= \sum_{k<l} (A_k A_l - A_l A_k) dx_k \wedge dx_l \\
&= \sum_{k<l} [A_k, A_l] dx_k \wedge dx_l.
\end{aligned}$$

□

So, $F_D = 0$ iff $F_A = 0$ for all A iff $\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} + [A_k, A_l] = 0$ for all $k < l$.

Suppose that $\{e_1', \dots e_r'\}$ is related to $\{e_1, \dots, e_r\}$ by $h : U \to \mathrm{GL}\,(r, \mathbb{R})$ so that its connection matrix is

$$A' = h^{-1} A h + h^{-1} dh$$

If $A = \sum_{k=1}^{n} A_k dx_k$ and $A' = \sum_{k=1}^{n} A'_k dx_k$, then:

$$A'_k = h^{-1} A_k h + h^{-1} \frac{\partial h}{\partial x_k}.$$

Therefore, if there exists a lcoal frame $\{e'_1, \ldots, e'_r\}$ with respect to which $A' = 0$ then there exists $h : U \to \mathrm{GL}\,(r, \mathbb{R})$ such that

$$h^{-1} A_k h + h^{-1} \frac{\partial h}{\partial x_k}.$$

Proof. ($\Longleftarrow$) If there is a vector bundle atlas such that $A_\alpha = 0$ for all α, then $F_{A_\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha = 0$. ($\Longrightarrow$) Suppose that $F_D = 0$, so that $F_A = 0$ for any connection matrix A. Let us first assume that B is a hypercube: $B = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_i| \leqslant 1\}$. Then E is trivial on B, so there exists a global vector bundle chart $\varphi : E \to B \times \mathbb{R}^r$ and a corresponding global frame $\left\{e_i = \varphi^{-1}(-, \vec{e}_i)\right\}_{i=1}^{r}$. Let A be the connection matrix of D with respect to theis frame and lets us write it:

$$A = \sum_{k=1}^{n} A_k dx_k$$

with each $A_k : U \to \mathfrak{gl}(r, \mathbb{R})$ smooth for all $k = 1, \ldots, n$. Then $F_A = 0$, which implies

$$\frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k} + [A_k, A_l] = 0 \quad (*).$$

We want to fund $h : B \to \mathrm{GL}\,(r, \mathbb{R})$ smooth such that

$$h^{-1} A_k h + h^{-1} \frac{\partial h}{\partial x_k}.$$

We do this in several steps by finding smooth maps $B \to \mathrm{GL}\,(r, \mathbb{R})$ that take A to a connection matrix $\tilde{A}$ with $\tilde{A}_1 = 0$, then $\tilde{A}_2 = 0$, etc.

- Can we find $h : B \to \mathrm{GL}\,(r, \mathbb{R})$ smooth such Mathematics

  $$\tilde{A}_1 = h^{-1} A_1 h + h^{-1} \frac{\partial h}{\partial x_1} \iff A_1 h + \frac{\partial h}{\partial x_1} = 0.$$

  This is a linear ODE for h in the variable x_1 with $x_2, \ldots, x_n$ fixed (but also with the equation varying smoothly in $x_2, \ldots, x_n$)). So there exists a smooth solution by the ODE theorem (exercise)

- Suppose that there is $h :\to \mathrm{GL}\,(r, \mathbb{R})$ smooth taking A to a connection matrix $\tilde{A}$ with $\tilde{A}_1, \ldots, \tilde{A}_p = 0$. Let us show that there is a new $\tilde{h} : B \to \mathrm{GL}\,(r, \mathbb{R})$ taking $\tilde{A}$ to $\tilde{\tilde{A}}$ with

  $$\tilde{\tilde{A}}_1, \ldots, \tilde{\tilde{A}}_p = 0.$$

  Then $\tilde{h}$ much satisfy

  $$\tilde{\tilde{A}}_k = \tilde{h}^{-1} \tilde{A}_k \tilde{h} + \tilde{h}^{-1} \frac{\partial \tilde{h}}{\partial x_k} = 0, \forall k = 1, \ldots, p+1$$
  $$\iff \begin{cases} \frac{\partial \tilde{h}}{\partial x_k} = 0 & \forall k = 1, \ldots, p \ (**) \\ \tilde{A}_{p+1} \tilde{h} + \frac{\partial \tilde{h}}{\partial x_{p+1}} = 0 & (***) \end{cases}$$

  As before, by the ODE theorem, there exists a solution $\tilde{h}$ to $(***)$. Also, since $F_{\tilde{A}} = 0$ by $(*)$, for all $k < p+1$, since D is flat we have

  $$\frac{\partial \tilde{A}_{p+1}}{\partial x_k} - \underbrace{\frac{\partial \tilde{A}_k}{\partial x_{p+1}}}_{=0} + [\tilde{A}_k, \underbrace{\tilde{A}_{p-1}}_{=0}] = 0$$
  $$\iff \frac{\partial \tilde{A}_{p+1}}{\partial x_k} = 0 \quad \forall k = 1, \ldots, p.$$

  So $\tilde{A}_{p+1}$ does not depend on $x_1, \ldots, x_p$. So $\tilde{h}$ satisfies $(**)$.

- Now for a general vector bundle, start with a vector bundle atlas whose open cover of B consists of open sets diffeomorphic to hypercubes, and replace every vector bundle chart by a chart with respect to which the connection matrix is 0, as above.

$\square$

We will end with a few more facts about curvature:

- We have see that if D_0 is a fixed connection on E, then the set of all connections on E is

$$\mathcal{A}(E) = \left\{D_0 + a \mid a \in \Omega^1\left(\mathrm{End}(E)\right)\right\}.$$

  One can show that

$$F_{D_0+a} = F_{D_0} + D_0(a) + a \wedge a$$

  for every $a \in \Omega^1\left(\mathrm{End}(E)\right)$, where D_0 also denotes the induced connection on $\mathrm{End}(E)$.

- **Bianchi identity.** Let D be a connection on E. Then,

$$F_D : \Gamma(E) \to \Omega^2(E)$$

  and is $C^\infty(B)$-linear. We can therefore think of F_D as an element of $\Omega^2\left(\mathrm{End}(E)\right)$.
  As an aside: In general, if E_1 and E_2 are vector bundles on B, then $\Gamma\left(\mathrm{Hom}\left(E_1, E_2\right)\right)$ is identified with the set

$$\{C^\infty(B) - \text{linear maps } \Gamma(E_1) \to \Gamma(E)_2\}$$

  Indeed, given $\psi \in \Gamma\left(\mathrm{Hom}\left(E_1, E_2\right)\right)$ so that

$$\psi : B \to \mathrm{Hom}\left(E_1, E_2\right) = \bigsqcup_{b\in B} \mathrm{Hom}\left((E_1)_b, (E_2)_b\right)$$

  so that $\psi(b) : (E_1)_b \to (E_2)_b$ is $\mathbb{R}$-linear. Then ψ induces

$$\begin{aligned} \tilde{\psi} : \Gamma(E_1) &\to \Gamma(E_2) \\ \sigma &\mapsto \tilde{\psi}(\sigma) \end{aligned}$$

  where

$$\begin{aligned} \tilde{\psi}(\sigma) : B &\to E_2 \\ b &\mapsto \psi(b)(\sigma(b)) \in (E_2)_b. \end{aligned}$$

  Conversely, let $\tilde{\psi} : \Gamma(E_1) \to \Gamma(E_2)$ be $C^\infty(B)$-linear. Set

$$\begin{aligned} \psi : B &\to \mathrm{Hom}\left(E_1, E_2\right) \\ b &\mapsto \psi(b) \in \mathrm{Hom}\left((E_1)_b, (E_2)_b\right) \end{aligned}$$

  where, for all $b \in B$,

$$\begin{aligned} \psi(b) : (E_1)_b &\to (E_2)_b \\ e = \sigma(b) &\mapsto \tilde{\psi}(\sigma)(b) \end{aligned}$$

  for some local section σ. One can show that this definition of $\psi(b)$ is independent of the choice of σ by the $C^\infty(B)$-lineaity of $\tilde{\psi}$ and $\psi(b)$ is $\mathbb{R}$-linear.

