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QUANTUM DISCOVERY

Quantum mechanics for quantum computing [2/2]

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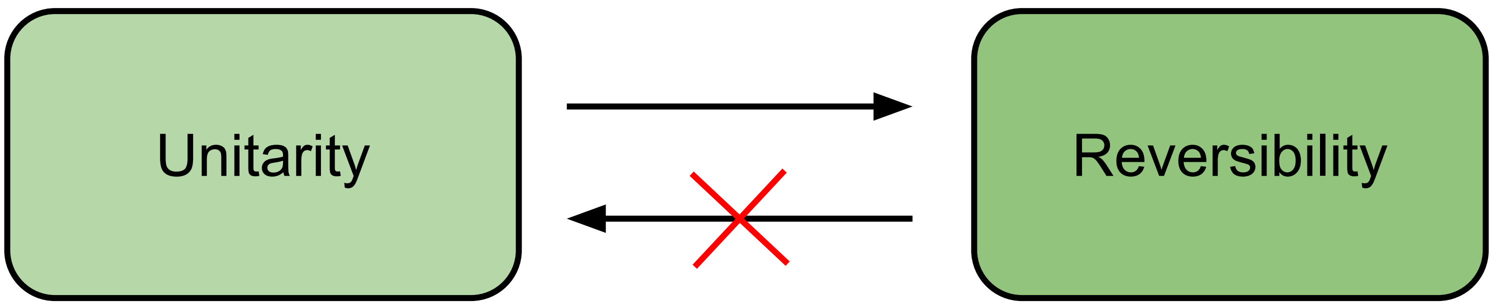
Postulates of quantum mechanics

Unitary evolution

<u>2nd postulate:</u> The time evolution of a quantum system is described by a unitary transformation

- State evolution: $$|\psi(t_2)\rangle = U_{12}\,|\psi(t_1)\rangle$$
- Unitarity: $$UU^\dagger = I$$



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Postulates of quantum mechanics

Unitary evolution

Examples of unitary operators (the Pauli operators):

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow XX^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \longrightarrow YY^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longrightarrow ZZ^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

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Postulates of quantum mechanics

Unitary evolution

Examples of unitary operators (the Pauli operators):

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow X\left(\alpha|0\rangle + \beta|1\rangle\right) = \beta|0\rangle + \alpha|1\rangle$$

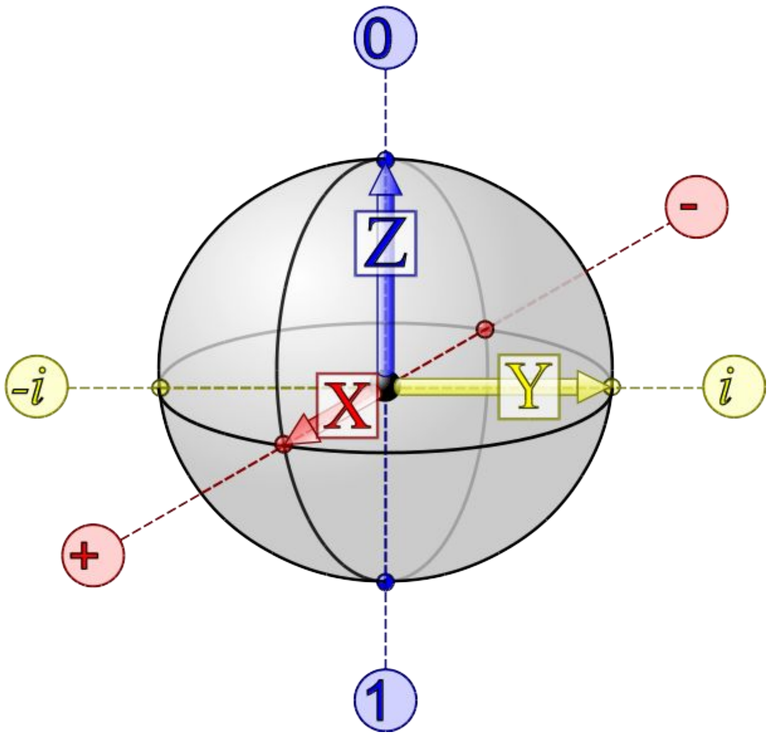
$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \longrightarrow Y\left(\alpha|0\rangle + \beta|1\rangle\right) = -i\left(\beta|0\rangle - \alpha|1\rangle\right)$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longrightarrow Z\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha|0\rangle - \beta|1\rangle$$

