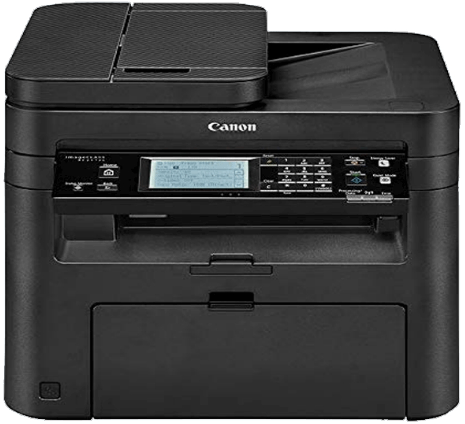
Ch5: No-Cloning, Teleportation and Superdense Coding

Theorem statement

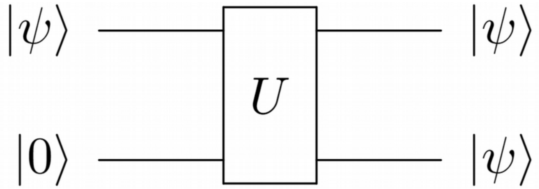
In terms of computer architecture, it's convenient to copy the state of a bit in order to create a new bit exactly the same.



The protocol that seems really simple in the classical scenario is nonetheless proven impossible to perform quantumly. In physics, the no-cloning theorem states that it is impossible to create an identical copy of an arbitrary unknown quantum state.

The cloning problem

Let's formalize the problem. You are given a qubit that can be in any state $|\psi\rangle$, and you want to duplicate the qubit's state through a transformation that acts on your qubit and a second qubit initialized in $|0\rangle$.



We want a unitary transformation U as a map $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$:

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \xrightarrow{U} (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) \quad \forall \alpha, \beta \in \mathbb{C}$$

Proof of the theorem

Second qubit replicating the first qubit means $|00\rangle \xrightarrow{U} |00\rangle$ and $|10\rangle \xrightarrow{U} |11\rangle$ which leads to the transformation of the left-hand side:

$$\begin{aligned} \alpha|00\rangle + \beta|10\rangle \quad &\xrightarrow{U} \quad \alpha \cdot U|00\rangle + \beta \cdot U|10\rangle \\ &= \qquad \alpha|00\rangle + \beta|11\rangle \end{aligned}$$

On the other hand, the expanded right-hand side is:

$$(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

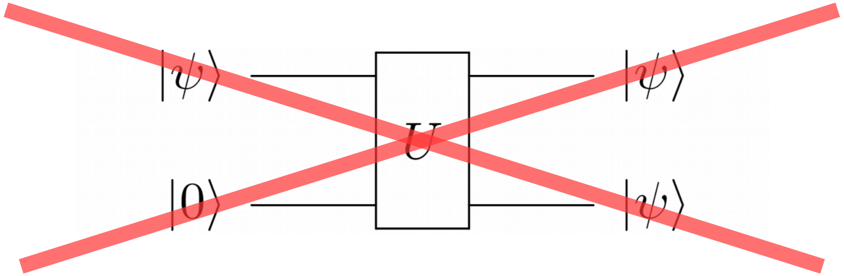
So we have a contradiction as

$$\alpha|00\rangle + \beta|11\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

only when $\alpha = 1$ or $\beta = 1$.

Proof of the theorem

Thus it's impossible to clone a qubit state unless the qubit is in the classical state 0 or 1.



The no-cloning theorem introduces a fundamental barrier in quantum information processing as there isn't a universal machine that handles the cloning problem.

No-cloning and entanglement

The state of one system can be entangled with the state of another system without violation of the theorem as this is not cloning.

Cloning is a process, the result of which is a separable state with identical factors. No well-defined state can be attributed to a subsystem of an entangled state.

Consequences

Imagining that you are free to clone an arbitrary qubit with no idea about its state, i.e. there exists a cloning unitary, you could overcome the post-measurement collapse of a qubit by performing as many experiments as you desire on the clones to figure out the true state of that qubit.

As we will see later, the no-cloning theorem is a corner stone of quantum cryptography protocols.

The no-cloning theorem also prevents the use of classical error correction techniques on quantum states. For example, backup copies of a state in the middle of a quantum computation cannot be created and used for correcting subsequent errors.

Imperfect cloning

Even though it is impossible to make perfect copies of an unknown quantum state, it is possible to produce imperfect copies.

This can be done by coupling a larger auxiliary system to the system that is to be cloned, and applying a unitary transformation to the combined system. If the unitary transformation is chosen correctly, several components of the combined system will evolve into approximate copies of the original system.