  Proposition 2.5. For any connection D on E,

$$D(F_D) = 0$$

  where D also denotes the induced connection on $\mathrm{End}(E)$.

  Proof. $F_D \in \Omega^2\left(\mathrm{End}(E)\right)$ and for all $\psi \in \Gamma\left(\mathrm{End}(E)\right)$, then induced connection on $\mathrm{End}(E)$is such that for all $\sigma \in \Gamma(E)$,

$$D(\psi)(\sigma) := D(\psi(\sigma)) - \psi(D(\sigma)).$$

  In general if $\tau \in \Omega^k\left(\mathrm{End}(E)\right)$, for all $\sigma \in \Gamma(E)$,

$$D(\tau)(\sigma) = D(\tau(\sigma)) - \tau(D(\sigma)).$$

  So we have

$$\begin{aligned} D(F_D)(\sigma) &= D(F_D(\sigma)) - F_D(D(\sigma)) \\ &= D \circ D \circ D(\sigma) - D \circ D \circ D(\sigma) \\ &= 0. \end{aligned}$$

  □

Lecture 13 --- February 25, 2020

2.1.3 Affine Connections

Let M be a smooth manifold. An affine connection is a linear connection on TM:

$$\begin{aligned}\nabla : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y\end{aligned}$$

such that it

- is $C^\infty(M)$-linear in X
- satisfies Leibniz in Y: For all $f \in C^\infty(M)$, $\nabla(X, fY) = X(f)Y + f\nabla_X Y$.

Note. If we think of the connection as $D : \Gamma(TM) \to \Omega^1(TM)$ such that D is $\mathbb{R}$-linear and satisfies Leibniz: for all $Y \in \Gamma(TM)$ and for every $f \in C^\infty(M)$, we have that

$$D(fY) = df \otimes Y + fD(Y),$$

then

$$\nabla(X, Y) = D_X(Y) = D(Y)(X).$$

(i) **Torsion.** For all $X, Y \in \Gamma(TM)$,

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

This is $C^\infty(M)$-linear in X and Y, and also is skew. We say that ∇ is *torsion-free* if $T^\nabla \equiv 0$ iff

$$\nabla_X Y - \nabla_Y X = [X, Y] \ \forall X, Y \in \Gamma(TM) \quad (*).$$

$(*)$ is very useful in formulae and in proofs.

Torsion-free connections are 'symmetric': Let $x_1, \dots, x_n$ be local coordinates on M so that $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$ is a local frame of TM. Them for all i, j, we have

$$\begin{aligned}&\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \in \Gamma\left(TM\big|_U\right) \\ \implies &\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}.\end{aligned}$$

If $T^\nabla \equiv 0$, then by (*),

$$\begin{aligned}&\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \left[\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k}\right] = 0 \\ \iff &\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} \\ \iff &\sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k} = \sum_{k=1}^{n} \Gamma_{ji}^k \frac{\partial}{\partial x_k} \\ \iff &\Gamma_{ij}^k = \Gamma_{ji}^k\end{aligned}$$

So the *Christoffel symbols* Γ_{ij}^k are symmetric in i, j.

(ii) **Curvature.** For all $X, Y, Z \in \Gamma(TM)$,

$$R_{X,Y}^\nabla(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

- R^∇ is $C^\infty(M)$-linear in X, Y and Z.
- It is also skew in X, Y.

A direct computation gives that

$$\underbrace{F_D(Z)}_{\in \Omega^2(TM)}(X, Y) = R_{X,Y}^\nabla(Z)$$

for every $X, Y, Z \in \Gamma(TM)$. Note that F_D is zero if and only if R^∇ is zero. We say that ∇ is flat if and only if $R^\nabla \equiv 0$, which happen if and only if $F_D \equiv 0$, so ∇ is flat if and only if D is flat.

2.2 Connections on a Fibre Bundle

Let (E, B, π, F) be a fibre bundle. Here, the notion of a connection is given by an appropriate splitting of TE.

For all $e \in E$, set

$$\begin{aligned} V_e &:= \left\{ \text{ the set of tangent vectors to } E \text{ at } e \text{ that are tangent to } E_{\pi(e)} \right\} \\ &= \textit{vertical tangent space at } e. \end{aligned}$$

Recall that $\pi_* : TE \to TB$ is a submersion so that $E_b \subset E$ is a submanifold for all $b \in B$. and

$$\pi_{*,e} : T_e E \to T_{\pi(e)} B$$

is surjective for all $e \in E$. set

$$V_e = \ker\left(\pi_{*,e} : T_e E \to T_{\pi(e)} B\right).$$

This is a vector space of dimension $\dim E - \dim B = \dim F$.

Let (U, φ) be a bundle chart of E with $e \in U$ so that

$$\varphi : E_U \to U \times F$$

Then $\pi_* = (\mathrm{pr}_1)_* \circ \varphi_*$. For all $e \in E_U$, set $\varphi(e) = (\pi(e), \overline{\varphi}(e))$ with $\overline{\varphi}(e) \in F$. Then,

$$\begin{aligned} T_{\overline{\varphi}(e)} F &= \ker\left((\mathrm{pr}_1)_{*,(\pi(e),\overline{\varphi}(e))}\right) \\ &\cong \ker(\pi_{*,e}) \end{aligned}$$

So we have a subspace $V_e \subseteq T_e E$ of dimension $\dim F$. If we set

$$VE = \bigsqcup_{e \in E} V_e$$

is a smooth vector bundle on E. This bundle is called the *vertical bundle of E*.

Definition 2.6. An *(Ehnresmann) connection* or a *fibre bundle connection* on (E, B, π, F) is a collection $\{H_e \mid e \in E\}$ with each H_e a subspace of $T_e E$ of dimension $\dim B$ for all $e \in E$, called the *horizontal subspaces*, such that

- the assignment $e \mapsto H_e$ depends smoothly on $e \in E$, and
- for all $e \in E$, $T_e E = V_e \oplus H_e$.

Note. In other words,

$$HE = \bigsqcup_{e \in E} H_e$$

is a smooth vector bundle on E called the *horizontal bundle of E*.

In other words, an Ehnresmann connection on E is a smooth distribution on E such that $E = VE \oplus HE$.

Example 2.5. $E = B \times F$. In this case, suppose that $\{x_1, \ldots, x_n\}$ are local coordinates on B and $\{y_1, \ldots, y_r\}$ local coordinates on F. Then:

$$T_e = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}\right\}$$

and

$$V_e = \left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}.$$

If we set $H_e = \operatorname{span}\left\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}\right\}$, then $T_e E = V_e \oplus H_e$ for all $e \in E$ and the corresponding Ehnresmann conneciton is called the *trivial connection*.

Definition 2.7. An Ehnresmann connection is called *flat* if it is given by an integrable smooth distribution HE on E.

(By Frobenius, this means that $[H_e, H_e] \subset H_e$ for all $e \in E$). This means that H_e are tangent to submanifolds of E.

Note. An Ehnresmann connection is flat if and only if for all $e \in E$, there is a chart (U, φ) such that φ takes HE on E_U to the trivial connection on $U \times F$.