Postulates of quantum mechanics

Unitary evolution

Examples of unitary operators (the Pauli operators):



$$\begin{cases} X|+\rangle = |+\rangle \\ X|-\rangle = -|-\rangle \end{cases}$$

$$\begin{cases} Y|+i\rangle = |+i\rangle \\ Y|-i\rangle = -|-i\rangle \end{cases}$$

$$\begin{cases} Z|0\rangle = |0\rangle \\ Z|1\rangle = -|1\rangle \end{cases}$$

X-basis

$$|\pm\rangle \equiv \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$$

$$|\pm i\rangle \equiv \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}$$

Y-basis

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Postulates of quantum mechanics

Unitary evolution

2nd postulate (alternative statement): The time evolution of a quantum system is governed by the Schrödinger's equation

$$i\hbar \frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = H(t)|\psi(t)\rangle \xrightarrow{\textit{Hamiltonian}} H = \begin{bmatrix} E_0 & & & \\ & E_1 & & \\ & & \ddots & \\ & & & E_n \end{bmatrix} = H^\dagger$$

$$E_0 \leq E_1 \leq \ldots \leq E_n$$

Postulates of quantum mechanics

Unitary evolution

2nd postulate (alternative statement): The time evolution of a quantum system is governed by the Schrödinger's equation

$$i\hbar \frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = H(t)|\psi(t)\rangle \xrightarrow{\textit{Hamiltonian}} H = \begin{bmatrix} E_0 & & & \\ & E_1 & & \\ & & \ddots & \\ & & & E_n \end{bmatrix} = H^\dagger$$

$E_0 \leq E_1 \leq \ldots \leq E_n$

Time independent Hamiltonian

$$|\psi(t_2)\rangle = \exp\left(\frac{-iH(t_2 - t_1)}{\hbar}\right)|\psi(t_1)\rangle = U_{12}|\psi(t_1)\rangle$$

Time dependent Hamiltonian

$$|\psi(t_2)\rangle = \exp\left(\frac{-i\int_{t_1}^{t_2} H(t)\mathrm{d}t}{\hbar}\right)|\psi(t_1)\rangle = U_{12}|\psi(t_1)\rangle$$

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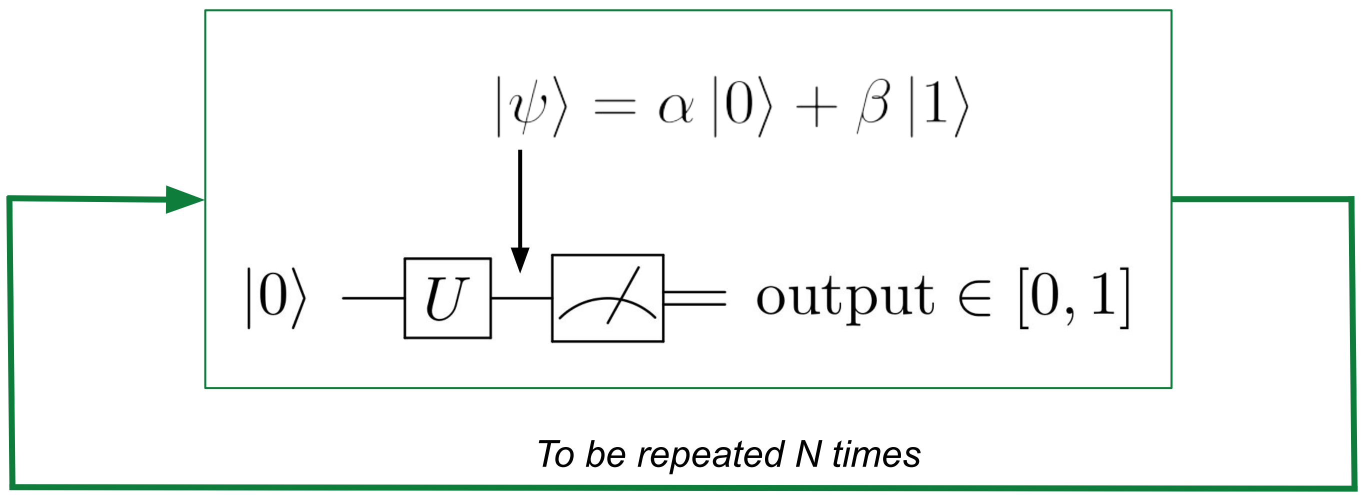
Postulates of quantum mechanics

