PARALLELS IN GEOMETRY

WITH TEACHING NOTES

MATH 1166: SPRING 2015

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Preface

These notes are designed with future middle grades mathematics teachers in mind. While most of the material in these notes would be accessible to an accelerated middle grades student, it is our hope that the reader will find these notes both interesting and challenging. In some sense we are simply taking the topics from a middle grades class and pushing "slightly beyond" what one might typically see in schools. In particular, there is an emphasis on the ability to communicate mathematical ideas. Three goals of these notes are:

- To enrich the reader's understanding of both numbers and algebra. From the basic algorithms of arithmetic—all of which have algebraic underpinnings—to the existence of irrational numbers, we hope to show the reader that numbers and algebra are deeply connected.

- To place an emphasis on problem solving. The reader will be exposed to problems that "fight-back." Worthy minds such as yours deserve worthy opponents. Too often mathematics problems fall after a single "trick." Some worthwhile problems take time to solve and cannot be done in a single sitting.

- To challenge the common view that mathematics is a body of knowledge to be memorized and repeated. The art and science of doing mathematics is a process of reasoning and personal discovery followed by justification and explanation. We wish to convey this to the reader, and sincerely hope that the reader will pass this on to others as well.

In summary—you, the reader, must become a doer of mathematics. To this end, many questions are asked in the text that follows. Sometimes these questions are answered; other times the questions are left for the reader to ponder. To let the reader know which questions are left for cogitation, a large question mark is displayed:

?

The instructor of the course will address some of these questions. If a question is not discussed to the reader's satisfaction, then we encourage the reader to put on a thinking-cap and think, think, think! If the question is still unresolved, go to the World Wide Web and search, search, search!

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Please report corrections, suggestions, gripes, complaints, and criticisms to Bart Snapp at `snapp@math.osu.edu` or Brad Findell at `findell.2@osu.edu`

Thanks and Acknowledgments

This document has a somewhat lengthy history. In the Fall of 2005 and Spring of 2006, Bart Snapp gave a set of lectures at the University of Illinois at Urbana-Champaign. His lecture notes were typed and made available as an open-source textbook. During subsequent semesters, those notes were revised and modified under the supervision of Alison Ahlgren and Bart Snapp. Many people have made contributions, including Tom Cooney, Melissa Dennison, and Jesse Miller. A number of students also contributed to that document by either typing original hand-written notes or suggesting problems. They are: Camille Brooks, Michelle Bruno, Marissa Colatosti, Katie Colby, Anthony 'Tino' Forneris, Amanda Genovise, Melissa Peterson, Nicole Petschenko, Jason Reczek, Christina Reincke, David Seo, Adam Shalzi, Allice Son, Katie Strle, and Beth Vaughn.

In 2009, Greg Williams, a Master of Arts in Teaching student at Coastal Carolina University, worked with Bart Snapp to produce an early draft of the chapter on isometries.

In the Winter of 2010 and 2011, Bart Snapp gave a new set of lectures at the Ohio State University. In this course the previous lecture notes were heavily modified, resulting in a new text *Parallels in Geometry*. In 2012 and 2013, Bart Snapp and Brad Findell continued revising these notes, and in 2014, text and activities were added to address ideas from the Common Core State Standards.

Contents

1 Proof by Picture

A picture is worth a thousand words.

—Unknown

1.1 Basic Set Theory

The word *set* has more definitions in the dictionary than any other word. In our case we'll use the following definition:

Definition A **set** is any collection of elements for which we can always tell whether an element is in the set or not.

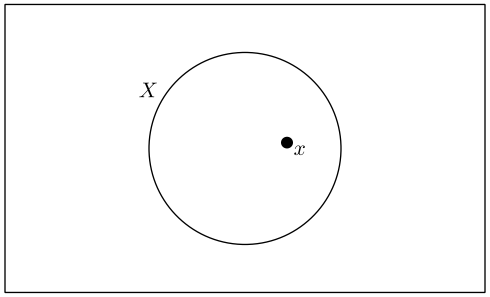
Question What are some examples of sets? What are some examples of things that are not sets?

?

If we have a set X and the element x is inside of X, we write:

$$x \in X$$

This notation is said “x in X.” Pictorially we can imagine this as:



Sometimes the elements of a set can be listed or described by words or formulas. In such cases, we often use curly braces { and } to enclose the elements of the set or a description of the set. For example, if $X = \{2, 3, 7\}$, and $Y = \{\text{even numbers}\}$, then each of the following statements are true:

$$2 \in X \qquad 4 \notin X \qquad 6 \in Y \qquad 9 \notin Y.$$

Definition A **subset** Y of a set X is a set Y such that every element of Y is also an element of X. We denote this by:

$$Y \subseteq X$$

If Y is contained in X, we will sometimes loosely say that X is *bigger* than Y.

Question Can you think of a set X and a subset Y where saying X is bigger than Y is a bit misleading?

?

Sometimes it is useful to list a set of sets. For example, if $X = \{2, 3, 7\}$, then

$$Y = \{\{2\}, \{2, 7\}, \{3, 7\}\}$$

is a set containing a few subsets of X.

Question How many elements are in the set Y?

?

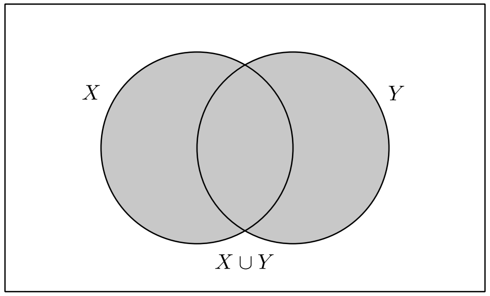
Question How is the meaning of the symbol $\in$ different from the meaning of the symbol $\subseteq$?

?

1.1.1 Union

Definition Given two sets X and Y, X **union** Y is the set of all the elements in X or Y. We denote this by $X \cup Y$.

Pictorially, we can imagine this as:



Warning Note that this definition uses the *inclusive or*. In everyday language, it is common to use the word "or" in an exclusive sense, meaning, "but not both." But in mathematics, the word "or" is almost always used inclusively.

Thus, the phrase "A or B" allows for the possibility of both.

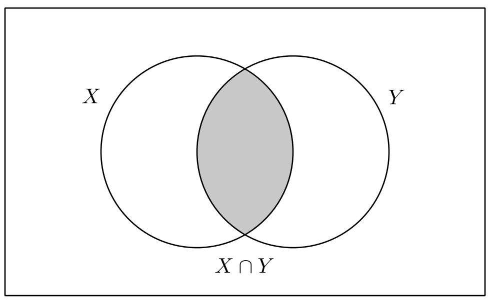
Question What about the above picture shows that "or" is used inclusively in the definition of union?

?

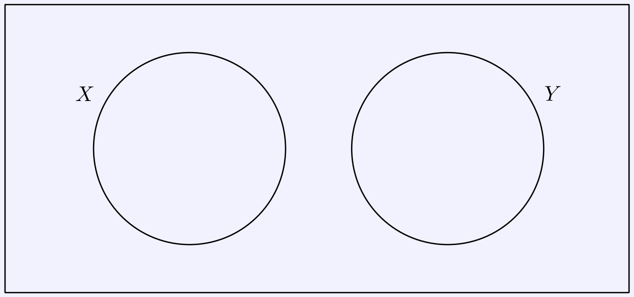
1.1.2 Intersection

Definition Given two sets X and Y, X **intersect** Y is the set of all the elements that are simultaneously in X and in Y. We denote this by $X \cap Y$.

Pictorially, we can imagine this as:



Question Consider the sets X and Y below:



What is $X \cap Y$?

I'll take this one: Nothing! The set with no elements is called the **empty set**. We sometimes denote the empty set as $\{\}$, but it is more common to denote the empty set with the symbol $\varnothing$.

Question How is $\{\varnothing\}$ different from $\varnothing$?

?

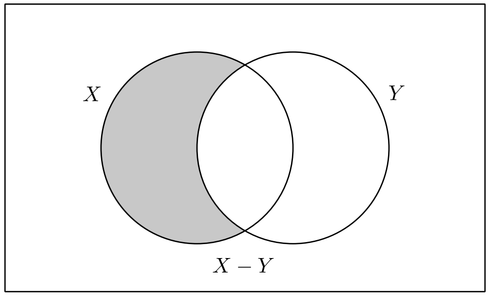
Question The empty set is a subset of every set. Why does this makes sense? Why does it make sense to say *the* empty set rather than *an* empty set?

?

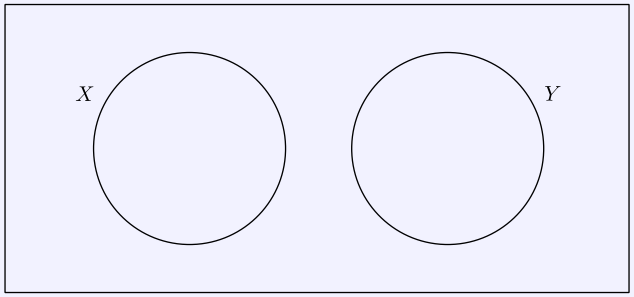
1.1.3 Complement

Definition Given two sets X and Y, X **complement** Y is the set of all the elements that are in X and are not in Y. We denote this by $X - Y$.

Pictorially, we can imagine this as:



Question Check out the two sets below:



What is $X - Y$? What is $Y - X$?

?

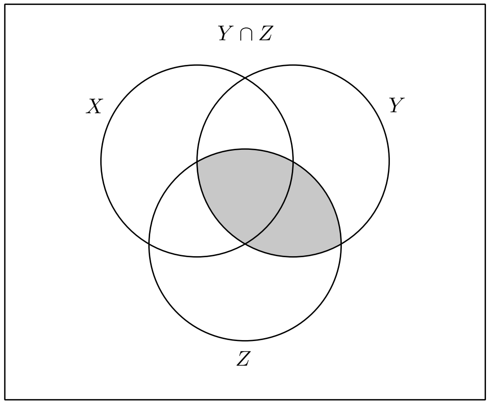
1.1.4 Putting Things Together

OK, let's try something more complex:

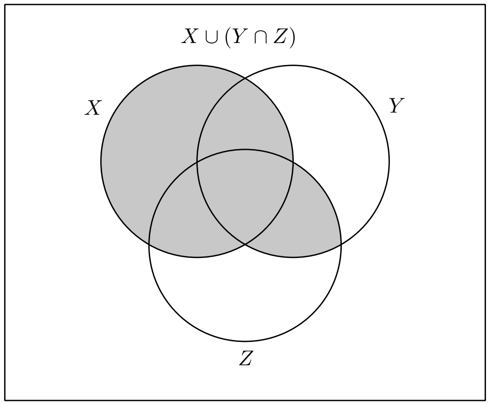
Question Prove that:

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

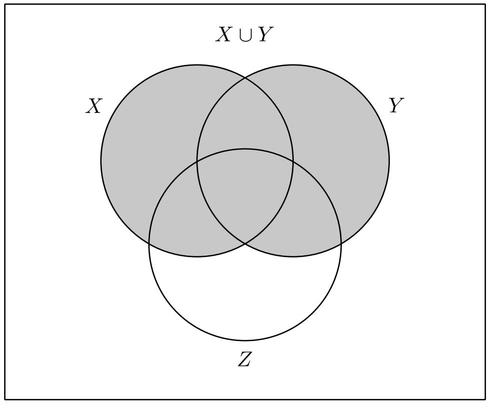
Proof *Look at the left-hand side of the equation first. We can represent the elements in $Y \cap Z$ with shaded region in the following diagram:*



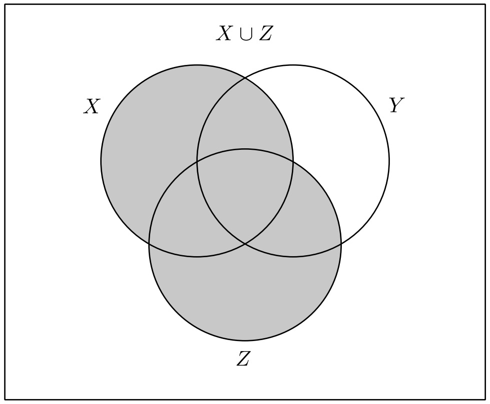
So the union of this region with X is represented the shaded region in this diagram.



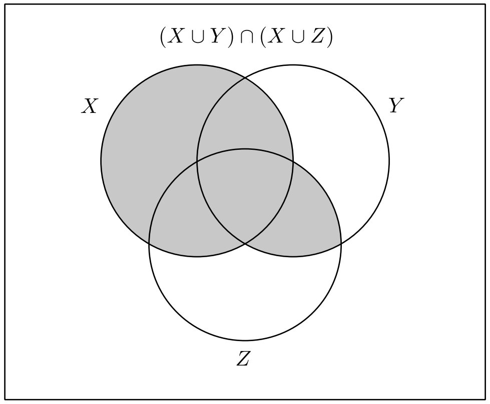
Now, looking at the right-hand side of the equation, $X \cup Y$ is represented by this shaded region:



And $X \cup Z$ is represented by this shaded region:



The region shaded in both of the diagrams, which is the intersection of $X \cup Y$ and $X \cup Z$, is represented by the shaded region below.



Comparing the diagrams representing the left-hand and right-hand sides of the equation, we see that the same regions are shaded, and so we are done.

Problems for Section 1.1

(1) Given two sets X and Y, explain what is meant by $X \cup Y$.

(2) Given two sets X and Y, explain what is meant by $X \cap Y$.

(3) Given two sets X and Y, explain what is meant by $X - Y$.

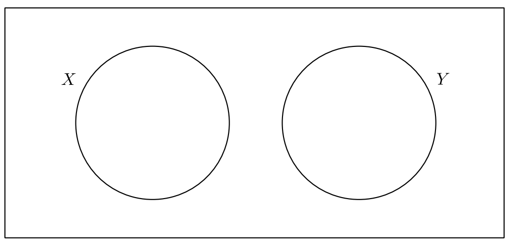
(4) Explain the difference between the symbols $\in$ and $\subseteq$.

(5) How is $\{\varnothing\}$ different from $\varnothing$?

(6) List all the subsets of the set $X = \{2, 3, 5, 7\}$. In general, how many subsets are there of an n-element set? Explain why this makes sense.

(7) Draw a Venn diagram for the set of elements that are in X or Y *but not both*. How does it differ from the Venn diagram for $X \cup Y$?

(8) If we let X be the set of "right triangles" and we let Y be the set of "equilateral triangles" does the picture below show the relationship between these two sets?



Explain your reasoning.

(9) If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{3, 4, 5, 6\}$ find:

(a) $X \cup Y$
(b) $X \cap Y$
(c) $X - Y$
(d) $Y - X$

In each case explain your reasoning.

(10) Let $n\mathbb{Z}$ represent the integer multiples of n. So for example:

$$3\mathbb{Z} = \{\dots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \dots\}$$

Compute the following:

(a) $3\mathbb{Z} \cap 4\mathbb{Z}$
(b) $2\mathbb{Z} \cap 5\mathbb{Z}$
(c) $3\mathbb{Z} \cap 6\mathbb{Z}$
(d) $4\mathbb{Z} \cap 6\mathbb{Z}$
(e) $4\mathbb{Z} \cap 10\mathbb{Z}$

In each case explain your reasoning.

(11) Make a general rule for intersecting sets of the form $n\mathbb{Z}$ and $m\mathbb{Z}$. Explain why your rule works.

(12) Prove that:
$$X = (X \cap Y) \cup (X - Y)$$

(13) Prove that:
$$X - (X - Y) = (X \cap Y)$$

(14) Prove that:
$$X \cup (Y - X) = (X \cup Y)$$

(15) Prove that:
$$X \cap (Y - X) = \varnothing$$

(16) Prove that:
$$(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$$

(17) Prove that:
$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

(18) Prove that:
$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$

(19) Prove that:
$$X - (Y \cap Z) = (X - Y) \cup (X - Z)$$

(20) Prove that:
$$X - (Y \cup Z) = (X - Y) \cap (X - Z)$$

(21) If $X \cup Y = X$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

(22) If $X \cap Y = X$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

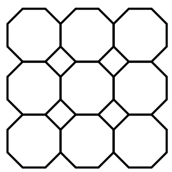
(23) If $X - Y = \varnothing$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

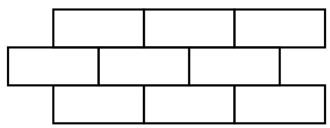
1.2 Tessellations

Go to the internet and look up M.C. Escher. He was an artist. Look at some of his work. When you do your search be sure to include the word “tessellation” OK? Back already? Very good. Sometimes Escher worked with tessellations. What’s a tessellation? I’m glad you asked:

Definition A **tessellation** is a pattern of polygons fitted together to cover the entire plane without overlapping.

While it is impossible to actually cover the entire plane with shapes, if we give you enough of a tessellation, you should be able to continue it’s pattern indefinitely. Here are pieces of tessellations:

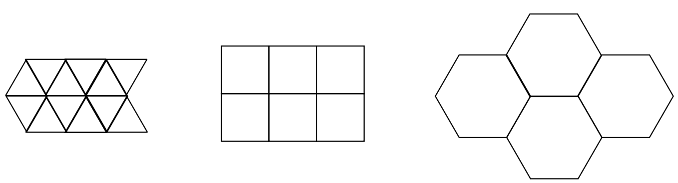




On the left we have a tessellation of a square and an octagon. On the right we have a “brick-like” tessellation.

Definition A tessellation is called a **regular tessellation** if it is composed of copies of a single regular polygon and these polygons meet vertex to vertex.

Example 1.2.1) Here are some examples of regular tessellations:



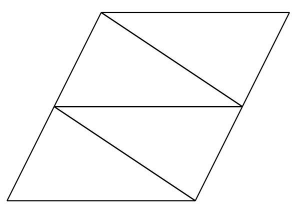
Johannes Kepler, who lived from 1571–1630, was one of the first people to study tessellations. He certainly knew the next theorem:

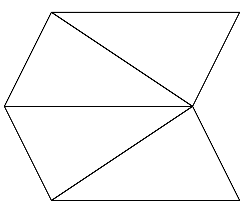
Theorem 1.2.2 *There are only* 3 *regular tessellations.*

Question Why is the theorem above true?

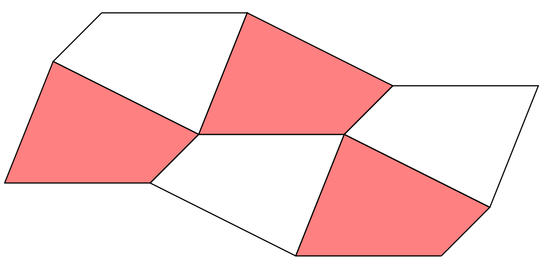
?

Since one can prove that there are only three regular tessellations, and we have shown three above, then that is all of them. On the other hand there are lots of nonregular tessellations. Here are two different ways to tessellate the plane with a triangle:





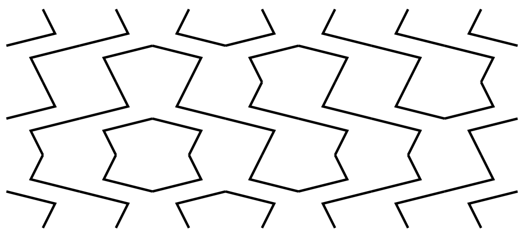
Here is a way that you can tessellate the plane with any old quadrilateral:



1.2.1 Tessellations and Art

How does one make art with tessellations? To start, a little decoration goes a long way. Check this out: Decorate two squares as such:

Tessellate them randomly in the plane to get this lightning-like picture:



Question What sort of picture do you get if you tessellate these decorated squares randomly in a plane?

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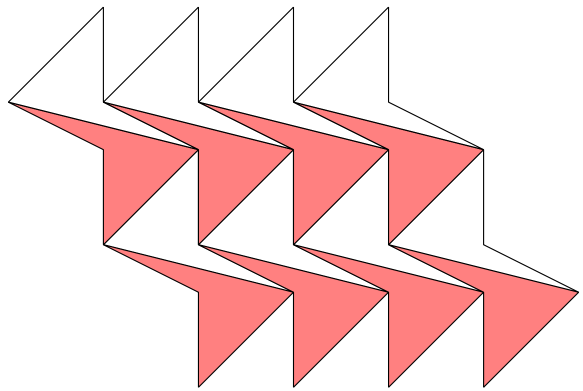
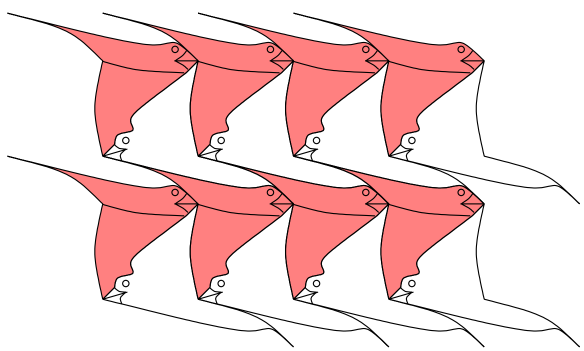
Another way to go is to start with your favorite tessellation:



Then you modify it a bunch to get something different:



Question What kind of art can you make with tessellations?

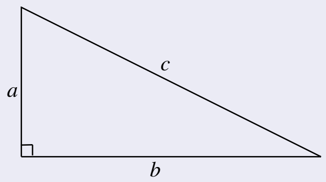
?

I'm not a very good artist, but I am a mathematician. So let's use a tessellation to give a proof! Let me ask you something:

Question What is the most famous theorem in mathematics?

Probably the Pythagorean Theorem comes to mind. Let's recall the statement of the Pythagorean Theorem:

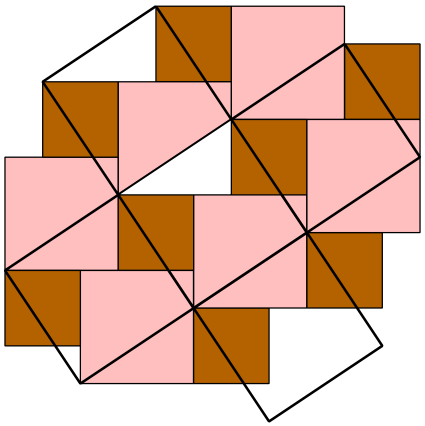
Theorem 1.2.3 (Pythagorean Theorem) *Given a right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. Symbolically, if a and b represent the lengths of the legs and c is the length of the hypotenuse,*



then

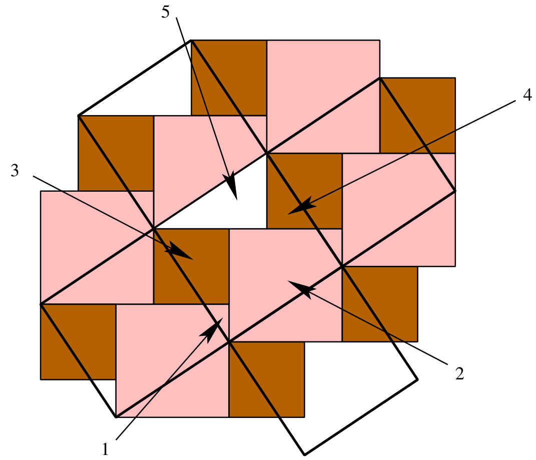
$$a^2 + b^2 = c^2.$$

Let's give a proof! Check out this tessellation involving 2 squares:



Question How does the picture above "prove" the Pythagorean Theorem?

Proof (Solution) *The white triangle is our right triangle. The area of the middle overlaid square is c^2, the area of the small dark squares is a^2, and the area of the medium lighter square is b^2. Now label all the "parts" of the large overlaid square:*



From the picture we see that

$$a^2 = \{3 \text{ and } 4\}$$
$$b^2 = \{1, 2, \text{ and } 5\}$$
$$c^2 = \{1, 2, 3, 4, \text{ and } 5\}$$

Hence

$$c^2 = a^2 + b^2$$

Since we can always put two squares together in this pattern, this proof will work for any right triangle.

Problems for Section 1.2

(1) Show two different ways of tessellating the plane with a given scalene triangle. Label your picture as necessary.

(2) Show how to tessellate the plane with a given quadrilateral. Label your picture.

(3) Show how to tessellate the plane with a nonregular hexagon. Label your picture.

(4) Give an example of a polygon with 9 sides that tessellates the plane.

(5) Give examples of polygons that tessellate and polygons that do not tessellate.

(6) Give an example of a triangle that tessellates the plane where both 4 and 8 angles fit around each vertex.

(7) True or False: Explain your conclusions.

  (a) There are exactly 5 regular tessellations.

  (b) Any quadrilateral tessellates the plane.

  (c) Any triangle will tessellate the plane.

  (d) If a triangle is used to tessellate the plane, then it is always the case that exactly 6 angles will fit around each vertex.

  (e) If a polygon has more than 6 sides, then it cannot tessellate the plane.

(8) Given a regular tessellation, what is the sum of the angles around a given vertex?

(9) Given that the regular octagon has 135 degree angles, explain why you cannot give a regular tessellation of the plane with a regular octagon.

(10) Fill in the following table:

Regular n-gon	Does it tessellate?	Measure of an angle	If it tessellates, how many surround each vertex?
3-gon			
4-gon			
5-gon			
6-gon			
7-gon			
8-gon			
9-gon			
10-gon			

Hint: A regular n-gon has interior angles of $180(n-2)/n$ degrees.

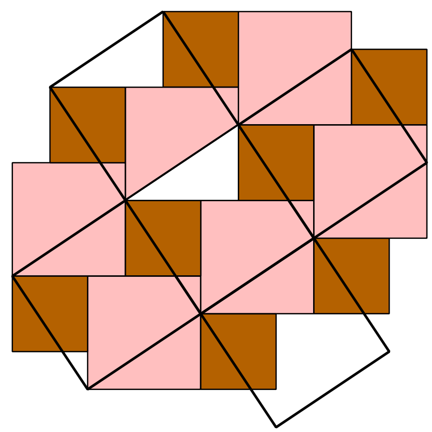
  (a) What do the shapes that tessellate have in common?

  (b) Make a graph with the number of sides of an n-gon on the horizontal axis and the measure of a single angle on the vertical axis. Briefly describe the relationship between the number of sides of a regular n-gon and the measure of one of its angles.

  (c) What regular polygons *could* a bee use for building hives? Give some reasons that bees seem to use hexagons.

(11) Considering that the regular n-gon has interior angles of $180(n-2)/n$ degrees, and Problem (10) above, prove that there are only 3 regular tessellations of the plane.

(12) Explain how the following picture “proves” the Pythagorean Theorem.



1.3 Proof by Picture

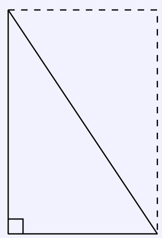
Fixnote: Citation omitted until it works.

Pictures generally do not constitute a proof on their own. However, a good picture can show insight and communicate concepts better than words alone. In this section we will show you pictures giving the idea of a proof and then ask you to supply the words to finish off the argument.

1.3.1 Proofs Involving Right Triangles

Let's start with something easy:

Question Explain how the following picture "proves" that the area of a right triangle is half the base times the height.



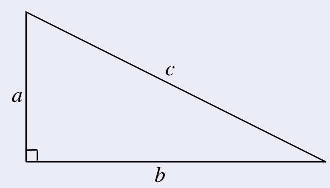
?

That wasn't so bad was it? Now for a game of *whose-who*:

Question What is the most famous theorem in mathematics?

Probably the Pythagorean Theorem comes to mind. Let's recall the statement of the Pythagorean Theorem:

Theorem 1.3.1 (Pythagorean Theorem) *Given a right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. Symbolically, if a and b represent the lengths of the legs and c is the length of the hypotenuse,*



then

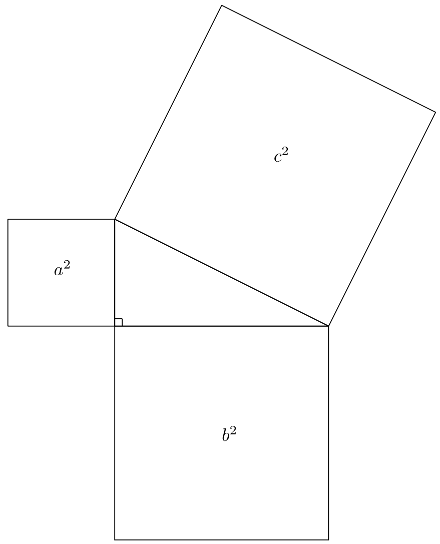
$$a^2 + b^2 = c^2.$$

Question What is the converse to the Pythagorean Theorem? Is it true? How do you prove it?

?

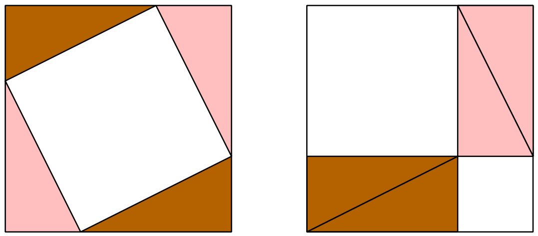
While everyone may know the Pythagorean Theorem, not as many know how to prove it. Euclid's proof goes kind of like this:

Consider the following picture:



Now, cut up the squares a^2 and b^2 in such a way that they fit into c^2 perfectly. When you give a proof that involves cutting up the shapes and putting them back together, it is called a **dissection proof**. The trick to ensure that this is actually a proof is in making sure that your dissection will work no matter what right triangle you are given. Does it sound complicated? Well it can be.

Is there an easier proof? Sure, look at:



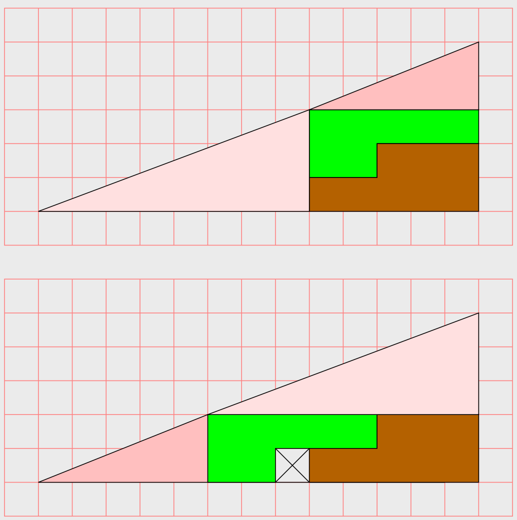
Question How does the picture above "prove" the Pythagorean Theorem?