Finally, let us give an equivalent way of defining an Ehresmann connection: An Ehresmann connection can be defined as a vector bundle map

$$K : TE \to TE$$

such that $K \circ K = K$ and such that $K(T_e E) = V_e$. We recover the previous definition by setting $H_e = \ker K\big|_{T_e E}$ for every $e \in E$.

Remark. If (E, B, π, F) is a vector bundle, we will see that any linear connection $D : \Gamma(E) \to \Omega^1(E)$ gives rise to an Ehresmann connection, but not all Ehresmann connections on E come from linear connections.

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Definition 2.8. Let (E, B, π, F) be a fibre bundle. For all $e \in E$, $V_e = \ker(\pi_{*,e} : T_eE \to T_{\pi e}B) \subseteq T_eE$ is called the *vertical subspace*. A *horizontal subspace* at e is a subspace $H_e \subseteq T_eE$ such that $T_eE = V_e \oplus H_e$. An *Ehresmann connection on E* is a connection $\{H_e \mid e \in E\}$ such that

- the assignment $e \mapsto H_e$ varies smoothly in e, and
- for all $e \in E$, H_e is a horizontal subspace.

Note. $\dim V_e = \dim F$ and $\dim H_e = \dim B$ for all $e \in E$. $HE = \bigsqcup_{e\in E} H_e$ is a smooth distribution on E.

Another way of defining H_e is as a vector bundle map $K : TE \to TE$ such that $K \circ K = K$ and such that $K(TE) = VE = \bigsqcup_{e\in E} V_e$. Then we set $HE = \bigsqcup_{e\in E} \ker(\pi_{*,e} : T_eE \to T_eE)$.

This map K can be interpreted as a 1-form on E with values in TE, i.e., as an element of $\Omega^1(TE)$, which is called the *connection 1-form* of the Ehresmann connection K.

How can we see this explicitly? Let (U, φ) be a bundle chart for E so that $\varphi : E_U \to U \times F$ is a diffeomorphism with $U \subseteq B$ open and $\mathrm{pr}_1 \circ \varphi = \pi$. Then, for all $e \in E_U$,

$$\varphi_{*,e} : T_eE \to T_{\varphi(e)}(U \times F)$$

is an isomorphism. Pick local coordinates $(x_1, \dots, x_n)$ on U and $(y_1, \dots, y_r)$ on F. (assume they are defined on some open set $W \subseteq U \times F$). Then,

$$T_{\varphi(e)}(U \times F) = \operatorname{span}\left\{ \frac{\partial}{\partial x_1}\Big|_{\varphi(e)}, \dots, \frac{\partial}{\partial x_n}\Big|_{\varphi(e)}, \frac{\partial}{\partial y_1}\Big|_{\varphi(e)}, \dots, \frac{\partial}{\partial y_r}\Big|_{\varphi(e)} \right\}.$$

so we set

$$\frac{\partial}{\partial x_i}\Big|_e = \varphi_{*,e}^{-1}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(e)}\right) \quad \text{(and)}$$
$$\frac{\partial}{\partial y_j}\Big|_e = \varphi_{*,e}^{-1}\left(\frac{\partial}{\partial y_j}\Big|_{\varphi(e)}\right)$$

so that $T_eE = \operatorname{span}\left\{ \frac{\partial}{\partial x_j}\big|_e, \frac{\partial}{\partial y_j}\big|_e \right\}$. Also,

$$\begin{aligned}\pi_{*,e}\left(\frac{\partial}{\partial x_i}\Big|_e\right) &= (\mathrm{pr}_1)_{*,\varphi(e)}\left(\varphi_{*,e}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(e)}\right)\right) \\ &= \frac{\partial}{\partial x_i}\Big|_{\pi(e)}\end{aligned}$$

and

$$\pi_{*,e}\left(\frac{\partial}{\partial y_j}\Big|_e\right) = 0.$$

So $V_e = \operatorname{span}\left\{ \frac{\partial}{\partial y_j}\big|_e \right\}$.

Recall that $K : TE \to TE$ is a vector bundle map such that

- $K \circ K = K$
- $K(TE) = VE$

So for all $j = 1, \dots, r$, since $\frac{\partial}{\partial y_j}\big|_e \in V_e$,

$$K\left(\frac{\partial}{\partial y_j}\Big|_e\right) = \frac{\partial}{\partial y_j}\Big|_e$$

and for all $i = 1, \dots, n$,

$$K\left(\frac{\partial}{\partial x_i}\Big|_e\right) \in V_e$$
$$\implies K\left(\frac{\partial}{\partial x_i}\Big|_e\right) = \sum_{j=1}^{r} b_{ij}(e) \frac{\partial}{\partial y_j}\Big|_e$$

for some $b_{ij}(e) \in \mathbb{R}$.

Thus, we have

$$\begin{cases} K\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^r b_{ij} \frac{\partial}{\partial y_j} & \text{for some } b_{ij} \in C^\infty(\varphi^{-1}(W)) \\ K\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial y_j}. \end{cases}$$

Thus, K correspondts to the 1-form with values in TE given by

$$\tau := \sum_{j=1}^{r} \left(\left(\sum_{i=1}^{n} b_{ij} dx_i \right) + dy_j \right) \otimes \frac{\partial}{\partial y_j}.$$

This is called the *connection 1-form of K*. Also,

$$\begin{aligned} H_e &= \ker\left(\pi_{*,e} : T_e E \to T_e E\right) \\ &= \text{span} \underbrace{\left\{ \frac{\partial}{\partial x_i}\Big|_e - \sum_{j=1}^{r} b_{ij}(e) \frac{\partial}{\partial y_j}\Big|_e \right\}}_{\text{linearly independent}}. \end{aligned}$$

Curvature of an Ehresmann connection. Let HE be an Ehresmann connection on E so that $TE = HE \oplus VE$. So, for all $X \in \Gamma(E)$ we can uniquely write

$$X = X_v + X_h$$

with $X_v \in \Gamma(VE)$ and $X_h \in \Gamma(HE)$.

Definition 2.9. The *curvature* of HE is a 2-form on E with values in TE (i.e., an element of $\Omega^2(TE)$) defined by: For all $X, Y \in \Gamma(TE)$,

$$R(X,Y) = [X_h, Y_h]_v \in \Gamma(VE) \subset \Gamma(TE).$$

We see that

$$\begin{aligned} R \equiv 0 &\iff [X_h, V_h]_v = 0 \ \forall X, Y \in \Gamma(TE) \\ &\iff [X_h, V_h] \in HE \ \forall X, Y \in \Gamma(TE) \\ &\iff [HE, HE] \subset HE \\ &\iff HE \text{ is flat.} \end{aligned}$$

Example 2.6. $E = \mathbb{R}^2 \times \mathbb{R}$, where the first factor is the base and the second is the fibre. Pick local coordinates $(x_1, x_2) \in \mathbb{R}^2$ and $y \in \mathbb{R}$. $T_e E = \text{span}\left\{\frac{\partial}{\partial x_1}\right\}\big|_e, \frac{\partial}{\partial x_2}\big|_e$ and $V_e = \text{span}\left\{\frac{\partial}{\partial y}\big|_e\right\}$.

1. Set $H_e = \text{span}\left\{\frac{\partial}{\partial x_1}\big|_e, \frac{\partial}{\partial x_2}\big|_e\right\}$. Then $[HE, HE] \subset HE$, so HE is flat. Here, HE is the trivial connection.

2. Set $HE = \text{span}_{C^\infty(E)}\left\{\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial y}, \frac{\partial}{\partial x_2}\right\}$. Since

$$\left[\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial y}, \frac{\partial}{\partial x_2}\right] = -\frac{\partial}{\partial y} \notin HE,$$

HE is not flat. Note that $R\left(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial y}, \frac{\partial}{\partial x_2}\right) = -\left(\frac{\partial}{\partial y}\right)_v = -\frac{\partial}{\partial y} \neq 0.$

How does this relate to linear connections where E is a vector bundle?

