Métodos Numéricos 2019 - Obligatorio 1

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1 Ejercicio 1

1.1 Representación de f en $\mathbb{R}^2$

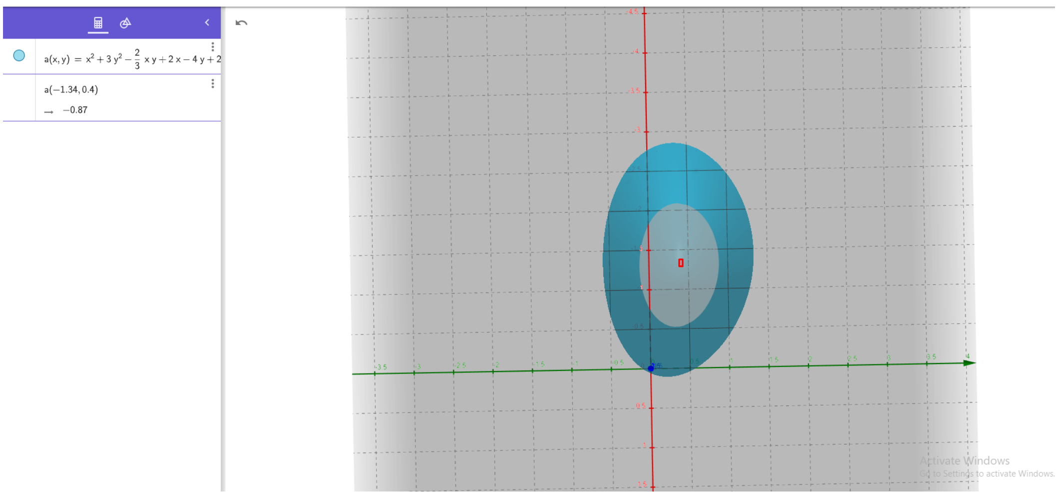

Fig. 1. isocurvas de $f(x, y)$

Se adjuntan dos imágenes 1 y 2 donde se puede ver la función $f(x, y)$. Su forma parece ser convexa con un mínimo global cerca de cero.

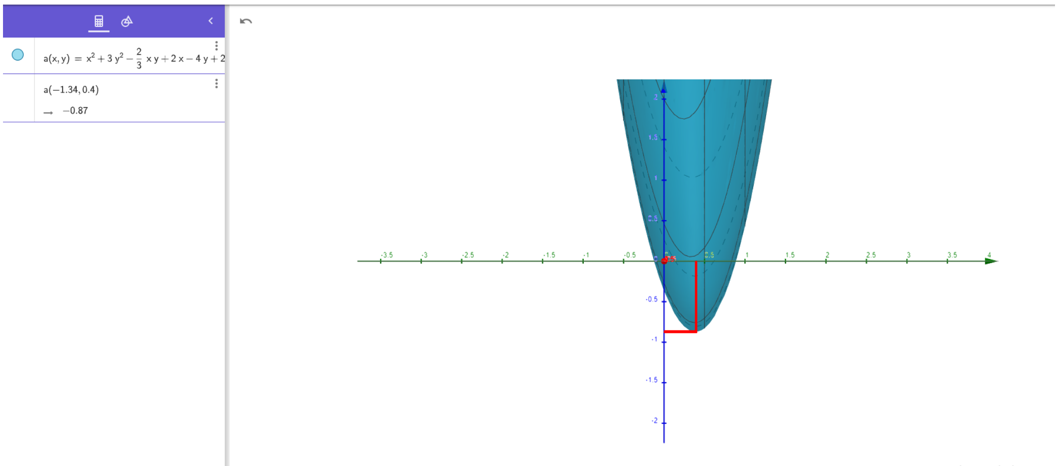

Fig. 2. corte transversal de $f(x, y)$

1.2 Hallar Q y b

$Q \in \mathbb{R}^{2\times2}$ y $b \in \mathbb{R}^{2\times1}$ tienen la siguiente forma

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \tag{1}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{2}$$

Sea

$$\begin{gathered} f(z) = (z^T Q z - 2b^T z) + 2e^{z_x + z_y} \\ z = (z_x, z_y)^T \in \mathbb{R}^{2\times1}. \end{gathered} \tag{3}$$

Desarrollando

$$f(z) = q_{11} z_x^2 + q_{22} z_y^2 + (q_{12} + q_{21}) z_x z_y - 2b_1 z_x - 2b_2 z_y + 2e^{z_x + z_y} \tag{4}$$

Tenemos el siguiente sistema

$$\begin{cases} q_{11} = 1 \\ q_{12} + q_{21} = -\frac{2}{3} \\ q_{22} = 3 \\ b_1 = -1 \\ b_2 = 2 \end{cases} \tag{5}$$

Resolviendolo se tiene que $Q = \begin{pmatrix} 1 & -\frac{1}{3} \\ -\frac{1}{3} & 3 \end{pmatrix}$ y $b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

1.3 $F(z) = (Qz - b) + e^{x+y}$

1.4

1.5

Let us develop an ILP for the problem under study. The key idea is to consider connectivity requirement $r_q = 2$ for every pair of terminals q from the backbone, while $r_q = 1$ otherwise. Let $Q = \{q = (i, j), \forall i \neq j, i, j \in T \subseteq V\}$. Consider the following set of binary variables:

- $z_{ij} = 1$ iff $(i, j) \in E$ is in the backbone;
- $y_{ij} = 1$ iff $(i, j) \in E$ is in the access network;
- x_{ij}^q is the $i - j$ flow for every pair of terminals q;
- $p_i = 1$ iff the i is included in the access network.