Proof (Solution) *Both of the large squares above are the same size. Moreover both the unshaded regions above must have the same area. The large white square on the left has an area of* c^2 *and the two white squares on the right have a combined area of* $a^2 + b^2$*. Thus we see that:*

$$c^2 = a^2 + b^2$$

Now a paradox:

Paradox What is wrong with this picture?



Question How does this happen?

?

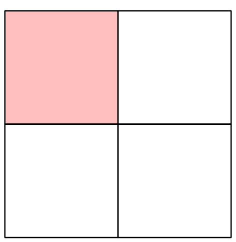
1.3.2 Proofs Involving Boxy Things

Consider the problem of *Doubling the Cube*. If a mathematician asks us to double a cube, he or she is asking us to double the **volume** of a given cube. One may be tempted to merely double each side, but this doesn't double the volume!

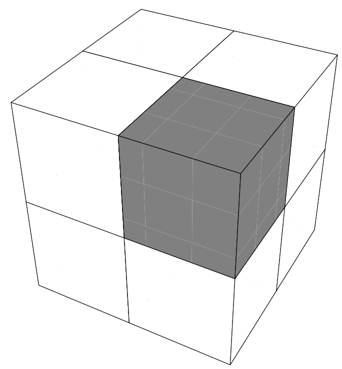
Question Why doesn't doubling each side of the cube double the volume of the cube?

?

Well, let's answer an easier question first. How do you double the area of a square? Does taking each side and doubling it work?



No! You now have four times the area. So you **cannot** double the area of a square merely by doubling each side. What about for the cube? Can you double the volume of a cube merely by doubling the length of every side? Check this out:



Ah, so the answer is again no. If you double each side of a cube you have 8 times the volume.

Question What happens to the area of a square if you multiply the sides by an arbitrary integer? What about the volume of a cube? Can you explain what is happening here?

?

1.3.3 Proofs Involving Infinite Sums

As is our style, we will start off with a question:

Question Can you add up an infinite number of terms and still get a finite number?

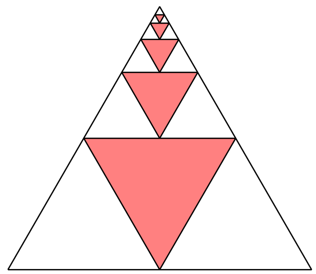
Consider 1/3. Actually, consider the decimal notation for 1/3:

$$\frac{1}{3} = .33333333333333333333333333333\ldots$$

But this is merely the sum:

$$.3 + .03 + .003 + .0003 + .00003 + .000003 + \cdots$$

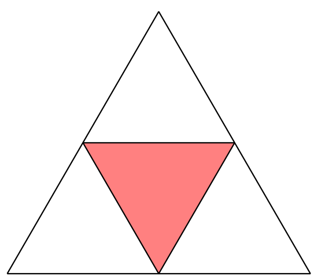
It stays less than 1 because the terms get so small so quickly. Are there other infinite sums of this sort? You bet! Check out this picture:



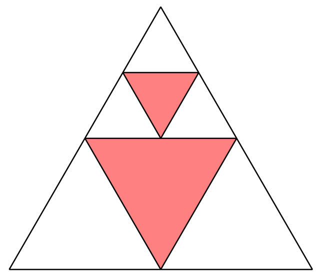
Question Explain how the picture above "proves" that:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \cdots = \frac{1}{3}$$

Proof (Solution) *Let's take it in steps. If the big triangle has area* 1*, the area of the shaded region below is* $1/4$*.*



We also see that the area of the shaded region below



is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2$$

Continuing on in this fashion we see that the area of all the shaded regions is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \cdots$$

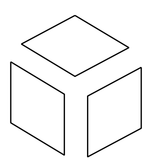
But look, the unshaded triangles have twice as much area as the shaded triangle. Thus the shaded triangles must have an area of $1/3$.

1.3.4 Thinking Outside the Box

A *calisson* is a French candy that sort of looks like two equilateral triangles stuck together. They usually come in a hexagon-shaped box.

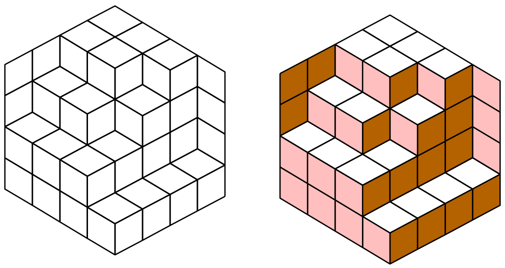
Question How do the calissons fit into their hexagon-shaped box?

If you start to put the calissons into a box, you quickly see that they can be placed in there with exactly three different orientations:



Theorem 1.3.2 *In any packing, the number of calissons with a given orientation is exactly one-third the total number of calissons in the box.*

Look at this picture:



Question How does the picture above "prove" Theorem 1.3.2? Hint: Think outside the box!

?

Problems for Section 1.3

(1) Explain the rule

$$\text{even} + \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(2) Explain the rule

$$\text{odd} + \text{even} = \text{odd}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(3) Explain the rule

$$\text{odd} + \text{odd} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(4) Explain the rule

$$\text{even} \cdot \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(5) Explain the rule

$$\text{odd} \cdot \text{odd} = \text{odd}$$

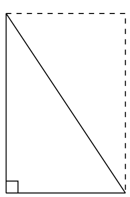
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(6) Explain the rule

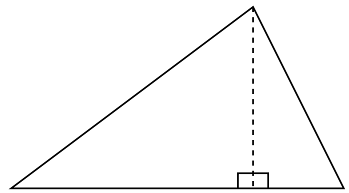
$$\text{odd} \cdot \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

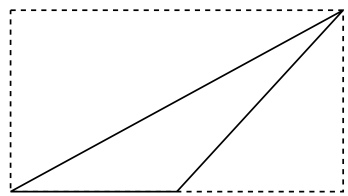
(7) Explain how the following picture "proves" that the area of a right triangle is half the base times the height.



(8) Suppose you know that the area of a **right** triangle is half the base times the height. Explain how the following picture "proves" that the area of **every** triangle is half the base times the height.

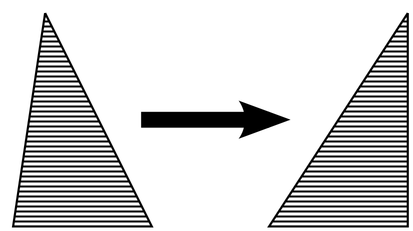


Now suppose that a student, say *Geometry Giorgio* attempts to solve a similar problem. Again knowing that the area of a right triangle is half the base times the height, he draws the following picture:

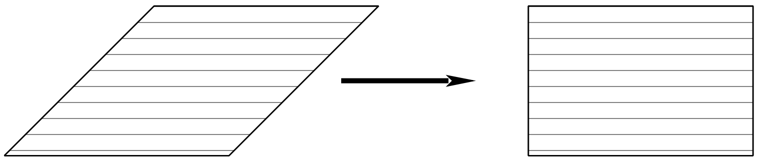


Geometry Giorgio states that the diagonal line cuts the rectangle in half, and thus the area of the triangle is half the base times the height. Is this correct reasoning? If so, give a complete explanation. If not, give correct reasoning based on *Geometry Giorgio*'s picture.

(9) Suppose you know that the area of a **right** triangle is half the base times the height. Explain how the following picture "proves" that the area of any triangle is half the base times the height. Note, this way of thinking is the basis for Cavalieri's Principle.

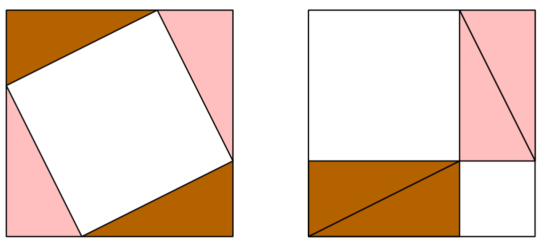


(10) Explain how the following picture “proves” that the area of any parallelogram is base times height. Note, this way of thinking is the basis for Cavalieri’s Principle.

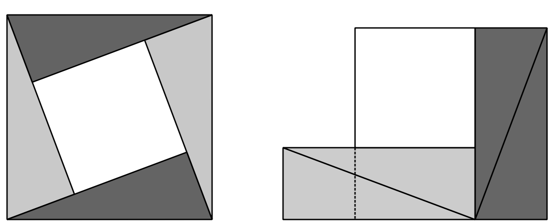


(11) Explain how to use a picture to “prove” that a triangle of a given area could have an arbitrarily large perimeter.

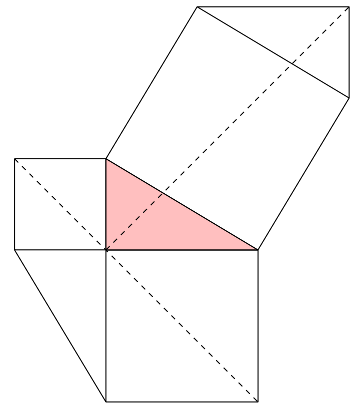
(12) Give two explanations of how the following picture “proves” the Pythagorean Theorem, one using algebra and one without algebra.



(13) Give two explanations of how the following picture “proves” the Pythagorean Theorem, one using algebra and one without algebra.

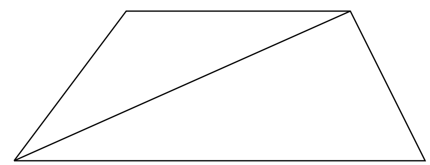


(14) Explain how the following picture “proves” the Pythagorean Theorem.



Note: This proof is due to Leonardo da Vinci.

(15) Recall that a trapezoid is a quadrilateral with two parallel sides. Consider the following picture:

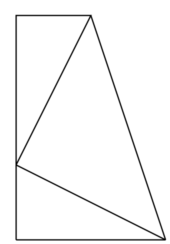


How does the above picture prove that the area of a trapezoid is

$$\text{area} = \frac{h(b_1 + b_2)}{2},$$

where h is the height of the trapezoid and b_1, b_2, are the lengths of the parallel sides?

(16) Explain how the following picture “proves” the Pythagorean Theorem.

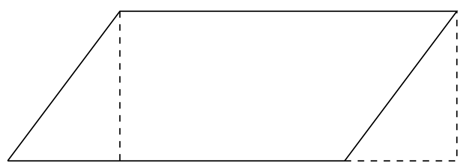


Note: This proof is due to James A. Garfield, the 20th President of the United States.

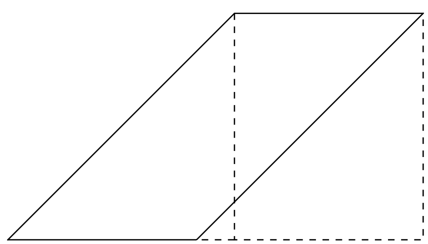
(17) Look at Problem (15). Can you use a similar picture to prove that the area of a parallelogram

is the length of the base times the height?

(18) Explain how the following picture "proves" that the area of a parallelogram is base times height.



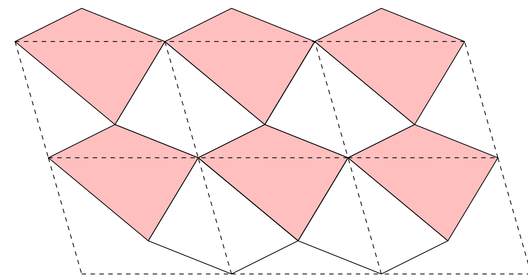
Now suppose that a student, say *Geometry Giorgio* attempts to solve a similar problem. In an attempt to prove the formula for the area of a parallelogram, *Geometry Giorgio* draws the following picture:



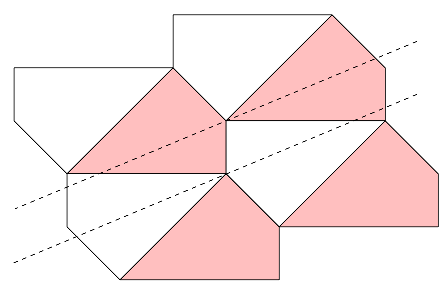
At this point *Geometry Giorgio* says that he has proved the formula for area of a parallelogram. What do you think of his picture? Give a complete argument based on his picture.

(19) Which of the above "proofs" for the formula for the area of a parallelogram is your favorite? Explain why.

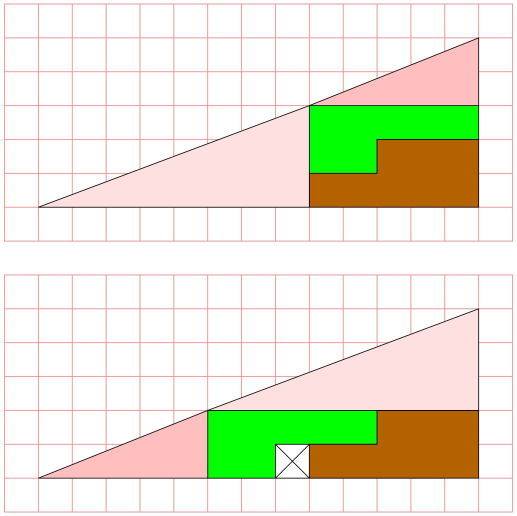
(20) Explain how the following picture "proves" that the area of a quadrilateral is equal to half of the area of the parallelogram whose sides are parallel to and equal in length to the diagonals of the original quadrilateral.



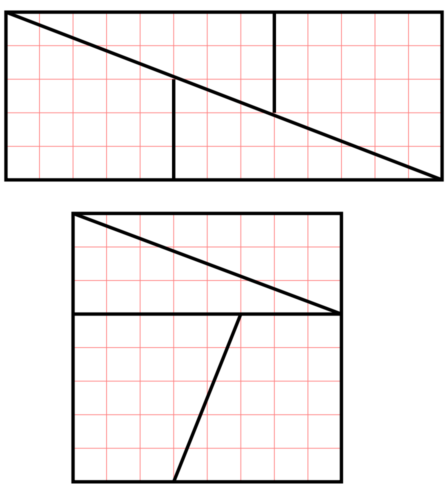
(21) Explain how the following picture "proves" that if a quadrilateral has two opposite angles that are equal, then the bisectors of the other two angles are parallel or on top of each other.



(22) Why might someone find the following picture disturbing? How would you assure them that actually everything is good and well in the geometrical world?



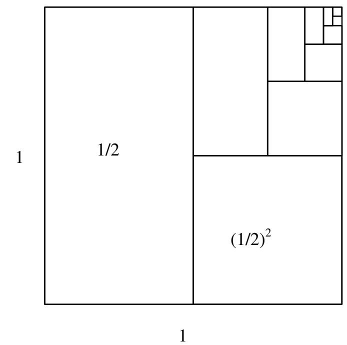
(23) Why might someone find the following picture disturbing? How would you assure them that actually everything is good and well in the geometrical world?



(24) How could you explain to someone that doubling the lengths of each side of a cube does not double the volume of the cube?

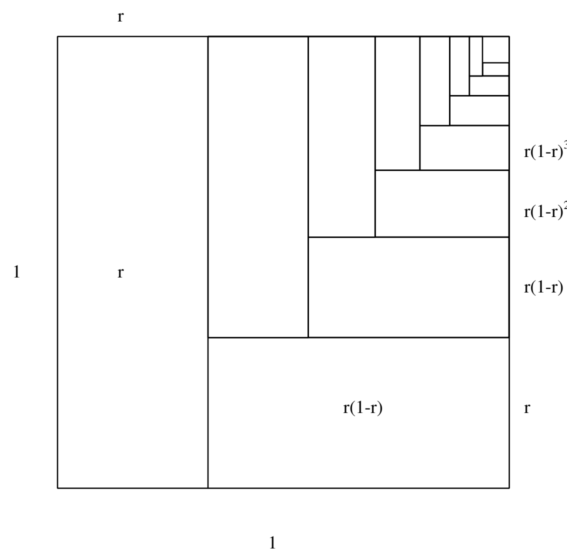
(25) Explain how the following picture "proves" that:

$$\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\cdots=1$$



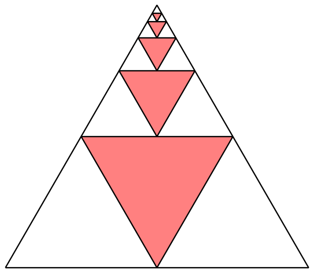
(26) Explain how the following picture "proves" that if $0<r<1$:

$$r+r(1-r)+r(1-r)^2+r(1-r)^3+\cdots=1$$



(27) Explain how the following picture "proves" that:

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\left(\frac{1}{4}\right)^5+\cdots=\frac{1}{3}$$

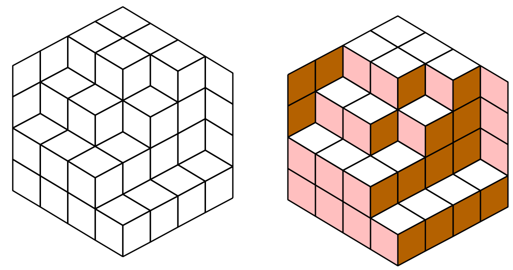


(28) Considering Problem (25), Problem (26), and Problem (27) can you give a new picture "proving" that:

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\left(\frac{1}{4}\right)^5+\cdots=\frac{1}{3}$$

Carefully explain the connection between your picture and the mathematical expression above.

(29) Explain how the following picture "proves" that in any packing, the number of calissons with a given orientation is exactly one-third the total number of calissons in the box.



2 Compass and Straightedge Constructions

Mephistopheles: I must say there is an obstacle
That prevents my leaving:
It's the pentagram on your threshold.
Faust: The pentagram impedes you?
Tell me then, you son of hell,
If this stops you, how did you come in?
Mephistopheles: Observe! The lines are poorly drawn;
That one, the outer angle,
Is open, the lines don't meet.

—Göthe, *Faust* act I, scene III

2.1 Constructions

About a century before the time of Euclid, Plato—a student of Socrates—declared that the compass and straightedge should be the only tools of the geometer. Why would he do such a thing? For one thing, both the the compass and straightedge are fairly simple instruments. One draws circles, the other draws lines—what else could possibly be needed to study geometry? Moreover, rulers and protractors are far more complex in comparison and people back then couldn't just walk to the campus bookstore and buy whatever they wanted. However, there are other reasons:

(1) Compass and straightedge constructions are **independent of units**.

(2) Compass and straightedge constructions are **theoretically correct**.

(3) Combined, the compass and straightedge seem like **powerful tools**.

Compass and straightedge constructions are **independent of units**. Whether you are working in centimeters or miles, compass and straightedge constructions work just as well. By not being locked to set of units, the constructions given by a compass and straightedge have certain generality that is appreciated even today.

Compass and straightedge constructions are **theoretically correct**. In mathematics, a correct method to solve a problem is more valuable than a correct solution. In this sense, the compass and straightedge are ideal tools for the mathematician. Easy enough to use that the rough drawings that they produce can be somewhat relied upon, yet simple enough that the tools themselves can be described theoretically. Hence it is usually not too difficult to connect a given construction to a formal proof showing that the construction is correct.

Combined, the compass and straightedge seem like **powerful tools**. No tool is useful unless it can solve a lot of problems. Without a doubt, the compass and straightedge combined form a powerful tool. Using a compass and straightedge, we are able to solve many problems exactly. Of the problems that we cannot solve exactly, we can always produce an approximate solution.

We'll start by giving the rules of compass and straightedge constructions:

Rules for Compass and Straightedge Constructions

(1) You may only use a compass and straightedge.

(2) You must have two points to draw a line.

(3) You must have a point and a line segment to draw a circle. The point is the center and the line segment gives the radius.

(4) Points can only be placed in two ways:

   (a) As the intersection of lines and/or circles.

   (b) As a **free point**, meaning the location of the point is not important for the final outcome of the construction.

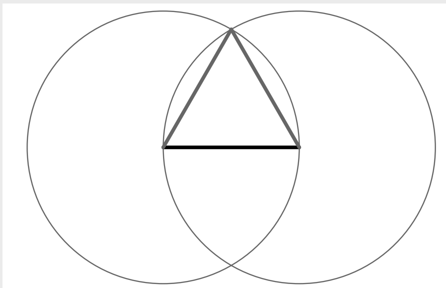
Our first construction is also Euclid's first construction:

Construction (Equilateral Triangle) We wish to construct an equilateral triangle given the length of one side.

(1) Open your compass to the width of the line segment.

(2) Draw two circles, one with the center being each end point of the line segment.

(3) The two circles intersect at two points. Choose one and connect it to both of the line segment's endpoints.



Euclid's second construction will also be our second construction:

Construction (Transferring a Segment) Given a segment, we wish to move it so that it starts on a given point, on a given line.

(1) Draw a line through the point in question.

(2) Open your compass to the length of the line segment and draw a circle with the given point as its center.

(3) The line segment consisting of the given point and the intersection of the circle and the line is the transferred segment.

If you read *The Elements*, you'll see that Euclid's construction is much more complicated than ours. Apparently, Euclid felt the need to justify the ability to move a distance. Many sources say that Euclid used what is called a *collapsing compass*, that is a compass that collapsed when it was picked up. However, I do not believe that such an invention ever existed. Rather this is something that lives in the conservative geometer's head.

Regardless of whether the difficulty of transferring distances was theoretical or physical, we need not worry when we do it. In fact, Euclid's proof of the above theorem proves that our modern way of using the compass to transfer distances is equivalent to using the so-called collapsing compass.

Question Exactly how would one prove that the modern compass is equivalent to the collapsing compass? Hint: See Euclid's proof.

?

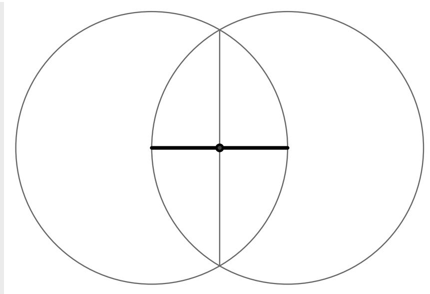
Construction (Bisecting a Segment) Given a segment, we wish to cut it in half.

(1) Open your compass to the width of the segment.

(2) Draw two circles, one with the center being at each end point of the line segment.

(3) The circles intersect at two points. Draw a line through these two points.

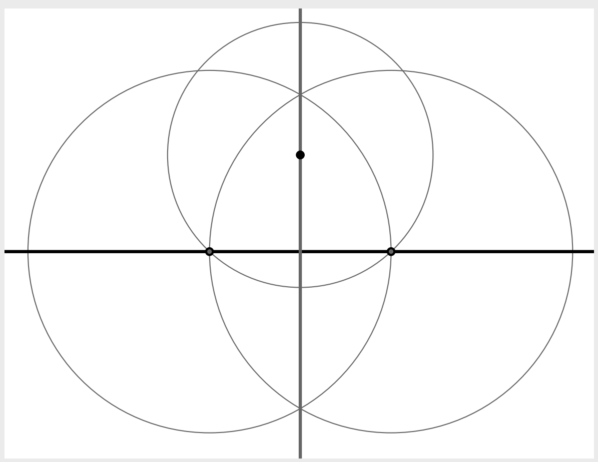
(4) The new line bisects the original line segment.



Construction (Perpendicular to a Line through a Point) Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.

(1) Draw a circle centered at the point large enough to intersect the line in two distinct points.

(2) Bisect the line segment. The line used to do this will be the desired line.

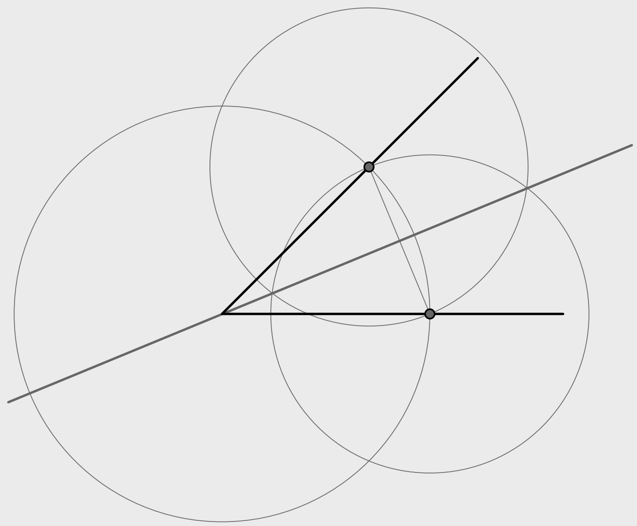


Construction (Bisecting an Angle) We wish to divide an angle in half.

(1) Draw a circle with its center being the vertex of the angle.

(2) Draw a line segment where the circle intersects the lines.

(3) Bisect the new line segment. The bisector will bisect the angle.



We now come to a very important construction:

Construction (Copying an Angle) Given a point on a line and some angle, we wish to copy the given angle so that the new angle has the point as its vertex and the line as one of its edges.

(1) Open the compass to a fixed width and make a circle centered at the vertex of the angle.

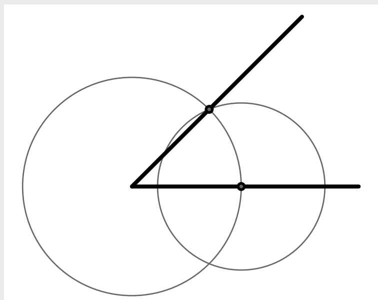
(2) Make a circle of the same radius on the line with the point.

(3) Open the compass so that one end touches the 1st circle where it hits an

edge of the original angle, with the other end of the compass extended to where the 1st circle hits the other edge of the original angle.

(4) Draw a circle with the radius found above with its center where the second circle hits the line.

(5) Connect the point to where the circles meet. This is the other leg of the angle we are constructing.

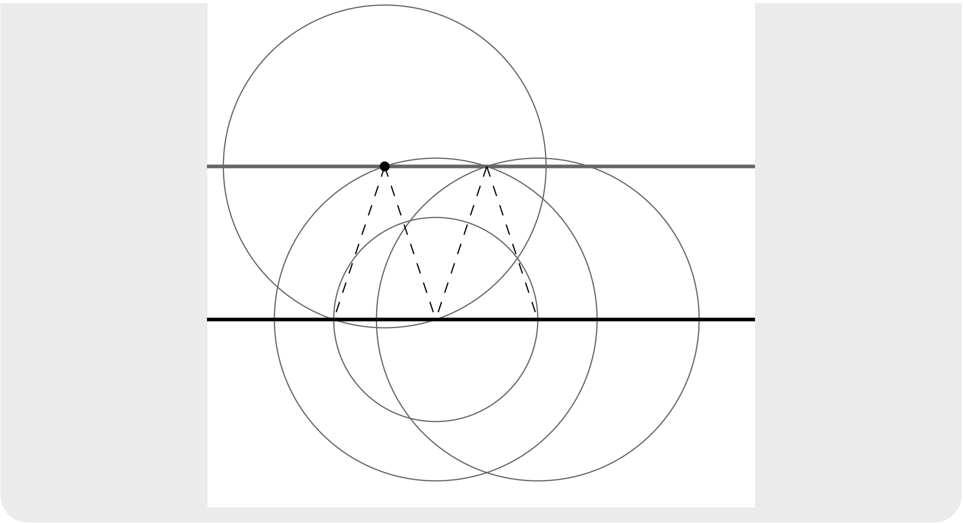


Construction (Parallel to a Line through a Point) Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.

(1) Draw a circle around the given point that passes through the given line at two points.

(2) We now have an isosceles triangle, duplicate this triangle.

(3) Connect the top vertexes of the triangles and we get a parallel line.



Question Can you give another different construction?

?

Problems for Section 2.1

(1) What are the rules for compass and straightedge constructions?

(2) What is a collapsing compass? Why don't we use them or worry about them any more?

(3) Prove that the collapsing compass is equivalent to the modern compass.

(4) Given a line segment, construct an equilateral triangle whose edge has the length of the given segment. Explain the steps in your construction and how you know it works.

(5) Use a compass and straightedge to bisect a given line segment. Explain the steps in your construction and how you know it works.

(6) Given a line segment with a point on it, construct a line perpendicular to the segment that passes through the given point. Explain the steps in your construction and how you know it works.

(7) Use a compass and straightedge to bisect a given angle. Explain the steps in your construction and how you know it works.

(8) Given an angle and some point, use a compass and straightedge to copy the angle so that the new angle has as its vertex the given point. Explain the steps in your construction and how you know it works.

(9) Given a point and line, construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction and how you know it works.

(10) Given a point and line, construct a line parallel to the given line that passes through the given point. Explain the steps in your construction and how you know it works.

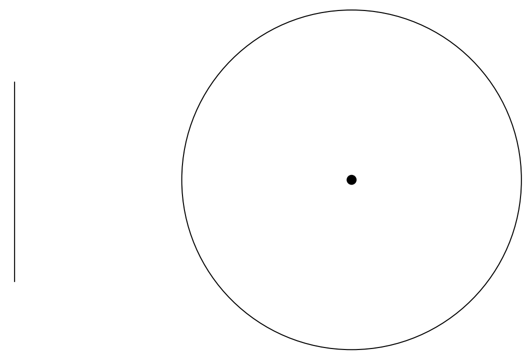
(11) Given a length of 1, construct a triangle whose perimeter is a multiple of 6. Explain the steps in your construction and how you know it works.

(12) Construct a 30-60-90 right triangle. Explain the steps in your construction and how you know it works.

(13) Given a length of 1, construct a triangle with a perimeter of $3 + \sqrt{5}$. Explain the steps in your construction and how you know it works.

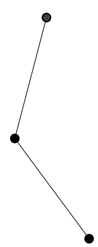
(14) Given a length of 1, construct a triangle with a perimeter that is a multiple of $2 + \sqrt{2}$. Explain the steps in your construction and how you know it works.

(15) Here is a circle and here is the side length of an inscribed regular 5-gon.



Construct the regular 5-gon. Explain the steps in your construction and how you know it works.

(16) Here is a piece of a regular 7-gon.



Construct the entire regular 7-gon. Explain the steps in your construction and how you know it works.

2.2 Anatomy of Figures

In studying geometry we seek to discover the points that can be obtained given a set of rules. In our case the set of rules consists of the rules for compass and straightedge constructions.

Question In regards to compass and straightedge constructions, what is a *point*?

?

Question In regards to compass and straightedge constructions, what is a *line*?

?

Question In regards to compass and straightedge constructions, what is a *circle*?

?

OK, those are our basic figures, pretty easy right? Now I'm going to quiz you about them:

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

2.2.1 Lines Related to Triangles

Believe it or not, in mathematics we often try to study the simplest objects as deeply as possible. After the objects listed above, triangles are among the most basic of geometric figures, yet there is much to know about them. There are several lines that are commonly associated to triangles. Here they are:

- Perpendicular bisectors of the sides.
- Bisectors of the angles.
- Altitudes of the triangle.
- Medians of the triangle.