Suppose that $(E, B, \pi, \mathbb{R}^r)$ is a vector bundle and $D : \Gamma(E) \to \Omega^1(E)$ is a linear connection on E. Without loss of generality, assume that $E = B \times \mathbb{R}^r$ (otherwise, work with a vector bundle atlas on E) with frame $\{e_1, \ldots, e_r\}$ where $e_i(b)L = (b, \vec{e}_i)$. Also, suppse $D = d + A$ where $A = (a_{ij})$ is the connection matrix of D with respect to the frame $\{e_1, \ldots, e_r\}$. Choose local coordinates $(x_1, \ldots, x_n)$ on B and coordinates $(y_1, \ldots y_r)$ on $\mathbb{R}^r$. Set

$$b_{ij}(x_1, \ldots, x_n, y_1, \ldots, y_r) = \sum_{l=1}^{r} a_{jl}\left(\frac{\partial}{\partial x_i}\right) y_l.$$

Here, $a_{jl}\left(\frac{\partial}{\partial x_i}\right) \in C^\infty(B)$ *and* note that b_{ij} is a linear function in $y_j's$. Thus, we set

$$\begin{aligned} K &: TE \to TE \\ \frac{\partial}{\partial x_i} &\mapsto \sum_{k=1}^{r} b_{ij} \frac{\partial}{\partial y_j} \\ \frac{\partial}{\partial y_j} &\mapsto \frac{\partial}{\partial y_j}. \end{aligned}$$

IMPORTANT. Not all Ehresmann connections on the vector bundle E come from a linear connection D, because the smooth functions b_{ij} need not be linear in the y_j's.

What is the geometric interpretation of the Ehresmann connection obtained from D?

Definition 2.10. Let $(E, B, \pi, \mathbb{R}^r)$ be a vector bundle and $D : \Gamma(E) \to \Omega^1(E)$ be a linear conneciton on E. $\sigma \in \Gamma(E)$ is called *flat* or *covariantly constant* if $D\sigma = 0$.

Note that $D = 0$ if and only if $D_X\sigma = 0$ for all $X \in \Gamma(TB)$.

Example 2.7. Let $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be the trivial line bundle on $\mathbb{R}^2$. Choose coordinates $(x_1, x_2) \in \mathbb{R}^2$ and $y \in \mathbb{R}$ (the former being the base and the latter the fibre). Then

$$\begin{aligned} e : \mathbb{R}^2 &\to \mathbb{R}^2 \times \mathbb{R} \\ (x_1, x_2) &\mapsto (x_1, x_2, 1) \end{aligned}$$

is a frame for E. Then for any $\sigma \in E$, $\sigma = \overline{\sigma}e$ for $\overline{\sigma} : \mathbb{R}^2 \to \mathbb{R}$ smooth.

- If $D = d =$ the trivial connection, then $D\sigma = d\overline{\sigma} \otimes e$. So $D\sigma = 0$ if and only if $\overline{\sigma}$ is a constant function on $\mathbb{R}^2$.
- If $D = d + A$ where $A = (a_{11}) = (dx_1)$ (remember, A is 1×1). Then $D(e) = a_{11} \otimes e = dx_1 \otimes e$. Then

$$\begin{aligned} D(\sigma) &= D(\overline{\sigma}e) \\ &= d\overline{\sigma} \otimes e + \overline{\sigma}D(e) \\ &= d\overline{\sigma} \otimes e + \overline{\sigma}dx_1 \otimes e \\ \implies D(\sigma) = 0 &\iff d\overline{\sigma} + \overline{\sigma}dx_1 = 0 \\ &\iff d\overline{\sigma} = -\overline{\sigma}dx_1. \end{aligned}$$

  This has solution $\overline{\sigma} = Ce^{-x_1}$.

What about along a curve?

Definition 2.11. Let $\gamma : I = (-\varepsilon, \varepsilon) \subset \mathbb{R} \to B$ be smooth. Let $\sigma \in \Gamma(E)$. Then σ is said to be *covariantly constant along γ* if

$$D_{\dot{\gamma}(t)}\sigma = 0$$

for all $t \in I$.

Given the linear connection D and corresponding Ehresmann connection $HE \subset TE$, one can show that

$$H_e = \left\{\dot{\xi}(0) \mid \xi(t) = \sigma(\gamma(t)) \text{ for } \sigma \in \Gamma(E) \text{ such that } D\sigma = 0,\ \gamma : I \to \mathbb{R} \text{ smooth.}\right\}$$

Lecture 15 --- March 3, 2020

2.3 Metric Connections

2.3.1 Metrics

Let $(E, B, \pi, \mathbb{R}^r)$ be a real vector bundle. Also, denote $\underline{\mathbb{R}} = B \times \mathbb{R}$ the trivial line bundle on B.

Definition 2.12. A *Riemannian metric* on E is a section

$$g \in \Gamma\left(\mathrm{Hom}\left(E \otimes E, \underline{\mathbb{R}}\right)\right)$$

such that g is symmetric and positive-definite. I.e., for every $b \in B$,

$$g_b = g(b) E_b \otimes E_b \to \underline{\mathbb{R}}_b = \{b\} \times \mathbb{R}$$

such that, setting $g_b(e, e') = g_b(e \otimes e')$ for all $e, e' \in E_b$:

- $g_b : E_b \times E_b \to \mathbb{R}$ is bilinear
- $g_b(e, e') = g_b(e', e)$
- $g_b(e, e') \geqslant 0$ and $g_b(e, e) = 0 \iff e = 0$.

Moreover, a *Riemannian manifold* is a smooth manifold M together with a Riemannian metric on its tangent bundle.

Remark.

1. For each $b \in B$, $g_b : E_b \times E_b \to \mathbb{R}$ is an inner product on E_b. So, Riemannian metrics can be thought of as a smooth choice of inner products on the fibres of E.
2. A Riemannian metric g can also be interpreted as a $C^\infty(B)$-linear map $g : \Gamma\left(E \otimes E\right) \to \Gamma\left(\underline{\mathbb{R}}\right) = C^\infty(B)$. To be precise, given $\sigma_1, \sigma_2 \in \Gamma\left(E\right)$ so that $\sigma_1 \otimes \sigma_2 \in \Gamma\left(E \otimes E\right)$, and set

   $$g(\sigma_1, \sigma_2)(b) = g_b(\sigma_1(b), \sigma_2(b))$$

   for all $b \in B$. In fact, we also denote by $g(\sigma, \sigma_2) := g(\sigma_1 \otimes \sigma_2)$, which is a $C^\infty(B)$-bilinear map.
3. One can try to understand g in terms of a local frame $\{e_1, \ldots, e_r\}$ on some open set $U \subseteq B$. For every $b \in U$, $\{e_1(b), \ldots, e_r(b)\}$ is a basis of E_b so that $g_b : E_b \times E_b \to \mathbb{R}$ is completely determined by

   $$g_{ij}(b) = g_b(e_i(b), e_j(b)).$$

   Then $(g_{ij}(b))$ is an $r \times r$-matrix that is symmetric and positive-definite. Let $\{e_1^*, \ldots, e_r^*\}$ be the dual frame of E^* over U so that $\{e_1^*(b), \ldots, e_r^*(b)\}$ is the dual basis of E_b^*. We can then write

   $$g = \sum_{i,j=1}^{r} g_{ij} e_i^* \otimes e_j^*$$

   on U. Locally, g is specified by a matrix (g_{ij}) where $g_{ij} : U \to \mathbb{R}$ and the matrix $(g_{ij}(b))$ is symmetric and positive-definite for all $b \in U$.

