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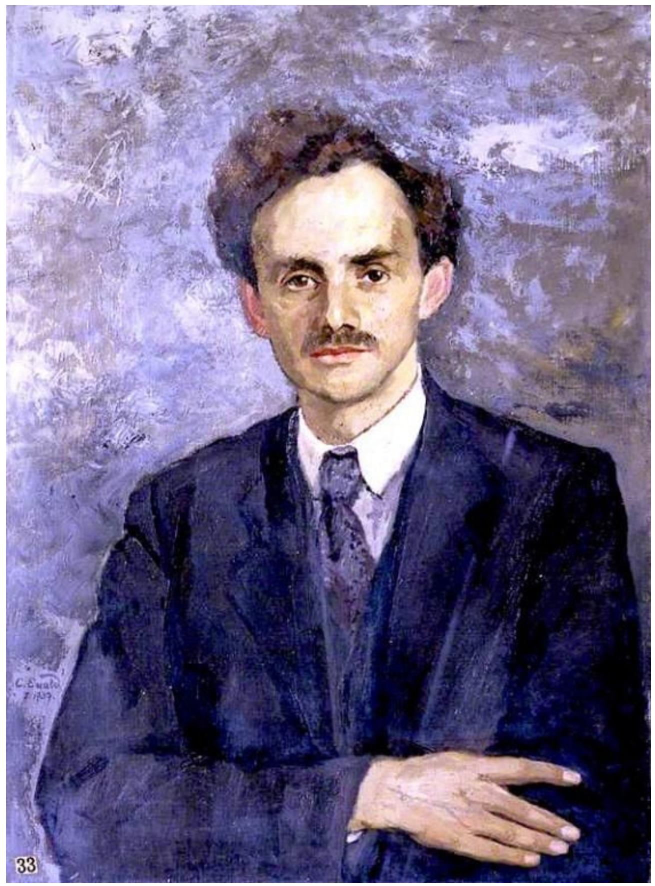
QUANTUM DISCOVERY

Quantum mechanics for quantum computing [1/2]

PASQAL
www.pasqal.com
office@pasqal.com
7 rue Léonard de Vinci
91300 Massy
France

Dirac notations

A portrait of Paul. A. M. Dirac



Dirac notations

Ket-vector

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$$

Bra-vector

$$\langle\psi| = (\psi_1^*, \psi_2^*, \ldots, \psi_N^*)$$

Hermitian conjugate

$$|\psi\rangle^\dagger = \langle\psi|$$

Dirac notations

Inner product

$$\langle\phi|\psi\rangle = (\phi_1^*, \phi_2^*, \cdots, \phi_N^*) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} = \phi_1^*\psi_1 + \phi_2^*\psi_2 + \cdots + \phi_N^*\psi_N$$

Dirac notations

Outer product

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} (\phi_1^*, \phi_2^*, \cdots, \phi_N^*) = \begin{pmatrix} \psi_1\phi_1^* & \psi_1\phi_2^* & \cdots & \psi_1\phi_N^* \\ \psi_2\phi_1^* & \psi_2\phi_2^* & \cdots & \psi_2\phi_N^* \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N\phi_1^* & \psi_N\phi_2^* & \cdots & \psi_N\phi_N^* \end{pmatrix}$$

Dirac notations

Tensor product of operators

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$$

$$A \otimes B =$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,2} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \\ & \\ a_{2,1} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{2,2} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}$$

$$A \otimes B \neq B \otimes A$$

Dirac notations

Tensor product of vectors

$$|\psi\rangle \otimes |\phi\rangle = |\psi, \phi\rangle = \begin{pmatrix} \psi_1 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \\ \psi_2 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \\ \vdots \\ \psi_N \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \psi_1\phi_1 \\ \psi_1\phi_2 \\ \vdots \\ \psi_2\phi_1 \\ \psi_2\phi_2 \\ \vdots \\ \psi_N\phi_N \end{pmatrix}$$

Postulates of quantum mechanics

Postulate / Axiom	Theorem
A concept that is taken to be true even without proof	Shown to be true by using proof

Postulates of quantum mechanics

1st postulate = State space → 2nd postulate = Unitary evolution → 3rd postulate = Measurement

Postulates of quantum mechanics

State space

1st postulate: Associated to any quantum physical system is a complex Hilbert space known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.

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Postulates of quantum mechanics

State space

<u>1st postulate:</u> Associated to any quantum physical system is a complex Hilbert space known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.

The case of a single qubit

- State vector: $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
- Probability amplitudes: $\alpha, \beta \in \mathbb{C}$ →
- Normalization: $|\alpha|^2 + |\beta|^2 = 1$
- Orthonormal set of basis vector: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3rd postulate

$$p(0) = |\alpha|^2 = \|\langle 0|\psi\rangle\|^2$$

$$p(1) = |\beta|^2 = \|\langle 1|\psi\rangle\|^2$$

Postulates of quantum mechanics

Global phase

A global phase – a complex exponential in front of a whole quantum state – has no effect on measurement results:

$$|\psi'\rangle = e^{i\theta}\,|\psi\rangle$$

$$p(0) = \|\langle 0|\psi'\rangle\|^2 = \langle\psi'|0\rangle\langle 0|\psi'\rangle = \langle\psi|0\rangle\, e^{-i\theta}e^{i\theta}\,\langle 0|\psi\rangle = \|\langle 0|\psi\rangle\|^2$$

$$p(1) = \|\langle 1|\psi'\rangle\|^2 = \langle\psi'|1\rangle\langle 1|\psi'\rangle = \langle\psi|1\rangle\, e^{-i\theta}e^{i\theta}\,\langle 1|\psi\rangle = \|\langle 1|\psi\rangle\|^2$$