An ILP formulation for the 2NCSP-SN can be expressed as follows:

$$\min_{H\subseteq G} c(H) = \sum_{ij\in E} c_{ij}.z_{ij} + \sum_{s\in S} a_s.p_s + \sum_{ij\in E} d_{ij}.y_{ij} \tag{1}$$

$$\text{s.t.} \sum_{j:(j,i)\in E} x^q_{ji} - \sum_{j:(i,j)\in Ed^q} x^q_{ij} = I(i).r_i \ \forall i \in V,\ \forall q = (q_o, q_d) \in Q \tag{2}$$

$$I(i) = 1\ \forall i \in V \backslash \{q_o\},\ I(q_o) = -1 \tag{3}$$

$$r_i = 0\ \forall i \in V \backslash \{q_o, q_d\} \tag{4}$$

$$\max(1, p_{q_o} + p_{q_d}) \leq r_i \leq 1 + \min(p_{q_o}, p_{q_d}),\ \forall i \in \{q_o, q_d\},\ \forall q \in Q \tag{5}$$

$$x^q_{ij} + x^q_{ji} \leq z_{ij} + y_{ij},\ \forall ij \in E,\ \forall q \in Q \tag{6}$$

$$\sum_{j\in\delta(i)} y_{ij} \leq 1 + Mp_i,\ \forall i \in T \tag{7}$$

$$\sum_{j\in\delta(i)} (z_{ij} + y_{ij}) \leq Mp_i,\ \forall i \in S \tag{8}$$

$$y_{ij} \leq 2 - p_i - p_j,\ \forall ij \in E, i, j \in V \tag{9}$$

$$z_{ij} \leq \min(p_i, p_j),\ \forall ij \in E, i, j \in V \tag{10}$$

$$z_{ij} + y_{ij} \leq 1\ \forall ij \in E, i, j \tag{11}$$

$$2p_i \leq \sum_{j\in\delta(i)} z_{ij} \leq Mp_i,\ \forall i \in V \tag{12}$$

Where $\delta(i)$ is neighbor-set for node i, and M is an arbitrarily large integer. The objective function (1) is the contribution of internal/external connections and Steiner nodes. Constraints (2)-(5) ensure connectivity using Kirchhoff equations. Constraints (6) and (7) force one-way flow. By Constraint (8), optional Steiner nodes belong to the backbone, if needed. The definitions of binary variables y_{ij} and z_{ij} are captured by Constraints (9) and (10). Constraints (11) state that either y_{ij} or z_{ij} can be set to 1, but not both. Constraints (12) state that a terminal from the backbone could have multilinks, but nodes from the access network have one link.

2 Methodology

From now on, we assume that the internal/external costs are positive and internal costs satisfy the triangle inequality. Without loss of generality, a complete graph $G = (V, E)$ is considered. Let $\alpha_e = \frac{c_e}{d_e}, \forall e \in E$ be the primary/secondary cost ratio for each arc. In this section we build an approximation algorithm for the the

2NCSP-SN of factor 4α, being α:

$$\alpha = \max_{e \in E} \{\alpha_e\} \tag{13}$$

Recall that Christofides's algorithm is a 3/2-factor for the metric TSP. The key concept of our approximation algorithm is Christofides in order to span the terminal-set with an elementary cycle. Greedy augmentations of the solution including Steiner nodes also takes place, whenever the cost is reduced.

In Line 1, $Christofides$ is called in order to build an elementary cycle C that spans the terminal-set T. The corresponding solution is updated in Lines 2-3, where the backbone is C and the access network is empty yet. In the while-loop (Lines 4-11), Steiner nodes are greedily included in the backbone, whenever the cost is reduced (Line 5). If this happens, some terminal node v is included in the access network, and the evidence $s \in S$ is added to the backbone (Lines 6-7). Observe that candidate terminals $t \in T$ are iteratively checked (Line 9), and the condition $|J| \geq 3$ forces to have a cycle in the backbone. The corresponding feasible solution F is finally returned (Line 12).