Example 2.8. beginenumerate

$E = B \times \mathbb{R}^r$ and $\{e_1, \ldots, e_r\}$ is the standard frame. Then

$$g = \sum_{i,j} g_{ij} e_i^* \otimes e_j^*$$

where $g_{ij}(b)$ is positive-definite and symmetric for all $b \in B$.

M is a smooth manifolds and g is a Riemannian metric on M. In local coordinates $(x_1, \ldots, x_n)$ on M. Then

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

In particular, if $M = \mathbb{R}^n$ so that $TM = \mathbb{R}^n \times \mathbb{R}^n$, Then

$$g = \sum_{i} dx_i \otimes dx_i$$

is the Euclidean metric.

Let M be a smooth manifold and S an embedded submanifold of M with inclusion map $\iota : S \to M$. Given any Riemannian metric g on M, one defines the *induced metric* g_S on S by specifying for $p \in S$ and $X, Y \in T_pM$,

$$g_{S,p}(X,Y) = g_{\iota(p)}(\iota_{*,p}X, \iota_{*,p}Y)$$

Note. One can think of the induced metric as the restriction of g to tangent vectors to S.

Proposition 2.6. For any vector bundle E, Riemannian metrics always exist.

Proof. We have already seen that they exist locally. Use a partition of unity to construct one globally. □

Note. Can also define *pseudo-Riemannian metrics* where g_b is symmetric but non-degenerate, not necessarily positive-definite. For example, the Minkowski metric on $\mathbb{R}^4$ given by

$$g = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

ii. On a complex vector bundle $(E, B, \pi, \mathbb{C}^r)$, we conider *Hermitian metrics* which are a choice of Hermitian inner product on each fibre E_b that varies smoothly with b. More on this later.

Definition 2.13. Let g be a Riemannian metric on E. A set of sections $\{\sigma_1, \ldots, \sigma_l\}$ on E is called *orthonormal* if, at every point $b \in B$, $\{\sigma_1(b), \ldots, \sigma_l(b)\}$ is an orthonormal set with respect to g_b. I.e.,

$$g_b(e_i(b), e_j(b)) = \delta_{ij}$$

for all $i, j \in \{1, \ldots, r\}$ and $b \in B$. A frame is called an *orthonormal frame* if it is an orthonormal set of sections.

Note. With respect to an orthonormal frame $\{e_1, \ldots, e_r\}$,

$$g = \sum_{i=1}^{r} e_i^* \otimes e_i^*.$$

Proposition 2.7. For any Riemannian metric g on E and any point $b \in B$, there exists an open neighbourhood $U \ni b$ on which there is an orthonormal frame of E with respect to g.

Proof. Start with any local frame, and then apply Gram-Schmidt.

Warning. If $E = TM$, and $\{x_1, \ldots, x_n\}$ are local coordinates, then it may not be the case that $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ is an orthonormal frame.

2.3.2 Metric Connections

Let $(E, B, \pi, \mathbb{R}^r)$ be a real vector bundle and g be a Riemannian metric on E. Then, for all $\sigma_1, \sigma_2 \in \Gamma(E)$, $g(\sigma_1, \sigma_2) \in C^\infty(B)$. It is natural to consider the rate of change of the smooth function $g(\sigma_1, \sigma_2)$: $dg(\sigma_1, \sigma_2)$ $ORX(g(\sigma_1, \sigma_2))$ for all $X \in \Gamma(TB)$. Assume that $E = B \times \mathbb{R}^r$ and $g_b = I_{r \times r}$. Let $\sigma_1, \sigma_2 \in \Gamma(E)$ so that $\sigma_i(b) = (b, \overline{\sigma}_i(b))$ for some smooth $\overline{\sigma}_i : B \to \mathbb{R}$. Then, for all $b \in B$,

$$g(\sigma_1, \sigma_2)(b) = \overline{\sigma}_1(b) \cdot \overline{\sigma_2}(b).$$

And if $x_1, \ldots, x_n$ are local coordinates on B,

$$\frac{\partial}{\partial x_i}\left(g(\sigma_1, \sigma_2)\right) = \frac{\partial}{\partial x_i}\left(\overline{\sigma}_1\right) \cdot \overline{\sigma}_2 + \overline{\sigma}_1 \cdot \frac{\partial}{\partial x_i}\left(\overline{\sigma}_2\right).$$

In general, if E is any vector bundle:

Definition 2.14. Let E be a vector bundle with Riemannian metric g. We say that a linear connection D on E is *compatible* with g if, for all $X \in \Gamma(TM)$ and $\sigma_1, \sigma_2 \in \Gamma(E)$, we have

$$X(g(\sigma_1, \sigma_2)) = g\left(D_X\sigma_1, \sigma_2\right) + g(\sigma_1, D_X\sigma_2).$$

If D is compatible with g, then D is called a *metric connection*.

Proposition 2.8. For any Riemannin metric g on E, there exists at least one linear connection D compatible with it.

Proof. Given any point $b \in B$, we know that there exists an orthonormal frame $\{e_1, \ldots, e_r\}$ of E on an open neighbourhood of b. Suppose that there is a connection D that is compatible with g, and let $A = (a_{ij})$ be the connection matrix of D with respect to this frame. This forces $a_{ij} = -a_{ji}$. Then, connections that are compatible with g always exist locally. Then use a partition of unity to stitch it up to a global connection that is compatible with g. □

Example 2.9. Let $E = B \times \mathbb{R}^r$ with Riemannian metric g. Let $\{e_1, \ldots, e_r\}$ be an orthonormal frame with respect to this metric. Then $A = (a_{ij})$ with $a_{ij} \in \Omega^1(B)$ and $a_{ij} = -a_{ji}$. Then the connection $D = d + A$ is compatible with g. So, there exist *many* connections which are compatible with g. But, if one imposes additional conditions on D, one can obtain uniqueness as well. For example, the Levi-Civita connection.

Proposition 2.9. Let M be a smooth manifold and let g be a Riemannian metric on M. Then there exists a unique affine connection $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ that is compatible with g and is torsion-free. This connection is called the *Levi-Civita* connection of (M, g).

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Recall. Let $(E, B, \pi, \mathbb{R}^r)$ be a real vector bundle. $g \in \Gamma(\mathrm{Hom}(E \otimes E, \underline{\mathbb{R}}))$ (where $\underline{\mathbb{R}} = B \times \mathbb{R}$ is the trivial line bundle over B) is called a *Riemannian metric*. For all $b \in B$,

$$g_b : E_b \times E_b \to \mathbb{R}$$

is

- bilinear
- symmetric: $g_b(e, e') = g_b(e', e)$ for all $e, e' \in E_b$
- positive definite: $g_b(e, e) \geqslant 0$ for all $e \in E_b$ with equality iff $e = 0$.

Given a Riemannian metric g on E, a connectino $D : \Gamma(E) \to \Omega^1(E)$ is called a *metric connection* if it is *compatible* with g, in the sense that for all $\sigma_1, \sigma_2 \in \Gamma(E)$ and for all $X \in \Gamma(TB)$,

$$X(g(\sigma_1, \sigma_2)) = g(D_X\sigma_1, \sigma_2) + g(\sigma_1, D_X\sigma_2)$$

Proposition 2.10. Let M be a smooth manifold and g be a Riemannian metric on M. Then there exists a unique connection ∇ on TM that is compatible with g and is torsion-free. This conneciton is called the *Levi-Civita connection* of g.

Proof. Uniqueness: Let ∇ be a conneciton on TM that is torsion-free and compatible with g.