The first two lines above are self-explanatory. The next two need definitions.

Definition An **altitude** of a triangle is a line segment originating at a vertex of the triangle that meets the line containing the opposite side at a right angle.

Definition A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side.

Question The intersection of any two lines containing the altitudes of a triangle is called an **orthocenter**. How many orthocenters does a given triangle have?

?

Question The intersection of any two medians of a triangle is called a **centroid**. How many centroids does a given triangle have?

?

Question What is the physical meaning of a centroid?

?

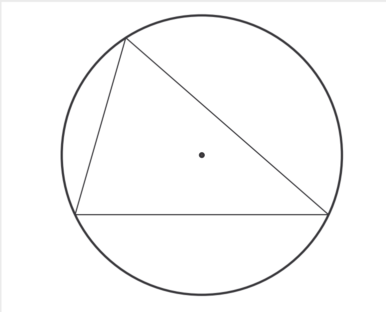
2.2.2 Circles Related to Triangles

There are also two circles that are commonly associated to triangles. Here they are:

- The circumcircle.
- The incircle.

These aren't too bad. Check out the definitions.

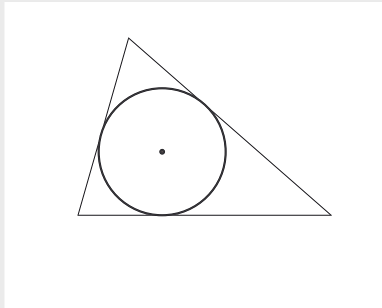
Definition The **circumcircle** of a triangle is the circle that contains all three vertexes of the triangle. Its center is called the **circumcenter** of the triangle.



Question Does every triangle have a circumcircle?

?

Definition The **incircle** of a triangle is the largest circle that will fit inside the triangle. Its center is called the **incenter** of the triangle.



Question Does every triangle have an incircle?

?

Question Are any of the lines described above related to these circles and/or centers? Clearly articulate your thoughts.

?

Problems for Section 2.2

(1) Compare and contrast the idea of “intersecting sets” with the idea of “intersecting lines.”

(2) Place three points in the plane. Give a detailed discussion explaining how they may or may not be on a line.

(3) Place three lines in the plane. Give a detailed discussion explaining how they may or may not intersect.

(4) Explain how a perpendicular bisector is different from an altitude. Draw an example to illustrate the difference.

(5) Explain how a median is different from an angle bisector. Draw an example to illustrate the difference.

(6) What is the name of the point that is the same distance from all three sides of a triangle? Explain your reasoning.

(7) What is the name of the point that is the same distance from all three vertexes of a triangle? Explain your reasoning.

(8) Could the circumcenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(9) Could the orthocenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(10) Could the incenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(11) Could the centroid be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(12) Are there shapes that do not contain their centroid? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(13) Draw an equilateral triangle. Now draw the lines containing the altitudes of this triangle. How many orthocenters do you have as intersections of lines in your drawing? Hints:

  (a) More than one.

  (b) How many triangles are in the picture you drew?

(14) Given a triangle, construct the circumcenter. Explain the steps in your construction.

(15) Given a triangle, construct the orthocenter. Explain the steps in your construction.

(16) Given a triangle, construct the incenter. Explain the steps in your construction.

(17) Given a triangle, construct the centroid. Explain the steps in your construction.

(18) Given a triangle, construct the incircle. Explain the steps in your construction.

(19) Given a triangle, construct the circumcircle. Explain the steps in your construction.

(20) Given a circle, give a construction that finds its center.

(21) Where is the circumcenter of a right triangle? Explain your reasoning.

(22) Where is the orthocenter of a right triangle? Explain your reasoning.

(23) Can you draw a triangle where the circumcenter, orthocenter, incenter, and centroid are all the same point? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(24) True or False: Explain your conclusions.

  (a) An altitude of a triangle is always perpendicular to a line containing some side of the triangle.

  (b) An altitude of a triangle always bisects some side of the triangle.

  (c) The incenter is always inside the triangle.

  (d) The circumcenter, the centroid, and the orthocenter always lie in a line.

  (e) The circumcenter can be outside the triangle.

  (f) The orthocenter is always inside the triangle.

  (g) The centroid is always inside the incircle.

(25) Given 3 distinct points not all in a line, construct a circle that passes through all three points. Explain the steps in your construction.

2.3 Trickier Constructions

Question How do you construct regular polygons? In particular, how do you construct regular: 3-gons, 4-gons, 5-gons, 6-gons, 7-gons, 8-gons, 10-gons, 12-gons, 17-gons, 24-gons, and 144-gons?

?

Well the equilateral triangle is easy. It was the first construction that we did. What about squares? What about regular hexagons? It turns out that they aren't too difficult. What about pentagons? Or say n-gons? We'll have to think about that. Let's leave the difficult land of n-gons and go back to thinking about nice, three-sided triangles.

Construction (SAS Triangle) Given two sides with an angle between them, we wish to construct the triangle with that angle and two adjacent sides.

(1) Transfer the one side so that it starts at the vertex of the angle.

(2) Transfer the other side so that it starts at the vertex.

(3) Connect the end points of all moved line segments.

The "SAS" in this construction's name spawns from the fact that it requires two sides with an angle *between* them. The SAS Theorem states that we can obtain a unique triangle given two sides and the angle between them.

Construction (SSS Triangle) Given three line segments we wish to construct the triangle that has those three sides if it exists.

(1) Choose a side and select one of its endpoints.

(2) Draw a circle of radius equal to the length of the second side around the chosen endpoint.

(3) Draw a circle of radius equal to the length of the third side around the other endpoint.

(4) Connect the end points of the first side and the intersection of the circles. This is the desired triangle.

Question Can this construction fail to produce a triangle? If so, show how. If not, why not?

?

Question Remember earlier when we asked about the converse to the Pythagorean Theorem? Can you use the construction above to prove the converse of the Pythagorean Theorem?

?

Question Can you state the SSS Theorem?

?

Construction (SAA Triangle) Given a side and two angles, where the given side does not touch one of the angles, we wish to construct the triangle that has this side and these angles if it exists.

(1) Start with the given side and place the adjacent angle at one of its endpoints.

(2) Move the second angle so that it shares a leg with the leg of the first angle–not the leg with the side.

(3) Extend the side past the first angle, forming a new angle with the leg of the second angle.

(4) Move this new angle to the other endpoint of the side, extending the legs of this angle and the first angle will produce the desired triangle.

Question Can this construction fail to produce a triangle? If so, show how. If not, why not?

?

Question Can you state the SAA Theorem?

?

Question What about other combinations of S's and A's?

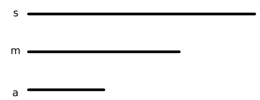
SSS, SSA, SAS, SAA, ASA, AAA

?

2.3.1 Challenge Constructions

Question How can you construct a triangle given the length of one side s, the length of the median to that side m, and the length of the altitude of the opposite angle a?

Proof (Follow-Along) *Use these lengths and follow the directions below.*



(1) *Start with the given side.*

(2) *Since the median hits our side at the center, bisect the given side.*

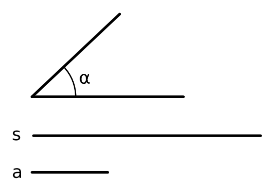
(3) *Make a circle of radius equal to the length of the median centered at the bisector of the given side.*

(4) *Construct a line parallel to our given line of distance equal to the length of the given altitude away.*

(5) *Where the line and the circle intersect is the third point of our triangle. Connect the endpoints of the given side and the new point to get the triangle we want.*

Question How can you construct a triangle given one angle α, the length of an adjacent side s, and the altitude to that side a?

Proof (Follow-Along) *Use these and follow the directions below.*



(1) *Start with a line containing the side.*

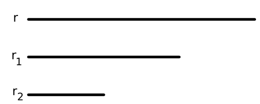
(2) *Put the angle at the end of the side.*

(3) *Draw a parallel line to the side of the length of the altitude away.*

(4) *Connect the angle to the parallel side. This is the third vertex. Connect the endpoints of the given side and the new point to get the triangle we want.*

Question How can you construct a circle with a given radius tangent to two other circles?

Proof (Follow-Along) *Use these and follow the directions below.*



(1) *Let r be the given radius, and let r_1 and r_2 be the radii of the given circles.*

(2) *Draw a circle of radius $r_1 + r$ around the center of the circle of radius r_1.*

(3) *Draw a circle of radius $r_2 + r$ around the center of the circle of radius r_2.*

(4) *Where the two circles drawn above intersect is the center of the desired circle.*

Question Place two tacks in a wall. Insert a sheet of paper so that the edges hit the tacks and the corner passes through the imaginary line between the tacks. Mark where the corner of the piece of paper touches the wall. Repeat this process, sliding the paper around. What curve do you end up drawing?

?

Question How can you construct a triangle given an angle and the length of the opposite side?

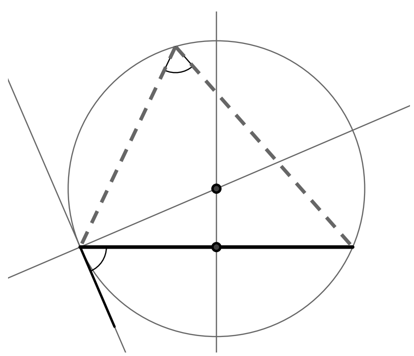
Proof (Solution) *We really can't solve this problem completely because the information given doesn't uniquely determine a triangle. However, we can still say something. Here is what we can do:*

(1) *Put the known angle at one end of the line segment. Note in the picture below, it is at the left end of the line segment and it is opening downwards.*

(2) *Construct the perpendicular bisector of the given segment.*

(3) *See where the bisector in Step 2 intersects the perpendicular of the other leg of the angle drawn from the vertex of the angle.*

(4) *Draw arc centered at the point found in Step 3 that touches the endpoints of the original segment.*



Every point on the arc is a valid choice for the vertex of the triangle.

Question Why does the above method work?

?

Question You are on a boat at night. You can see three lighthouses, and you know their position on a map. Also you know the angles of the light rays between the lighthouses as measured from the boat. How do you figure out where you are?

?

2.3.2 Problem Solving Strategies

The harder constructions discussed in this section can be difficult to do. There is no rote method to solve these problems, hence you must rely on your brain. Here are some hints that you may find helpful:

Construct what you can. You should start by constructing anything you can, even if you don't see how it will help you with your final construction. In doing so you are "chipping away" at the problem just as a rock-cutter chips away at a large boulder. Here are some guidelines that may help when constructing triangles:

(1) If a side is given, then you should draw it.

(2) If an angle is given and you know where to put it, draw it.

(3) If an altitude of length ℓ is given, then draw a line parallel to the side that the altitude is perpendicular to. This new line must be distance ℓ from the side.

(4) If a median is given, then bisect the segment it connects to and draw a circle centered around the bisector, whose radius is the length of the median.

(5) If you are working on a figure, construct any "mini-figures" inside the figure you are trying to construct. For example, many of the problems below ask you to construct a triangle. Some of these constructions have right-triangles inside of them, which are easier to construct than the final figure.

Sketch what you are trying to find. It is a good idea to try to sketch the figure that you are trying to construct. Sketch it accurately and label all pertinent parts. If there are special features in the figure, say two segments have the same length or there is a right-angle, make a note of it on your sketch. Also mark what is unknown in your sketch. We hope that doing this will help organize your thoughts and get your "brain juices" flowing.

Question Why are the above strategies good?

?

Problems for Section 2.3

(1) Construct a square. Explain the steps in your construction.

(2) Construct a regular hexagon. Explain the steps in your construction.

(3) Your friend Margy is building a clock. She needs to know how to align the twelve numbers on her clock so that they are equally spaced on a circle. Explain how to use a compass and straightedge construction to help her out. Illustrate your answer with a construction and explain the steps in your construction.

(4) Construct a triangle given two sides of a triangle and the angle between them. Explain the steps in your construction.

(5) State the SAS Theorem.

(6) Construct a triangle given three sides of a triangle. Explain the steps in your construction.

(7) State the SSS Theorem.

(8) Construct a triangle given a side and two angles where one of the angles does not touch the given side. Explain the steps in your construction.

(9) State the SAA Theorem.

(10) Construct a triangle given a side between two given angles. Explain the steps in your construction.

(11) State the ASA Theorem.

(12) Explain why when given an isosceles triangle, that two of its angles have equal measure. Hint: Use the SAS Theorem.

(13) Construct a figure showing that a triangle cannot always be uniquely determined when given an angle, a side adjacent to that angle, and the side opposite the angle. Explain the steps in your construction and explain how your figure shows what is desired. Explain what this says about the possibility of a SSA theorem. Hint: Draw many pictures to help yourself out.

(14) Give a construction showing that a triangle is uniquely determined if you are given a right-angle, a side touching that angle, and another side not touching the angle. Explain the steps in your construction and explain how your figure shows what is desired.

(15) Construct a triangle given two adjacent sides of a triangle and a median to one of the given sides. Explain the steps in your construction.

(16) Construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

(17) Construct a triangle given a side, the median to the side, and the angle opposite to the side. Explain the steps in your construction.

(18) Construct a triangle given an altitude, and two angles not touching the altitude. Explain the steps in your construction.

(19) Construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(20) Construct a triangle, given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(21) Construct a circle with a given radius tangent to two other given circles. Explain the steps in your construction.

(22) Does a given angle and a given opposite side uniquely determine a triangle? Explain your answer.

(23) You are on the bank of a river. There is a tree directly in front of you on the other side of the river. Directly left of you is a friend a known distance away. Your friend knows the angle starting with them, going to the tree, and ending with you. How wide is the river? Explain your work.

(24) You are on a boat at night. You can see three lighthouses, and you know their position on a map. Also you know the angles of the light rays from the lighthouses. How do you figure out where you are? Explain your work.

(25) Construct a triangle given an angle, the length of a side adjacent to the given angle, and the length of the angle's bisector to the opposite side. Explain the steps in your construction.

(26) Construct a triangle given an angle, the length of the opposite side, and the length of the altitude of the given angle. Explain the steps in your construction.

(27) Construct a triangle given one side, the length of the altitude of the opposite angle, and the radius of the circumcircle. Explain the steps in your construction.

(28) Construct a triangle given one side, the length of the altitude of an adjacent angle, and the radius of the circumcircle. Explain the steps in your construction.

(29) Construct a triangle given one side, the length of the median connecting that side to the opposite angle, and the radius of the circumcircle. Explain the steps in your construction.

(30) Construct a triangle given one angle and the lengths of the altitudes to the two other angles. Explain the steps in your construction.

(31) Construct a circle with a given radius tangent to two given intersecting lines. Explain the steps in your construction.

(32) Given a circle and a line, construct another circle of a given radius that is tangent to both the original circle and line. Explain the steps in your construction.

(33) Construct a circle with three smaller circles of equal size inside such that each smaller circle is tangent to the other two and the larger outside circle. Explain the steps in your construction.

3 Folding and Tracing Constructions

> We don't even know if Foldspace introduces us to one universe or many. . .
>
> —Frank Herbert

3.1 Constructions

While origami as an art form is quite ancient, folding and tracing constructions in mathematics are relatively new. The earliest mathematical discussion of folding and tracing constructions that I know of appears in T. Sundara Row's book *Geometric Exercises in Paper Folding*, , first published near the end of the Nineteenth Century. In the Twentieth Century it was shown that every construction that is possible with a compass and straightedge can be done with folding and tracing. Moreover, there are constructions that are possible via folding and tracing that are *impossible* with compass and straightedge alone. This may seem strange as you can draw a circle with a compass, yet this seems impossible to do via paper-folding. We will address this issue in due time. Let's get down to business—here are the rules of folding and tracing constructions:

Rules for Folding and Tracing Constructions

(1) You may only use folds, a marker, and semi-transparent paper.

(2) Points can only be placed in two ways:

 (a) As the intersection of two lines.

(b) By marking "through" folded paper onto a previously placed point. Think of this as when the ink from a permanent marker "bleeds" through the paper.

(3) Lines can only be obtained in three ways:

(a) By joining two points—either with a drawn line or a fold.

(b) As a crease created by a fold.

(c) By marking "through" folded paper onto a previously placed line.

(4) One can only fold the paper when:

(a) Matching up points with points.

(b) Matching up a line with a line.

(c) Matching up two points with two intersecting lines.

Now we are going to present several basic constructions. Compare these to the ones done with a compass and straightedge. We will proceed by the order of difficulty of the construction.

Construction (Transferring a Segment) Given a segment, we wish to move it so that it starts on a given point, on a given line.

Construction (Copying an Angle) Given a point on a line and some angle, we wish to copy the given angle so that the new angle has the point as its vertex and the line as one of its edges.

Transferring segments and copying angles using folding and tracing without a "bleeding marker" can be tedious. Here is an easy way to do it:

Use 2 sheets of paper and a pen that will mark through multiple sheets.

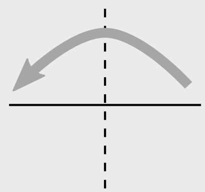
Question Can you find a way to do the above constructions without using a marker whose ink will pass through paper?

?

Construction (Bisecting a Segment) Given a segment, we wish to cut it in half.

(1) Fold the paper so that the endpoints of the segment meet.

(2) The crease will bisect the given segment.



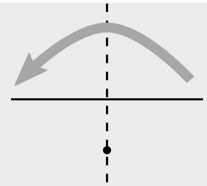
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Perpendicular through a Point) Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.

(1) Fold the given line onto itself so that the crease passes though the given point.

(2) The crease will be the perpendicular line.



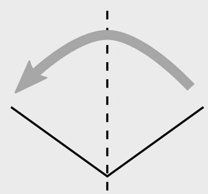
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Bisecting an Angle) We wish to divide an angle in half.

(1) Fold a point on one leg of the angle to the other leg so that the crease passes though the vertex of the angle.

(2) The crease will bisect the angle.



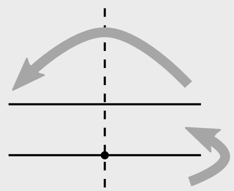
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Parallel through a Point) Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.

(1) Fold a perpendicular line through the given point.

(2) Fold a line perpendicular to this new line through the given point.



Now there may be a pressing question in your head:

Question How the heck are we going to fold a circle?

First of all, remember the definition of a circle:

Definition A **circle** is the set of points that are a fixed distance from a given point.

Question Is the center of a circle part of the circle?

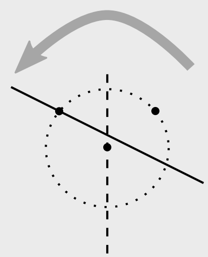
?

Secondly, remember that when doing compass and straightedge constructions we can **only** mark points that are intersections of lines and lines, lines and circles, and circles and circles. Thus while we technically draw circles, we can only actually mark certain points on circles. When it comes to folding and tracing constructions, drawing a circle amounts to marking points a given distance away from a given point—that is exactly what we can do with compass and straightedge constructions.

Construction (Intersection of a Line and a Circle) We wish to construct the points where a given line meets a given circle. Note: A circle is given by a point on the circle and the central point.

(1) Fold the point on the circle onto the given line so that the crease passes through the center of the circle.

(2) Mark this point though both sheets of paper onto the line.



Question Which rule for folding and tracing constructions are we using above?

?

Question How could you check that your folding and tracing construction is correct?

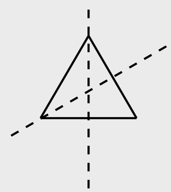
?

Construction (Equilateral Triangle) We wish to construct an equilateral triangle given the length of one side.

(1) Bisect the segment.

(2) Fold one end of the segment onto the bisector so that the crease passes through the other end of the segment. Mark this point onto the bisector.

(3) Connect the points.



Question Which rules for folding and tracing constructions are we using above?

?

Construction (Intersection of Two Circles) We with to intersect two circles, each given by a center point and a point on the circle.

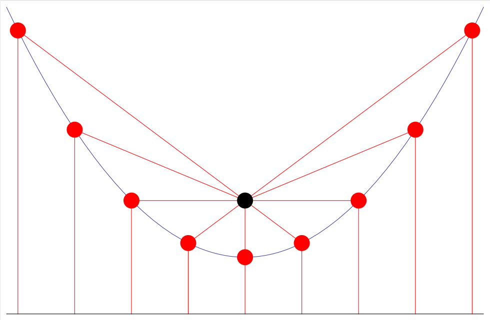
(1) Use four sheets of tracing paper. On the first sheet, mark the centers of both circles. On the next two sheets, mark the center and point on each of the circle—one circle per sheet.

(2) Simply move the two sheets with the centers and points on the circles, so that the centers are over the centers from the first sheet, and the points on the circles coincide. Now on the fourth sheet, mark all points.

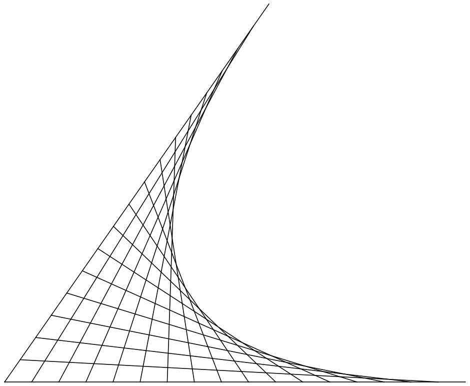
?

Think about the definition of a circle. In a similar fashion we can define other common geometric figures:

Definition Given a point and a line, a **parabola** is the set of points such that each of these points is the same distance from the given point as it is from the given line.



We can also form a parabola from an *envelope of tangents*:



Using a similar idea we can essentially obtain a parabola using folding and tracing.

Construction (Parabola) Given a point and a line we wish to construct a parabola.

(1) Make a series of equally spaced marks on your line.

(2) Fold the point onto the marks.

(3) Repeat the above step until an envelope of tangents forms.

Question Considering the definition of the parabola, can you explain why the above construction makes sense?

?

Question Can you give a compass and straightedge construction of a parabola?

?

Our final basic folding and tracing construction is one that **cannot** be done with compass and straightedge alone.

This construction was discovered by S.T. Gormsen and verified by S.H. Kung.

Construction (Angle Trisection) We wish to divide an angle into thirds.

(1) Bisect the given angle.

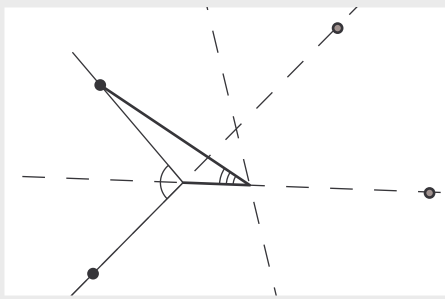
(2) Find two points (one on each leg of the angle) equidistant from the vertex of the angle.

(3) Fold the two points found above so that one of them lands on the extension (behind the angle) of the angle bisector and one lands on the line containing the other leg of the triangle—this will be behind the vertex. You are basically folding the angle back over itself.

(4) The crease from the last step will intersect the angle bisector at some point, mark it.

(5) The angle with the above mark as its vertex, the bisector found above as one of its legs, and the line to either of the points found in step 2 above will be

one third of the starting angle.



Problems for Section 3.1

(1) What are the rules for folding and tracing constructions?

(2) Use folding and tracing to bisect a given line segment. Explain the steps in your construction.

(3) Given a line segment with a point on it, use folding and tracing to construct a line perpendicular to the segment that passes through the given point. Explain the steps in your construction.

(4) Use folding and tracing to bisect a given angle. Explain the steps in your construction.

(5) Given a point and line, use folding and tracing to construct a line parallel to the given line that passes through the given point. Explain the steps in your construction.

(6) Given a point and line, use folding and tracing to construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction.

(7) Given a circle (a center and a point on the circle) and line, use folding and tracing to construct the intersection. Explain the steps in your construction.

(8) Given a line segment, use folding and tracing to construct an equilateral triangle whose edge has the length of the given segment. Explain the steps in your construction.

(9) Explain how to use folding and tracing to transfer a segment.

(10) Given an angle and some point, use folding and tracing to copy the angle so that the new angle has as its vertex the given point. Explain the steps in your construction.

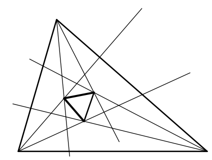
(11) Explain how to use folding and tracing to construct envelope of tangents for a parabola.

(12) Explain how to use folding and tracing to trisect a given angle.

(13) Use folding and tracing to construct a square. Explain the steps in your construction.

(14) Use folding and tracing to construct a regular hexagon. Explain the steps in your construction.

(15) Morley's Theorem states: If you trisect the angles of any triangle with lines, then those lines form a new equilateral triangle inside the original triangle.



Give a folding and tracing construction illustrating Morley's Theorem. Explain the steps in your construction.

(16) Given a length of 1, construct a triangle whose perimeter is a multiple of 6. Explain the steps in your construction.

(17) Construct a 30-60-90 right triangle. Explain the steps in your construction.

(18) Given a length of 1, construct a triangle with a perimeter of $3 + \sqrt{5}$. Explain the steps in your construction.

3.2 Anatomy of Figures Redux

Remember, in studying geometry we seek to discover the points that can be obtained given a set of rules. Now the set of rules consists of the rules for folding and tracing constructions.

Question In regards to folding and tracing constructions, what is a *point*?

?

Question In regards to folding and tracing constructions, what is a *line*?

?

Question In regards to folding and tracing constructions, what is a *circle*?

?

OK, those are our basic figures, pretty easy right? Now I'm going to quiz you about them (I know we've already gone over this, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

Problems for Section 3.2

(1) In regards to folding and tracing constructions, what is a *circle*? Compare and contrast this to a naive notion of a circle.

(2) Explain how a perpendicular bisector is different from an altitude. Use folding and tracing to illustrate the difference.

(3) Explain how a median different from an angle bisector. Use folding and tracing to illustrate the difference.

(4) Given a triangle, use folding and tracing to construct the circumcenter. Explain the steps in your construction.

(5) Given a triangle, use folding and tracing to construct the orthocenter. Explain the steps in your construction.

(6) Given a triangle, use folding and tracing to construct the incenter. Explain the steps in your construction.

(7) Given a triangle, use folding and tracing to construct the centroid. Explain the steps in your construction.

(8) Could the circumcenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(9) Could the orthocenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

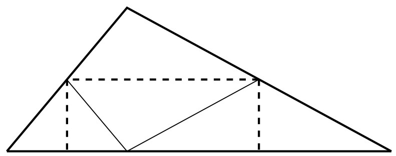
(10) Could the incenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(11) Could the centroid be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(12) Where is the circumcenter of a right triangle? Explain your reasoning and illustrate your ideas with folding and tracing.

(13) Where is the orthocenter of a right triangle? Explain your reasoning and illustrate your ideas with folding and tracing.

(14) The following picture shows a triangle that has been folded along the dotted lines:



Explain how the picture "proves" the following statements:

(a) The interior angles of a triangle sum to 180°.

(b) The area of a triangle is given by $bh/2$.

(15) Use folding and tracing to construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(16) Use folding and tracing to construct a triangle given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(17) Use folding and tracing to construct a triangle given one angle and the altitudes to the other two angles. Explain the steps in your construction.

(18) Use folding and tracing to construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

4 Toward Congruence and Similarity

4.1 Transformations, Symmetry, and Congruence

In school mathematics, transformations and symmetry have typically been small niche topics, separate from each other, separate from most of the rest of school mathematics, and receiving little curricular attention. Congruence, on the other hand, is a more prominent idea that begins informally in the elementary grades as "same shape, same size" and culminates in high school with axioms, theorems, and proofs.

In this section, we demonstrate how transformations can undergird both symmetry and congruence, thereby strengthening all three topics and also establishing groundwork for an analogous approach to similarity.

4.1.1 Transformations

Informally, a transformation of the plane is a "motion," such as a rotation or a stretch of the plane. More formally, a transformation is a function that takes points in the plane as inputs and gives points as outputs.[G-CO.2] In school mathematics, we consider only transformations that take lines to lines, so that key geometric features are "preserved." For example a triangle remains a triangle when it is rotated and even when it is stretched.

CCSS G-CO.2: Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

Transformations are often specified using a coordinate system, but coordinates are not necessary. For now, we will explore transformations without a coordinate system. Later, we will use coordinates, along with matrices and vectors, to describe transformations.

Definition Transformations that preserve distances and angles are called *isometries*, and the most important of these are *basic rigid motions*: translations, rotations, and reflections.

Question Is a transformation that stretches the plane an isometry? Explain.

?

Through exploration with transparencies, tracing paper, software,[G-CO.2] it is not hard to see that the basic rigid motions have important properties.[8.G.1 8.G.1a 8.G.1b 8.G.1c] Based on such explorations, we write careful definitions of translation, reflection, rotation, by considering what is required to specify each transformation.[G-CO.4]

Definition The *identity transformation*, sometimes called the "do nothing" transformation, doesn't move the plane at all. As a function, the identity transformation takes a point to itself: The output is identical to the input.

Question Is the identity transformation a translation, rotation, or reflection? Explain.

?

4.1.2 Symmetry

A *symmetry* of a figure is a transformation that takes the figure onto itself, so that the figure is "preserved" by the transformation. In everyday language, we sometimes say a figure is "symmetrical," but mathematically we can be more precise by specifying the symmetry transformation(s) of the figure.[G-CO.3]

CCSS G-CO.2: Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

CCSS 8.G.1: Verify experimentally the properties of rotations, reflections, and translations:

CCSS 8.G.1a: Lines are taken to lines, and line segments to line segments of the same length.

CCSS 8.G.1b: Angles are taken to angles of the same measure.

CCSS 8.G.1c: Parallel lines are taken to parallel lines.

CCSS G-CO.4: Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

CCSS G-CO.3: Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

Question What are the symmetries of a rectangle? Be sure to specify the transformations.

?

4.1.3 Congruence

Congruence is often defined using angles and side lengths. But such a definition cannot apply to figures that are not polygons. A more inclusive definition is as follows:

Definition Two figures (in the plane) are said to be *congruent* to one another if there is a sequence of basic rigid motions that takes one figure onto the other.

The idea behind this definition is sometimes called the *principle of superposition*, which states that congruent figures can be placed exactly on top of one another. The above definition is more precise than superposition because it calls for an explicit sequence of basic rigid motions (e.g., translations, rotations, and reflections) rather than merely "movement" of one figure onto the other.

Question When we say that two polygons are congruent, why is the order of labeling the vertices important? For example, if we know $\triangle ABC \simeq \triangle XYZ$, does it follow that $\triangle ABC \simeq \triangle YXZ$? Explain. (Hint: Which angle of $\triangle XYZ$ corresponds to $\angle A$? Which side of $\triangle ABC$ corresponds to $\overline{XZ}$?)