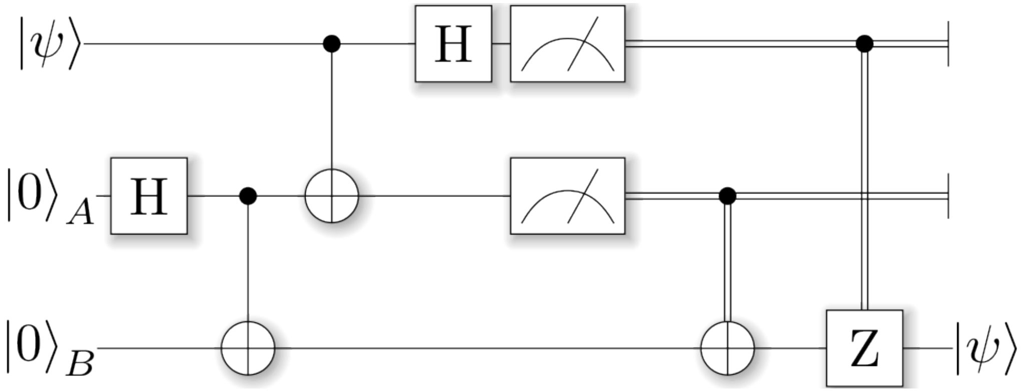
In 1996, V. Buzek and M. Hillery showed that a universal cloning machine can make a clone of an unknown state with the surprisingly high fidelity of 5/6.

Basic idea

Quantum teleportation is a process in which quantum information can be transmitted from one location to another, with the help of classical communication and previously shared quantum entanglement between the sending and receiving location.

Teleporting an unknown qubit state

Below is an example quantum circuit for teleporting a qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:



Let's see what's going on here step by step...

Teleporting an unknown qubit state

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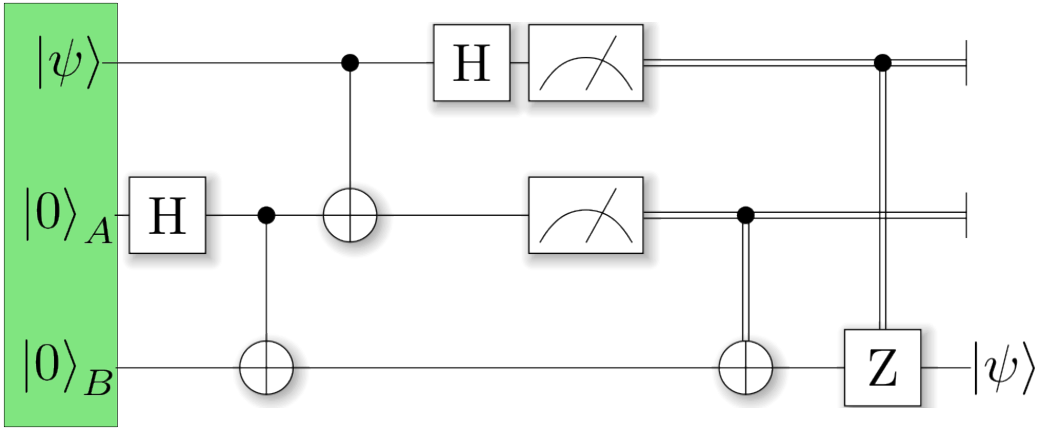