A quantum state is always defined up to a global phase. And without loss of generality:

$$|\psi\rangle = |\alpha|\,|0\rangle + |\beta|e^{i\phi}\,|1\rangle$$

where: $\phi = \arg(\beta) - \arg(\alpha)$,

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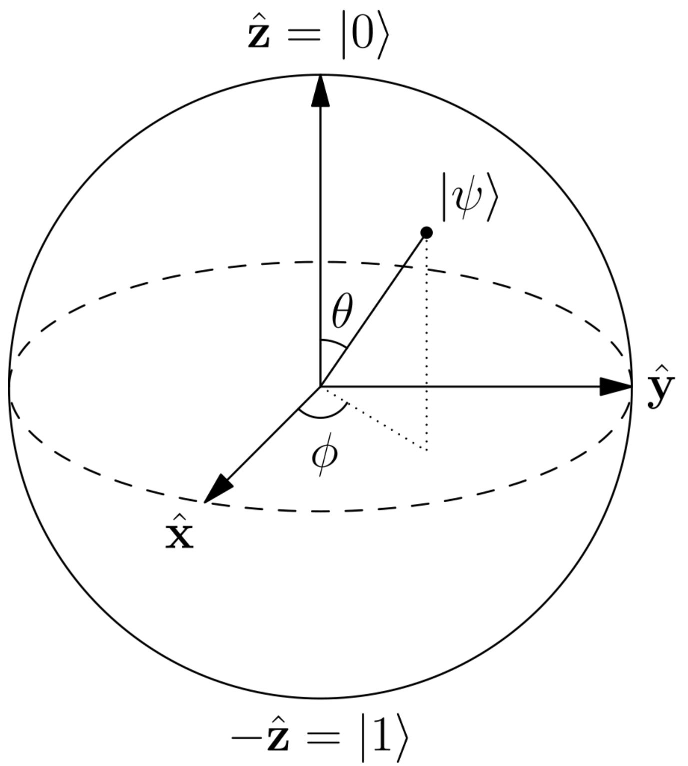
Postulates of quantum mechanics

Bloch sphere representation of a single qubit

A qubit state can also be seen as a vector pointing at a sphere surface:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

- Amplitude parameter: $0 \leq \theta \leq \pi$
- Phase parameter: $0 \leq \phi \leq 2\pi$
- Bloch vector coordinates: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$



Postulates of quantum mechanics

Extending the computational basis

$$|00\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|01\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|10\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}1\\0\end{pmatrix}\\1\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|11\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}0\\1\end{pmatrix}\\1\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

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Postulates of quantum mechanics

Pair of qubits

- State vector: $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$
- Probability amplitudes: $\alpha, \beta, \gamma, \delta \in \mathbf{C}$
- Normalization: $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$

3rd postulate

$$p(0,0) = |\alpha|^2 \qquad p(1,0) = |\gamma|^2$$

$$p(0,1) = |\beta|^2 \qquad p(1,1) = |\delta|^2$$

Postulates of quantum mechanics

Pair of qubits

Example bipartite state:

$$|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

$$\alpha = \beta = \gamma = \delta = \frac{1}{2}$$

$$p(0,0) = p(0,1) = p(1,0) = p(1,1) = \frac{1}{4}$$

Postulates of quantum mechanics

Product states

A state that can be factored as a tensor product of monopartite states is called a separable state, or a product state. In the case of two qubits, a product state writes:

$$|\psi\rangle = (\alpha_a|0\rangle + \beta_a|1\rangle) \otimes (\alpha_b|0\rangle + \beta_b|1\rangle)$$

<u>Example product state:</u>

$$|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

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Postulates of quantum mechanics

Entangled states

Example entangled state:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \longrightarrow \begin{bmatrix} \alpha = 1/\sqrt{2} \\ \beta = 0 \\ \gamma = 0 \\ \delta = 1/\sqrt{2} \end{bmatrix} \overset{?}{=} \begin{bmatrix} \alpha_a \alpha_b \\ \alpha_a \beta_b \\ \beta_a \alpha_b \\ \beta_a \beta_b \end{bmatrix}$$

No

A state that cannot be factored is called an entangled state. Such a state exhibits strong correlations.

Postulates of quantum mechanics

Entangled states

<u>Example entangled state:</u>

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \begin{bmatrix} \alpha = 1/\sqrt{2} \\ \beta = 0 \\ \gamma = 0 \\ \delta = 1/\sqrt{2} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \alpha_a \alpha_b \\ \alpha_a \beta_b \\ \beta_a \alpha_b \\ \beta_a \beta_b \end{bmatrix}$$

No

A state that cannot be factored is called an entangled state. Such a state exhibits strong correlations.

$$p(a=0|b=0) = p(b=0|a=0) = p(a=1|b=1) = p(b=1|a=1) = 1$$

$$p(a=0|b=1) = p(b=0|a=1) = p(a=1|b=0) = p(b=1|a=0) = 0$$

Conclusion

- ➔ Algebraïc calculations can be written in a short and concise way using Dirac notations
- ➔ The quantum axioms give a mathematical formulation of quantum mechanics in terms of operators acting in a Hilbert space
- ➔ A qubit is mathematically described as a linear combination - or superposition - of two orthonormal basis vector in a Hilbert space
- ➔ A qubit can also be seen as a vector pointing at a sphere surface
- ➔ Adding more qubits increase the dimensionality of the state space as 2 to the power of number of qubits
- ➔ Multiqubit systems can be either in a separable state or in an entangled state, or even in a mix of both as some qubits can be entangled and some other not