Projective measurements

Sampling a quantum state through a series of measures in the computational basis:



Bitstring of size N: 011001...100100101

$$p(0) = \frac{\#0}{N}$$

$$p(1) = \frac{\#1}{N}$$

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Postulates of quantum mechanics

Projective measurements

Probabilities can be calculated with projectors:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$\begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$p(0) = |\alpha|^2 = |\langle 0|\psi\rangle|^2 = \langle\psi|0\rangle\langle 0|\psi\rangle = \langle\psi|P_{|0\rangle}|\psi\rangle$$

$$p(1) = |\beta|^2 = |\langle 1|\psi\rangle|^2 = \langle\psi|1\rangle\langle 1|\psi\rangle = \langle\psi|P_{|1\rangle}|\psi\rangle$$

$$\begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

Projector onto zero

$$P_{|0\rangle} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

Projector onto one

$$P_{|1\rangle} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

Postulates of quantum mechanics

Projective measurements

Example use of projectors:

$$|\psi\rangle_{ab} = \frac{|00\rangle + |10\rangle + |11\rangle}{\sqrt{3}}$$

When measuring the first qubit only, result 0 occurs with probability one third:

$$p(a=0) = \sum_{b=0}^{1} |\langle 0b|\psi\rangle|^2 = |\langle 00|\psi\rangle|^2 + |\langle 01|\psi\rangle|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 + |0|^2 = \frac{1}{3}$$

and result 1 with probability two thirds:

$$p(a=1) = \sum_{b=0}^{1} |\langle 1b|\psi\rangle|^2 = |\langle 10|\psi\rangle|^2 + |\langle 11|\psi\rangle|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

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Postulates of quantum mechanics

Projective measurements

Example use of projectors:

$$|\psi\rangle_{ab} = \frac{|00\rangle + |10\rangle + |11\rangle}{\sqrt{3}} \xrightarrow{\quad a = 1 \quad} |\psi'\rangle = \frac{|10\rangle + |11\rangle}{\sqrt{2}}$$

Why so ?

The second qubit was not measured

Because we got a one when measuring the first qubit

$$|\psi'\rangle = \frac{\left(\hat{P}_{|1\rangle} \otimes I\right)|\psi\rangle}{\sqrt{p(a=1)}} = \frac{\left(|1\rangle\langle 1| \otimes I\right)\frac{|00\rangle + |10\rangle + |11\rangle}{\sqrt{3}}}{\sqrt{\frac{2}{3}}} = \frac{\frac{|10\rangle + |11\rangle}{\sqrt{3}}}{\sqrt{\frac{2}{3}}} = \frac{|10\rangle + |11\rangle}{\sqrt{2}}$$

To ensure the output state is a unit vector

Postulates of quantum mechanics

Projective measurements

Example use of projectors:

$$|\psi'\rangle = \frac{\boxed{\left(\hat{P}_{|1\rangle} \otimes I\right)|\psi\rangle}}{\sqrt{p(a=1)}} = \frac{\left(|1\rangle\langle 1| \otimes I\right)\frac{|00\rangle + |10\rangle + |11\rangle}{\sqrt{3}}}{\sqrt{\frac{2}{3}}} = \frac{\frac{|10\rangle + |11\rangle}{\sqrt{3}}}{\sqrt{\frac{2}{3}}} = \frac{|10\rangle + |11\rangle}{\sqrt{2}}$$

$$\left(\hat{P}_{|1\rangle} \otimes I\right)|\psi\rangle = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{|10\rangle + |11\rangle}{\sqrt{3}}$$

Postulates of quantum mechanics

Projective measurements

For an operator to be a projector it has to verify the following properties:

$$P_m^\dagger = P_m \qquad\qquad P_m^2 = P_m \qquad\qquad P_m P_n = P_m \delta_{m,n}$$

where the index *m* refers to the measurement outcome that may occur in the experiment

The probability that result *m* occurs is: $p(m) = \langle\psi|P_m|\psi\rangle$

The state of the system after the measurement is: $\dfrac{P_m|\psi\rangle}{\sqrt{p(m)}}$

The measurement operators satisfy the completeness equation: $\sum_m P_m = I$

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Postulates of quantum mechanics

General measurements

Third postulate: Quantum measurements are described by a collection of measurement operators. These are operators acting on the state space of the system being measured.