Require: $G = (T \cup S, E)$,$c(e), d(e) \forall e \in E$,
1: $C \leftarrow Christofides(G, c)$
2: $L \leftarrow \emptyset, I \leftarrow T, E' \leftarrow E(C), J \leftarrow T$
3: $F \leftarrow (L \cup I, E')$
4: **while** $|J| \geq 3$ **do**
5:     **if** there are $s \in S$, $(t, v), (v, w) \in F$: $c(t, v) + c(v, w) > d(v, s) + c(t, s) + c(s, w)$ **then**
6:         $L \leftarrow L \cup \{v\}, I \leftarrow I \cup \{s\} \setminus \{v\}, J \leftarrow J \setminus \{t, v\}$
7:         $E' \leftarrow E' \cup \{(t, s), (s, v), (s, w)\} \setminus \{(t, v), (v, w)\}$
8:     **else**
9:         $J \leftarrow J \setminus \{t\}$
10:     **end if**
11: **end while**
12: **return** $F = (I \cup L, E')$

Lemma 2.1 *If $L(F) \neq \emptyset$ then $\alpha > 1/2$*

Proof. By the triangle inequality, $c_{(t,v)} \leq c_{(t,s)} + c_{(s,v)}$ and $c_{(v,w)} \leq c_{(v,s)} + c_{(s,w)}$, so $c_{(t,v)} + c_{(v,w)} \leq c_{(t,s)} + 2.c_{(s,v)} + c_{(s,w)}$. If $L \neq \emptyset$, there exists $v \in L$, $s \in S \cap I, t, w \in I$ such that $c_{(t,v)} + c_{(v,w)} > d_{(v,s)} + c_{(t,s)} + c_{(s,w)}$,
Therefore $d_{(v,s)} = c_{(v,s)}/\alpha_{v,s} < 2.c_{(s,v)}$, so there exists $e = (v,s)$ such that $\alpha_e > 1/2$, that is $\alpha > 1/2$ with $\alpha = \max_{e \in E} \{\alpha_e\}$.

□

Theorem 2.2 $c(F) \leq \max\{2, 4\alpha\} \times OPT$.

Proof. Let $G^* = (S^* \cup T, E^*)$ be the optimal solution, H the cheapest Hamilton tour spanning T and H^S the cheapest Hamilton tour spanning $T \cup S^*$. Analogously, let us denote TNC (TEC) to the optimal 2-node (resp. 2-edge) connected spanning subgraph for $T \cup S^*$. Recall that F is the output and C the cycle obtained using Christofides algorithm. Combining Monma and Christofides theorems:

$$c(G^*) \leq c(F) \leq c(C) \leq \frac{3}{2}c(H) \leq \left(\frac{3}{2}\right)\left(\frac{4}{3}\right)c(TNC) = 2c(TNC) \tag{14}$$

Let $G^* = B \cup L$, being B its backbone. Consider an augmentation F' for G^*, doubling edges from L with cost $c_{r,j} = \alpha_{r,j} d_{r,j}$ and adding them. F' is 2-edge connected and

$$c(F') = c(B) + 2\sum_{r \in L} c_{r,j} \leq c(B) + 2\alpha \sum_{r \in L} d_{r,j} = 2\alpha c(G^*) + (1 - 2\alpha)c(B), \tag{15}$$

If $1 - 2\alpha > 0$, $c(F') \leq 2\alpha c(G^*) + 2(1 - 2\alpha)c(G^*) \leq 2c(G^*)$. In this case a factor 2 is provided, and by Lemma 2.1 F consists of an elementary cycle.

Otherwise, combining (14) and (15) we have that:

$$\begin{aligned} c(G^*) &\leq c(F) \leq 2c(TNC) = 2c(TEC) \leq 2c(F') \\ &\leq 4\alpha c(G^*) + 2(1 - 2\alpha)c(B) \\ &\leq 4\alpha c(G^*). \end{aligned}$$

□

3 Experimental Analysis

In order to highlight the effectiveness of our approximation algorithm, a sensibility analysis with respect to the ratio α is carried out. We consider a single instance from TSPLIB named berlin52.tsp. This is the case of a real-life network with Euclidean costs. In order to find the globally optimum solution, an induced subgraph with 22 nodes is considered (with 10 terminal-nodes and 12 Steiner nodes). The ILP has been executed in CPLEX 12.6.3 MIP solver using an Intel i7 processor, 2.30 GHz, 8GB RAM. Table 1 illustrates the performance $c(F)/OPT$ as a function of α. The cycle C obtained using Christofides algorithm is $c(C) = 3164.8$, while the cheapest Hamiltonian tour H spanning the terminal-set has a cost $c(H) = 2826.5$. Naturally, since F considers greedy augmentations of C, we get that $c(F) \leq c(C)$ in all cases.

Table 1
Sensibility Analysis as a function of α

α	OPT	$c(F)$	$c(F)/OPT$
10	485.78	3117.3	6.42
4	1110.48	3164.8	2.85
2	2115.02	3164.8	1.50
4/3	2611.45	3164.8	1.21
1	2786.80	3164.8	1.14
4/5	2811.63	3164.8	1.13
2/3	2822.88	3164.8	1.12
4/7	2825.60	3164.8	1.12

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