$$|\Psi_0\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle$$

Teleporting an unknown qubit state

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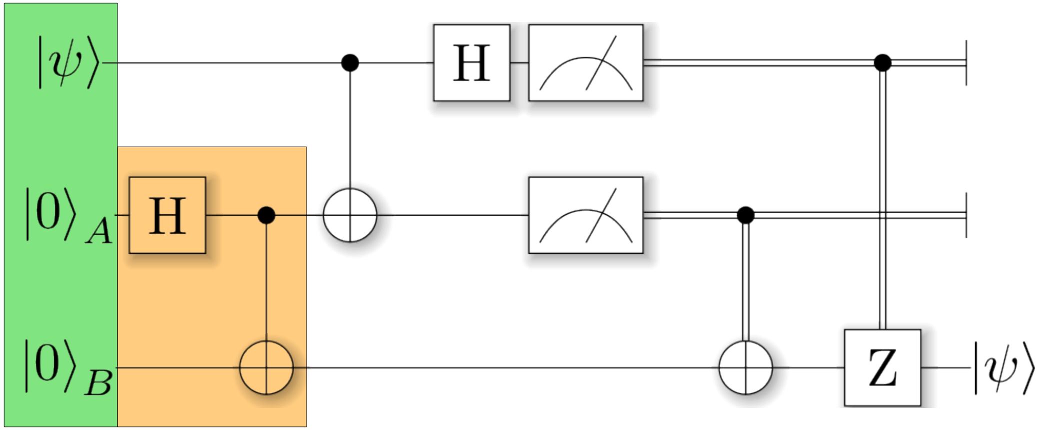
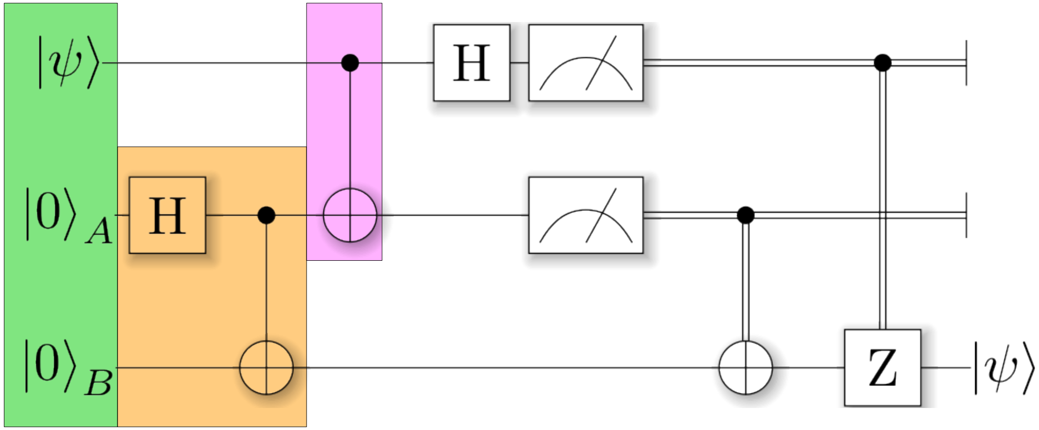


$$|\Psi_0\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle$$

$$\begin{aligned}|\Psi_1\rangle &= (\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \\ &= \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle\end{aligned}$$

Teleporting an unknown qubit state

Below is an example quantum circuit for teleporting a qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

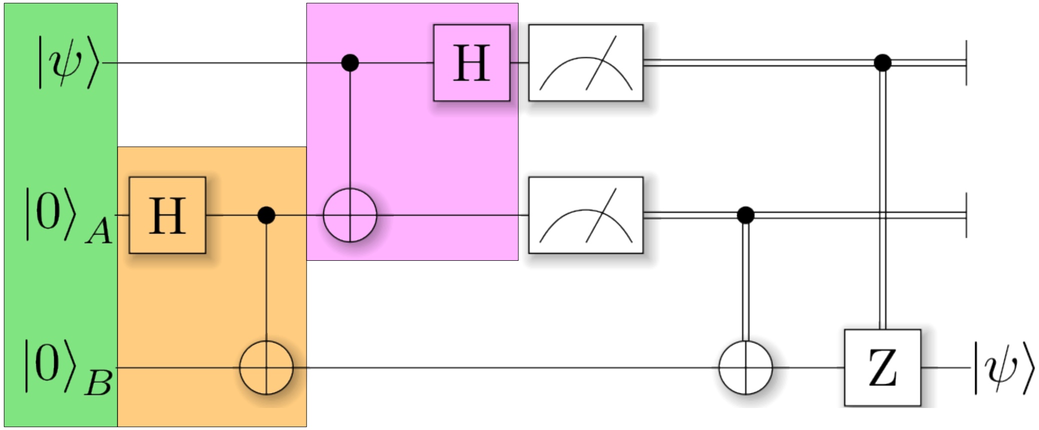


$$
\begin{aligned}
|\Psi_1\rangle &= (\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \\
&= \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle
\end{aligned}
$$

$$
|\Psi_{1.5}\rangle = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|110\rangle + \frac{\beta}{\sqrt{2}}|101\rangle
$$