?

The above definition of congruence helps us in two directions.[8.G.2] First, if we have a sequence of basic rigid motions that takes one figure onto another, then we know the two figures are congruent. Furthermore, the sequence of basic rigid motions sets up the correspondences between various parts of the figures. Conversely, if

CCSS 8.G.2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

two figures are congruent, then we know it is possible to find a sequence of basic rigid motions that takes one figure onto the other. And the sequence of basic rigid motions often takes advantage of corresponding parts that are already known to be congruent.

For triangles, we still have the familiar congruence criteria, such as side-side-side (SSS), side-angle-side (SAS), and angle-side-angle (ASA). The key idea is that although triangles have six measures of sides and angles, most of the time (but not always) just three of these measures are sufficient to determine the triangle uniquely. Students can develop intuition about these criteria by drawing triangles from given conditions.[7.G.2] The next step is to show, first, that the above definition fits with traditional notions of triangle congruence[G-CO.7], and, second, to prove that the triangle congruence criteria follow from the properties of the basic rigid motions.[G-CO.8]

CCSS 7.G.2: Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

CCSS G-CO.7: Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

CCSS G-CO.8: Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Problems for Section 4.1

(1) What is required to specify a translation?

(2) What is required to specify a rotation?

(3) What is required to specify a reflection?

Sometimes a sequence of transformations can be described as a single translation, rotation, or reflection.

(4) What kind of transformation is a translation followed by a translation? Explain. Be sure to consider any special cases.

(5) What kind of transformation is a rotation followed by a rotation? Explain. Be sure to consider any special cases.

(6) What kind of transformation is a reflection followed by another reflection? Explain. Be sure to consider any special cases.

(7) Will the letter P look like a P after a reflection? What about after a sequence of two reflections? What about after a sequence of 73 or 124 reflections? Explain your reasoning.

(8) How will your answer to the previous problem change if you use a capital D? Explain.

(9) Given a figure and its image after a translation, how do find the direction and distance of the translation? How many points and images do you need?

(10) Given a figure and its image after a reflection, how do you find the line of reflection? How many points and images do you need?

(11) Given a figure and its image after a rotation, how do you find the center and the angle of the rotation? How many points and images do you need?

(12) Categorize the capital letters of the alphabet by their symmetries.

(13) Write the words COKE and PEPSI in capital letters so that they read vertically. Use a mirror to look at a reflection of the words. What is different about the reflections of the two words? Explain.

(14) Describe all of the symmetries of the following figures:

(a) An equilateral triangle

(b) An isosceles triangle that is not equilateral

(c) A square

(d) A rectangle that is not a square

(e) A rhombus that is not a square

(f) A (non-special) parallelogram

(g) A regular $n-$gon

(15) What are the symmetries of a circle?

(16) How can you use the symmetries of a circle to determine whether a figure is indeed a circle?

(17) What are the symmetries of a line?

(a) Describe all translation symmetries.

(b) Describe all rotation symmetries.

(c) Describe two types of reflection symmetries.

(d) Given a line, describe a rotation symmetry and a reflection symmetry that have the same effect on a line. How do the corresponding transformations differ in what they do to the surrounding space?

(18) How can you use the symmetries of a line to determine whether a figure is indeed a line?

(19) Find some tessellations. For each tessellation, describe all of its symmetries.

4.2 Euclidean and non-Euclidean Geometries

The geometry of school mathematics is called *Euclidean Geometry* for it is the geometry organized and detailed by Euclid more than 2,000 years ago. To better understand the assumptions that underlie Euclidean geometry and the results that follow, it helps to be aware of non-Euclidean geometries. Perhaps the most accessible of these is spherical geometry, because we can make use of basketballs that we can hold in our hands, and we can take advantage of our experience traveling on our (approximately spherical) Earth, modeled by a globe.

Question Before we talk about spheres, what does it mean to say that a plane is two-dimensional and space is three-dimensional? What is "dimension"?

?

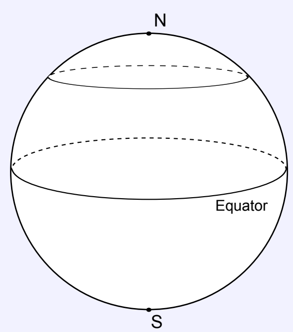
To think about spherical geometry, it helps to imagine a bug crawling on the surface of a sphere. From the bug's perspective, the surface of the sphere is very much the same as the surface of a Euclidean plane. Both surfaces are two-dimensional in the sense that the bug has two degrees of freedom: forward/backward and left/right. Any other movement can be expressed as a combination of these. (We are assuming the bug must stay *on the surface*: It can neither fly away from nor burrow underneath the surface.) Whereas the surface of a Euclidean plane is infinite and flat, the surface of a sphere is finite and curved. But if the sphere is reasonably large (compared to the bug), then even a very smart bug might have trouble determining whether she or he was walking on a sphere or on a flat plane.

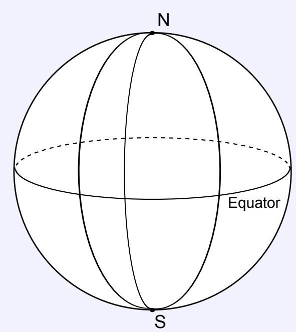
Question Explain in your own words how to think about the surface of a sphere as two-dimensional.

?

Points in spherical geometry are taken to be points on the surface of the sphere. But "lines" present more of a challenge: We want lines to be "straight", but any path on the surface of a sphere curves with the surface. Suppose the bug travels forward along a path that is as straight as possible, being very careful to veer neither right nor left. Alternatively, because lines should determine “shortest paths” between two points, stretch a rubber band between two points on a basketball or on a globe to find the shortest path. (Try this!) In both cases, you will find that best answer is that a “line” in spherical geometry is a *great circle*, which is to say a circle that is as big as possible on the sphere. From a three-dimensional perspective, the center of a great circle is the same as the center of the sphere.

Question Consider the pictures below.



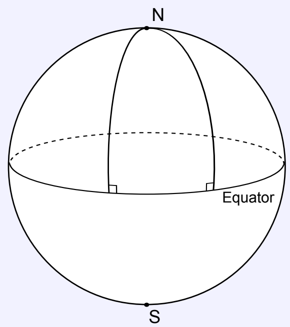


Are longitude lines on the earth “lines” in spherical geometry? What about latitude lines? Explain your reasoning.

?

In non-Euclidean geometries, many familiar results no longer hold. In spherical geometry, for example, there are no parallel lines because any two “lines” (i.e., great circles) intersect in two points, and the sum of the angles in a triangle is greater than 180°.

Question Use the following picture to explain that the sum of the angles in a triangle in spherical geometry can be greater than 180°.



?

Other non-Euclidean geometries are even stranger than spherical geometry! In hyperbolic geometry, for example, parallel lines are not a fixed distance apart, and the sum of the angles in a triangle is less than 180°.

The following statements characterize three different types of geometries:

- **Euclidean geometry**: Given a line and a point not on the line, there is *exactly one line* parallel to the given line.
- **Spherical geometry**: Given a line and a point not on the line, there are *no lines* through the point parallel to the given line.
- **Hyperbolic geometry**: Given a line and a point not on the line, there is *more than one line* parallel to the given line.

In this course, we explore neither spherical nor hyperbolic geometry in detail, but keep these contrasting ideas in mind as we continue to dig into Euclidean geometry.

Problems for Section 4.2

(1) From the above statements about angle sums in triangles, what can you conclude about angle sums in quadrilaterals in spherical and hyperbolic geometries?

(2) In Euclidean geometry, a rectangle is a quadrilateral with four right angles.

  (a) What can you conclude about rectangles in spherical and hyperbolic geometries? Explain.

  (b) What does this imply about the usefulness of familiar (Euclidean) area formulas in these other geometries? Explain your reasoning.

(3) In Euclidean geometry, when three distinct points A, B, and C, lie on a line, it is easy to tell which point is between the other two. Does this work in spherical geometry? Explain your reasoning.

(4) A bear goes traveling. She walks due south for one mile, turns left 90°, and walks due east for one mile. She again turns left 90°, and then walks due north for one mile, ending in the place where she started. What color is the bear? Explain your reasoning.

(5) When walking on a sphere, how could a bug check whether she or he was traveling straight.

(6) In Euclidean geometry, any two distinct points determine a unique line. This is sometimes (but not always) true in spherical geometry. What can you say about two distinct points that do not lie on a unique line in spherical geometry?

(7) In Euclidean geometry, given a line and a point, there is a unique perpendicular to the given line through the given point. Describe how this sometimes fails in spherical geometry.

(8) Can the Euclidean definition of a circle make sense on a sphere? Be sure that the center of the circle is a point on the sphere. How would you measure the radius of the circle?

4.3 Assumptions in Mathematics

Every area of mathematics is based on a set of assumptions, sometimes called axioms or postulates,• which are merely statements that are accepted without proof. They serve as the foundation of the theory being developed, and all other facts are proven beginning with these assumptions. This approach is called the *axiomatic method*.

. . . Or at least that's how mathematics is imagined to work. In practice, because mathematics is so vast and interconnected, most mathematical reasoning and problem solving starts "in the middle"• from a collection of accepted facts, with little worry about which statements were taken as assumptions and which were proven as theorems.

• In classical mathematics, ``axioms'' were self-evident statements that were common to many areas of science (including mathematics), whereas ``postulates'' were common-sense facts drawn from experience in specific areas, such as geometry. In modern mathematics, this distinction is no longer seen as significant, and most assumptions are merely called axioms. In deference to Euclid's *Elements*, the word postulate is used almost exclusively to discuss key assumptions in geometry, as you will see below.

• In this course, we started in the middle. In this section, we are examining the foundation.

Question In school mathematics we can "explain" the properties of whole or rational numbers by appealing to models and to meanings of the arithmetic operations. But in advanced mathematics courses, the real numbers are usually specified via axioms, some of the axioms have names.

What are the names of the following axioms:

(1) $a + b = b + a$

(2) $a(bc) = (ab)c$

(3) $a(b + c) = ab + ac$

(4) If $a = b$ and $b = c$ then $a = c$

?

Chances are you used the word "property" or "law" rather than "axiom" in your responses. Some properties of arithmetic have important names, such as the *distributive property of multiplication over addition*. The fourth property above is called the transitive property of equality. But in school mathematics, it is neither necessary to instructive to insist that every such property have a name that students are expected to recall.

4.3.1 Assumptions for School Geometry

We propose the following set of assumptions• for school geometry:

(A1) Through two distinct points passes a unique line.

(A2) Given a line and a point not on the line, there is exactly one line passing through the point which is parallel to the given line (Parallel postulate).

(A3) The points on a line can be placed in one-to-one correspondence with the real numbers so that differences measure distances (Ruler postulate).

(A4) The rays with a common endpoint can be numbered so that differences measure angles and so that straight angles measure 180° (Protractor postulate).

(A5) Every basic rigid motion (rotation, reflection, or translation) has the following properties:

(i) It maps a line to a line, a ray to a ray, and a segment to a segment.

(ii) It preserves distance and angle measure.

(A6) Areas of geometric figures have the following properties:

(i) Congruent figures enclose equal areas.

(ii) Area is additive, i.e., the area of the union of two regions that overlap only at their boundaries is the sum of their areas.

(iii) A rectangle with side-lengths a and b has area ab, where a and b can be any non-negative real numbers.

• In addition to these geometric assumptions, we of course assume the properties of the algebra of real numbers.

These formal axioms, we should be clear, are intended not for students but for teachers. And even teachers need not memorize them. Instead, we suggest that teachers remember them informally in the following chunks:

- Points, lines, and parallel lines behave as they should (A1 and A2)
- Distance and angle measure behave as they should (A3 and A4)
- Basic rigid motions behave as they should (A5)
- Area behaves as it should (A6)

We are almost ready to use these axioms to prove some basic results. First, we need a crucial definition.

Definition Two distinct lines are said to be **parallel** if they have no point in common.

Most of the time, of course, two distinct lines will have exactly one point in common.

Question Can the two distinct lines have more than one point in common? Use the above axioms to explain your reasoning.

?

The ruler postulate gives us a definition of betweenness, which allows to to define line segment and ray.

Definition If points A, X, and B are on a line l, we say that X is *between* A and B if $AX + XB = AB$.

Question Use the concept of betweenness to define line segment $\overline{AB}$. Now use the concept of betweenness to define ray $\overrightarrow{AB}$.

?

Question Use the protractor postulate to provide a definition of adjacent angles, analogous to betweenness for distances.

?

Theorem 4.3.1 *Let l be a line and O be a point on l. Let R be the 180° rotation around O. Then R maps l to to itself.*

Question Can you prove this theorem? (Hint: Pick points P and Q on l so that O is between them, and consider the straight angle $\angle POQ$.)

?

Theorem 4.3.2 *Let l be a line and O be a point* not *lying on l. Let R be the 180° rotation around O. Then R maps l to a line parallel to itself.*

Note: The following proof uses function notation to describe the images under the rotation R. Thus $R(l)$ is the image of line l, and $R(Q)$ is the image of point Q.

Proof *Suppose $R(l)$ is not parallel to l. Then $R(l)$ and l have a point Q in common. Because Q is on $R(l)$, there is a point P on l, so that $R(P) = Q$. Because R is a 180° rotation around O, the three points P, Q, and O lie in a line m. But Q is by assumption also a point in l, so l and m have two distinct points in common: P and Q. But l and $R(l)$ are distinct because O is on m but not on l. We have a contradiction, and thus $R(l)$ must be parallel to l.*

Problems for Section 4.3

(1) Use adjacent angles to prove that vertical angles are equal.

(2) Now use rotations to prove that vertical angles are equal.

(3) Prove that alternate interior angles and corresponding angles of a transversal with respect to a pair of parallel lines are equal.

(4) Prove that the sum of the interior angles of a triangle is 180°.

(5) Prove: If a pair of alternate interior angles or a pair of corresponding angles of a transversal with respect to two lines are equal, then the lines are parallel.

4.4 Dilations, Scaling, and Similarity

In a previous section, we saw how transformations can be used as a foundation for describing congruence and explaining the triangle congruence criteria. In this section, we show how transformations can be used to describe similarity. Because the basic rigid motions all preserve distances, we need a new kind of transformation: a dilation.

Definition Given a point O and a positive number r, a *dilation* about O by scale factor r, is a mapping that takes a point P to a point P' so that $OP' = r \cdot OP$.

With this definition, rubber bands are natural tools for exploring dilations. Through explorations with rubber bands and with geometry software, we observe that a dilation has the following properties:[G-SRT.1]

CCSS G-SRT.1: Verify experimentally the properties of dilations given by a center and a scale factor:

(i) It maps lines to lines, rays to rays, and segments to segments.

(ii) It changes distance by a factor of r, where r is the scale factor of the dilation.

(iii) It maps every line passing through the center of dilation to itself, and it maps every line not passing through the center of the dilation to a parallel line.

(iv) It preserves angle measure.

We could assume these properties, just as we have assumed the properties of the basic rigid motions. Instead, we use our assumptions about area to prove some of these properties. These are the Side-Splitter Theorems.

Now we are ready to define similarity.[8.G.4]

CCSS 8.G.4: Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

Definition A geometric figure is *similar* to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations.

To Be Continued.

Fixnote: Fold in Bart's notes on similarity, which are in a file called similarity.tex in the chapters directory

Problems for Section 4.4

(1) Compare and contrast the ideas of *equal triangles*, *congruent triangles*, and *similar triangles*.

(2) Explain why all equilateral triangles are similar to each other.

(3) Explain why all isosceles right triangles are similar to each other.

(4) Explain why when given a right triangle, the altitude of the right angle divides the triangle into two smaller triangles each similar to the original right triangle.

(5) The following sets contain lengths of sides of similar triangles. Solve for all unknowns—give all solutions. In each case explain your reasoning.

(a) $\{3, 4, 5\}$, $\{6, 8, x\}$

(b) $\{3, 3, 5\}$, $\{9, 9, x\}$

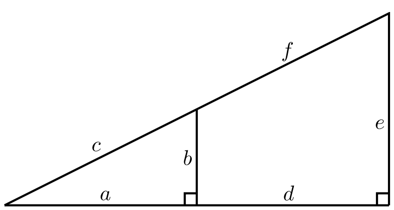
(c) $\{5, 5, x\}$, $\{10, 4, y\}$

(d) $\{5, 5, x\}$, $\{10, 8, y\}$

(e) $\{3, 4, x\}$, $\{4, 5, y\}$

(6) A *Pythagorean Triple* is a set of three positive integers $\{a, b, c\}$ such that $a^2 + b^2 = c^2$. Write down an infinite list of Pythagorean Triples. Explain your reasoning and justify all claims.

(7) Here is a right triangle, note it is **not** drawn to scale:



Solve for all unknowns in the following cases.

(a) $a = 3$, $b =?$, $c =?$, $d = 12$, $e = 5$, $f =?$

(b) $a =?$, $b = 3$, $c =?$, $d = 8$, $e = 13$, $f =?$

(c) $a = 7$, $b = 4$, $c =?$, $d =?$, $e = 11$, $f =?$

(d) $a = 5$, $b = 2$, $c =?$, $d = 6$, $e =?$, $f =?$

In each case explain your reasoning.

(8) Suppose you have two similar triangles. What can you say about the area of one in terms of the area of the other? Be specific and explain your reasoning.

(9) During a solar eclipse we see that the apparent diameter of the Sun and Moon are nearly equal. If the Moon is around 240000 miles from Earth, the Moon's diameter is about 2000 miles, and the Sun's diameter is about 865000 miles how far is the Sun from the Earth?

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(10) When jets fly above 8000 meters in the air they form a vapor trail. Cruising altitude for a commercial airliner is around 10000 meters. One day I reached my arm into the sky and measured the length of the vapor trail with my hand—my hand could just span the entire trail. If my hand spans 9 inches and my arm extends 25 inches from my eye, how long is the vapor trail? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(11) David proudly owns a 42 inch (measured diagonally) flat screen TV. Michael proudly owns a 13 inch (measured diagonally) flat screen TV. Dave sits comfortably with his dog Fritz at a distance of 10 feet. How far must Michael stand from his TV to have the "same" viewing experience? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(12) You love IMAX movies. While the typical IMAX screen is 72 feet by 53 feet, your TV is only a 32 inch screen—it has a 32 inch diagonal. How close do you have to sit to your screen to simulate the IMAX format? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(13) David proudly owns a 42 inch (measured diagonally) flat screen TV. Michael proudly owns a 13 inch (measured diagonally) flat screen TV. Michael stands and watches his TV at a distance of 2 feet. Dave sits comfortably with his dog Fritz at a distance of 10 feet. Whose TV appears bigger to the respective viewer? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(14) Here is a personal problem: Suppose you are out somewhere and you see that when you stretch out your arm, the width of your thumb is the same apparent size as a distant object. How far away is the object if you know the object is:

(a) 6' long (as tall as a person).

(b) 16' long (as long as a car).

(c) 40' long (as long as a school bus).

(d) 220' long (as long as a large passenger airplane).

(e) 340' long (as long as an aircraft carrier).

Explain your reasoning.

(15) I was walking down Woody Hayes Drive, standing in front of St. John Arena when a car pulled up and the driver asked, "Where is Ohio Stadium?" At this point I was a bit perplexed, but nevertheless I answered, "Do you see the enormous concrete building on the other side of the street that looks like the Roman Colosseum? That's it."

The person in the car then asked, "Where are the Twin-Towers then?" Looking up, I realized that the towers were in fact just covered by top of Ohio Stadium. I told the driver to just drive around the stadium until they found two enormous identical towers—that would be them. They thanked me and I suppose they met their destiny.

I am about 2 meters tall, I was standing about 100 meters from the Ohio Stadium and Ohio Stadium is about 40 meters tall. If the Towers are around 500 meters from the rotunda (the front entrance of the stadium), how tall could they be and still be obscured by the stadium? Explain your reasoning—for the record, the towers are about 80 meters tall.

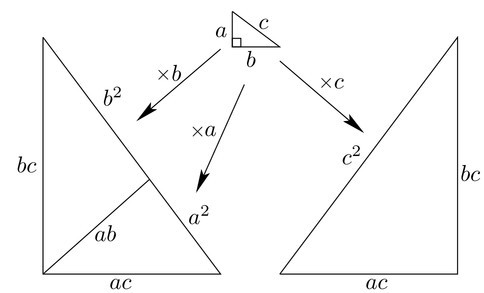
(16) Explain how to use the notion of similar triangles to multiply numbers with your answer expressed as a segment of the appropriate length.

(17) Explain how to use the notion of similar triangles to divide numbers with your answer expressed as a segment of the appropriate length.

(18) Consider the following combinations of S's and A's. Which of them produce a *Congruence Theorem*? Which of them produce a *Similarity Theorem*? Explain your reasoning.

SSS, SSA, SAS, SAA, ASA, AAA

(19) Explain how the following picture "proves" the Pythagorean Theorem.



4.5 Length, Area, and Volume Under Scaling

To be written.

Fixnote: Include problems on making scale drawings and on using scale drawings to solve problems about length and area.

5 Coordinate Constructions

> As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
>
> —Joseph Louis Lagrange

5.1 Constructions

One of the deepest and powerful aspects of mathematics is that it allows one to see connections between disparate areas. So far we have used different physical techniques (compass and straightedge constructions along with origami constructions) to solve similar problems. Take a minute and reflect upon that—isn't it cool that similar problems can be solved by such different methods? You back? OK—so let's see if we can solidify these connections through abstraction and in the process, make a third connection. We are going to see the algebra behind the geometry we've done. Making these connections isn't easy and can be scary. Thankfully, you are a fearless (yet gentle) reader.

Rules for Coordinate Constructions

(1) A point is an ordered pair (x, y) of real numbers x and y. Points can only be placed as the intersection of lines and/or circles.

(2) Lines are defined as all points (x, y) that are solutions to equations of the form

$$ax + by = c \qquad \text{for given } a, b, c.$$

(3) Circles centered at (a, b) of radius c are defined as all solutions to equations of the form
$$(x-a)^2 + (y-b)^2 = c^2 \qquad \text{for given } a, b, c.$$

(4) The distance between two points $A = (a_x, a_y)$ and $B = (b_x, b_y)$ is given by
$$d(A, B) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}.$$

Just as we have done before, we will present several basic constructions. Compare these to the ones done with a compass and straightedge and the ones done by folding and tracing. We will proceed by the order of difficulty of the construction.

Construction (Bisecting a Segment) Given a segment, we wish to cut it in half.

(1) Let (x_1, y_1) and (x_2, y_2) be the endpoints of your segment.

(2) We claim the midpoint is:
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Question Can you explain why this works?

?

Construction (Parallel through a Point) Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.

(1) Let $ax + by = c$ be the line and let (x_0, y_0) be the point.

(2) Set $c_0 = ax_0 + by_0$.

(3) The line $ax + by = c_0$ is the desired parallel line.

Question Can you explain why this works?

?

Construction (Perpendicular through a Point) Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.

(1) Let (x_0, y_0) be the given point and let $ax + by = c$ be the given line.

(2) Find $c_0 = bx_0 - ay_0$.

(3) The desired line is $bx + (-a)y = c_0$.

Question Can you explain why this works? Can you give some examples of it in action?

?

Construction (Line between two Points) Given two points, we wish to give the line connecting them.

(1) Call the two points (x_1, y_1) and (x_2, y_2).

(2) Write

$$\begin{aligned} ax_1 + by_1 &= c, \\ ax_2 + by_2 &= c. \end{aligned}$$

(3) Solve for $-a/b$ and c.

Example 5.1.1) Suppose you want to find the line between the points $(3, 1)$ and $(2, 5)$. Write

$$\begin{aligned} a \cdot 3 + b \cdot 1 &= c, \\ a \cdot 2 + b \cdot 5 &= c, \end{aligned}$$

and subtract these equations to get:

$$a - b \cdot 4 = 0$$

Now we see

$$\begin{aligned} -b \cdot 4 &= -a, \\ -4 &= -a/b. \end{aligned}$$

Now we can take **any** values of a and b that make the equation above true, and plug them back in to $a \cdot 3 + b = c$ to obtain c. **You should explain why this works!** I choose $a = 4$ and $b = 1$. From this I see that $c = 13$ so the line we desire is:

$$4x + y = 13$$

Construction (Intersection of a Line and a Circle) We wish to find the points where a given line meets a given circle.

(1) Let $ax + by = c$ be the given line.

(2) Let $(x - x_0)^2 + (y - y_0)^2 = r^2$ be the given circle.

(3) Solve for x and y.

Question Can you give an example and draw a picture of this construction?

?

Construction (Bisecting an Angle) We wish to divide an angle in half.

(1) Find two points on the angle equidistant from the vertex.

(2) Bisect the segment connecting the point above.

(3) Find the line connecting the vertex to the bisector above.

Question Can you give an example and draw a picture of this construction?

?

Construction (Intersection of Two Circles) Given two circles, we wish to find the points where they meet.

(1) Let $(x - a_1)^2 + (y - b_1)^2 = c_1^2$ be the first circle.

(2) Let $(x - a_2)^2 + (y - b_2)^2 = c_2^2$ be the second circle.

(3) Solve for x and y.

Question Can you give an example and draw a picture of this construction? How many examples should you give for "completeness" sake?

?

Question We wish to construct an equilateral triangle given the length of one side. Can you do this?

?

Problems for Section 5.1

(1) What are the rules for coordinate constructions?

(2) Explain how to transfer a segment using coordinate constructions.

(3) Explain how to copy an angle using coordinate constructions (but don't actually do it!)

(4) Given two points, use coordinate constructions to construct a line between both points. Explain the steps in your construction.

(5) Given segment, use coordinate constructions to bisect the segment. Explain the steps in your construction.

(6) Given a point and line, use coordinate constructions to construct a line parallel to the given line that passes through the given point. Explain the steps in your construction.

(7) Given a point and line, use coordinate constructions to construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction.

(8) Given a line and a circle, use coordinate constructions to construct the intersection of these figures. Explain the steps in your construction.

(9) Use coordinate constructions to bisect a given angle. Explain the steps in your construction.

(10) Given two circles, use coordinate constructions to construct the intersection of these figures. Explain the steps in your construction.

(11) Use algebra to help explain why lines intersect in zero, one, or infinitely many points.

(12) Use algebra to help explain why circles and lines intersect in zero, one, or two points.

(13) Use algebra to help explain why circles intersect in zero, one, two, or infinitely many points.

(14) Use coordinate constructions to construct an equilateral triangle. Explain the steps in your construction.

(15) Use coordinate constructions to construct a square. Explain the steps in your construction.

(16) Use coordinate constructions to construct a regular hexagon. Explain the steps in your construction.

5.2 Brave New Anatomy of Figures

Once more, in studying geometry we seek to discover the points that can be obtained given a set of rules. Now the set of rules consists of the rules for coordinate constructions.

Question In regards to coordinate constructions, what is a *point*?

?

Question In regards to coordinate constructions, what is a *line*?

?

Question In regards to coordinate constructions, what is a *circle*?

?

Now I'm going to quiz you about them (I know we've already gone over this *twice*, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

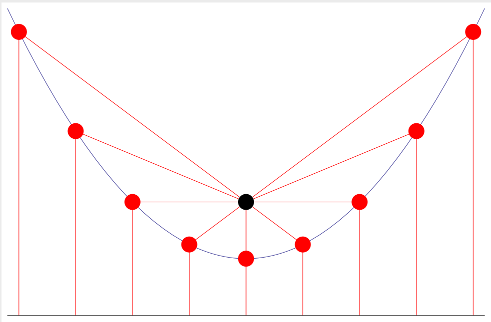
Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

5.2.1 Parabolas

Recall the definition of a *parabola*:

Definition Given a point and a line, a **parabola** is the set of points such that each of these points is the same distance from the given point as it is from the given line.



Fancy folks call the point the **focus** and they call the line the **directrix**.

However I know that you—being rather cosmopolitan in your knowledge and experience—know that from a coordinate geometry point of view that the formula for a parabola should be *something* like:

$$y = ax^2 + bx + c$$

Question How do you rectify these two different notions of a parabola?

I'm feeling chatty, so let me take this one. What would be really nice is if we could extract the focus and directrix from any formula of the form $y = ax^2 + bx + c$. I think we'll work it for a specific example. Consider:

$$y = 3x^2 + 6x - 7$$

Step 1 Complete the square. Write:

$$\begin{aligned} y &= 3x^2 + 6x - 7 \\ &= 3(x^2 + 2x) - 7 \\ &= 3(x^2 + 2x + 1 - 1) - 7 \\ &= 3(x^2 + 2x + 1) - 3 - 7 \\ &= 3(x + 1)^2 - 10 \end{aligned}$$

Step 2 Compare with the following basic form:

$$y = u(x - v)^2 + w$$

Given a parabola in the form above, we have that

$$\text{focus} : \left(v, w + \frac{1}{4u}\right) \qquad \text{and} \qquad \text{directrix} : y = w - \frac{1}{4u}.$$

So in our case the focus is at

$$\left(-1, -10 + \frac{1}{12}\right)$$

and our directrix is the line

$$y = -10 - \frac{1}{12}.$$

Question Can you use the distance formula to show that every point on the parabola is the same distance from focus as it is from the directrix?

?

Problems for Section 5.2

(1) In regards to coordinate constructions, what is a *point*? Compare and contrast this to a naive notion of a point.

(2) In regards to coordinate constructions, what is a *line*? Compare and contrast this to a naive notion of a line.

(3) In regards to coordinate constructions, what is a *circle*? Compare and contrast this to a naive notion of a circle. In particular, explain how the formula for the circle arises.

(4) Explain what is meant by the *focus* of a parabola.

(5) Explain what is meant by the *directrix* of a parabola.

(6) Will the following formula

$$y = ax^2 + bx + c$$

really plot *any* parabola in the plane? If so why? If not, can you give a formula that will? Explain your reasoning.

(7) For each parabola given, find the focus and directrix:

(a) $y = x^2$
(b) $y = 7x^2$
(c) $y = -2x^2$
(d) $y = x^2 - 4x$
(e) $y = x^2 - 12$
(f) $y = x^2 - x + 1$
(g) $y = x^2 + 2x - 5$
(h) $y = 2x^2 - 3x - 7$
(i) $y = -17x^2 + 42x - 3$
(j) $x = y^2 - 5y$
(k) $x = 3y^2 - 23y + 17$

In each case explain your reasoning.