The probability that result *m* occurs is: $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$

The state of the system after the measurement is: $\dfrac{M_m|\psi\rangle}{\sqrt{p(m)}}$

The measurement operators satisfy the completeness equation: $\sum_m M_m^\dagger M_m = I$

Postulates of quantum mechanics

Expectation value and standard deviation

In statistics, the expectation value of a random variable X is defined the following way:

$$\mathbb{E}\{X\} = \sum_{k \in X(\Omega)} kP(X = k)$$

From the expectation value we can then compute the variance:

$$\mathbb{V}\{X\} = \mathbb{E}\{X^2\} - \mathbb{E}^2\{X\} = \sum_{k \in X(\Omega)} k^2 P(X = k) - \left(\sum_{k \in X(\Omega)} kP(X = k) \right)^2$$

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Postulates of quantum mechanics

Expectation value and standard deviation

Example (uniform distribution):

k	1	2	3	4	5	6
$P(X=k)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

→ $$\mathbb{E}\{X\} = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{7}{2}$$

→ $$\mathbb{V}\{X\} = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + \cdots + 6^2 \times \frac{1}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

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Postulates of quantum mechanics

Expectation value and standard deviation

In quantum mechanics, the expectation value can be thought of as an average of all the possible outcomes of a measurement as weighted by their likelihood:

$$\boxed{M|m\rangle = m|m\rangle} \longrightarrow \mathbb{E}\{M\} = \sum_m m p(m)$$

Example expectation value:

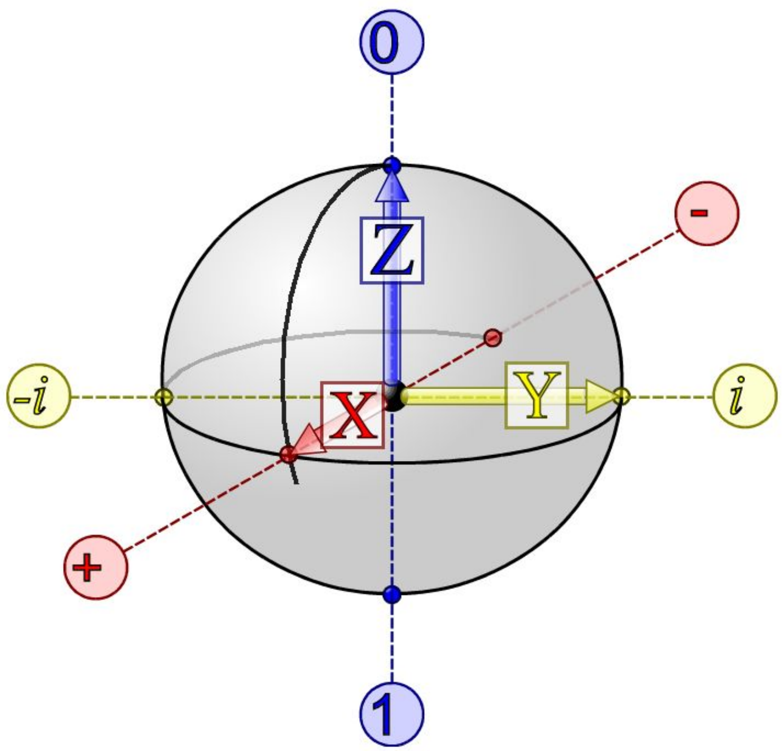
$$|\psi\rangle = \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle \qquad \begin{aligned} Z|0\rangle &= +1|0\rangle \\ Z|1\rangle &= -1|1\rangle \end{aligned}$$

$$\mathbb{E}\{Z\} = +1 \times p(0) - 1 \times p(1) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Postulates of quantum mechanics

Expectation value and standard deviation

The technique known as quantum state tomography consists in calculating the expectation value of Pauli operators X, Y, and Z to reconstruct the Bloch vector:



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \langle X \rangle & = & p(+) - p(-) \\ \langle Y \rangle & = & p(+i) - p(-i) \\ \langle Z \rangle & = & p(0) - p(1) \end{bmatrix}$$

Postulates of quantum mechanics

Expectation value and standard deviation

Applying rules of projective measurements:

$$\boxed{P_{|m\rangle} = |m\rangle\langle m|}$$

$$\boxed{M|m\rangle = m|m\rangle} \longrightarrow \mathbb{E}\{M\} = \sum_m m p(m)$$

$$= \sum_m m \langle\psi|P_{|m\rangle}|\psi\rangle$$

$$= \langle\psi|\left(\sum_m m P_{|m\rangle}\right)|\psi\rangle$$

$$= \langle\psi|M|\psi\rangle \equiv \langle M\rangle$$

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Postulates of quantum mechanics