- Torsion-free:

$$\begin{aligned} & T(X,Y) = 0 \text{ for all } X, Y \in \Gamma(TM) \\ \iff & \nabla_X Y - \nabla_Y X - [X,Y] = 0 \text{ for all } X, Y \in \Gamma(TM) \\ \iff & \nabla_X Y = \nabla_Y X + [X,Y] \text{for all } X, Y \in \Gamma(TM). \end{aligned}$$

- Compatibility with g: For all $X, Y, Z \in \Gamma(TM)$:

$$\begin{aligned} X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y(g(Z,X)) &= g(\nabla_Y Z, Z) + g(Z, \nabla_Y X) \\ Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(Z, \nabla_Z Y) \end{aligned}$$

$$X(g(Y,Z)) + Y(g(Z,X)) + Z(g(X,Y)) = 2g(\nabla_X Y, Z) + g(Y, [X,Z]) + g(Z, [Y,Z]) - g(X, [Y,Z])$$

Thus

$$g(\nabla_X Y, Z) = \text{ expression that only involves } g, X, Y, Z\ .$$

Suppose that there exist two connections ∇^1 and ∇^2 that are torsion-free and compatible with g. Then, forall $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} & g\left(\nabla^1_X Y, Z\right) = g\nabla^2_X Y, Z \\ \implies & \\ \implies & \nabla^1 = \nabla^2 \end{aligned}$$

program

as $X, Y, Z \in \Gamma(TM)$ were arbitrary, by the non-degeneracy of g.

Existence: Let $(x_1, \ldots, x_n)$ be local coordinates on M and consider the local frame $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$. Then:

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$$

where $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. (g_{ij}) is invertible at every point and we denote the inverse by (g^{ij}). We set

$$\Gamma^k_{ij} := \frac{1}{2}\sum_l g^{kl}\left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij}\right)$$

This clearly satisfies $\Gamma^k_{ij} = \Gamma^k_{ji}$, so ∇ is torsion-free. It patches well together when changing coordinate frames, and it is compatible with g. $\square$

2.3.3 Subbundles and Orthogonal Complements

Definition 2.15. A *subbundle* of a vector bundle $(E, B, \pi, \mathbb{R}^r)$ is a subset $V \subseteq E$ such that $\pi\big|_V : V \to B$ is the projeciton map of a vector bundle with total space V and base space B and $V_b := \left(\pi\big|_V\right)^{-1}(b)$ is a linear subspace of E_b for all $b \in B$.

Note. Since $\pi\big|_V : V \to B$ is a vector bundle, V_b has the same dimension for each $b \in B$.

Example 2.10. $E = B \times \mathbb{R}^3$, where we consider $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$, with coordinates (x, y) on the first summand and z on the second. Then $V = \{(b, (x, y, 0)) \mid b \in B\}$ is a subbundle of E.

2. If $E = B \times \mathbb{R}^r$ and $\{e_1, \dots, e_r\}$ is any frame of E. Then picking

$$V := \operatorname{span}_{C^\infty(B)} \{e_{i_1}, \dots, e_{i_l}\}$$

is a subbundle of E of rank l, where $\{i_1, \dots, i_l\} \subset \{1, \dots, r\}$ with $i_s \neq i_t$ if $s \neq t$.

3. If E is a vector bundle and $V := \bigsqcup_{b \in B} V_b$ with $V_b \subset E_b$ for all $b \in B$, then V is a subbundle of E of rank l if and only if for every $b \in B$, there exists an open neighbourhood $U \ni b$ and smooth local sections $\{\sigma_1, \dots, \sigma_l\} \subset \Gamma(U, E)$ such that $\{\sigma_1(q), \dots, \sigma_l(q)\} = V_q$ for all $q \in U$.

Proposition 2.11. If V is a subbundle of E, then V is an embedded submanifold of E.

Proof. One can show that the inclusion map $V \hookrightarrow E$ is an embedding. Suppose that $\{e_1, \dots, e_l\}$ is a local frame of V on an open neighbourhood U of some point $b \in B$. then, one can complete $\{e_1, \dots, e_l\}$ to a local frame $\{e_1, \dots, e_r\}$ of E around b as follows: $\{e_1(b), \dots, e_l(b)\}$ is a basis for V_b, which is a linear subspace of E_b. So we can complete it to a basis $\{e_1(b), \dots, e_l(b), \tilde{e}_{l+1}, \dots, \tilde{e}_r\}$. Pick a local chart φ of E on an open neighbourood $\tilde{U}$ of B. Set

$$e_i := \varphi^{-1}\left(-, \overline{\varphi}(\tilde{e}_i)\right)$$

for $i = l + 1, \dots, k$. Thus $\{e_1(b), \dots, e_r b\}$ is linearly independent, so that

$$\det\left(e_1(b) | \dots | e_r(b)\right) \neq 0.$$

By the property of the determinant function, we have that $\det\left(e_1 | \dots | e_r\right) \neq 0$ on a neighbourhood W of b. In terms of this frame, we have the following local charts of V and E:

$$\begin{aligned} \varphi_E : E_{U \cap W} &\to (U \cap W) \times \mathbb{R}^r \\ E_b \ni e = a_1 e_1(b) + \dots a_r e_r(b) &\mapsto (b, (a_1, \dots, a_r)) \end{aligned}$$

and

$$\begin{aligned} \varphi_V : V_{U \cap W} &\to (U \cap W) \times \mathbb{R}^l \\ V_b \ni e = a_1 e_1(b) + \dots + a_l e_l(b) &\mapsto (b, (a_1, \dots, a_l)) \end{aligned}$$

. then

$$(U \cap W) \times \mathbb{R}^l \xrightarrow{\varphi_V^{-1}} V_{U \cap V} \hookrightarrow E_{U \cap V} \xrightarrow{\varphi_E} (U \cap W) \times \mathbb{R}^r$$

with the maps

$$(b, (a_1, \dots, a_l)) \mapsto (a_1 e_1(b) + \dots +) a_l e_l(b) \mapsto a_1 e_1(b) + \dots a_l e_l(b) + 0 \mapsto (b, (a_1, \dots, a_l, 0, \dots, 0))$$

The inclusion map is clearly an embedding. ∎

Remark. A subbundle V of E is an embedded submanifold $V \subset E$ such that $V_b := V \cap E_b \subseteq E_b$ is a linear subspace. Some authors define subbundles this way.

Orthogonal Complements.

Definition 2.16. Let $(E, B, \pi, \mathbb{R}^r)$ be a real vector bundle and V be a subbundle of E. Let g be a Riemannian metric on E. We define the *orthogonal complement* $V^\perp$ of V as

$$V^\perp = \bigsqcup_{b \in B} V_b^\perp$$

where $V_b^\perp = \{e \in E_b \mid g(e, v) = 0 \text{ for all } v \in V_b\}$.

Proposition 2.12. $V^\perp$ is a subbundle of E of rank $r - l$ and

$$E = V \oplus V^\perp.$$

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Let $(E, B, \pi, \mathbb{R}^r)$ be a vector bundle and g a Riemannian metric on E. For any subbundle V of E,

$$V^\perp = \bigsqcup_{b \in B} V_b^\perp \subset E$$

where $V_b^\perp = \{e \in E_b \mid g(e, e') = 0 \text{ for all } e' \in V_b \}$.
Then we have

Proposition 2.13. $V^\perp$ is a subbundle of E such that $E = V \oplus V^\perp$.

Example 2.11. M is a smooth manfiold and $S \subset M$ an embedded submanifold of M. Let g be a Riemannian metric on M (i.e., on TM). We define

$$NS := (TS)^\perp$$

is the *normal bundle of S in M*, which is a subbundle of $TM\big|_S = \bigsqcup_{x \in S} T_x M$. We have $TM\big|_S = TS \oplus NS$. The metric g gives a natural splitting of $TM\big|_S$, with one component given by TS.