Teleporting an unknown qubit state

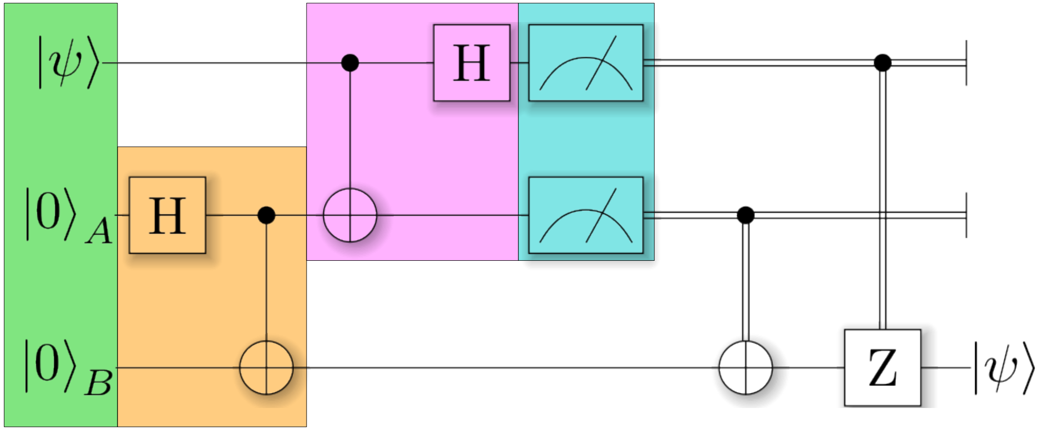
Below is an example quantum circuit for teleporting a qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:



$$\begin{aligned}
|\Psi_2\rangle &= \frac{\alpha}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)(|00\rangle + |11\rangle) + \frac{\beta}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)(|10\rangle + |01\rangle) \\
&= \frac{\alpha}{2}(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \frac{\beta}{2}(|010\rangle + |001\rangle - |110\rangle - |101\rangle) \\
&= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)
\end{aligned}$$

Teleporting an unknown qubit state

Below is an example quantum circuit for teleporting a qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:



$b_0 b_1$	BOTTOM QUBIT
00	$\alpha|0\rangle + \beta|1\rangle$
01	$\alpha|1\rangle + \beta|0\rangle$
10	$\alpha|0\rangle - \beta|1\rangle$
11	$\alpha|1\rangle - \beta|0\rangle$

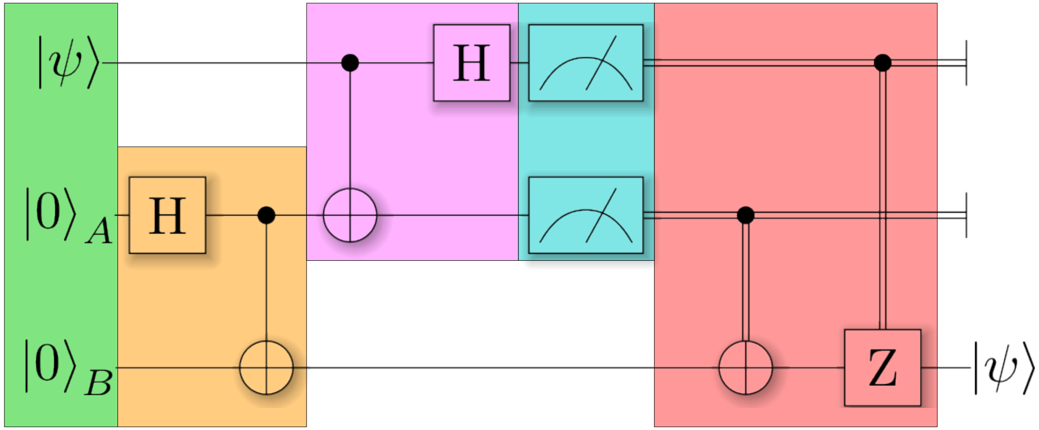
$$|\Psi_2\rangle = \frac{\alpha}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)(|00\rangle + |11\rangle) + \frac{\beta}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right.$$

$$= \frac{\alpha}{2}(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \frac{\beta}{2}(|010\rangle + |001\rangle - |110\rangle - |101\rangle)$$

$$= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$$

Teleporting an unknown qubit state

Below is an example quantum circuit for teleporting a qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:



$b_0 b_1$	BOTTOM QUBIT
00	$\alpha\|0\rangle + \beta\|1\rangle \xrightarrow{I} \alpha\|0\rangle + \beta\|1\rangle$
01	$\alpha\|1\rangle + \beta\|0\rangle \xrightarrow{X} \alpha\|0\rangle + \beta\|1\rangle$
10	$\alpha\|0\rangle - \beta\|1\rangle \xrightarrow{Z} \alpha\|0\rangle + \beta\|1\rangle$
11	$\alpha\|1\rangle - \beta\|0\rangle \xrightarrow{ZX} -\alpha\|0\rangle - \beta\|1\rangle$

$$
\begin{aligned}
|\Psi_2\rangle &= \frac{\alpha}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)(|00\rangle + |11\rangle) + \frac{\beta}{\sqrt{\ }} \\
&= \frac{\alpha}{2}(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \frac{\beta}{2}(\ldots - |110\rangle - |101\rangle) \\
&= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)
\end{aligned}
$$

Teleportation and no-cloning

No, teleportation doesn't violate the no-cloning theorem discussed in 11.1 as we perform a measurement on the top register in order to transfer information at a distance.