Expectation value and standard deviation

Example expectation values (1/2):

$$\langle X \rangle = \left(\sqrt{\frac{2}{3}}\langle 0| + \sqrt{\frac{1}{3}}\langle 1|\right) X \left(\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle\right)$$

$$= \left(\sqrt{\frac{2}{3}}\langle 0| + \sqrt{\frac{1}{3}}\langle 1|\right)\left(\sqrt{\frac{2}{3}}|1\rangle + \sqrt{\frac{1}{3}}|0\rangle\right)$$

$$= \frac{2}{3}\langle 0|1\rangle + \frac{1}{3}\langle 1|0\rangle + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}}\langle 0|0\rangle + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}}\langle 1|1\rangle$$

$$= 0 + 0 + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}}$$

$$\langle X \rangle = \frac{2\sqrt{2}}{3}$$

Postulates of quantum mechanics

Expectation value and standard deviation

Example expectation values (2/2):

$$\langle Y \rangle = \left(\sqrt{\frac{2}{3}} \langle 0| + \sqrt{\frac{1}{3}} \langle 1| \right) Y \left(\sqrt{\frac{2}{3}} |0\rangle + \sqrt{\frac{1}{3}} |1\rangle \right)$$

$$= \left(\sqrt{\frac{2}{3}} \langle 0| + \sqrt{\frac{1}{3}} \langle 1| \right) \left(i\sqrt{\frac{2}{3}} |1\rangle - i\sqrt{\frac{1}{3}} |0\rangle \right)$$

$$= \frac{2i}{3} \langle 0|1\rangle - \frac{i}{3} \langle 1|0\rangle - i\sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}} \langle 0|0\rangle + i\sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}} \langle 1|1\rangle$$

$$= 0 - 0 - i\sqrt{\frac{2}{9}} + i\sqrt{\frac{2}{9}}$$

$$\langle Y \rangle = 0$$

Postulates of quantum mechanics

Expectation value and standard deviation

In quantum mechanics, the standard deviation quantifies the uncertainty that exists on the outcome of a measurement:

$$\Delta M = \sqrt{\langle M^2 \rangle - \langle M \rangle^2}$$

Postulates of quantum mechanics

Expectation value and standard deviation

Example standard deviation (1/2):

$$|\psi\rangle = |0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$\langle X\rangle = p(+) - p(-) = \frac{1}{2} - \frac{1}{2} = 0$$

$$\begin{aligned}\langle X\rangle &= \langle 0|X|0\rangle \\ &= \langle 0|1\rangle \\ \langle X\rangle &= 0\end{aligned}$$

$$\begin{aligned}\Delta X &= \sqrt{\langle X^2\rangle - \langle X\rangle^2} \\ &= \sqrt{\langle 0|X^2|0\rangle - 0^2} \\ &= \sqrt{\langle 0|I|0\rangle} \\ &= \sqrt{\langle 0|0\rangle} \\ \Delta X &= 1\end{aligned}$$

Postulates of quantum mechanics

Expectation value and standard deviation

Example standard deviation (2/2):

$$|\psi\rangle = |+\rangle$$

$$\langle X \rangle = p(+) - p(-) = 1 - 0 = 1$$

$$\begin{aligned} \langle X \rangle &= \langle + | X | + \rangle \\ &= \langle + | + \rangle \\ \langle X \rangle &= 1 \end{aligned}$$

$$\begin{aligned} \Delta X &= \sqrt{\langle X^2 \rangle - \langle X \rangle^2} \\ &= \sqrt{\langle + | X^2 | + \rangle - 1^2} \\ &= \sqrt{\langle + | I | + \rangle - 1} \\ &= \sqrt{\langle + | + \rangle - 1} \\ &= \sqrt{1 - 1} \\ \Delta X &= 0 \end{aligned}$$

Conclusion

➔ In quantum mechanics, the Schrödinger equation describes how a system changes with time

➔ Transformations on quantum system are necessarily unitary, thus reversible

➔ In quantum mechanics, measurements are non unitary transformations, for that reason there is a specific postulate describing rules of measurements

➔ Measurements in quantum computing are usually projective measurements, we describe such measurement with orthogonal projectors

➔ Through repeated measurements on systems prepared in a same state, we can reconstruct a probability distributions associated to this state

➔ From probability distributions one can then access average values and standard deviations