(8) Explain in general terms (without appealing to an example) how to find the focus and directrix of a parabola $y = ax^2 + bx + c$.

(9) Use coordinate constructions to construct the circle that passes through the points:

$$A = (0, 0), \qquad B = (3, 3), \qquad C = (4, 0).$$

Sketch this situation and explain your reasoning.

(10) Consider the points

$$A = (1, 1) \qquad \text{and} \qquad B = (5, 3).$$

(a) Find the midpoint between A and B.
(b) Find the line the connects A and B. Use algebra to show that the midpoint found above is actually on this line.
(c) Use algebra to show that this midpoint is equidistant from both A and B.

Sketch this situation and explain your reasoning in each step above.

(11) Consider the parabola $y = x^2/4 + x + 2$.

(a) Find the focus and directrix of this parabola.
(b) Sketch the parabola by plotting points.
(c) Use folding and tracing to fold the envelope of tangents of the parabola.

Present the above items simultaneously on a single graph. Explain the steps in your work.

(12) Consider the following line and circle:

$$x - y = -1 \qquad \text{and} \qquad (x-1)^2 + (y-1)^2 = 5$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(13) Consider the following two circles:

$$x^2 + y^2 = 5 \qquad \text{and} \qquad (x-1)^2 + (y-1)^2 = 5$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(14) Consider the following two circles:

$$(x+1)^2 + (y-1)^2 = 9 \qquad \text{and} \qquad (x-3)^2 + (y-2)^2 = 4$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(15) Explain how to find the minimum or maximum of a parabola of the form:

$$y = ax^2 + bx + c$$

(16) Given a triangle, use coordinate constructions to construct the circumcenter. Explain the steps in your construction.

(17) Given a triangle, use coordinate constructions to construct the orthocenter. Explain the steps in your construction.

(18) Given a triangle, use coordinate constructions to construct the incenter. Explain the steps in your construction.

(19) Given a triangle, use coordinate constructions to construct the centroid. Explain the steps in your construction.

(20) Use coordinate constructions to construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(21) Use coordinate constructions to construct a triangle given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(22) Use coordinate constructions to construct a triangle given one angle and the altitudes to the other two angles. Explain the steps in your construction.

(23) Use coordinate constructions to construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

5.3 Constructible Numbers

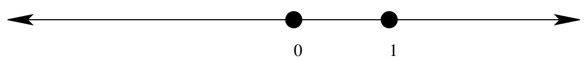
We've now practiced three types of constructions:

(1) Compass and straightedge constructions.

(2) Folding and Tracing constructions.

(3) Coordinate constructions.

You may be wondering what is meant by the words "constructible numbers." Imagine a line with two points on it:



Label the left point 0 and the right point 1. If we think of this as a starting point for a number line, then a **constructible number** is nothing more than a point we can obtain on the above number line using one of the construction techniques above starting with the points 0 and 1.

(1) Denote the set of numbers constructible by compass and straightedge with $\mathbb{C}$. We'll call $\mathbb{C}$ the set of *constructible numbers*.

(2) Denote the set of numbers constructible by folding and tracing with $\mathcal{F}$. We'll call $\mathcal{F}$ the set of *folding and tracing numbers*.

(3) Denote the set of numbers constructible by coordinate constructions with $\mathcal{D}$. We'll call $\mathcal{D}$ the set of *Descartes numbers*.

Be warned, this notion of so-called "Descartes numbers" is unique to these pages.

Mostly in this chapter we'll be talking about $\mathbb{C}$. You'll have to deal with $\mathcal{F}$ and $\mathcal{D}$ yourself.

Question Exactly what numbers are in $\mathbb{C}$?

How do we attack this question? Well first let's get a bit of notation. Recall that we use the symbol "$\in$" to mean *is in*. So we know that 0 and 1 are *in* the set of constructible numbers. So we write

$$0 \in \mathbb{C} \qquad \text{and} \qquad 1 \in \mathbb{C}.$$

Question Is this true for $\mathcal{F}$, the set of folding and tracing numbers? What about $\mathcal{D}$, the set of Descartes numbers?

?

If we could use constructions to make the operations $+$, $-$, $\cdot$, and $\div$, then we would be able to say a lot more. In fact we will do just this.

Question How does one add and subtract using a compass and straightedge?

?

Question Starting with 0 and 1, what numbers could we add to our number line by simply adding and subtracting?

At this point we have all the positive whole numbers, zero, and the negative whole numbers. We have a special name for this set, we call it the **integers** and denote it by the letter $\mathbb{Z}$:

$$\mathbb{Z} = \{\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}.$$

Question Are the integers contained in $\mathcal{F}$, the set of folding and tracing numbers? Are the integers contained in $\mathcal{D}$, the set of Descartes numbers?

?

We still have some more operations:

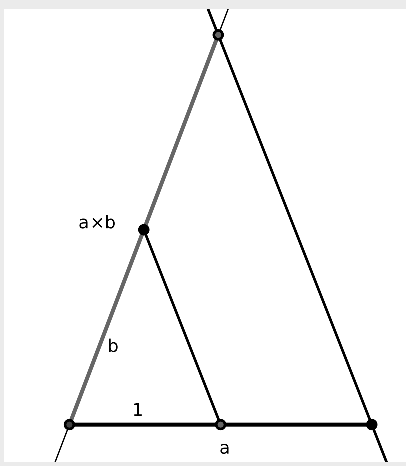
Construction (Multiplication) This construction is based on the idea of similar triangles. Start with given segments of length a, b, and 1:

(1) Make a small triangle with the segment of length 1 and segment of length b.

(2) Now place the segment of length a on top of the unit segment with one end at the vertex.

(3) Draw a line parallel to the segment connecting the unit to the segment of length b starting at the other end of segment of length a.

(4) The length from the vertex to the point that the line containing b intersects the line drawn in Step 3 is of length $a \cdot b$.



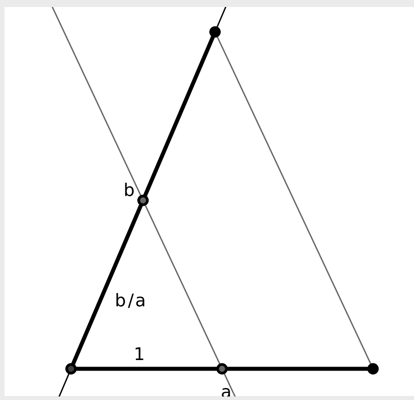
Construction (Division) This construction is also based on the idea of similar triangles. Again, you start with given segments of length a, b, and 1:

(1) Make a triangle with the segment of length a and the segment of length b.

(2) Put the unit along the segment of length a starting at the vertex where the segment of length a and the segment of length b meet.

(3) Make a line parallel to the third side of the triangle containing the segment of length a and the segment of length b starting at the end of the unit.

(4) The distance from where the line drawn in Step 3 meets the segment of length b to the vertex is of length b/a.



Question What does our number line look like at this point?

Currently we have $\mathbb{Z}$, the integers, and all of the fractions. In other words:

$$\mathbb{Q} = \left\{\frac{a}{b} \text{ such that } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0\right\}$$

Fancy folks will replace the words *such that* with a colon ":" to get:

$$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0\right\}$$

We call this set the **rational numbers**. The letter $\mathbb{Q}$ stands for the word *quotient*, which should remind us of fractions.

In mathematics we study sets of numbers. In any field of science, the first step to understanding something is to classify it. One sort of classification that we have is the notion of a *field*.

Definition A **field** is a set of numbers, which we will call F, that is closed under two associative and commutative operations $+$ and $\cdot$ such that:

(1) (a) There exists an additive identity $0 \in F$ such that for all $x \in F$,

$$x + 0 = x.$$

(b) For all $x \in F$, there is an additive inverse $-x \in F$ such that

$$x + (-x) = 0.$$

(2) (a) There exists a multiplicative identity $1 \in F$ such that for all $x \in F$,

$$x \cdot 1 = x.$$

(b) For all $x \in F$ where $x \neq 0$, there is a multiplicative inverse x^{-1} such that

$$x \cdot x^{-1} = 1.$$

(3) Multiplication distributes over addition. That is, for all $x, y, z \in F$

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Now, a word is in order about three tricky words I threw in above: *closed*, *associative*, and *commutative*:

Definition A set F is **closed** under an operation $*$ if for all $x, y \in F$, $x * y \in F$.

Example 5.3.1) The set of integers, $\mathbb{Z}$, is closed under addition, but is not closed under division.

Definition An operation $*$ is **associative** if for all x, y, and z

$$x * (y * z) = (x * y) * z.$$

Definition An operation $*$ is **commutative** if for all x, y

$$x * y = y * x.$$

Question Is $\mathbb{Z}$ a field? Is $\mathbb{Q}$ a field? Can you think of other fields? What about the set of constructible numbers $\mathbb{C}$? What about the folding and tracing numbers $\mathcal{F}$? What about the Descartes numbers $\mathcal{D}$?

?

From all the constructions above we see that the set of constructible numbers $\mathbb{C}$ is a field. However, which field is it? In fact, the set of constructible numbers is bigger than $\mathbb{Q}$!

Construction (Square-Roots) Start with given segments of length a and 1:

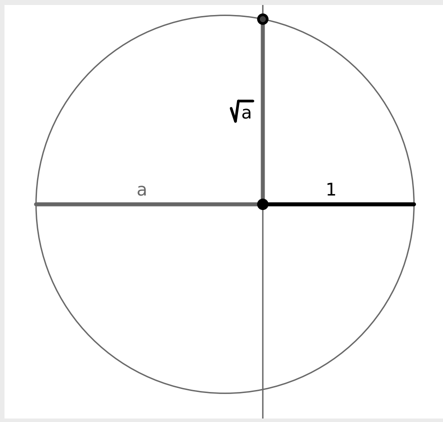
(1) Put the segment of length a immediately to the left of the unit segment on a line.

(2) Bisect the segment of length $a + 1$.

(3) Draw an arc centered at the bisector that starts at one end of the line segment of length $a + 1$ and ends at the other end.

(4) Construct the perpendicular at the point where the segment of length a meets the unit.

(5) The line segment connecting the meeting point of the segment of length a and the unit to the arc drawn in Step 3 is of length $\sqrt{a}$.



This tells us that square-roots are constructible. In particular, the square-root of two is constructible. But the square-root of two is not rational! That is, there is no fraction

$$\frac{a}{b} = \sqrt{2} \qquad \text{such that } a, b \in \mathbb{Z}.$$

Question Can you remind me, how do we know that $\sqrt{2}$ is not rational?

?

Question Are square-roots found in $\mathcal{F}$, the set of folding and tracing numbers? What about $\mathcal{D}$, the set of Descartes numbers?

?

OK, so how do we talk about a field that contains both $\mathbb{Q}$ and $\sqrt{2}$? Simple, use this notation:

$$\mathbb{Q}(\sqrt{2}) = \{\text{the smallest field containing both } \mathbb{Q} \text{ and } \sqrt{2}\}$$

So the set of constructible numbers contains all of $\mathbb{Q}(\sqrt{2})$. Does the set of constructible numbers contain even more numbers? Yes! In fact the $\sqrt{3}$ is also not rational, but is constructible. So here is our situation:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{C}$$

So all the numbers in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ are also in $\mathbb{C}$. But is this all of $\mathbb{C}$? Hardly! We could keep on going, adding more and more square-roots 'til the cows come home, and we still will not have our hands on all of the constructible numbers. But all is not lost. We can still say something:

Theorem 5.3.2 *The use of compass and straightedge alone on a field F can at most produce numbers in a field $F(\sqrt{a})$ where $a \in F$.*

Question Can you explain why the above theorem is true? Big hint: What is the relationship between $\mathbb{C}$ and $\mathcal{D}$?

?

The upshot of the above theorem is that the only numbers that are constructible are expressible as a combination of rational numbers and the symbols:

$$+ \quad - \quad \cdot \quad \div \quad \sqrt{}$$

So what are examples of numbers that are not constructible? Well to start $\sqrt[3]{2}$ is not constructible. Also π is not constructible. While both of these facts can be carefully explained, we will spare you gentle reader—for now.

The most accessible discussion of this fact that I know of can be found in .

Question Which of the following numbers are constructible?

$$3.1415926, \qquad \sqrt[16]{5}, \qquad \sqrt[3]{27}, \qquad \sqrt[6]{27}.$$

?

Problems for Section 5.3

(1) Explain what the set denoted by $\mathbb{Z}$ is.

(2) Explain what the set denoted by $\mathbb{Q}$ is.

(3) Explain what the set $\mathbb{C}$ of constructible numbers is.

(4) Given two line segments a and b, construct $a+b$. Explain the steps in your construction.

(5) Given two line segments a and b, construct $a-b$. Explain the steps in your construction.

(6) Given three line segments 1, a, and b, construct $a \cdot b$. Explain the steps in your construction.

(7) Given three line segments 1, a, and b, construct a/b. Explain the steps in your construction.

(8) Given a unit, construct $4/3$. Explain the steps in your construction.

(9) Given a unit, construct $3/4$. Explain the steps in your construction.

(10) Use the construction for multiplication to explain why when multiplying two numbers between 0 and 1, the product is always still between 0 and 1.

(11) Explain why the construction for multiplication works.

(12) Use the construction for division to explain why when dividing a positive number by a number between 0 and 1, the quotient is always larger than the initial positive number.

(13) Explain why the construction for division works.

(14) Given a unit, construct $\sqrt{2}$. Explain the steps in your construction.

(15) Use algebra to help explain why the construction for square-roots works.

(16) Give relevant and revealing examples of numbers in the set $\mathbb{Z}$.

(17) Give relevant and revealing examples of numbers in the set $\mathbb{Q}$.

(18) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2})$.

(19) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(20) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

(21) Which of the following are constructible numbers? Explain your answers.

(a) 3.141

(b) $\sqrt[3]{5}$

(c) $\sqrt{3+\sqrt{17}}$

(d) $\sqrt[8]{5}$

(e) $\sqrt[10]{37}$

(f) $\sqrt[16]{37}$

(g) $\sqrt[3]{28}$

(h) $\sqrt[3]{27}$

(i) $\sqrt{13+\sqrt[3]{2}+\sqrt{11}}$

(j) $3+\sqrt[5]{4}$

(k) $\sqrt{3+\sqrt{19}+\sqrt{10}}$

(22) Is $\sqrt{7}$ a rational number? Is it a constructible number? Explain your reasoning.

(23) Is $\sqrt{8}$ a rational number? Is it a constructible number? Explain your reasoning.

(24) Is $\sqrt{9}$ a rational number? Is it a constructible number? Explain your reasoning.

(25) Is $\sqrt[3]{7}$ a rational number? Is it a constructible number? Explain your reasoning.

(26) Is $\sqrt[3]{8}$ a rational number? Is it a constructible number? Explain your reasoning.

(27) Is $\sqrt[3]{9}$ a rational number? Is it a constructible number? Explain your reasoning.

5.4 Impossibilities

Oddly enough, the importance of compass and straightedge constructions is not so much what we can construct, but what we cannot construct. It turns out that classifying what we cannot construct is an interesting question. There are three classic problems which are impossible to solve with a compass and straightedge alone:

(1) Doubling the cube.

(2) Squaring the circle.

(3) Trisecting the angle.

5.4.1 Doubling the Cube

The goal of this problem is to double the volume of a given cube. This boils down to trying to construct roots to the equation:

$$x^3 - 2 = 0$$

But we can see that the only root of the above equation is $\sqrt[3]{2}$ and we already know that this number is not constructible.

Question Why does doubling the cube boil down to constructing a solution to the equation $x^3 - 2 = 0$?

?

5.4.2 Squaring the Circle

Given a circle of radius r, we wish to construct a square that has the same area. Why would someone want to do such a thing? Well to answer this question you must ask yourself:

Question What is area?

?

So what is the deal with this problem? Well suppose you have a circle of radius 1. Its area is now π square units. How long should the edge of a square be if it has the same area? Well the square should have sides of length $\sqrt{\pi}$ units. In 1882, it was proved that π is not the root of any polynomial equation, and hence $\sqrt{\pi}$ is not constructible. Therefore, it is impossible to square the circle.

5.4.3 Trisecting the Angle

This might sound like the easiest to understand, but it's a bit subtle. Given any angle, the goal is to trisect that angle. It can be shown that this cannot be done using a compass and straightedge. In particular, it is impossible to trisect a 60 degree angle with compass and straightedge alone. However, we are not saying that you cannot trisect some angles with compass and straightedge alone, in fact there are *special* angles which can be trisected using a compass and straightedge. However the methods used to trisect those special angles will fail miserably in nearly all other cases.

Question Can you think of any angles that can be trisected using a compass and straightedge?

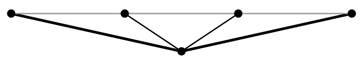
?

Just because it is impossible to trisect an arbitrary angle with compass and straightedge alone does not stop people from trying.

Question If you did not know that it was impossible to trisect an arbitrary angle with a compass and straightedge alone, how might you try to do it?

?

One common way that people try to trisect angles is to take an angle, make an isosceles triangle using the angle, and divide the line segment opposite the angle into three equal parts. While you can divide the opposite side into three equal parts, it in fact **never** trisects the angle. When you do this procedure to acute angles, it *seems* to work, though it doesn't really. You can see that it doesn't by looking at an obtuse angle:



Trisecting the line segment opposite the angle clearly leaves the middle angle much larger than the outer two angles. This happens regardless of the measure of the angle. This mistake is common among people who think that they can trisect an angle with compass and straightedge alone.

5.4.4 Folding and Tracing's Time to Shine

We know that:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{C} = \mathcal{D}$$

Where does the set of folding and tracing numbers $\mathcal{F}$ fit into the parade? I'll tell you, if you promise not to tell anybody that I did. . . $\mathcal{F}$ is the leader of the pack! We already know that you can trisect angles using folding and tracing constructions. In fact you can even solve cubic equations! We'll show you how to do this.

Construction (Solving Cubic Equations) We wish to solve equations of the form:

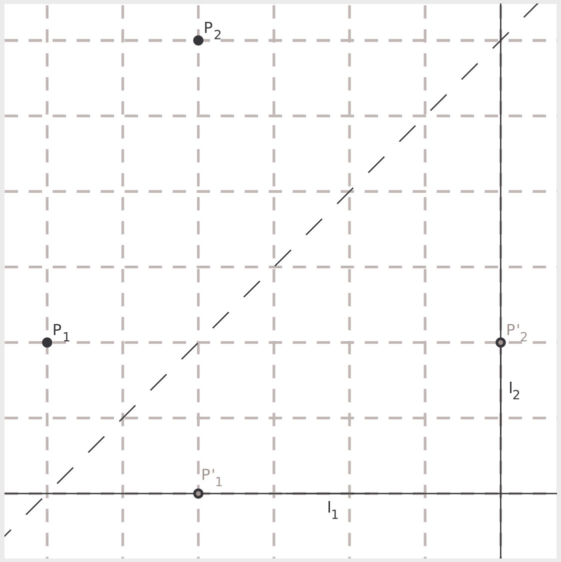
$$x^3 + ax^2 + bx + c = 0$$

(1) Plot the points: $P_1 = (a, 1)$ and $P_2 = (c, b)$.

(2) Plot the lines: $\ell_1 : y = -1$ and $\ell_2 : x = -c$.

(3) With a single fold, place P_1 onto ℓ_1 and P_2 onto ℓ_2.

(4) The slope of the crease is a solution to $x^3 + ax^2 + bx + c = 0$.



Question How do we get the “solution” from the slope?

?

Since folding and tracing constructions can duplicate every compass and straightedge construction and more, we have that $\mathbb{C} \subseteq \mathcal{F}$.

Problems for Section 5.4

(1) Explain the three classic problems that cannot be solved with a compass and straightedge alone.

(2) Use a compass and straightedge construction to trisect an angle of 90°. Explain the steps in your construction.

(3) Use a compass and straightedge construction to trisect an angle of 135°. Explain the steps in your construction.

(4) Use a compass and straightedge construction to trisect an angle of 45°. Explain the steps in your construction.

(5) Use a compass and straightedge construction to trisect an angle of 67.5°. Explain the steps in your construction.

(6) Use folding and tracing to construct an angle of 20°. Explain the steps in your construction.

(7) Use folding and tracing to construct an angle of 10°. Explain the steps in your construction.

(8) Is it possible to use compass and straightedge constructions to construct an angle of 10°? Why or why not?

(9) We have seen that:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{C} \subseteq \mathcal{F}$$

Give explicit examples showing that the set inclusions above are strict—none of them are set equality. Explain your reasoning.

(10) Use folding and tracing to find a solution to the following cubic equations:

(a) $x^3 - x^2 - x + 1 = 0$

(b) $x^3 - 2x^2 - x + 2 = 0$

(c) $x^3 - 3x - 2 = 0$

(d) $x^3 - 4x^2 + 5x - 2 = 0$

(e) $x^3 - 2x^2 - 5x + 6 = 0$

Explain the steps in your constructions.

5.5 Functions and More Functions

To be written.

Fixnote: This section is about coordinates, functions, and parametric equations.

6 City Geometry

> I always like a good math solution to any love problem.
>
> —Carrie Bradshaw

6.1 Welcome to the City

The approach taken in this section was adapted from
.

One day I was walking through the city—that's right, New York City. I had the most terrible feeling that I was lost. I had just passed a *Starbucks Coffee* on my left and a *Sbarro Pizza* on my right, when what did I see? Another *Starbucks Coffee* and *Sbarro Pizza*! Three options occurred to me:

(1) I was walking in circles.

(2) I was at the nexus of the universe.

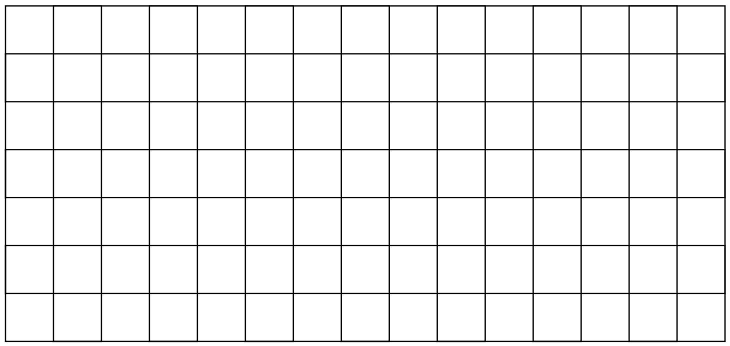
(3) New York City had way too many *Starbucks* and *Sbarro Pizzas*!

Regardless, I was lost. My buddy Joe came to my rescue. He pointed out that the city is organized like a grid.

"Ah! city geometry!" I exclaimed. At this point all Joe could say was "Huh?"

Question What the heck was I talking about?

Let me tell you: *Euclidean geometry* is regular old plane (not plain!) geometry. It is the geometry that we've been exploring thus far in our journey. In *city geometry* we have points and lines, just like in Euclidean geometry. However, most cities can be viewed as a grid of city blocks

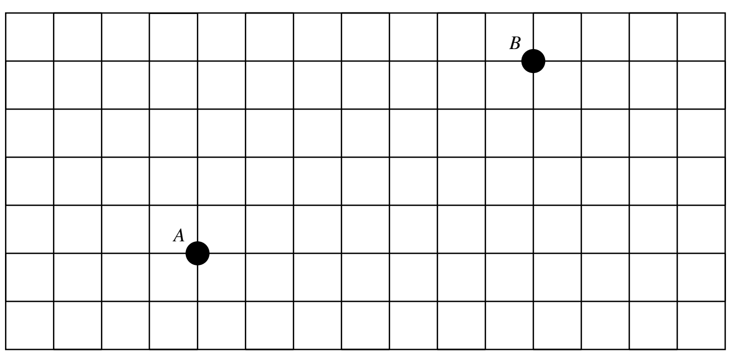


and when we travel in a city, we can only travel on the streets—we can't cut through the blocks. This means that we don't measure distance as the crow flies. Instead we use the *taxicab distance*:

Definition Given two points $A = (a_x, a_y)$ and $B = (b_x, b_y)$, we define the **taxicab distance** as:

$$d_T(A, B) = |a_x - b_x| + |a_y - b_y|$$

Example 6.1.1) Consider the following points:



Let $A = (0, 0)$. Now we see that $B = (7, 4)$. Hence

$$\begin{aligned} d_T(A, B) &= |0 - 7| + |0 - 4| \\ &= 7 + 4 \\ &= 11. \end{aligned}$$

Of course in real life, you would want to add in the appropriate units to your final answer.

Question How do you compute the distance between A and B as the crow flies?

?

Definition The geometry where points and lines are those from Euclidean geometry but distance is measured via taxicab distance is called **city geometry**.

Question Compare and contrast the notion of a line in Euclidean geometry and in city geometry. In either geometry is a line the unique shortest path between any two points?

?

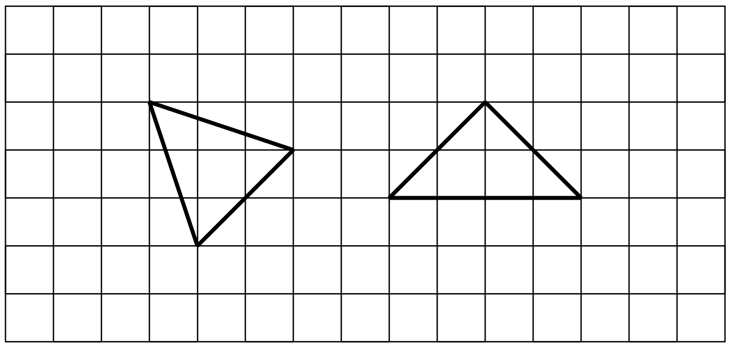
6.1.1 (Un)Common Structures

How different is life in city geometry from life in Euclidean geometry? Let's find out!

Triangles If we think back to Euclidean geometry, we may recall some lengthy discussions on triangles. Yet so far, we have not really discussed triangles in city geometry.

Question What does a triangle look like in city geometry and how do you measure its angles?

I'll take this one. Triangles look the same in city geometry as they do in Euclidean geometry. Also, you measure angles in exactly the same way. However, there is one minor hiccup. Consider these two triangles in city geometry:



Question What are the lengths of the sides of each of these triangles? Why is this odd?

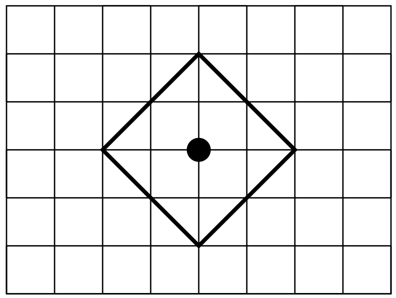
?

Hence we see that triangles are a bit funny in city geometry.

Circles Circles are also discussed in many geometry courses and this course is no different. However, in city geometry the circles are a little less round. The first question we must answer is the following:

Question What is a circle?

Well, a circle is the collection of all points equidistant from a given point. So in city geometry, we must conclude that a circle of radius 2 would look like:



Question What sort of shape should a city geometry compass draw?

?

Question How many points are there at the intersection of two circles in Euclidean geometry? How many points are there at the intersection of two circles in city geometry?

?

Problems for Section 6.1

(1) Given two points A and B in city geometry, does $d_T(A,B) = d_T(B,A)$? Explain your reasoning.

(2) It was once believed that Euclid's five postulates

(a) A line can be drawn from a point to any other point.

(b) A finite line can be extended indefinitely.

(c) A circle can be drawn, given a center and a radius.

(d) All right angles are ninety degrees.

(e) If a line intersects two other lines such that the sum of the interior angles on one side of the intersecting line is less than the sum of two right angles, then the lines meet on that side and not on the other side.

were sufficient to completely describe plane geometry. Explain how city geometry shows that Euclid's five postulates are **not** enough to determine all of the familiar properties of the plane.

(3) In Euclidean geometry are all equilateral triangles congruent assuming they have the same side length? Is this true in city geometry? Explain your reasoning.

(4) How many points are there at the intersection of two circles in Euclidean geometry? How many points are there at the intersection of two circles in city geometry? Explain your reasoning.

(5) What sort of shape should a city geometry compass draw? Explain your reasoning.

(6) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for an equilateral triangle using a city geometry compass.

(7) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for bisecting a segment using a city geometry compass.

(8) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for a perpendicular through a point using a city geometry compass.

(9) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for bisecting an angle using a city geometry compass.

(10) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for copying an angle using a city geometry compass.

(11) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for a parallel through a point using a city geometry compass.

6.2 Anatomy of Figures and the City

When we study geometry, what do we seek? That's right—we wish to discover the points that can be obtained given a set of rules. With city geometry, the major rule involved is the taxicab distance. Let's answer these questions!

Question In regards to city geometry, what is a *point*?

?

Question In regards to city geometry, what is a *line*?

?

Question In regards to city geometry, what is a *circle*?

?

Now I'm going to quiz you about them (I know we've already gone over this *twice*, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a city geometry circle containing these points? If no, why not? If yes, how do you know?

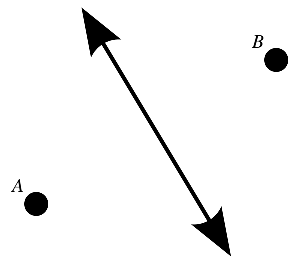
?

Midsets

Definition Given two points A and B, their **midset** is the set of points that are an equal distance away from both A and B.

Question How do we find the midset of two points in Euclidean geometry? How do we find the midset of two points in city geometry?

In Euclidean geometry, we just take the the following line:



If we had no idea what the midset should look like in Euclidean geometry, we could start as follows:

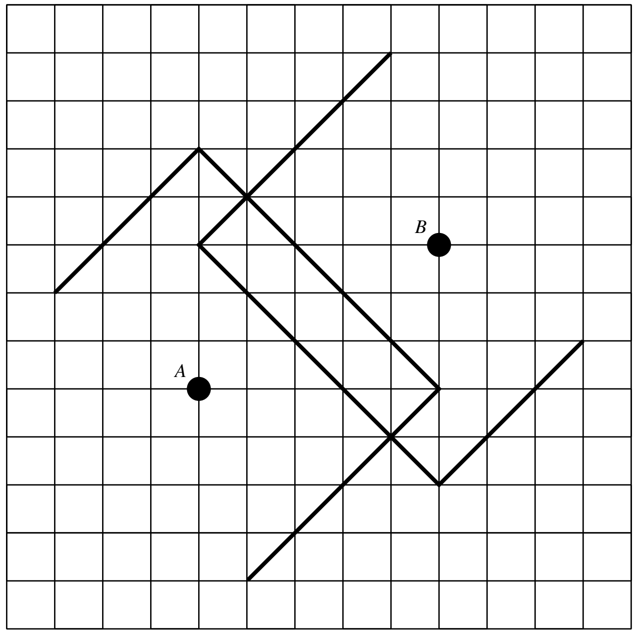
- Draw circles of radius r_1 centered at both A and B. If these circles intersect, then their points of intersection will be in our midset. (Why?)
- Draw circles of radius r_2 centered at both A and B. If these circles intersect, then their points of intersection will be in our midset.
- We continue in this fashion until we have a clear idea of what the midset looks like. It is now easy to check that the line in our picture is indeed the midset.