Teleportation thus doesn't mean duplicating a qubit or knowning its state, but just bringing it to another location.

Teleportation and relativity

There is no faster than light communication in the teleportation protocol because it is restricted by the speed of light as it uses an ordinary classical channel as a necessary requirement to transmit measurement results from the measurement station to the down register.

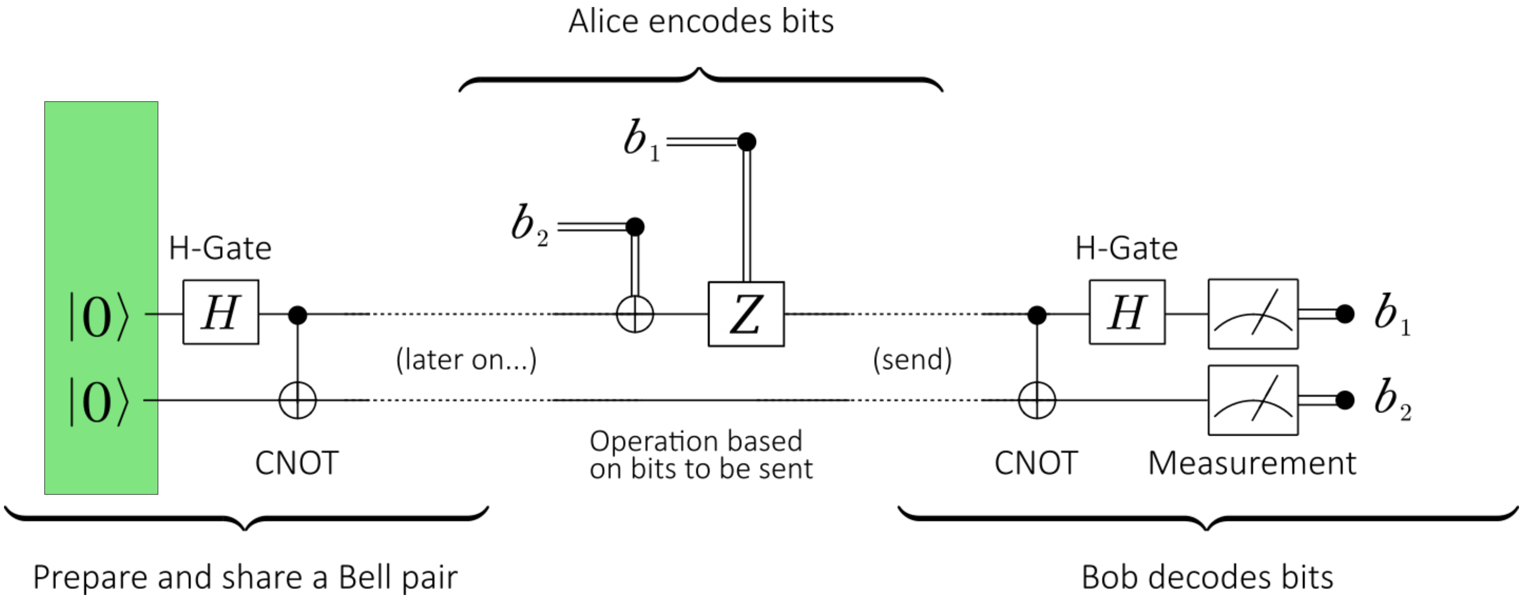
Basic idea

Superdense coding protocol can be thought of as the opposite of quantum teleportation.

Superdense coding is a quantum communication protocol using quantum entanglement to increase the rate at which information may be sent through a quantum channel.

