

A. Robust Quickest Change Detection

In many real world cases, the linear relationship of the system, which is depicted as $\mathbf{H} \in \mathbb{R}^{M \times N}$ a.k.a system matrix in this paper, can only be estimated. Therefore, the system matrix can be inaccurate, uncertain, or time-varying (Yang et al., 2014). In this section, by exploiting the robust optimization technique, we investigate the robust quickest change detection problem in the presence of uncertainties in \mathbf{H} . We next show that such a problem can also be cast as an MIQP problem. And therefore, both proposed detectors can be applied.

Before taking into account the uncertainties in \mathbf{H} , we first derive an equivalent form of our MIQP problem (16). In (16) the constraint $(\mathbf{I} - \mathbf{P}_{\mathbf{H}}^{\perp})(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}) = \mathbf{0}$ is equivalent to $\mathbf{H}^T(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}) = \mathbf{0}$, since they are essentially ensuring that $(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)})$ lie in the complementary space of $\mathcal{R}(\mathbf{H})$. And based on this constraint and the definition of $\tilde{\mathbf{x}}^{(t)}$ in (32), the item $(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)})^T \tilde{\mathbf{x}}^{(t)}$ in the objective function is equivalent to $(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)})^T \mathbf{x}^{(t)}$. Thus, an equivalent formulation of (16) can be written as follows:

$$\begin{aligned} \min \quad & \frac{1}{2\sigma^2} \left\{ \left\| (\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}) \right\|_2^2 - 2(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)})^T \mathbf{x}^{(t)} \right\} \\ \text{s.t.} \quad & \begin{cases} \mathbf{H}^T(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}) = \mathbf{0} \\ \rho_L \mathbf{u}^{(t)} \leq \boldsymbol{\mu}_+^{(t)} + \boldsymbol{\mu}_-^{(t)} \leq \rho_U \mathbf{u}^{(t)} \\ \boldsymbol{\mu}_+^{(t)} \geq \mathbf{0} \\ \boldsymbol{\mu}_-^{(t)} \geq \mathbf{0} \\ \boldsymbol{\mu}_+^{(t)} + \rho_U \mathbf{b}^{(t)} \leq \rho_U \mathbf{1} \\ \boldsymbol{\mu}_-^{(t)} - \rho_U \mathbf{b}^{(t)} \leq \mathbf{0} \end{cases} \\ \text{var : } & \mathbf{u}^{(t)} \in \{0, 1\}^M, \mathbf{b}^{(t)} \in \{0, 1\}^M, \boldsymbol{\mu}_+^{(t)}, \boldsymbol{\mu}_-^{(t)}. \end{aligned} \quad (\text{A.1})$$

We assume that the uncertainty set of \mathbf{H} is constraint-wise. Specifically, the uncertainties between different rows of \mathbf{H}^T are decoupled. This assumption is commonly used and can be applied to many practical problems (Yang et al., 2014) (Bertsimas & Sim, 2004). Let \mathbf{h}_i denote the i -th column of \mathbf{H} . And each \mathbf{h}_i belongs to an uncertainty set, i.e. $\mathbf{h}_i \in \mathcal{U}_i$. By simply relaxing the constraint $\mathbf{H}^T(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}) = \mathbf{0}$, we get the new constraints as follows:

$$-\varepsilon_i \leq \mathbf{h}_i^T (\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}) \leq \varepsilon_i, \quad \forall \mathbf{h}_i \in \mathcal{U}_i, \quad i = 1, \dots, N, \quad (\text{A.2})$$

where ε_i represents a small positive constant. And the above constraints can be equivalently transformed into several linear constraints, not only when \mathcal{U}_i represents some simple uncertain sets, including interval-based uncertainty set (Duchi, 2018), but also some very general uncertain sets (Yang et al., 2014), including polyhedron uncertainty set and D-norm uncertainty set. We take the general polyhedron uncertainty set as an example, i.e. $\mathcal{U}_i = \{\mathbf{h}_i \mid \mathbf{D}_i \mathbf{h}_i \leq \mathbf{d}_i\}$, $i = 1, \dots, N$. According to (Ben-Tal et al., 2009), by exploiting the robust optimization techniques, the constraints A.2 under the polyhedron uncertainty set can be equivalently reformulated as

$$\begin{aligned} \mathbf{p}_i^T \mathbf{d}_i &\leq \varepsilon_i, \quad \mathbf{D}_i^T \mathbf{p}_i = \boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}, \quad \mathbf{p}_i \geq \mathbf{0}, \quad i = 1, \dots, N, \\ \mathbf{n}_i^T \mathbf{d}_i &\leq \varepsilon_i, \quad -\mathbf{D}_i^T \mathbf{n}_i = \boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}, \quad \mathbf{n}_i \geq \mathbf{0}, \quad i = 1, \dots, N. \end{aligned} \quad (\text{A.3})$$

Thus, our MIQP problem where every column of \mathbf{H} belongs to the polyhedral uncertainty set is as follows:

$$\begin{aligned}
& \min \frac{1}{2\sigma^2} \left\{ \left\| (\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}) \right\|_2^2 - 2(\boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)})^T \mathbf{x}^{(t)} \right\} \\
& \text{s.t.} \quad \begin{cases} \mathbf{p}_i^T \mathbf{d}_i \leq \varepsilon_i, & \mathbf{D}_i^T \mathbf{p}_i = \boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}, \quad \mathbf{p}_i \geq 0, \quad i = 1, \dots, N \\ \mathbf{n}_i^T \mathbf{d}_i \leq \varepsilon_i, & -\mathbf{D}_i^T \mathbf{n}_i = \boldsymbol{\mu}_+^{(t)} - \boldsymbol{\mu}_-^{(t)}, \quad \mathbf{n}_i \geq 0, \quad i = 1, \dots, N \\ \rho_L \mathbf{u}^{(t)} \leq \boldsymbol{\mu}_+^{(t)} + \boldsymbol{\mu}_-^{(t)} \leq \rho_U \mathbf{u}^{(t)} \\ \boldsymbol{\mu}_+^{(t)} \geq \mathbf{0} \\ \boldsymbol{\mu}_-^{(t)} \geq \mathbf{0} \\ \boldsymbol{\mu}_+^{(t)} + \rho_U \mathbf{b}^{(t)} \leq \rho_U \mathbf{1} \\ \boldsymbol{\mu}_-^{(t)} - \rho_U \mathbf{b}^{(t)} \leq \mathbf{0} \end{cases} \quad (\text{A.4}) \\
& \text{var} : \mathbf{u}^{(t)} \in \{0, 1\}^M, \mathbf{b}^{(t)} \in \{0, 1\}^M, \boldsymbol{\mu}_+^{(t)}, \boldsymbol{\mu}_-^{(t)}, \{\mathbf{p}_i\}, \{\mathbf{n}_i\}.
\end{aligned}$$

And the linear constraints introduced by the uncertainty of \mathbf{H} do not change the nature of the problem, so that both our detectors can still be used. For convenient subsequent comparisons, we rename the SDPCUSUM and BBCUSUM considering the uncertainty of \mathbf{H} as RSDPCUSUM and RBBCUSUM, respectively.

B. Numerical Results

We compare the performance of different detectors on **Data I**, and the results show that RSDPCUSUM and RBBCUSUM achieve better performance when the matrix \mathbf{H} is inaccurate.

We assume the \mathbf{H} with the value given in (24) models the true linear relationship and is denoted as \mathbf{H}_r . The inaccurate estimate of \mathbf{H} is denoted as \mathbf{H}_e , and its value in our experiment is set to be

$$\mathbf{H} = \begin{bmatrix} 1.1 & -0.1 \\ 0.1 & 1.3 \\ -1.2 & 0.9 \\ 0.6 & 0.5 \end{bmatrix}. \quad (\text{B-1})$$

And we use the polyhedron uncertainty set to describe the uncertainty of \mathbf{H} , i.e.,

$$\mathbf{D}_i \mathbf{h}_i \leq \mathbf{d}_i, \quad i = 1, 2,$$

and

$$\mathbf{D}_1 = \mathbf{D}_2 = \begin{bmatrix} \mathbf{I} \\ -\mathbf{I} \end{bmatrix},$$

where \mathbf{I} represents the 4×4 identity matrix. \mathbf{d}_1 and \mathbf{d}_2 are set to be

$$\begin{aligned}
\mathbf{d}_1 &= [1.5, 0.5, -0.5, 1.5, -0.5, 0.5, 1.5, -0.5]^T, \\
\mathbf{d}_2 &= [0.5, 1.5, 1.5, 1.5, 0.5, -0.5, -0.5, -0.5]^T.
\end{aligned}$$

The experimental results are shown in TABLE 3. It is seen from TABLE 3 that under different (ρ_L, ρ_H) and different σ^2 of noise, the average detection delay of RSDPCUSUM and RBBCUSUM are relative smaller, which implies that RSDPCUSUM and RBBCUSUM achieve better performance in the presence of inaccurate \mathbf{H} . And it's worth mentioning that BBCUSUM and RGCUSUM show the almost same performance on our data when \mathbf{H} is inaccurate.

C. Performance Analysis

We have provided a guaranteed sub-optimality gap for SDPCUSUM. And it turns out that we can derive a similar sub-optimality gap for RSDPCUSUM.

Table 3. Performance Comparison on Simulated Data with Uncertain \mathbf{H}

	RGCUSUM AVERAGE DETECTION DELAY	BBCUSUM AVERAGE DETECTION DELAY	RSDPCUSUM AVERAGE DETECTION DELAY	RBBCUSUM AVERAGE DETECTION DELAY
$\rho_L = 0.2, \rho_H = 1.5$ $\sigma^2 = 0.5$ FALSE ALARM PERIOD = 500.0	389.5	389.1	357.2	343.4
$\rho_L = 0.1, \rho_H = 3.0$ $\sigma^2 = 2.0$ FALSE ALARM PERIOD = 500.0	439.2	439.0	409.2	404.0

Theorem C.1. Let \mathbf{s}_M denote the optimal solution to problem A.1 and \mathbf{s}_P denote the suboptimal solution generated by the RSDPCUSUM. Let the function $g : \mathbb{R}^M \rightarrow \mathbb{R}$ denote the objective function of problem A.1. The objective function value of the suboptimal solution \mathbf{s}_P can be bounded as follows :

$$g(\mathbf{s}_M) \leq g(\mathbf{s}_P) \leq g(\mathbf{s}_M) + \frac{2\sqrt{M}\rho_H}{\sigma^2} \|\mathbf{x}\|_2, \quad (\text{C-1})$$

where σ is the variance of noise, M is the dimension of the observation vector, and ρ_H is the upper bound of the detectable part of the injected anomaly.

The proof is similar to that of **Theorem 1**.

References

- Ben-Tal, A., El Ghaoui, L., and Nemirovski, A. *Robust optimization*, volume 28. Princeton university press, 2009.
- Bertsimas, D. and Sim, M. The price of robustness. *Operations research*, 52(1):35–53, 2004.
- Duchi, J. Optimization with uncertain data. *Notes*, 2018, 2018.
- Yang, K., Huang, J., Wu, Y., Wang, X., and Chiang, M. Distributed robust optimization (dro), part i: Framework and example. *Optimization and Engineering*, 15:35–67, 2014.