  How do we do it in city geometry? We do it basically the same way.

Example 6.2.1) Suppose you wished to find the midset of two points in city geometry.

We start by fixing coordinate axes. Considering the diagram below, if $A = (0, 0)$, then $B = (5, 3)$. We now use the same idea as in Euclidean geometry. Drawing circles of radius 3 centered at A and B respectively, we see that there are no points 3 points away from both A and B. Since $d_T(A, B) = 8$, this is to be expected. We will need to draw larger taxicab circles before we will find points in the midset. Drawing taxicab circles of radius 5, we see that the points $(1, 4)$

and $(4,-1)$ are both in our midset.

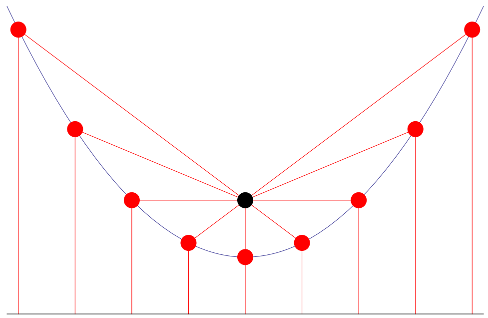


Now it is time to sing along. You draw circles of radius 6, to get two more points $(1,5)$ and $(4,-2)$. Drawing circles with larger radii yields more and more points "due north" of $(1,5)$ and "due south" of $(4,-2)$. However, if we draw circles of radius 4 centered at A and B respectively, their intersection is the line segment between $(1,3)$ and $(4,0)$. Unlike Euclidean circles, distinct city geometry circles can intersect in more than two points and city geometry midsets can be more complicated than their Euclidean counterparts.

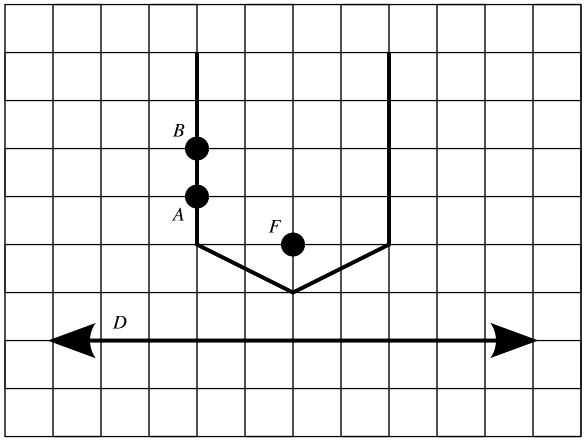
Question How do you draw the city geometry midset of A and B? What could the midsets look like?

?

Parabolas Recall that a parabola is a set of points such that each of those points is the same distance from a given point, F, as it is from a given line, D.



This definition still makes sense when we work with taxicab distance instead of Euclidean distance. To start, choose a value r and draw a line parallel to D at taxicab distance r away from D. Now draw a City circle of radius r centered at F. The points of intersection of this line and this circle will be r away from D and r away from F and so will be points on our City parabola. Repeat this process for different values of r.



Unlike the Euclidean case, the City parabola need not grow broader and broader as the distance from the line increases. In the picture above, as we go from A to B

on the parabola, both the taxicab and Euclidean distances to the line D increase by 1. The taxicab distance from the point F also increases by 1 as we go from A to B but the Euclidean distance increases by less than 1. For the Euclidean distance from F to the parabola to keep increasing at the same rate as the distance to the line D, the Euclidean parabola has to keep spreading to the sides.

Question How do you draw city geometry parabolas? What do different parabolas look like?

?

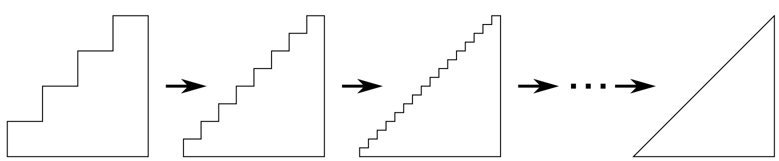
A Paradox To be completely clear on what a paradox is, here is the definition we will be using:

Definition A **paradox** is a statement that seems to be contradictory. This means it seems both true and false at the same time.

There are many paradoxes in mathematics. By studying them we gain insight—and also practice tying our brain into knots! Here is a paradox:

Paradox $\sqrt{2} = 2$.

Proof (False-Proof) *Consider the following sequence of diagrams:*



On the far right-hand side, we see a right-triangle. Suppose that the lengths of the legs of the right-triangle are one. Now by the Pythagorean Theorem, the *length of the hypotenuse is* $\sqrt{1^2 + 1^2} = \sqrt{2}$.

However, we see that the triangles coming from the left converge to the triangle on the right. In every case on the left, the stair-step side has length 2. *Hence when our sequence of stair-step triangles converges, we see that the hypotenuse of the right-triangle will have length* 2. *Thus* $\sqrt{2} = 2$.

Question What is wrong with the proof above?

?

Problems for Section 6.2

(1) Suppose that you have two triangles $\triangle ABC$ and $\triangle DEF$ in city geometry such that

(a) $d_T(A, B) = d_T(D, E)$.

(b) $d_T(B, C) = d_T(E, F)$.

(c) $d_T(C, A) = d_T(F, D)$.

Is it necessarily true that $\triangle ABC \equiv \triangle DEF$? Explain your reasoning.

(2) In city geometry, if all the angles of $\triangle ABC$ are 60°, is $\triangle ABC$ necessarily an equilateral triangle? Explain your reasoning.

(3) In city geometry, if two right triangles have legs of the same length, is it true that their hypotenuses will be the same length? Explain your reasoning.

(4) Considering that π is the ratio of the circumference of a circle to its diameter, what is the value of π in city geometry? Explain your reasoning.

(5) Considering that the area of a circle of radius r is given by πr^2, what is the value of π in city geometry? Explain your reasoning.

(6) When is the Euclidean midset of two points equal to their city geometry midset? Explain your reasoning.

(7) Find the city geometry midset of $(-2, 2)$ and $(3, 2)$.

(8) Find the city geometry midset of $(-2, 2)$ and $(4, -1)$.

(9) Find the city geometry midset of $(-2, 2)$ and $(2, 2)$.

(10) Draw the city geometry parabola determined by the point $(0, 2)$ and the line $y = 0$.

(11) Draw the city geometry parabola determined by the point $(3, 0)$ and the line $x = 0$.

(12) Draw the city geometry parabola determined by the point $(2, 0)$ and the line $y = x$.

(13) Find the distance in city geometry from the point $(3, 4)$ to the line $y = -1/3x$. Explain your reasoning.

(14) Draw the city geometry parabola determined by the point $(0, 4)$ and the line $y = x/3$. Explain your reasoning.

(15) Draw the city geometry parabola determined by the point $(0, 6)$ and the line $y = x/2$. Explain your reasoning.

(16) Draw the city geometry parabola determined by the point $(1, 4)$ and the line $y = 2x/3$. Explain your reasoning.

(17) Draw the city geometry parabola determined by the point $(3, 3)$ and the line $y = x/2$. Explain your reasoning.

(18) Find all points P such that $d_T(P, A) + d_T(P, B) = 8$. Explain your work. (In Euclidean geometry, this condition determines an *ellipse*. The solution to this problem could be called the *city geometry ellipse*.)

(19) True/False: Three noncollinear points lie on a unique Euclidean circle. Explain your reasoning.

(20) True/False: Three noncollinear points lie on a unique city geometry circle. Explain your reasoning.

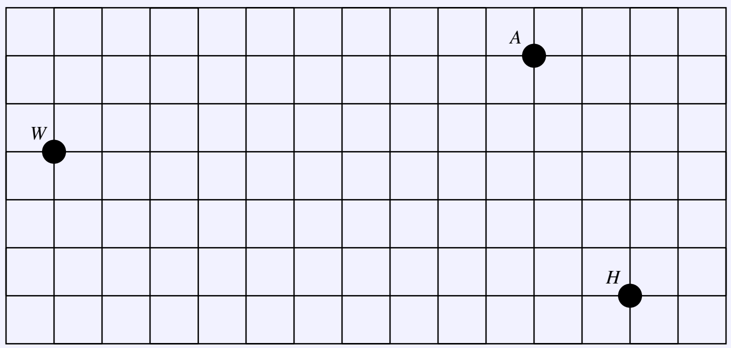
(21) Explain why no Euclidean circle can contain three collinear points. Can a city geometry circle contain three collinear points? Explain your conclusion.

(22) Can you find a false-proof showing that $\pi = 2$?

6.3 Getting Work Done

If you are interested in *real-world* types of problems, then maybe city geometry is the geometry for you. The concepts that arise in city geometry are directly applicable to everyday life.

Question Will just bought himself a brand new gorilla suit. He wants to show it off at three parties this Saturday night. The parties are being held at his friends' houses: the Antidisestablishment (A), Hausdorff (H), and the Wookie Loveshack (W). If he travels from party A to party H to party W, how far does he travel this Saturday night?



Proof (Solution) *We need to compute*

$$d_T(A, H) + d_T(H, W)$$

Let's start by fixing a coordinate system and making A the origin. Then H is $(2, -5)$ and W is $(-10, -2)$. Then

$$\begin{aligned} d_T(A, H) &= |0 - 2| + |0 - (-5)| \\ &= 2 + 5 \\ &= 7 \end{aligned}$$

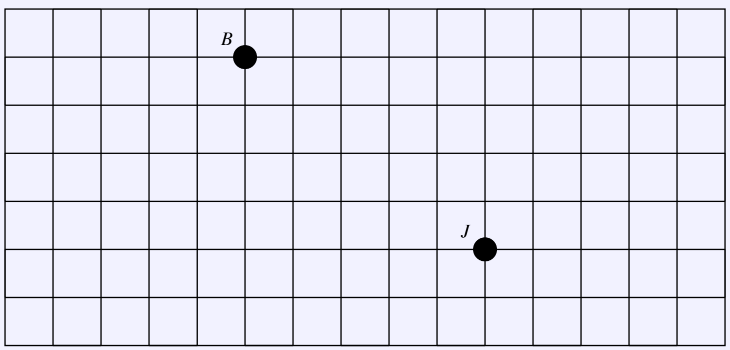
and

$$\begin{aligned} d_T(H, W) &= |2 - (-10)| + |-5 - (-2)| \\ &= 12 + 3 \\ &= 15. \end{aligned}$$

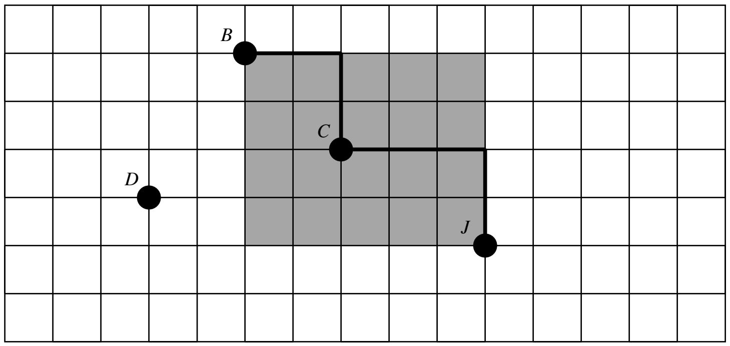
Will must trudge 7 + 15 = 22 *blocks in his gorilla suit.*

Okay, that's enough monkey business—I feel like pizza and a movie.

Question Brad and Melissa are going to downtown Champaign, Illinois. Brad wants to go to *Jupiter's* for pizza (J) while Melissa goes to *Boardman's Art Theater* (B) to watch a movie. Where should they park to minimize the total distance walked by both?



Proof (Solution) *Again, let's set up a coordinate system so that we can say what points we are talking about. If J is $(0, 0)$, then B is $(-5, 4)$.*



No matter where they park, Brad and Melissa's two paths joined together must make a path from B to J. This combined path has to be at least 9 blocks long since $d_T(B, J) = 9$. They should look for a parking spot in the rectangle formed by the points $(0, 0), (0, 4), (-5, 0)$, and $(-5, 4)$.

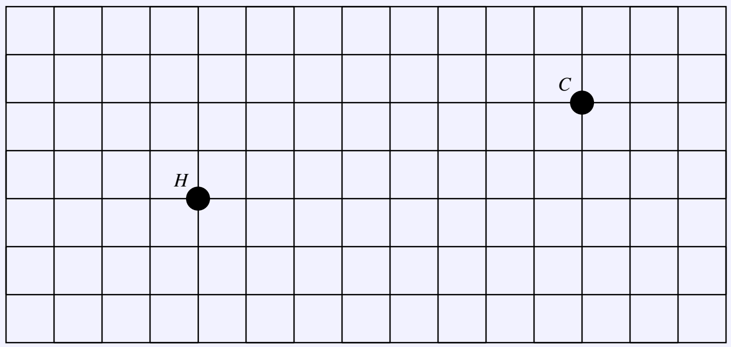
Suppose they park within this rectangle and call this point C. Melissa now walks 4 blocks from C to B and Brad walks 5 blocks from C to J. The two paths joined together form a path from B to J of length 9.

If they park outside the rectangle described above, for example at point D, then the corresponding path from B to J will be longer than 9 blocks. Any path from B to J going through D goes a block too far west and then has to backtrack a block to the east making it longer than 9 blocks.

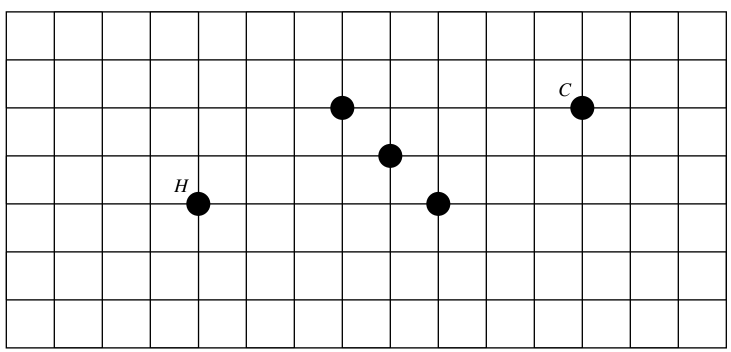
Question If we consider the same question in Euclidean geometry, what is the answer?

?

Question Tom is looking for an apartment that is close to Altgeld Hall (H) but is also close to his favorite restaurant, *Crane Alley* (C). Where should Tom live?



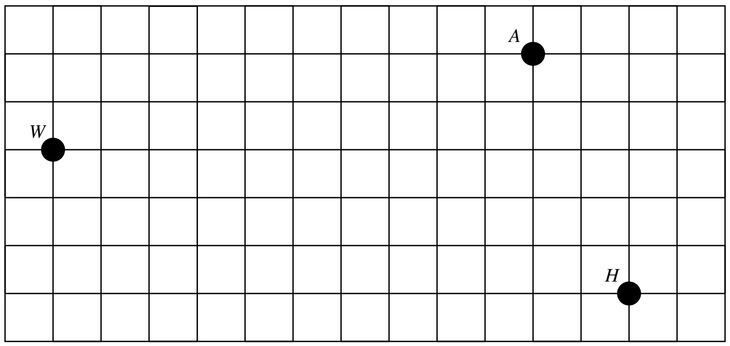
Proof (Solution) *If we fix a coordinate system with its origin at Altgeld Hall, H, then C is at $(8, 2)$. We see that $d_T(H, C) = 10$. If Tom wants to live as close as possible to both of these, he should look for an apartment, A, such that $d_T(A, H) = d_T(A, C) = 5$. He would then be living halfway along one of the shortest paths from Altgeld to the restaurant. Mark all the points 5 blocks away from H. Now mark all the points 5 blocks away from C.*



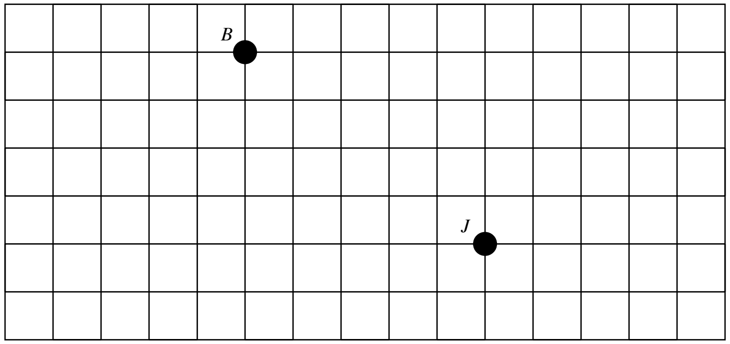
We now see that Tom should check out the apartments near $(5, 0)$, $(4, 1)$, and $(3, 2)$.

Problems for Section 6.3

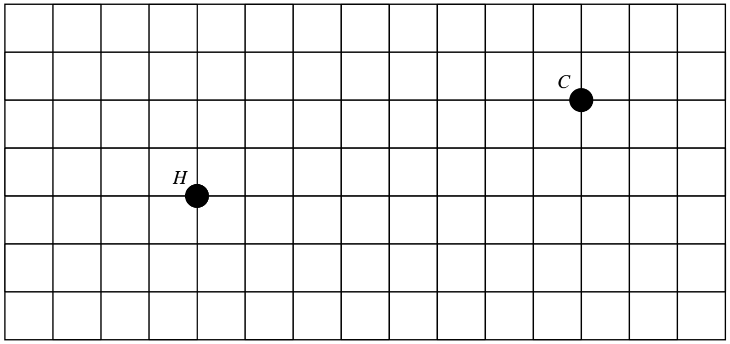
(1) Will just bought himself a brand new gorilla suit. He wants to show it off at three parties this Saturday night. The parties are being held at his friends' houses: the Antidisestablishment (A), Hausdorff (H), and the Wookie Loveshack (W). If he travels from party A to party H to party W, how far does he travel this Saturday night? Explain your reasoning.



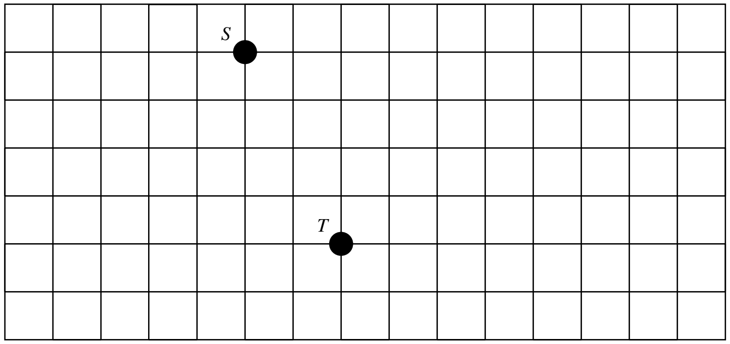
(2) Brad and Melissa are going to downtown Champaign, Illinois. Brad wants to go to *Jupiter's* for pizza (J) while Melissa goes to *Boardman's Art Theater* (B) to watch a movie. Where should they park to minimize the total distance walked by both? Explain your reasoning.



(3) Tom is looking for an apartment that is close to Altgeld Hall (H) but is also close to his favorite restaurant, *Crane Alley* (C). Where should Tom live? Explain your reasoning.

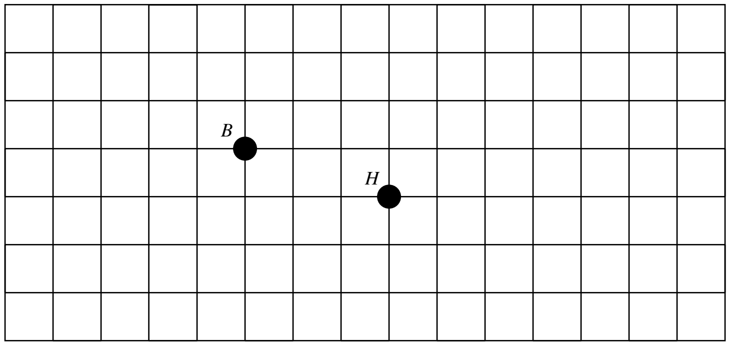


(4) Johann and Amber are going to German Village. Johann wants to go to *Schmidt's* (S) for a cream-puff while Amber goes to the *Thurman Cafe* (T) for some spicy wings. Where should they park to minimize the total distance walked by both if Amber insists that Johann should not have to walk a longer distance than her? Explain your reasoning.



(5) Han and Tom are going to downtown Clintonville. Han wants to go to get a haircut (H) and Tom wants to look at the bookstore (B). Where should they park to keep the total distance walked by both

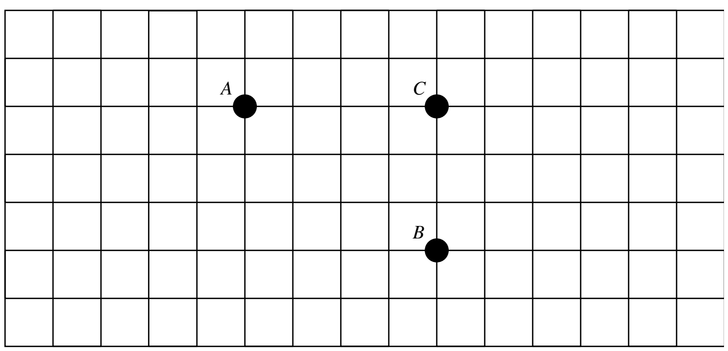
less than 8 blocks? Explain your reasoning.



(6) The university is installing emergency phones across campus. Where should they place them so that their students are never more than a block away from an emergency phone? Explain your reasoning.

(7) Tom and Ben have devised a ingenious *Puzzle-Stroll* (aka a *scavenger-hunt*). Here is one of the puzzles:

> To find what you seek, you must be one with the city—using it's distance, the treasure is 4 blocks from (A), 3 blocks from (B), and 2 blocks from (C).

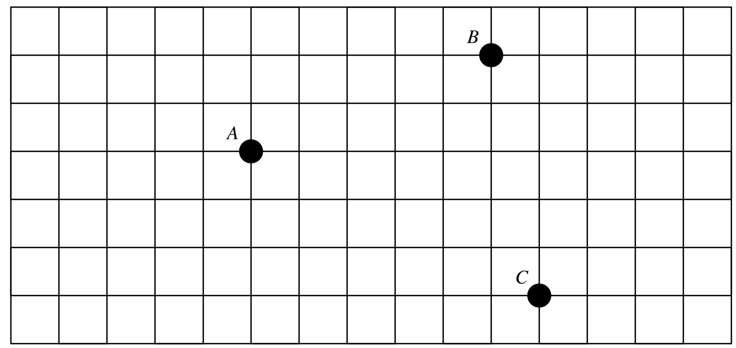


Where's the treasure? Explain your reasoning.

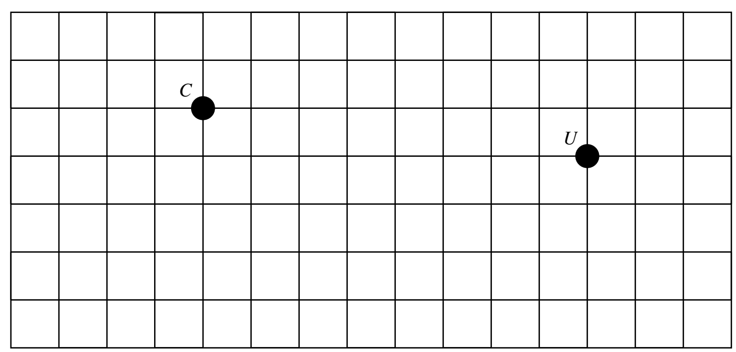
(8) Johann is starting up a new business, *Cafe Battle Royale*. He knows mathematicians drink a lot of coffee so he wants it to be near Altgeld Hall. Balancing this against how expensive rent is near campus, he decides the cafe should be 3 blocks from Altgeld Hall. Where should his cafe be located? Explain your reasoning.

(9) *Cafe Battle Royale, Inc.* is expanding. Johann wants his potential customers to always be within 4 blocks of one of his cafes. Where should his cafes be located? Explain your reasoning.

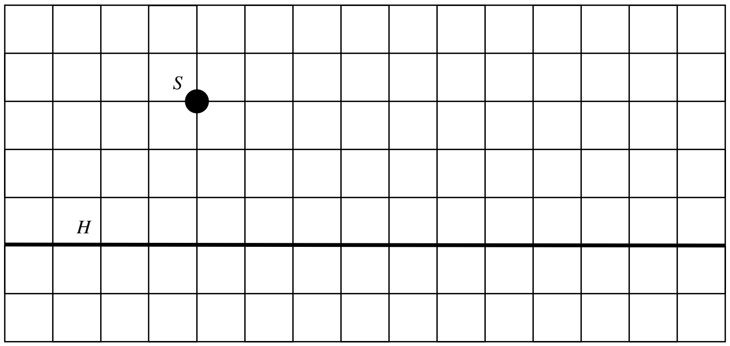
(10) There are hospitals located at A, B, and C. Ambulances should be sent to medical emergencies from whichever hospital is closest. Divide the city into regions in a way that will help the dispatcher decide which ambulance to send. Explain your reasoning.



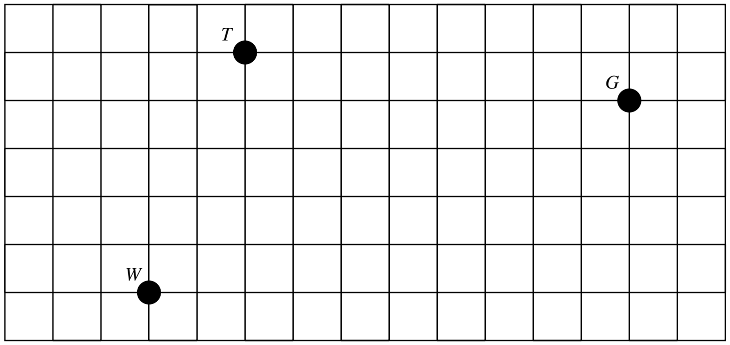
(11) Sylvia is going to open a new restaurant called *Grillvia's* where customers make their own food and then she grills it for them. She wants her restaurant to be equidistant from the heart of Champaign (C) and the heart of Urbana (U). Where should she put her restaurant? Explain your reasoning.



(12) Chris wants to live an equal distance from his favorite hangout *Studio 35* (S) and High Street (H) where he can catch the Number 2 bus. Where should he live? Explain your reasoning.



(13) Lisa just bought a 3-wheeled zebra-striped electric car and its range is limited. Suppose that each day Lisa likes to go to work (W), and then to the tea shop (T) **or** the garden shop (G) but not both, and then back home (H). Where should Lisa live? Give several options depending on how efficient her zebra-striped car is. Explain your reasoning.



7 The Algebra of Isometries

> And since you know you cannot see yourself, so well as by reflection, I, your glass, will modestly discover to yourself, that of yourself which you yet know not of.
>
> —William Shakespeare

7.1 Matrices as Functions

So far in this course, we have discussed isometries without coordinates. In previous algebra courses, you may have used matrices to solve systems of equations. In this chapter, we bring these ideas together, for when we have a coordinate system, an isometry can be represented by a matrix. Thus, we again use powerful techniques of algebra to do geometry.

We begin by revisiting the most important isometries, the basic rigid motions (translations, reflections, and rotations), this time with coordinates and matrices. To start us off, we need a little background on matrices.

Question What is a *matrix*?

?

You might think of a matrix as just a jumble of large brackets and numbers. However, we are going to think of matrices as *functions*. Just as we write $f(x)$ for a function f acting on a number x, we'll write:

$$\mathsf{M}\mathbf{p} = \mathbf{q}$$

to represent a matrix M mapping point $\mathbf{p}$ to point $\mathbf{q}$. A point $\mathbf{p}$ is often represented as an ordered pair of coordinates, $\mathbf{p} = (x, y)$. However, to make things work out nicely, we need to write our points all straight and narrow, with a little buddy at the end:

$$(x, y) \rightsquigarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Throughout this chapter, we will abuse notation slightly, freely interchanging several notations for a point:

$$\mathbf{p} \leftrightsquigarrow (x, y) \leftrightsquigarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

With this in mind, our work will be done via matrices and points that look like this:

$$\mathsf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Now recall the nitty gritty details of *matrix multiplication*:

$$\begin{aligned} \mathsf{M}\mathbf{p} &= \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} ax + by + c \cdot 1 \\ dx + ey + f \cdot 1 \\ 0 \cdot x + 0 \cdot y + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix} \end{aligned}$$

Question Fine, but what does this have to do with geometry?

In this chapter we are going to study a special type of functions, called *isometries*. These are function that preserves distances. Let's see what we mean by this:

Definition An **isometry** is a function M that maps points in the plane to other points in the plane such that

$$d(\mathbf{p}, \mathbf{q}) = d(\mathsf{M}\mathbf{p}, \mathsf{M}\mathbf{q}),$$

where d is the distance function.

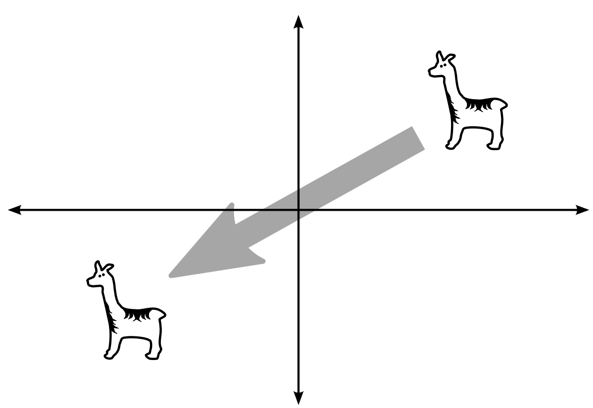
Question How do you compute the distance between two points again?

?

We're going to see that several ideas in geometry, specifically translations, reflections, and rotations which all seem very different, are actually all isometries. Hence, we will be thinking of these concepts as matrices.