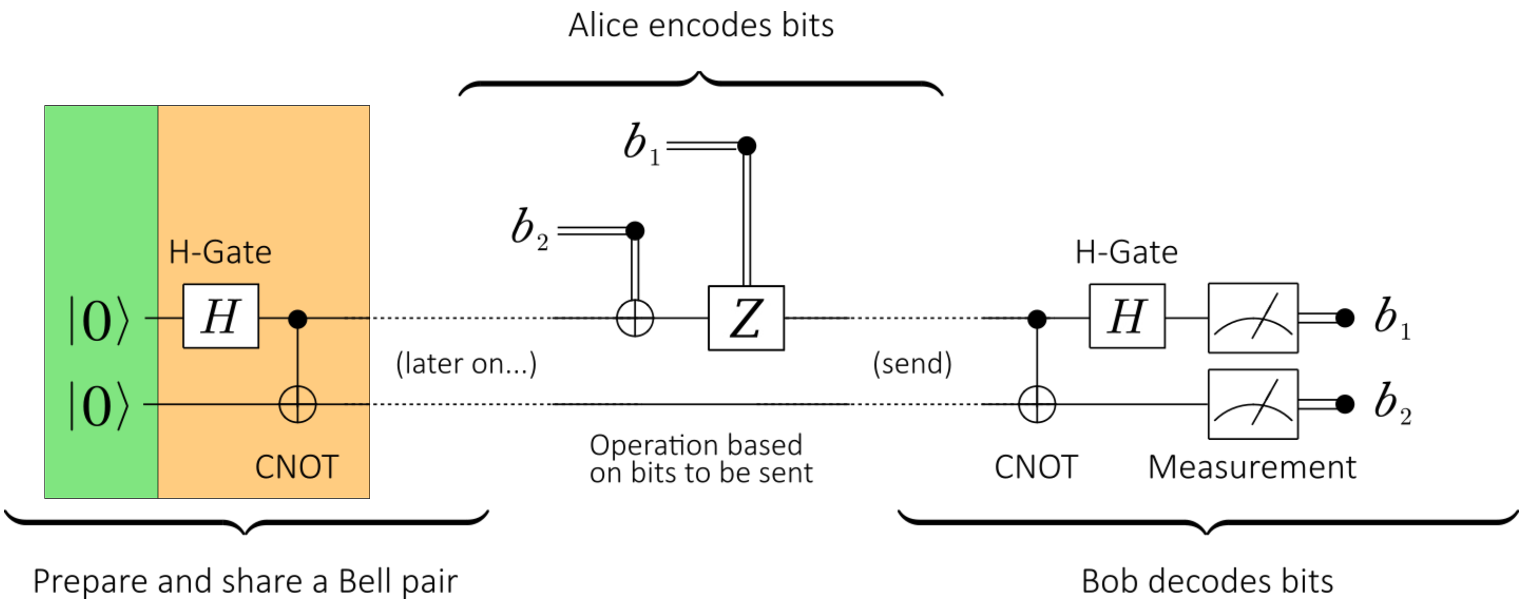
For maximally entangled states dense coding increases the maximum rate of transmitted information to two bits per qubit.