3 Characteristic Classes

Characteristic classes measure the extent to which a vector bundle fails to be trivial.

3.1 Stiefel-Whitney Classes

These are defined for real vector bundle and they take values in $\check{H}(B; \mathbb{Z}_2)$.
Recall.

- $\mathbb{Z}_2$ is thought of as an additive group
- $\{U_\alpha\}$ open cover of B such that ${}_\alpha \cap U_\beta$ is contractible whenever $U_\alpha \cap U_\beta \neq \varnothing$.

Define $C^k(B; \mathbb{Z}_2) = \{\{f_{\alpha_0 \ldots \alpha_k}\}$ with $f_{\alpha_0 \ldots \alpha_k} \in \mathbb{Z}_2$, $\forall \alpha_0, \ldots, \alpha_k$ such that $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} \neq \varnothing\}$. These are the *k-cochains*. We have the following map on cochains, called the coboundary map.

$$\begin{aligned} \delta : C^k(B; \mathbb{Z}_2) &\to C^{k+1}(B; \mathbb{Z}_2) \\ \sigma = \{f_{\alpha_0 \ldots \alpha_k}\} &\mapsto \delta\sigma \end{aligned}$$

where

$$(\delta\sigma)_{\beta_0 \ldots \beta_{k+1}} := \sum_{j=0}^{k+1} (-1)^j f_{\hat{\beta}_0 \ldots \hat{\beta}_j \ldots \beta_{k+1}}.$$

Here, the hat notation on the index means that we are removing the index β_j. Since each term in the above sum is in $\mathbb{Z}_2$, we have that $\delta\sigma = \sum_{j=0}^{k+1} f_{\hat{\beta}_0 \ldots \hat{\beta}_j \ldots \beta_{k+1}}$.

Note.

1. $\delta \circ \delta = 0$. We then get a complex
$$C^0(B; \mathbb{Z}_2) \xrightarrow{\delta} C^1(B; \mathbb{Z}_2) \xrightarrow{\delta} \ldots$$
2. One can add 2 k-cochains component-wise. Furthermore, $\delta(\sigma + \sigma') = \delta(\sigma) + \delta(\sigma')$ for any k-cochains σ, σ'.

Definition 3.1. $\mathrm{Z}^k(B; \mathbb{Z}_2) = \left\{\sigma \in C^k(B; \mathbb{Z}_2) \mid \delta\sigma = 0\right\} is the set of k-cocycles for \mathrm{k} \geqslant 0$.

$$B^k(B; \mathbb{Z}_2) = \begin{cases} \{0\} & \text{if } k = 0 \\ \left\{\sigma \in C^k(B; \mathbb{Z}_2) \mid \sigma = \delta\tau \text{ for some } \tau \in C^{k-1}(B; \mathbb{Z}_2)\right\} & \text{if } k > 0 \end{cases}$$

is the set of *k-coboundaries*.

Note.

1. For all $\sigma \in B^k(B; \mathbb{Z}_2)$, $k \geqslant 1$, $\sigma = \delta\tau$ for some $\tau \in C^{k-1}(B; \mathbb{Z}_2)$. Hence $\delta\sigma = \delta^2\tau$, and hence $B^k(B, \mathbb{Z}_2) \subset Z^k(B; \mathbb{Z}_2)$ for all $k \geqslant 1$.
2. $Z^k(B; \mathbb{Z}_2)$ and $B^k(B; \mathbb{Z}_2)$ are closed under addition.

3. $Z^0(B;\mathbb{Z}_2) = ?$ Let $\sigma \in Z^0(B;\mathbb{Z}_2)$ so that $\sigma = \{f_\alpha\}$. So

$$\begin{aligned}\delta\sigma = 0 &\iff (\delta\sigma)_{\alpha\beta} = f_\beta + f_\alpha = 0\\ &\iff f_\alpha = f_\beta \text{ for all } \alpha, \beta.\end{aligned}$$

Thus, to each connected component of B we associate a unique element in $\mathbb{Z}_2$. So

$$Z^0(B;\mathbb{Z}_2) = \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{\#\text{ of connected components of } B.}$$

Definition 3.2. $\check{H}^k(B;\mathbb{Z}_2) = Z^k(B;\mathbb{Z}_2)/B^k(B;\mathbb{Z}_2)$, for all $k \geqslant 0$, is the *kth Čech cohomology group with coefficients in* $\mathbb{Z}_2$.

Note.

1. $\check{H}^k(B;\mathbb{Z}_2) = Z^0(B;\mathbb{Z}_2) = \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{\#\text{ of connected components of } B.}$
2. If $B = \{\text{pt}\}$, then $\check{H}^0(B;\mathbb{Z}_2) = \mathbb{Z}_2$ and $\check{H}^k(B;\mathbb{Z}_2) = 0$ for $k > 0$.
3. $\check{H}^k(B;\mathbb{Z}_2)$ is a group under addition, where $+$ is defined as follows: $[\sigma], [\sigma'] \in \check{H}^k(B;\mathbb{Z}_2)$. Set

$$[\sigma] + [\sigma'] = [\sigma + \sigma'].$$

This is independent of the representative. Indeed, suppose $[\sigma] = [w]$ and $[\sigma'] = [w']$. Then $\sigma = w + \delta\tau$ and $\sigma' = w' + \delta\tau'$ for some τ, τ' in $B^k(B;\mathbb{Z}_2)$. Then $\sigma + \sigma' = w + w' + \delta(\tau + \tau')$. Hence $[\sigma + \sigma'] = [w + w']$.

Definition 3.3. Let $f : N \to B$ be a smooth map (where here, N is a smooth manifold). Then, $\left\{\tilde{U}_\alpha = f^{-1}(U_\alpha)\right\}$ is an open cover of f such that $\tilde{U}_\alpha \cap \tilde{U}_\beta$ is a disjoint union of contractibles sets when it is nonempty. For every $\sigma \in C^k(B;\mathbb{Z}_1)$, we define $f^*\sigma \in C^k(N;\mathbb{Z}_2)$ by

$$(f^*\sigma)_{\alpha_0\ldots\alpha_k} := \sigma_{\alpha_0\ldots\alpha_k}$$

for all $\alpha_0, \ldots, \alpha_k$ such that $\tilde{U}_{\alpha_0} \cap \tilde{U}_{\alpha_k} \neq$.

Note that $f^*(\sigma + \sigma') = f^*\sigma + f^*\sigma'$ and $f^*(\delta\sigma) = \delta(f^*\sigma)$ for all $\sigma, \sigma' \in C^k(B;\mathbb{Z}_2)$. Therefore,

$$f^*[\sigma] := [f^*\sigma]$$

is well-defined, giving us a map

$$\begin{aligned}f^* : \check{H}^k(B;\mathbb{Z}_2) &\to \check{H}^k(N;\mathbb{Z}_2)\\ [\sigma] &\mapsto [f^*\sigma]\end{aligned}$$

such that $f^*([\sigma] + [\sigma']) = f^*[\sigma] + f^*[\sigma']$, i.e., f^* is a homomorphism. If $f = \mathrm{id}_B$, then $f^*[\sigma] = [\sigma]$ for all $\sigma \in \check{H}^k(B;\mathbb{Z}_2)$.

Let $(E, B, \pi, \mathbb{R}^r)$ be a real vector bundle over B. Then there exist unique cohomology classes in $i(\check{B};\mathbb{Z}_2)$ satisfying the following four axioms:

Axiom 1: To each vector bundle E, there corresponds a sequence of cohomology classes

$$w_i(E) \in \check{H}^k(B;\mathbb{Z}_2)$$

called the *Stiefel-Whitney classes of E* such that $w_0(E) = 1 \in \check{H}^0(B;\mathbb{Z}_2)$ and $w_i(E) = 0$ for all $i > r = \operatorname{rank}(E)$.