7.1.1 Translations

Of all the isometries, *translations* are probably the easiest. With a translation, all we do is move our object in a straight line, that is, every point in the plane is moved the same distance and the same direction. Let's see what happens to Louie Llama when he is translated:

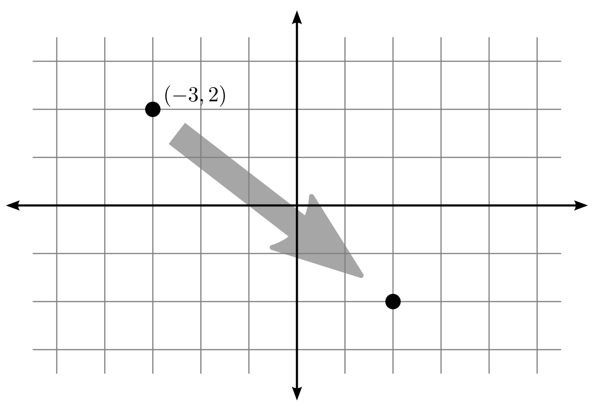


Pretty simple eh? We can give a more "mathematical" definition of a translation involving our newly-found knowledge of matrices! Check it:

Definition A **translation**, denoted by $\mathsf{T}_{(u,v)}$, is a function that moves every point a given distance u in the x-direction and a given distance v in the y-direction. We will use the following matrix to represent translations:

$$\mathsf{T}_{(u,v)} = \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}$$

Example 7.1.1) Consider the point $\mathbf{p} = (-3, 2)$. Use a matrix to translate $\mathbf{p}$ 5 units right and 4 units down.



Here is how you do it:

$$\begin{aligned}
\mathsf{T}_{(5,-4)}\mathbf{p} &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -3+0+5 \\ 0+2-4 \\ 0+0+1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence, we end up with the point $(2, -2)$. But you knew that already, didn't you?

Question Can you demonstrate with algebra why translations are isometries?

?

Question We know how to translate individual points. How do we move entire figures and other funky shapes?

?

7.1.2 Reflections

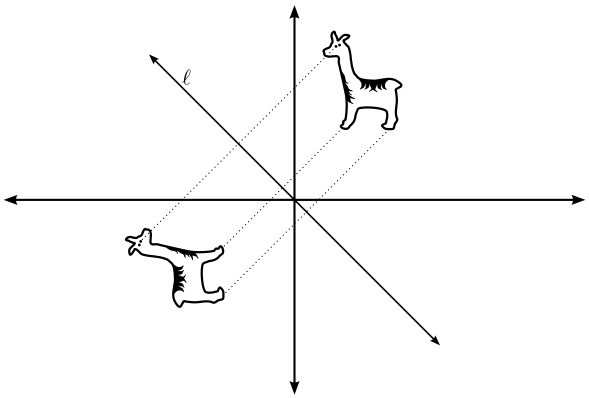
The act of reflection has fascinated humanity for millennia. It has a strong effect on our perception of beauty and has a defined place in art—not to mention how useful it is for the application of make-up. Here is our definition of a reflection:

Definition The **reflection** across a line ℓ, denoted by F_ℓ, is the function that maps a point $\mathbf{p}$ to a point $\mathsf{F}_\ell\mathbf{p}$ such that:

(1) If $\mathbf{p}$ is on ℓ, then $\mathsf{F}_\ell\mathbf{p} = \mathbf{p}$.

(2) If $\mathbf{p}$ is not on ℓ, then ℓ is the perpendicular bisector of the segment connecting $\mathbf{p}$ and $\mathsf{F}_\ell\mathbf{p}$.

You might be saying, "Huh?" It's not as hard as it looks. Check out this picture of the situation, again Louie Llama will help us out:



A Collection of Reflections We are going to begin with a trio of reflections. We'll start with a **horizontal reflection** across the y-axis. Using our matrix notation, we write:

$$\mathsf{F}_{x=0} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

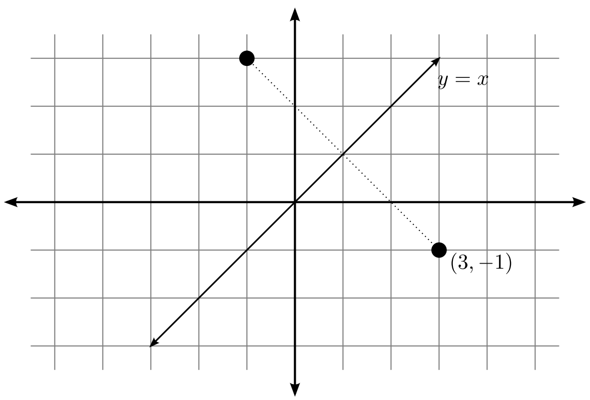
The next reflection in our collection is a **vertical reflection** across the x-axis. Using our matrix notation, we write:

$$\mathsf{F}_{y=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final reflection to add to our collection is a **diagonal reflection** across the line $y = x$. Using our matrix notation, we write:

$$\mathsf{F}_{y=x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 7.1.2) Consider the point $\mathbf{p} = (3, -1)$. Use a matrix to reflect $\mathbf{p}$ across the line $y = x$.



Here is how you do it:

$$\begin{aligned}
\mathsf{F}_{y=x}\mathbf{p} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0-1+0 \\ 3+0+0 \\ 0+0+1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence we end up with the point $(-1, 3)$.

Question Let $\mathbf{p}$ be some point in Quadrant I of the (x, y)-plane. What reflection will map this point to Quadrant II? What about Quadrant IV? What about Quadrant III?

?

Question Can you demonstrate with algebra why each of our reflections above are isometries?

?

7.1.3 Rotations

Imagine that you are on a swing set, going higher and higher until you are actually able to make a full circle. At the point where you are directly above where you would be if the swing were at rest, where is your head, comparatively? Your feet? Your hands?

Face it, I think we all dreamed of doing that when we were little—or in my case, last week.

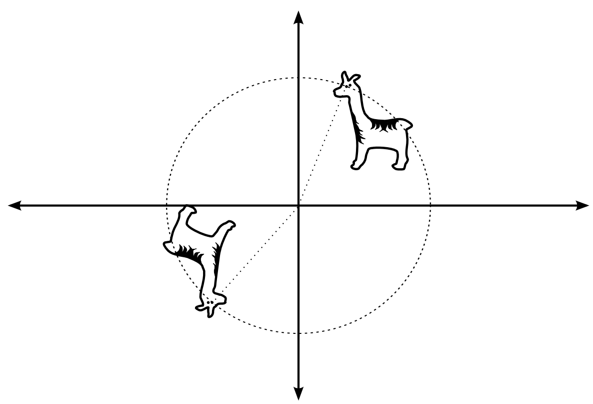
Rotations should bring circles to mind. This is not a coincidence. Check out our definition of a *rotation*:

Definition A **rotation** of ϑ degrees about the origin, denoted by R_ϑ, is a function that maps a point $\mathbf{p}$ to a point $\mathsf{R}_\vartheta\mathbf{p}$ such that:

(1) The points $\mathbf{p}$ and $\mathsf{R}_\vartheta\mathbf{p}$ are equidistant from the origin.

(2) An angle of ϑ degrees is formed by $\mathbf{p}$, the origin, and $\mathsf{R}_\vartheta\mathbf{p}$.

Louie Llama, can you do the honors?



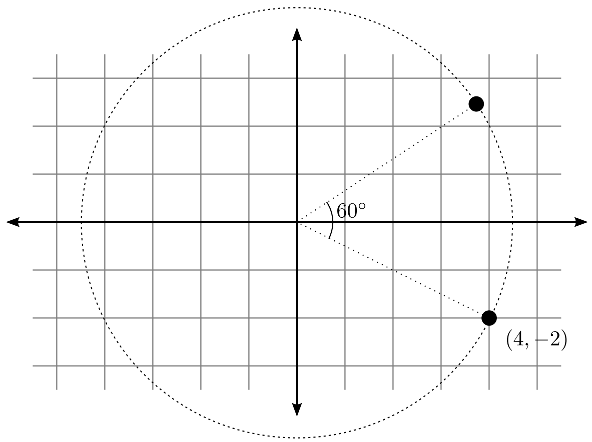
Warning Positive angles denote a counterclockwise rotation. Negative angles denote a clockwise rotation.

Looking back on trigonometry, there were some angles that kept on coming up.

Some of these were 90°, 60°, and 45°. We'll focus on these angles too.

$$R_{90} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{60} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{45} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 7.1.3) Consider the point $\mathbf{p} = (4, -2)$. Use a matrix to rotate $\mathbf{p}$ 60° about the origin.



Here is how you do it:

$$\begin{aligned} R_{60}\mathbf{p} &= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2+\sqrt{3}+0 \\ 2\sqrt{3}-1+0 \\ 0+0+1 \end{bmatrix} \\ &= \begin{bmatrix} 2+\sqrt{3} \\ 2\sqrt{3}-1 \\ 1 \end{bmatrix} \end{aligned}$$

Hence, we end up with the point $(2+\sqrt{3}, 2\sqrt{3}-1)$.

Question Do the numbers in the matrices above look familiar? If so, why?

?

Question How do you rotate a point 180 degrees?

?

Question Can you demonstrate with algebra why our rotations above are isometries?

?

Problems for Section 7.1

(1) How do you compute the distance between two points **p** and **q** in the plane?

(2) Use algebra to explain why:

$$d(\mathbf{p}, \mathbf{q}) = d(\mathbf{p} - \mathbf{q}, \mathbf{o}) = d(\mathbf{o}, \mathbf{p} - \mathbf{q})$$

where $\mathbf{o} = (0, 0)$.

(3) What is an isometry?

(4) What is a translation?

(5) What is a rotation?

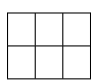
(6) What is a reflection?

(7) Reflecting back on this chapter, suppose I translate a point **p** to **p**′. Does it make any difference if I move the point **p** along a wiggly path

or a straight path? Explain your reasoning.

(8) Reflecting back on this chapter, is a rotation the continuous *act* of *moving* a point through an angle around some fixed point, or is it just a final picture compared to the initial one? Explain your reasoning.

(9) In the vector illustrator *Inkscape* there is an option to transform an image via a "matrix." If you select this tool, you are presented with 6 boxes to fill in with numbers:



Use what you've learned in this chapter to make a guess as to how this tool works.

(10) In what direction does a positive rotation occur?

(11) Is a 270° rotation the same as a −90° rotation? Explain your reasoning.

(12) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(13) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & 0 & 8 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(14) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(15) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & 2 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(16) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(17) Use a matrix to translate the point $(-1, 6)$ three units right and two units up. Sketch this situation and explain your reasoning.

(18) The matrix $\mathsf{T}_{(-2,6)}$ was used to translate the point **p** to $(-1, -3)$. What is **p**? Sketch this situation and explain your reasoning.

(19) Use a matrix to reflect the point $(5, 2)$ across the x-axis. Sketch this situation and explain your reasoning.

(20) Use a matrix to reflect the point $(-3, 4)$ across the y-axis. Sketch this situation and explain your reasoning.

(21) Use a matrix to reflect the point $(-1, 1)$ across the line $y = x$. Sketch this situation and explain your reasoning.

(22) Use a matrix to reflect the point $(1, 1)$ across the line $y = x$. Sketch this situation and explain your reasoning.

(23) The matrix $\mathsf{F}_{y=0}$ was used to reflect the point **p** to $(4, 3)$. What is **p**? Explain your reasoning.

(24) The matrix $\mathsf{F}_{y=0}$ was used to reflect the point **p** to $(0, -8)$. What is **p**? Explain your reasoning.

(25) The matrix $\mathsf{F}_{x=0}$ was used to reflect the point **p** to $(-5, -1)$. What is **p**? Explain your reasoning.

(26) The matrix $\mathsf{F}_{y=x}$ was used to reflect the point **p** to $(9, -2)$. What is **p**? Explain your reasoning.

(27) The matrix $\mathsf{F}_{y=x}$ was used to reflect the point **p** to $(-3, -3)$. What is **p**? Explain your reasoning.

(28) Considering the point $(3, 2)$, use a matrix to rotate this point 60° about the origin. Sketch this situation and explain your reasoning.

(29) Considering the point $(\sqrt{2}, -\sqrt{2})$, use a matrix to rotate this point 45° about the origin. Sketch this situation and explain your reasoning.

(30) Considering the point $(-7, 6)$, use a matrix to rotate this point 90° about the origin. Sketch this situation and explain your reasoning.

(31) Considering the point $(-1, 3)$, use a matrix to rotate this point 0° about the origin. Sketch this situation and explain your reasoning.

(32) Considering the point $(0, 0)$, use a matrix to rotate this point 120° about the origin. Sketch this situation and explain your reasoning.

(33) Considering the point $(1, 1)$, use a matrix to rotate this point -90° about the origin. Sketch this situation and explain your reasoning.

(34) The matrix R_{90} was used to rotate the point **p** to $(2, -5)$. What is **p**? Explain your reasoning.

(35) The matrix R_{60} was used to rotate the point **p** to $(0, 2)$. What is **p**? Explain your reasoning.

(36) The matrix R_{45} was used to rotate the point **p** to $(-\frac{1}{2}, \frac{5}{2})$. What is **p**? Explain your reasoning.

(37) The matrix R_{-90} was used to rotate the point **p** to $(4, 3)$. What is **p**? Explain your reasoning.

(38) If someone wanted to plot the graph of $y = x^2$, they might start by filling in the following table:

x	x^2
0	
1	
−1	
2	
−2	
3	
−3	

Reflect each point you obtain from the table above about the line $y = x$. Give a plot of this situation. What curve do you obtain? What is this new curve's relationship to $y = x^2$? Explain your reasoning.

(39) Some translation T was used to map point **p** to point **q**. Given $\mathbf{p} = (1, 2)$ and $\mathbf{q} = (3, 4)$, find T and explain your reasoning.

(40) Some translation T was used to map point **p** to point **q**. Given $\mathbf{p} = (-2, 3)$ and $\mathbf{q} = (2, 3)$, find T and explain your reasoning.

(41) Some reflection F was used to map point **p** to point **q**. Given $\mathbf{p} = (1, 4)$ and $\mathbf{q} = (1, -4)$, find F and explain your reasoning.

(42) Some reflection F was used to map point **p** to point **q**. Given $\mathbf{p} = (5, 0)$ and $\mathbf{q} = (0, 5)$, find F and explain your reasoning.

(43) Some rotation R was used to map point **p** to point **q**. Given $\mathbf{p} = (3, 0)$ and $\mathbf{q} = (0, 3)$, find R and explain your reasoning.

(44) Some rotation R was used to map point **p** to point **q**. Given $\mathbf{p} = (\sqrt{2}, \sqrt{2})$ and $\mathbf{q} = (0, 2)$, find R and explain your reasoning.

(45) Some matrix M maps

$$(0, 0) \mapsto (0, 0),$$
$$(1, 0) \mapsto (3, 0),$$
$$(0, 1) \mapsto (0, 5).$$

Find M and explain your reasoning.

(46) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (-1,1),\\ (1,0) &\mapsto (3,0),\\ (0,1) &\mapsto (0,5).\end{aligned}$$

Find M and explain your reasoning.

(47) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (1,1),\\ (1,0) &\mapsto (2,1),\\ (0,1) &\mapsto (1,2).\end{aligned}$$

Find M and explain your reasoning.

(48) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (2,2),\\ (1,1) &\mapsto (3,3),\\ (-1,1) &\mapsto (1,3).\end{aligned}$$

Find M and explain your reasoning.

(49) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (0,0),\\ (1,1) &\mapsto (0,3),\\ (-1,1) &\mapsto (5,0).\end{aligned}$$

Find M and explain your reasoning.

(50) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (1,2),\\ (1,1) &\mapsto (-3,1),\\ (-1,1) &\mapsto (2,-3).\end{aligned}$$

Find M and explain your reasoning.

7.2 The Algebra of Matrices

7.2.1 Matrix Multiplication

We know how to multiply a matrix and a point. Multiplying two matrices is a similar procedure:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj+bm+cp & ak+bn+cq & al+bo+cr \\ dj+em+fp & dk+en+fq & dl+eo+fr \\ gj+hm+ip & gk+hn+iq & gl+ho+ir \end{bmatrix}$$

Variables are all good and well, but let's do this with actual numbers. Consider the following two matrices:

$$\mathsf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{and} \qquad \mathsf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's multiply them together and see what we get:

$$\begin{aligned} \mathsf{MI} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot1+2\cdot0+3\cdot0 & 1\cdot0+2\cdot1+3\cdot0 & 1\cdot0+2\cdot0+3\cdot1 \\ 4\cdot1+5\cdot0+6\cdot0 & 4\cdot0+5\cdot1+6\cdot0 & 4\cdot0+5\cdot0+6\cdot1 \\ 7\cdot1+8\cdot0+9\cdot0 & 7\cdot0+8\cdot1+9\cdot0 & 7\cdot0+8\cdot0+9\cdot1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ &= \mathsf{M} \end{aligned}$$

Question What is IM equal to?

?

It turns out that we have a special name for I. We call it the **identity matrix**.

Warning Matrix multiplication is **not** generally commutative. Check it out:

$$\mathsf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When we multiply these matrices, we get:

$$\begin{aligned} \mathsf{FR} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 & 1 \cdot (-1) + 0 \cdot 0 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 \\ 0 \cdot 0 + (-1) \cdot 1 + 0 \cdot 0 & 0 \cdot (-1) + (-1) \cdot 0 + 0 \cdot 0 & 0 \cdot 0 + (-1) \cdot 0 + 0 \cdot 1 \\ 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 & 0 \cdot (-1) + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

On the other hand, we get:

$$\mathsf{RF} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question Can you draw some nice pictures showing geometrically that matrix multiplication is not commutative?

?

Question Is it always the case that $(\mathsf{LM})\mathsf{N} = \mathsf{L}(\mathsf{MN})$?

?

7.2.2 Compositions of Matrices

It is often the case that we wish to apply several isometries successively to a point. Consider the following:

$$\mathsf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{N} = \begin{bmatrix} g & h & i \\ j & k & l \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Now let's compute

$$\begin{aligned} \mathsf{M}(\mathsf{N}\mathbf{p}) &= \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} g & h & i \\ j & k & l \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} gx + hy + i \\ jx + ky + l \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} agx + ahy + ai + bjx + bky + bl + c \\ dgx + dhy + di + ejx + eky + el + f \\ 1 \end{bmatrix} \end{aligned}$$

Now *you* compute $(\mathsf{MN})\mathbf{p}$ and compare what *you* get to what we got above.

Compositions of Translations A composition of translations occurs when two or more successive translations are applied to the same point. Check it out:

$$
\begin{aligned}
\mathsf{T}_{(5,-4)}\mathsf{T}_{(-3,2)} &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \mathsf{T}_{(5+(-3),(-4)+2)} \\
&= \mathsf{T}_{(2,-2)}
\end{aligned}
$$

Theorem 7.2.1 *The composition of two translations $\mathsf{T}_{(u,v)}$ and $\mathsf{T}_{(s,t)}$ is equal to the translation $\mathsf{T}_{(u+s,v+t)}$.*

Question How do you prove the theorem above?

?

Question Can you give a single translation that is equal to the following composition?

$$\mathsf{T}_{(-7,5)}\mathsf{T}_{(0,-6)}\mathsf{T}_{(2,8)}\mathsf{T}_{(5,-4)}$$

?

Question Are compositions of translations commutative? Are they associative?

?

Compositions of Reflections A composition of reflections occurs when two or more successive reflections are applied to the same point. Check it out:

$$\begin{aligned}
\mathsf{F}_{y=0}\mathsf{F}_{y=x} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Question Is the composition $\mathsf{F}_{y=0}\mathsf{F}_{y=x}$ still a reflection?

?

Question Are compositions of reflections commutative? Are they associative?

?

Compositions of Rotations A composition of rotations occurs when two or more successive rotations are applied to the same point. Check it out:

$$\mathsf{R}_{60}\mathsf{R}_{60} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 7.2.2 *The product of two rotations* R_ϑ *and* R_φ *is equal to the rotation* $\mathsf{R}_{\vartheta+\varphi}$.

From this we see that:

$$\mathsf{R}_{120} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question What is the rotation matrix for a 360° rotation? What about a 405° rotation?

?

Question Are compositions of rotations commutative? Are they associative?

?

7.2.3 Mixing and Matching

Life gets interesting when we start composing translations, reflections, and rotations together. First we'll compose a reflection with a rotation:

$$
\begin{aligned}
F_{y=0}R_{60} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Question Does this result look familiar?

?

Now how about a rotation composed with a translation:

$$
\begin{aligned}
R_{90}T_{(3,-4)} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Question Does $R_{90}T_{(3,-4)} = T_{(3,-4)}R_{90}$?

?

Question Find a matrix that represents the reflection $\mathsf{F}_{y=-x}$.

I'll take this one. Note that

$$\begin{aligned}
\mathsf{F}_{y=-x} &= \mathsf{R}_{180}\mathsf{F}_{y=x} \\
&= \mathsf{R}_{90}\mathsf{R}_{90}\mathsf{F}_{y=x} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

OK looks good, but you, the reader, are going to have to check the above computation yourself.

Question How do we deal with reflections that are not across the lines $y = 0$, $x = 0$, or $y = x$? How would you reflect points across the line $y = 1$?

?

Problems for Section 7.2

(1) Give a single translation that is equal to $\mathsf{T}_{(-3,2)}\mathsf{T}_{(5,-1)}$. Explain your reasoning.

(2) Consider the two translations $\mathsf{T}_{(-4,8)}$ and $\mathsf{T}_{(4,-8)}$. Do these commute? Explain your reasoning.

(3) Give a single reflection that is equal to $\mathsf{F}_{x=0}\mathsf{F}_{y=0}$. Sketch this situation and explain your reasoning.

(4) Given any point $\mathbf{p} = (x, y)$, express $\mathsf{T}_{(4,2)}\mathsf{T}_{(6,-5)}\mathbf{p}$ as $\mathsf{T}_{(u,v)}\mathbf{p}$ for some values of u and v. Sketch this situation and explain your reasoning.

(5) Give a matrix for R_{-45}. Explain your reasoning.

(6) Give a matrix for R_{-60}. Explain your reasoning.

(7) Sam suggests that:

$$\mathsf{R}_{-90} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Why does he suggest this? Is it even true? Explain your reasoning.

(8) Give a matrix for $\mathsf{F}_{y=-x}$. Explain your reasoning.

(9) Given the point $\mathbf{p} = (-4, 2)$, use matrices to compute $\mathsf{F}_{y=0}\mathsf{F}_{y=x}\mathbf{p}$. Sketch this situation and explain your reasoning.

(10) Given the point $\mathbf{p} = (5, 0)$, use matrices to compute $\mathsf{F}_{y=x}\mathsf{F}_{y=-x}\mathbf{p}$. Sketch this situation and explain your reasoning.

(11) Give a single rotation that is equal to $\mathsf{R}_{45}\mathsf{R}_{60}$. Explain your reasoning.

(12) Given the point $\mathbf{p} = (1, 3)$, use matrices to compute $\mathsf{R}_{45}\mathsf{R}_{90}\mathbf{p}$. Sketch this situation and explain your reasoning.

(13) Given the point $\mathbf{p} = (-7, 2)$, use matrices to compute $\mathsf{R}_{45}\mathsf{R}_{-45}\mathbf{p}$. Sketch this situation and explain your reasoning.

(14) Given the point $\mathbf{p} = (-2, 5)$, use matrices to compute $\mathsf{R}_{90}\mathsf{R}_{-90}\mathsf{R}_{360}\mathbf{p}$. Sketch this situation and explain your reasoning.

(15) Given the point $\mathbf{p} = (5, 4)$, use matrices to compute $\mathsf{F}_{y=0}\mathsf{T}_{(2,-4)}\mathbf{p}$. Sketch this situation and explain your reasoning.

(16) Given the point $\mathbf{p} = (-1, 6)$, use matrices to compute $\mathsf{R}_{45}\mathsf{T}_{(0,0)}\mathbf{p}$. Sketch this situation and explain your reasoning.

(17) Given the point $\mathbf{p} = (11, 13)$, use matrices to compute $\mathsf{T}_{(-6,-3)}\mathsf{R}_{135}\mathbf{p}$. Sketch this situation and explain your reasoning.

(18) Given the point $\mathbf{p} = (-7, -5)$, use matrices to compute $\mathsf{R}_{540}\mathsf{F}_{x=0}\mathbf{p}$. Sketch this situation and explain your reasoning.

(19) Give a composition of matrices that will take a point and reflect it across the x-axis and then rotate the result 90° around the origin. Sketch this situation and explain your reasoning.

(20) Give a composition of matrices that will take a point and translate it three units up and 2 units left and then rotate it 90° clockwise around the origin. Sketch this situation and explain your reasoning.

(21) Give a composition of matrices that will take a point and rotate it 270° around the origin, reflect it across the line $y = x$, and then translate the result down 5 units and 3 units to the right. Sketch this situation and explain your reasoning.

(22) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $x = 1$. Sketch this situation and explain your reasoning.

(23) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $y = -4$. Sketch this situation and explain your reasoning.

(24) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $y = x + 5$. Sketch this situation and explain your reasoning.

(25) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and rotate it 45° around the point $(2, 3)$. Sketch this situation and explain your reasoning.

(26) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and rotate it 90° clockwise around the point $(-3, 4)$ Sketch this situation and explain your reasoning.

7.3 The Theory of Groups

One of the most fundamental notions in all of modern mathematics is that of a *group*. Sadly, many students never see a group in their education.

Definition A **group** is a set of elements (in our case matrices) which we will call $\mathcal{G}$ such that:

(1) There is an associative operation (in our case matrix multiplication).

(2) The set is closed under this operation (the product of any two matrices in the set is also in the set).

(3) There exists an identity $\mathsf{I} \in \mathcal{G}$ such that for all $\mathsf{M} \in \mathcal{G}$,

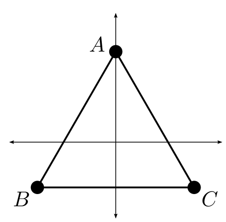
$$\mathsf{IM} = \mathsf{MI} = \mathsf{M}.$$

(4) For all $M \in \mathcal{G}$ there is an inverse $\mathsf{M}^{-1} \in \mathcal{G}$ such that

$$\mathsf{MM}^{-1} = \mathsf{M}^{-1}\mathsf{M} = \mathsf{I}.$$

7.3.1 Groups of Rotations

Let's see a group. Here we have an equilateral triangle centered at the origin of the (x, y)-plane:



Question The matrix R_{360} will rotate this triangle completely around the origin. What matrix will rotate this triangle one-third of a complete rotation?

As a gesture of friendship, I'll take this one. One-third of 360 is 120. So we see that R_{120} will rotate the triangle one-third of a full rotation. Do you remember this matrix? Here it is:

$$R_{120} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In spite of the fact that this matrix is messy and that matrix multiplication is somewhat tedious, you should realize that

$$R_{120}^2 = R_{240} \qquad \text{and} \qquad R_{120}^3 = R_{360}.$$

Let's put these facts (and a few more) together in what is called a *group table*. Remember multiplication tables from elementary school? Well, we're going to make something like a "multiplication table" of rotations. We'll start by listing the identity and powers of a one-third rotation along the top and left-hand sides. Setting $R = R_{120}$ we have:

$\circ$	I	R	R^2	R^3	$\cdots$
I					
R					
R^2					
R^3					
$\vdots$					

Since $\mathsf{R}^3 = \mathsf{I}$, we need only take our table to R^2. At this point we can write out the complete table:

$\circ$	I	R	R^2
I	I	R	R^2
R	R	R^2	I
R^2	R^2	I	R

Since matrix multiplication is associative, and we see from the table that every element has an inverse, we see that

$$\{\mathsf{I}, \mathsf{R}, \mathsf{R}^2\}$$

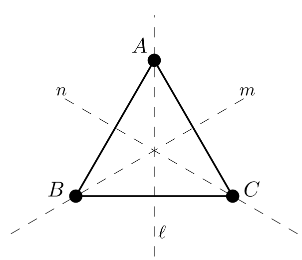
is a group.

Question What rotation matrices would we use when working with a square? A pentagon? A hexagon?

?

7.3.2 Groups of Reflections

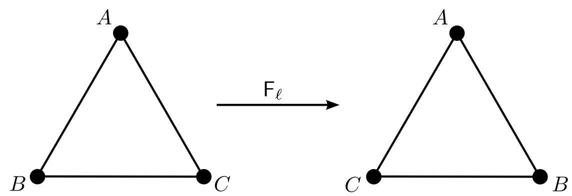
Let's see another group. Again consider an equilateral triangle. This time we are interested in the three lines of reflection that preserve this triangle:



Question Suppose that the triangle above is centered at the origin of the (x, y)-plane. What are equations for ℓ, m, and n?

?

The easiest of the reflections above is the reflection over F_ℓ.



We'll start our group table off with just two elements: I and $\mathsf{F} = \mathsf{F}_\ell$.

$\circ$	I	F
I	I	F
F	F	I

Notice that when we apply F twice we're right back where we started. Hence, $\mathsf{FF} = \mathsf{I}$. Since matrix multiplication is associative, we see that

$$\{\mathsf{I}, \mathsf{F}\}$$

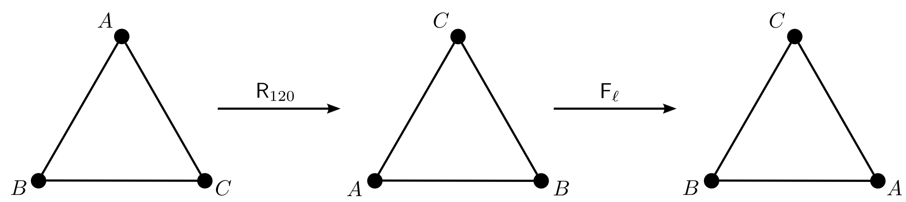
forms a group. Specifically this is a group of reflections of the triangle.

Question Above we used F_ℓ. What would happen if we used F_m or F_n? Also, what are the equations for the lines of symmetry of the square centered at the origin?

?

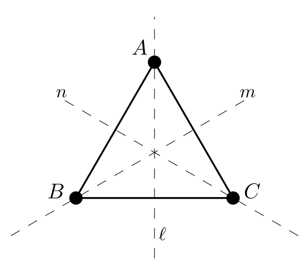
7.3.3 Symmetry Groups

Now let's mix our rotations and reflections. Consider our original triangle and apply $\mathsf{F}_\ell\mathsf{R}_{120}$:



What you may not immediately notice is that we obtain the same transformation by taking the original triangle and applying F_m.

As it turns out, every possible symmetry of the equilateral triangle can be represented using reflections and rotations. Each of these reflections and rotations can be expressed as a composition of a single reflection and a single rotation. The collection of all symmetries forms a group called the *symmetry group* of the equilateral triangle.