Quantum circuit representation



$$|\Psi_0\rangle = |00\rangle$$

Quantum circuit representation

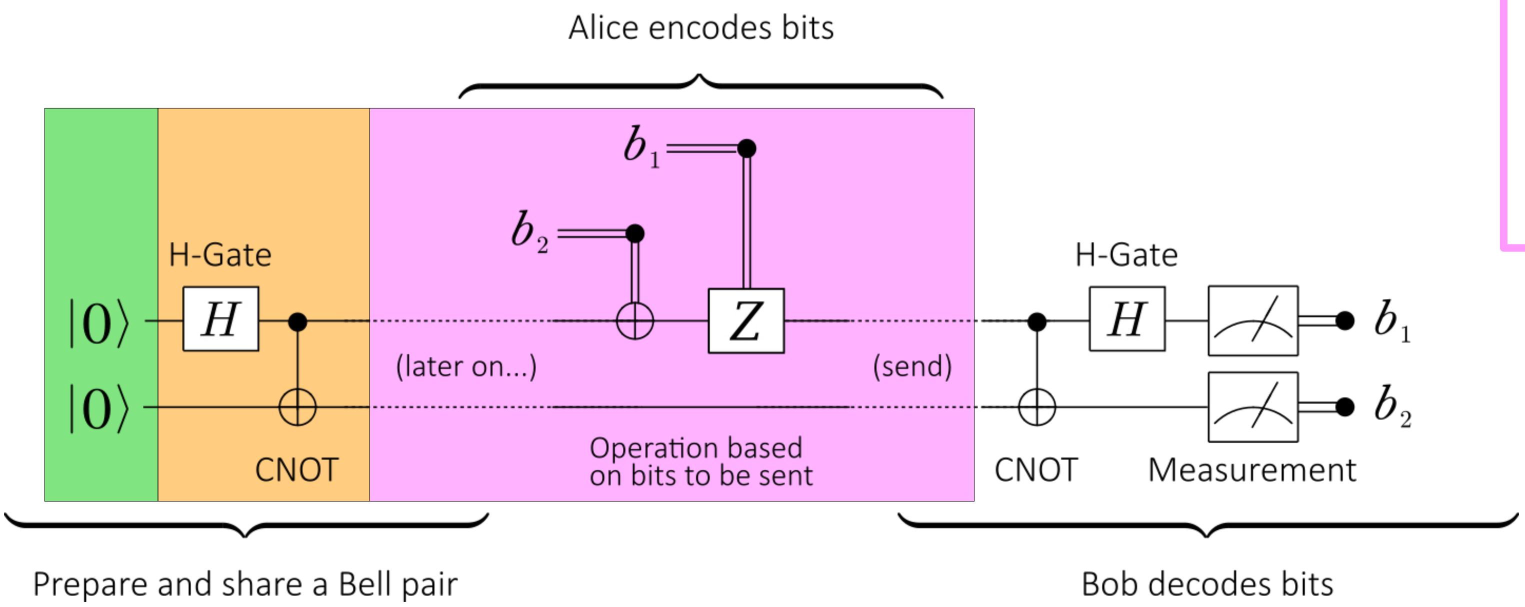


$$|\Psi_0\rangle = |00\rangle$$

$$|\Psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Quantum circuit representation

00 → I
01 → X
10 → Z
11 → XZ



$$|\Psi_0\rangle = |00\rangle$$

$$|\Psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\Psi_3\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$

$$|\Psi_3\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$$

$$|\Psi_3\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$$

$$|\Psi_3\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$$

11.3 Superdense coding

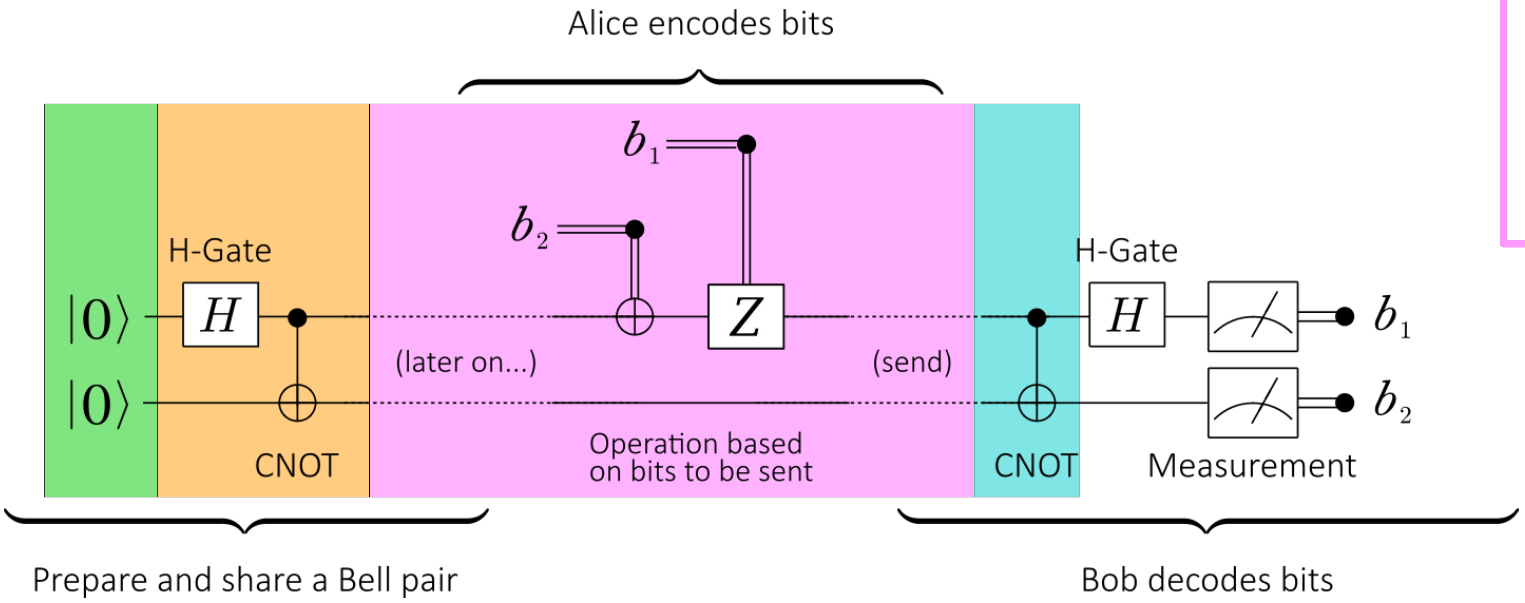
Quantum circuit representation

$00 \to I$

$01 \to X$

$10 \to Z$

$11 \to XZ$



$$|\Psi_3\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \qquad |\Psi_{3.5}\rangle = (|00\rangle + |10\rangle)/\sqrt{2}$$

$$|\Psi_3\rangle = (|10\rangle + |01\rangle)/\sqrt{2} \qquad |\Psi_{3.5}\rangle = (|11\rangle + |01\rangle)/\sqrt{2}$$

$$|\Psi_3\rangle = (|00\rangle - |11\rangle)/\sqrt{2} \qquad |\Psi_{3.5}\rangle = (|00\rangle - |10\rangle)/\sqrt{2}$$

$$|\Psi_3\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \qquad |\Psi_{3.5}\rangle = (|01\rangle - |11\rangle)/\sqrt{2}$$