Axiom 2: (Naturality). If $f : N \to B$ is a smooth map then

$$f^* w_i(E) = w_i(f^*E)$$

for every i.

Axiom 3: (The Whitney product Theorem). For any vector bundles E, E' on B,

$$w_i(E \oplus E') = \sum_{l+k=i} w_l(E) w_k(E')$$

Axiom 4: (Normalization). If γ_1^1 is the tautological line bundle on $\mathbb{P}^1$, then $w_1(\gamma_1^1) \neq 0$.

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Note.

1. $\check{H}^0(B;\mathbb{Z}_2) = \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{\#\text{ of connected components of } B}$
2. If $B = \{\text{pt}\}$ then

$$\check{H}^k(B;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

Let $(E, B, \pi, \mathbb{R}^r)$ be a real vector bundle. There exist unique cohomology classes in $\check{H}^i(B;\mathbb{Z}_2)$ that satisfy the following axioms:

Axiom 1: To each vector bundle E, there corresponds a sequence of cohomology classes $w_i(E) \in \check{H}^i(B;\mathbb{Z}_2)$ called the *Steifel-Whitney classes* of E. Also, $w_0(E) = 1 \in \check{H}^0(B;\mathbb{Z}_2)$ and $w_i(E) = 0$ for all $i > r = \text{rank}(E)$. Finally, we call $w_i(E)$ the i^{th} *Steifel-Whitney class of E*.

Axiom 2: (Naturality). If $f : N \to B$ is a smooth map with N a smooth manifold, then

$$w_i(f^*E) = f^*w_i(E)$$

for every i. In fact, if $(V, N, p, \mathbb{R}^l)$ is a real vector bundle and $F : V \to E$ is a vector bundle isomorphism covering the map $f : N \to B$, then

$$w_i(V) = f^*(w_i(E))$$

for all i.

Axiom 3: (The Whitney Product Theorem). If E and E' are vector bundles on B, then

$$w_i(E \oplus E') = \sum_{k+l=i} w_k(E)w_l(E').$$

In particular,

$$\begin{aligned} w_1(E \oplus E') &= w_0(E)w_1(E') + w_1(E)w_0(E') \\ &= w_1(E) + w_1(E') \\ w_2(E \oplus E') &= w_2(E) + w_1(E)w_1(E') + w_2(E'), \end{aligned}$$

etc.

Axiom 4: (Normalization). For the tautological line bundle $\gamma^1_{\mathbb{P}^1}$ over $\mathbb{P}^1$, we have $w_1(\gamma^1_{\mathbb{P}^1}) = 1$.

Note. Axiom 4 ensures that there exist vector bundles with non-zero i^{th} Steifel-Whitney classes for $i > 0$.

How does one prove that such classes exist? One shows that if the exist, they are unique, and then one constructs a set of cohomology classes that satisfy Axioms $1 - - - 4$.

3.2 Consequences of the 4 axioms

Proposition 3.1. If E, E' are vector bundles on B that are isomorphic, then

$$w_i(E) = w_i(E')$$

for all i.

Proof. From Axiom 2. There exists an isomorphism $F : E \to E'$ covering $\text{id}_B : B \to B$. Hence $w_i(E) = \text{id}_B^*(w_i(E')) = w_i(E')$ for all i. □

Proposition 3.2. Let $E = \underline{\mathbb{R}}^r = B \times \mathbb{R}^r$ be the trivial bundle of rank r over B. Then $w_i(\underline{\mathbb{R}}^r) = 0$ for all $i > 0$.

Proof. Let $f : B \to \{pt\}$ be the constant map and consider the trivial bundle $V = \{pt\} \times \mathbb{R}^r$. Then $\underline{\mathbb{R}}^r = f^*(V)$. By Axiom 2,

$$w_i(\underline{\mathbb{R}}^r) = f^*(w_i(V))$$

with $w_i(V) \in \check{H}^i(\{pt\}, \mathbb{Z}_2) = 0$ for all i. So $w_i(\underline{\mathbb{R}}^r) = 0$ for all i. □

Corollary 3.1. If $w_i(E) \neq 0$ for some $i > 0$, then E is not trivial.

Proposition 3.3. $w_i\left(\underline{\mathbb{R}}^s \oplus E\right) = w_i(E)$ for all i.

Proof. By Axiom 3, we have

$$\begin{aligned} w_i((\underline{\mathbb{R}}^s \oplus E)) &= \sum_{k+l=i} w_k(\underline{\mathbb{R}}^s) w_l(E) \\ &= \underbrace{w_0(\underline{\mathbb{R}}^s)}_{=1} w_i(E) + 0 \\ &= w_i(E). \end{aligned}$$

□

Proposition 3.4. Suppose that E has rank r and possesses a nowhere-vanishing section. Then $w_r(E) = 0$. More generally, if E has l linearly-independent sections, then $w_{r-l+1}(E) = \cdots = w_r(E) = 0$.

Proof. Since E has l linearly independent sections, they span a subbundle V of E of rank l that is trivial. Write $E = V \oplus V^\perp \cong \underline{\mathbb{R}}^l \oplus V^\perp$, by picking any Riemannian metric on E. Then by Proposition 3, $w_i(E) = w_i(V^\perp) = 0$ for all $i > r - l$. □

Note. The converse is not true: One can show that $w_i(TS^n) = 0$ for all $i > 0$ and $n \in \mathbb{N}$. But, when n is even, there is not even *one* nowhere-vanishing sections by the Hairy-Ball Theorem.

3.3 Orientability and the First Steifel-Whitney Class

Definition 3.4. E is orientable is *orientable* if and only if there exists a vector bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ such that $\det\left(\overline{g}_{\alpha\beta}\right) > 0$ for all α, β.

Example 3.1.

1. $E = B \times \mathbb{R}^r$ is orientable.
2. If M is a smooth manifold, then TM is orientable $\iff$ M is orientable.
3. Recall that a manifold M is orientable iff $\bigwedge^n T^*M$ is trivial. One has similarly that:

Proposition 3.5. E is orientable if and only if $\bigwedge^r E$ is trivial, where $r = \operatorname{rank}(E)$. In particular, a line bundle is orintable if and only if it is trivial.

Note. Not all vector bundles as orientable. For example, $\gamma^1_{\mathbb{P}^1}$ is not orientable (because it is non-trivial).

How can one define the first Steifel-Whitney class of a vector bundle E? Let $\{(U_\alpha, \varphi_\alpha)\}$ be a vector bundle atlas of E such that $U_\alpha \cap U_\beta$ are empty or contractible. We define $f_{\alpha\beta} = \operatorname{sgn}(\det(\overline{g}_{\alpha\beta})) \in \{\pm 1\}$. Set $w^1_{\alpha\beta} \in \mathbb{Z}_2$ to be the number such that $f_{\alpha\beta} = (-1)^{w^1_\beta}$. Then $\sigma = \left\{w^1_{\alpha\beta}\right\} \in C^1(B; \mathbb{Z}_2)$. Also,

$$(\delta\sigma)_{\alpha\beta\gamma} = w^1_{\beta\gamma} + w^1_{\alpha\gamma} + w^1_{\beta\gamma} = 0$$

because $\overline{g}_{\alpha\beta}\overline{g}_{\beta\gamma}\overline{g}_{\gamma\alpha} = \mathrm{id}$. Thus, $\det(\overline{g}_{\alpha\beta}\overline{g}_{\beta\gamma}\overline{g}_{\gamma\alpha}) = 1$. So $f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1$. Thus, $(-1)^{w^1_{\alpha\beta} + w^1_{\beta\gamma} + w^1_{\gamma\alpha}} = 1$.