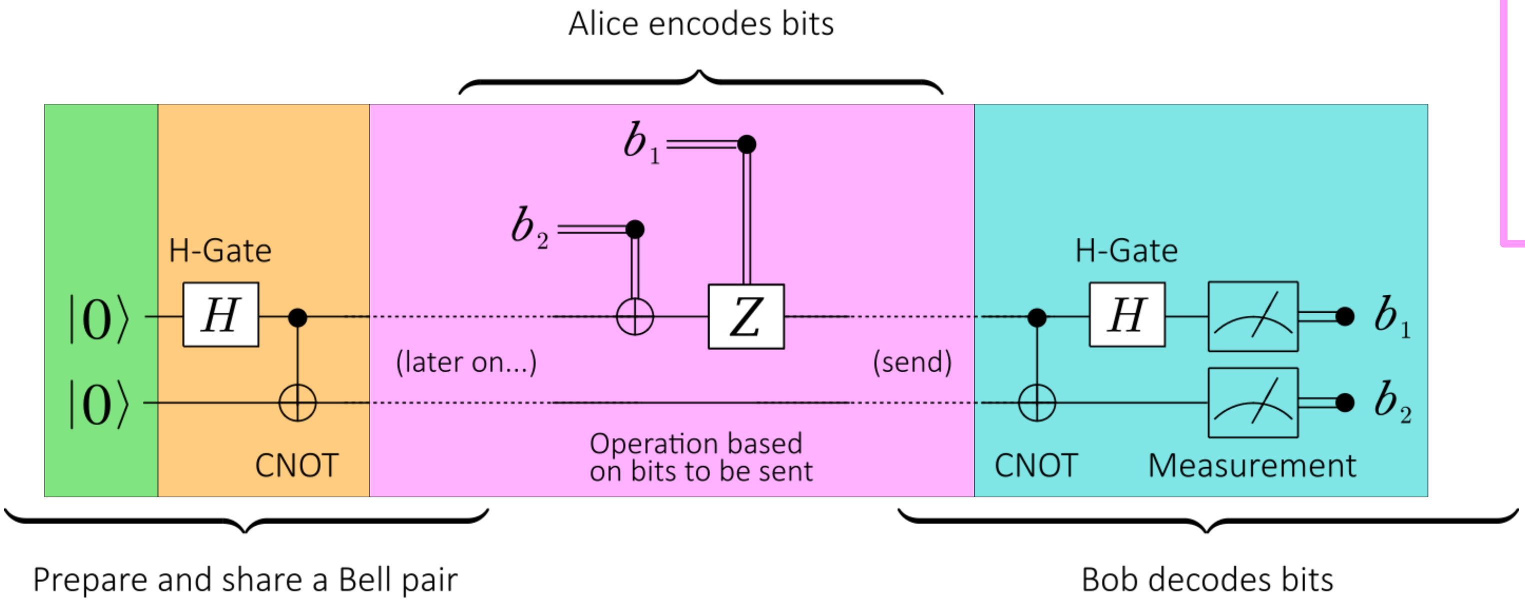
Quantum circuit representation

$00 \to I$

$01 \to X$

$10 \to Z$

$11 \to XZ$



$$|\Psi_3\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \qquad |\Psi_{3.5}\rangle = (|00\rangle + |10\rangle)/\sqrt{2} \qquad |\Psi_4\rangle = |00\rangle$$

$$|\Psi_3\rangle = (|10\rangle + |01\rangle)/\sqrt{2} \qquad |\Psi_{3.5}\rangle = (|11\rangle + |01\rangle)/\sqrt{2} \qquad |\Psi_4\rangle = |01\rangle$$

$$|\Psi_3\rangle = (|00\rangle - |11\rangle)/\sqrt{2} \qquad |\Psi_{3.5}\rangle = (|00\rangle - |10\rangle)/\sqrt{2} \qquad |\Psi_4\rangle = |10\rangle$$

$$|\Psi_3\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \qquad |\Psi_{3.5}\rangle = (|01\rangle - |11\rangle)/\sqrt{2} \qquad |\Psi_4\rangle = |11\rangle$$

Summary

Summarizing, Alice, interacting with only a single qubit, is able to transmit two bits of information to Bob. Of course, two qubits are involved in the protocol, but Alice never need to interact with the second qubit.

Classically, the task Alice accomplishes would have been impossible had she only transmitted a single classical bit.