Let's see the group table, note we'll let $\mathsf{R} = \mathsf{R}_{120}$ and $\mathsf{F} = \mathsf{F}_\ell$:

$\circ$	I	R	R^2	F	FR	FR^2
I	I	R	R^2	F	FR	FR^2
R	R	R^2	I	FR^2	F	FR
R^2	R^2	I	R	FR	FR^2	F
F	F	FR	FR^2	I	R	R^2
FR	FR	FR^2	F	R^2	I	R
FR^2	FR^2	F	FR	R	R^2	I

This table shows every symmetry of the triangle, including the identify I. By comparing the rows and columns of the group table, you can see that every element has an inverse. This combined with the fact that the matrix multiplication is associative shows that the symmetries of the triangle,

$$\{I, R, R^2, F, FR, FR^2\}$$

form a group.

Question Can you express the symmetries of squares in terms of reflections and rotations? What does the group table look like for the symmetry group of the square?

?

Problems for Section 7.3

(1) State the definition of a group of matrices.

(2) How many lines of reflectional symmetry exist for a square? Provide a drawing to justify your answer.

(3) What are the equations for the lines of reflectional symmetry that exist for the square? Explain your answers.

(4) How many lines of reflectional symmetry exist for a regular hexagon? Provide a drawing to justify your answer.

(5) What are the equations for the lines of reflectional symmetry for a regular hexagon? Explain your answers.

(6) How many consecutive rotations are needed to return the vertexes of a square to their original position? Provide a drawing to justify your answer, labeling the vertexes.

(7) How many degrees are in one-fourth of a complete rotation of the square? Explain your answer.

(8) How many degrees are in one-sixth of a complete rotation of the regular hexagon? Explain your answer.

(9) In this section, we've focused on a 3-sided figure, a 4-sided figure, and a 6-sided figure. Why do we not include the rotation group for the pentagon in this section? If we did, how many degrees would be in one-fifth of a complete rotation?

(10) With notation used in this section, draw pictures representing the action of the following isometries F_ℓ, R, RF_ℓ and $\mathsf{F}_\ell\mathsf{R}$ on the equilateral triangle.

(11) Consider a square centered at the origin. Draw pictures representing the action of $\mathsf{F}_{y=0}$, R_{90}, $\mathsf{R}_{90}\mathsf{F}_{y=0}$, and $\mathsf{F}_{y=0}\mathsf{R}_{90}$ on this square.

(12) Consider a hexagon centered at the origin. Draw pictures representing the action of $\mathsf{F}_{x=0}$, R_{60}, $\mathsf{R}_{60}\mathsf{F}_{x=0}$, and $\mathsf{F}_{x=0}\mathsf{R}_{60}$ on this hexagon.

(13) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is R_{120}. Explain and illustrate your answer.

(14) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is $\mathsf{F}_{x=0}$. Explain and illustrate your answer.

(15) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is R_{120}^2. Explain and illustrate your answer.

(16) Find two symmetries of the square, neither of which is the identity, such that their composition is R_{180}. Explain and illustrate your answer.

(17) Find two symmetries of the square, neither of which is the identity, such that their composition is F_ℓ. Explain and illustrate your answer.

(18) Find two symmetries of the square, neither of which is the identity, such that their composition is R_{270}. Explain and illustrate your answer.

(19) Use a group table to help you write out the symmetries of the equilateral triangle. List all elements that commute with every other element in the table. Explain your reasoning.

(20) Use a group table to help you write out the symmetries of the square. List all elements that commute with every other element in the table. Explain your reasoning.

(21) Use a group table to help you write out the symmetries of the regular hexagon. List all elements that commute with every other element in the table. Explain your reasoning.

(22) Let M be a symmetry of the equilateral triangle. Define

$$C(\mathsf{M}) = \{\text{all symmetries that commute with } \mathsf{M}\}.$$

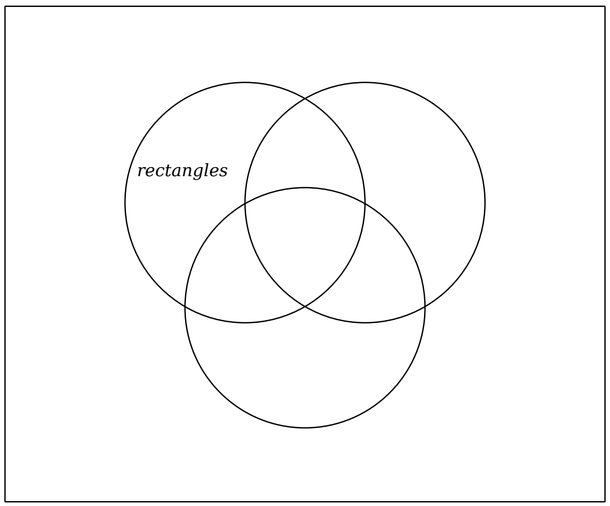
Write out $C(\mathsf{M})$ for every symmetry M of the equilateral triangle. Make some observations.

A Activities

A.1 It's What the Book Says

Teaching Note: This activity is mostly about careful definitions of these quadrilaterals. And we want to compare and contrast the two definitions of trapezoid.

Fifth graders were given the following task: Put the terms **square**, **rhombus**, and **parallelogram** in the Venn diagram below.



A.1.1) What are *squares*, *rhombuses*, and *parallelograms*?

A.1.2) Critique the question above based on mathematical content.

A.1.3) Supposing we know that a quadrilateral is a polygon with four sides, write clear and succinct definitions of each of the following terms:

(a) A *rectangle* is a quadrilateral

(b) A *parallelogram* is a quadrilateral

(c) A *rhombus* is a quadrilateral

(d) A *square* is a quadrilateral

(e) A *trapezoid* is a quadrilateral

(f) A *kite* is a quadrilateral

A.1.4) Create a Venn diagram showing the correct relationships among these quadrilaterals. Be ready to present and defend your diagram to your peers.

A.2 Forget Something?

A.2.1) Draw a Venn diagram with one set. List every possible relationship between an element and this set.

A.2.2) Draw a Venn diagram with two intersecting sets. List every possible relationship between an element and these sets.

A.2.3) Draw a Venn diagram with three intersecting sets. List every possible relationship between an element and these sets.

A.2.4) Describe and explain any patterns you see occurring.

A.2.5) Draw a Venn diagram with four intersecting sets. List every possible relationship between an element and these sets.

A.2.6) Are you **sure** that your diagram for Problem A.2.5 is correct? If so explain why. If not, draw a correct Venn diagram.

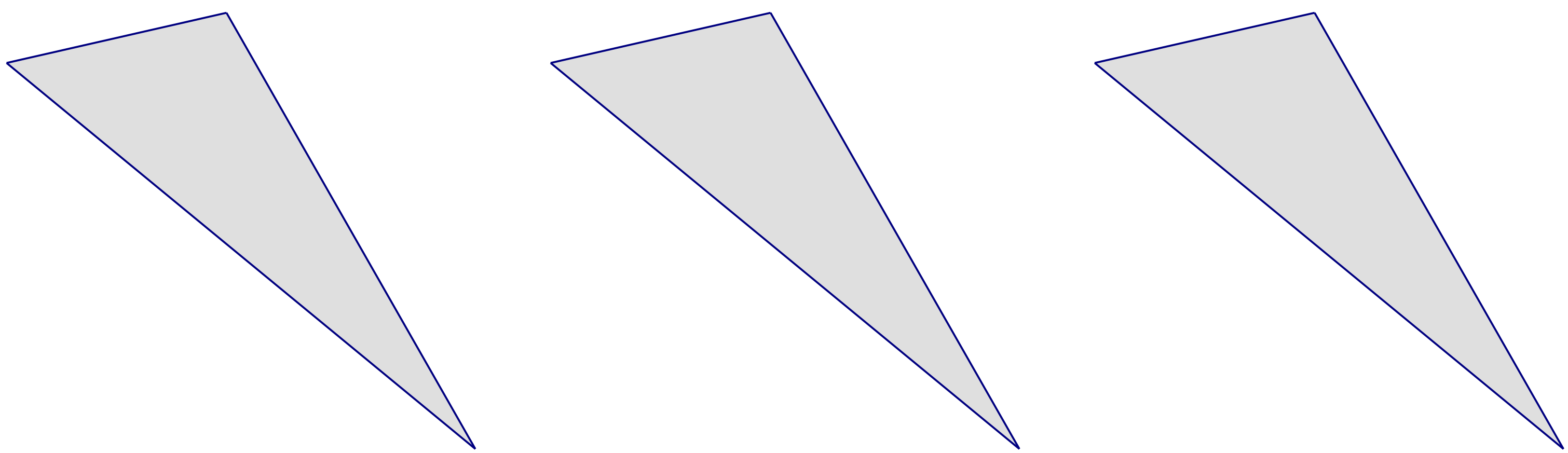
A.3 Measuring Area

A.3.1) Three congruent triangles are shown below.

(a) For each triangle, choose a base and use a ruler to draw carefully the corresponding height to that base. (Choose bases of different lengths.) Remember: A *height* is a measured on a line that is perpendicular to a base and containing the opposite vertex.

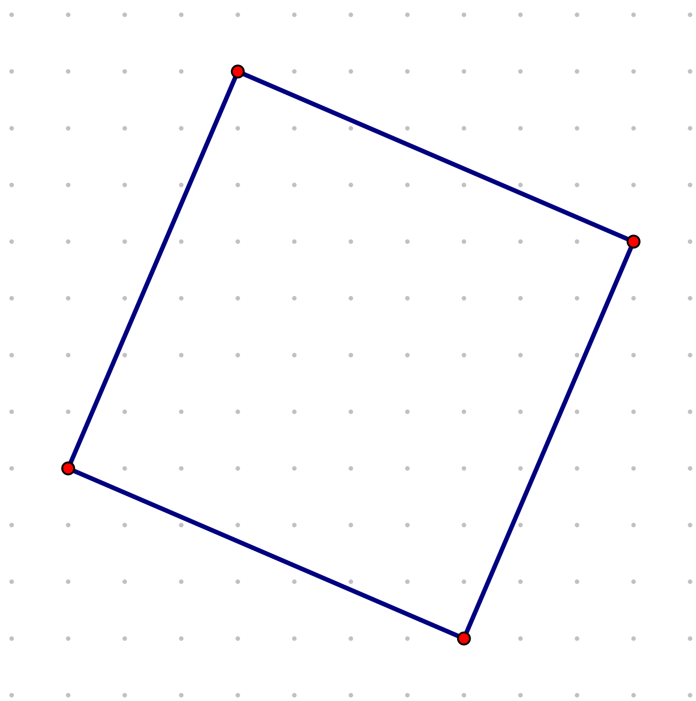
(b) Measure the heights and bases accurately, and compute the area of each triangle.

(c) What do your results demonstrate about the formula for the area of a triangle?



A.4 Tilted Square

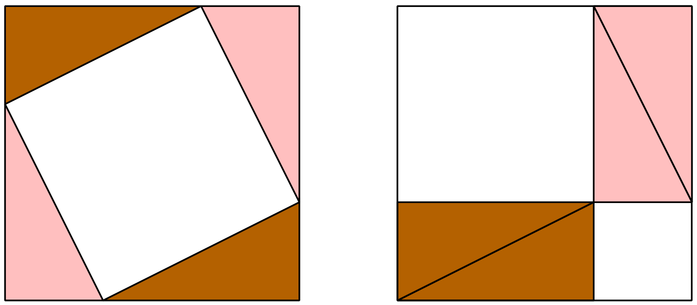
A.4.1) In the diagram below, the dots are 1 centimeter apart, both vertically and horizontally. The vertices of the square all lie exactly on such dots. Find the area of the square, *without computing the length of the side of the square*. Explain your method.



A.5 Pythagorean Theorem

A.5.1) Give two explanations of how the following picture "proves" the Pythagorean Theorem, one using algebra and one without algebra.[8.G.6]

CCSS 8.G.6: Explain a proof of the Pythagorean Theorem and its converse.

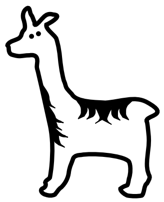


A.5.2) State the converse of the Pythagorean Theorem and prove it.

A.6 Louie Llama and the Triangle

> Fixnote: Also include something like Walking and Turning from Beckmann.

We are going to investigate why the interior angles of a triangle sum to 180°. We won't be alone on this journey, we'll have help. Meet Louie Llama:



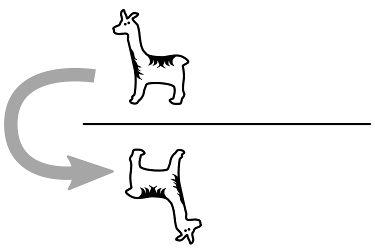
Louie Llama is rather radical for a llama and doesn't mind being rotated at all.

A.6.1) Draw a picture of Louie Llama rotated 90° counterclockwise.

A.6.2) Draw a picture of Louie Llama rotated 180° counterclockwise.

A.6.3) Draw a picture of Louie Llama rotated 360° counterclockwise.

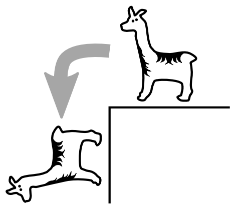
A.6.4) Sometimes Louie Llama likes to walk around lines he finds:



Through what angle did Louie Llama just rotate?

Now we're going to watch Louie Llama go for a walk. Draw yourself any triangle. Draw a crazy scalene triangle—those are the kind that Louie Llama likes best. Louie

Llama is going to parade proudly around this triangle. When Louie Llama walks around corners he rotates. Check it out:



Take your triangle and denote the measure of its angles as a, b, and c. Start Louie Llama out along a side adjacent to the angle of measure a. He should be on the outside of the triangle, his feet should be pointing toward the triangle, and his face should be pointing toward the angle of measure b.

A.6.5) Sketch Louie Llama walking to the angle of measure b. Walk him around the angle. As he goes around the angle his feet should always be pointing toward the triangle. Through what angle did Louie Llama just rotate?

A.6.6) Sketch Louie Llama walking to the angle of measure c. Walk him around the angle. Through what angle did Louie Llama just rotate?

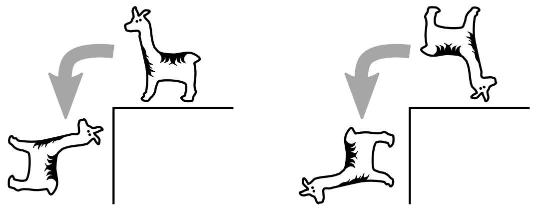
A.6.7) Finally sketch Louie Llama walking back to the angle of measure a. Walk him around the angle. He should be back at his starting point. Through what angle did Louie Llama just rotate?

A.6.8) All in all, how many degrees did Louie Llama rotate in his walk?

A.6.9) Write an equation where the right-hand side is Louie Llama's total rotation and the left-hand side is the sum of each rotation around the angle. Can you solve for $a + b + c$?

As you may have guessed, Louie Llama isn't your typical llama, for one thing he likes to walk backwards and on his head! He also like to do somersaults. Louie

Llama can somersault around corners in two different ways:



A.6.10) What does Louie Llama's somersault have to do with the angle of the corner? Can you precisely explain how Louie Llama rotates when he somersaults around corners?

A.6.11) Can you walk Louie Llama around your original triangle allowing him to walk backwards (or even on his head!), letting him do somersaults as he pleases around corners, and **directly** arrive at the equation

$$a + b + c = 180^\circ?$$

A.6.12) Can you rephrase what we did above in terms of *exterior angles* and *interior angles*?

A.6.13) Can you walk Louie Llama around other shapes and figure out what the sum of their interior angles are? Let's do this with a table:

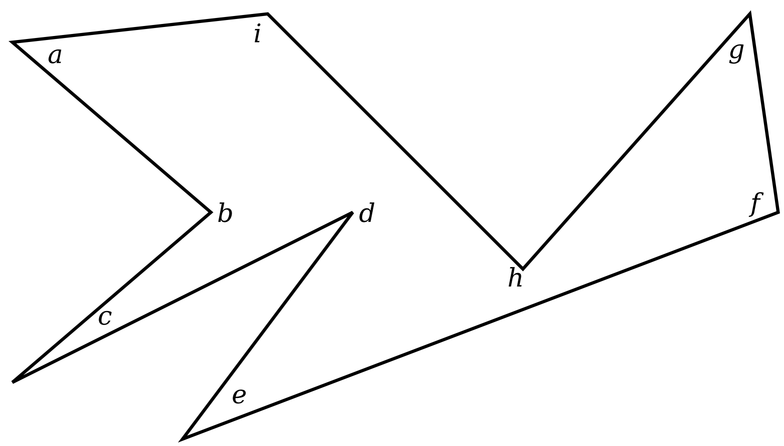
Fixnote: Do we want to include exterior angles in the table?

n-gon	sum of interior angles	interior angle of a regular n-gon
3		
4		
5		
6		
7		
8		
n		

A.7 Angles in a Funky Shape

We are going to investigate the sum of the interior angles of a funky shape.

A.7.1) Using a protractor, measure the interior angles of the crazy shape below:



Use this table to record your findings:

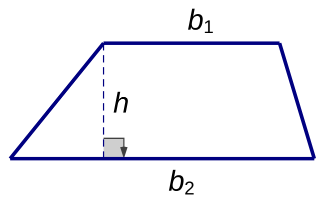
a	b	c	d	e	f	g	h	i

A.7.2) Find the sum of the interior angles of the polygon above.

A.7.3) What should the sum be? Explain your reasoning.

A.8 Trapezoid Area

A.8.1) In this activity, we explore several ways of justifying the formula for the area of a trapezoid, as labeled below.

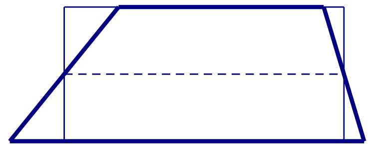
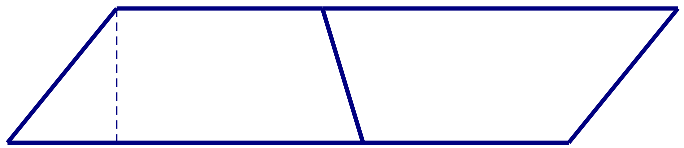
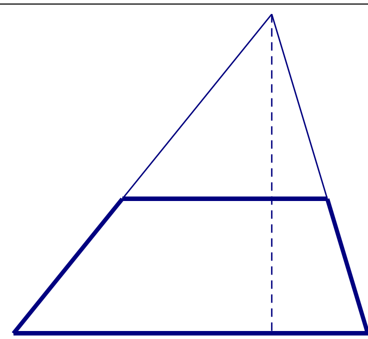


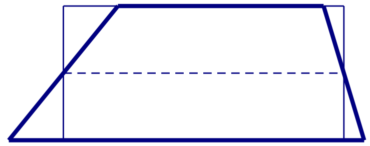
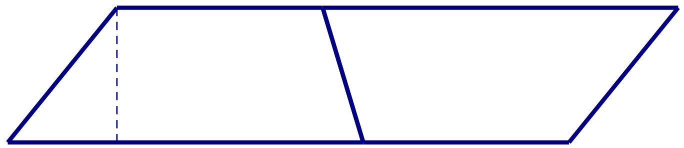
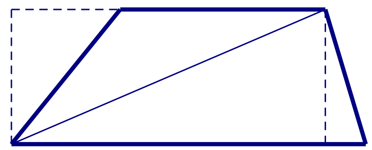
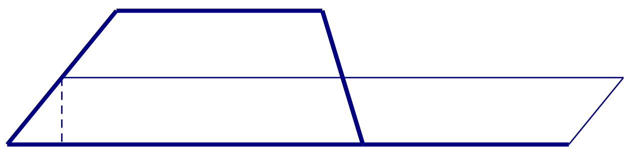
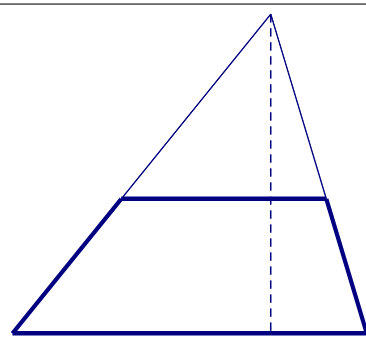
Complete the table on the following page so that in each row the explanation, the figure, and the area formula together describe a way of computing the area. For comparison purposes, each illustration should include a trapezoid congruent to the trapezoid above.

All of the area formulas will, of course, be equivalent to one another as expressions. But each way of expressing the area will make the most sense with figure and the explanation from the same row.

Teaching Note: The next page is for students and then a completed answer page follows.

Fixnote: For the student edition, comment out the answer page.

Explanation	Figure	Area Formula
Rectangle with width that is the average of the bases.		$\left(\frac{b_1 + b_2}{2}\right)h$
		
Two triangles with the same height and different bases.		
		$(b_1 + b_2)\frac{h}{2}$
		

Explanation	Figure	Area Formula
Rectangle with width that is the average of the bases.		$\left(\frac{b_1 + b_2}{2}\right)h$
Half of a large parallelogram.		$\frac{1}{2}(b_1 + b_2)h$
Two triangles with the same height and different bases.		$\frac{1}{2}b_1 h + \frac{1}{2}b_2 h$
A parallelogram with half the height.		$(b_1 + b_2)\frac{h}{2}$
Difference between two triangles, with x as height of small triangle.		$\frac{1}{2}b_2(x + h) - \frac{1}{2}b_1 x$, with $\frac{x}{b_1} = \frac{x+h}{b_2}$

A.9 Triangle Investigation

A.9.1) Use informal reasoning to draw triangles based on the conditions given below. You may use ruler, protractor, and/or compass if you wish, but your solutions should come from reflection and reasoning on you work. In each part, determine whether the information provided determines a unique $\triangle ABC$, more than one triangle, or no triangle.[7.G.2] Note: To check to see if two triangles are different, attempt to lay one directly on top of the other.

CCSS 7.G.2: Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

(a) $AB = 4$ and $BC = 5$

(b) $m\angle CAB = 25^\circ$, $m\angle ABC = 75^\circ$, $m\angle BCA = 80^\circ$

(c) $m\angle CAB = 25^\circ$, $m\angle ABC = 65^\circ$, $m\angle BCA = 80^\circ$

(d) $AB = 5$, $m\angle BAC = 30^\circ$, $m\angle ABC = 45^\circ$

(e) $AB = 5$, $BC = 4$, $m\angle ABC = 60^\circ$

(f) $BC = 7$, $CA = 8$, $AB = 9$

(g) $BC = 4$, $CA = 8$, $AB = 3$

(h) $m\angle ABC = 45^\circ$, $BC = 8$, $CA = 12$

(i) $m\angle ABC = 30^\circ$, $BC = 10$, $CA = 7$

(j) $m\angle ABC = 60^\circ$, $BC = 10$, $CA = 3$

A.10 UnMessUpable Figures

Suppose we draw or a construct a geometric figure with pencil, paper, compass, and straightedge. If we want to compare to another example of the geometric figure, we need to begin again from scratch. With dynamic geometry software, however, we can alter the original figure by "dragging" vertices and segments to create many other examples. For this to work properly, we want to *construct* the figure rather than merely *draw* it, so that a square, for example, remains a square even if we move its vertices. Some folks call such figures "UnMessUpable."

The following problems are intended to be explored using dynamic geometry software such as *Geogebra*, *Geometer's Sketchpad*, or *Cabri*. Before you begin, explore the menus and toolbars to see what tools the software provides. Notice that some several-step compass-and-straightedge constructions, such as perpendicular bisector, are available as a single-step tools in the software. Feel free to use these in your work below. But in this activity do not use tools for transformations (e.g., translations, reflections, or rotations) or that construct objects from measurements.

Begin each problem below in a new sketch or window.

A.10.1) Construct a segment between two points. Then construct an equilateral triangle with that segment as one of its sides. Be sure that the triangle remains equilateral when you drag its vertices. (Note: Do not use a "regular polygon" tool.)

A.10.2) Construct a segment between two points. Then construct a square with that segment as one of its sides. Be sure that it remains a square when you drag its vertices. (Note: Do not use a "regular polygon" tool.)

A.10.3) Construct an UnMessUpable parallelogram.

A.10.4) Construct a rectangle that, through dragging, can be long and thin, short and fat, or anything in between, but that is always a rectangle.

A.10.5) *Copy a segment.* Construct a segment and a line. Then copy the segment onto the line. Hide the line so that the segment alone is clear. Then drag the vertices that determine the initial segment to show that the copy is always congruent to it.

A.10.6) *Copy an angle.* Using the ray tool, construct an angle and a separate ray. Then copy the angle onto the other ray. Drag the vertices that determine the first angle to show that the copy is always congruent to it.

A.10.7) Construct a capital H so that the midline is always the perpendicular bisector of both sides.

A.10.8) Construct a quadrilateral so that one pair of opposite sides is always congruent.

A.11 Isosceles Bisectors

Theorem A.11.1 (Isosceles Triangle Theorem) *If two sides of a triangle are congruent, then the angles opposite those sides are congruent.*

A.11.1) Prove the Isosceles Triangle Theorem. (Hint: In your explorations, you have noticed that in most triangles the median, perpendicular bisector, angle bisector, and altitude to a side lie on four different lines. So if you draw a new line in your diagram, be sure to indicate which of these lines you are drawing.)

A.11.2) Prove the Isosceles Triangle Theorem without drawing another line. Hint: Is there a way in which the triangle is congruent to itself?

A.11.3) State the converse of the Isosceles Triangle Theorem and prove it.

A.11.4) Prove that the points on the perpendicular bisector of a segment are *exactly those* that are equidistant from the endpoints of the segment. Note that the phrase *exactly those* requires that we prove a simpler statement as well as its converse:

(a) Prove that a point on the perpendicular bisector of a segment is equidistant from the endpoints of that segment.

(b) Prove that a point that is equidistant from the endpoints of a segment lies on the perpendicular bisector of that segment.

A.11.5) Prove that the perpendicular bisectors of a triangle are concurrent. Hint: Name the intersection of two of the perpendicular bisectors and then show that it must also lie on the other two. (This is a standard approach for showing the concurrency of three lines.)

A.11.6) Draw a line and a point not on the line. Describe how to find the *exact* distance from the point to the line.

A.11.7) Prove that the points on an angle bisector are *exactly those* that are equidistant from the sides of the angle.

A.11.8) Prove that the angle bisectors of a triangle are concurrent.

A.12 About Medians

Here we explore several ways of thinking about the medians of triangles.

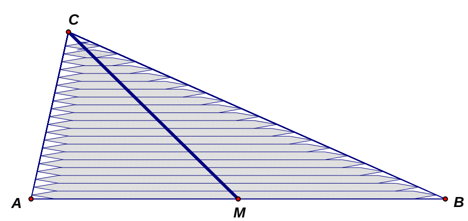
A.12.1) On cardstock, use a ruler to draw a medium-sized, non-right, non-isosceles triangle, and then cut it out as accurately as you can. Draw two of the medians on the cutout triangle. Draw the third median to make sure they are concurrent.

(a) Using a ruler, try balancing the triangle along each median. (Ask a partner to hold the ruler steady.)

(b) Now try balancing the triangle along a line that is *not* a median. How does your line relate to the intersection of the medians? Explain why this makes sense.

(c) Try balancing the triangle from a string at the intersection of the medians. (Use the point of your compass to make a hole in the cardstock.)

A.12.2) Imagine stacking toothpicks in a triangle, as shown below.



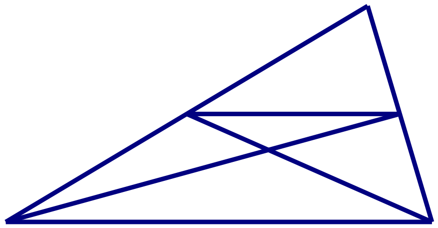
(a) Explain, using toothpicks, why the triangle would balance on a ruler placed along the median $\overline{CM}$.

(b) Explain, using a different collection of toothpicks, why the triangle would balance along the median to side $\overline{AC}$. Describe how the toothpicks would need be placed, relative to side $\overline{AC}$.

(c) The two medians will intersect at a point. Explain why the triangle (without toothpicks) should balance from a string or on a pencil point at the intersection of the two medians.

(d) Use a balancing argument to explain why the third median should contain the intersection of the first two.

A.12.3) Use the picture below to show that a pair of medians intersects at a point 2/3 of the way from the vertex to the opposite side. Then use that fact to argue that the three medians must be concurrent.



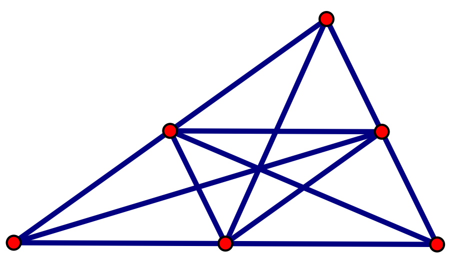
A.12.4) Imagine a triangle made of nearly weightless material with one-pound weights placed at each of the vertices, A, B, and C.

(a) Explain why the triangle will balance on a ruler along the median to side $\overline{AB}$.

(b) Explain why the triangle will continue to balance along the median when the masses at A and B are both moved to the midpoint of $\overline{AB}$.

(c) Now imagine trying to balance the triangle at a single point along the median. Where will it balance? Use the phrase "weighted average" to explain your reasoning.

A.12.5) Using the picture below, explain why the medians of the large triangle are also medians of the medial triangle. Then explain how repeating this process indefinitely proves that the medians are concurrent.



A.13 Triangle Centers

In this activity, we use *Geogebra* to explore the basic lines, centers, and circles related to triangles.

A.13.1) Here are some easy questions to get the brain-juices flowing!

(a) Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

(b) Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

(c) Place two lines randomly in the plane. How many points do you expect them to share?

(d) Place three lines randomly in the plane. How many points do you expect all three lines to share?

(e) Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

(f) Place four points randomly in the plane. Do you expect to be able to draw a circle containing all four at once? Explain your reasoning.

Definition Three (or more) distinct lines are said to be **concurrent** if they have a point in common.

A.13.2) In *Geogebra*, draw a triangle. Now construct the perpendicular bisectors of the sides. Describe what you notice. Does this work for every triangle?

A.13.3) In a new *Geogebra* sketch, draw a triangle. Now bisect the angles. Describe what you notice. Does this work for every triangle?

A.13.4) In a new *Geogebra* sketch, draw a triangle. Now construct the lines containing the altitudes. Describe what you notice. Does this work for every triangle?

A.13.5) In a new *Geogebra* sketch, draw a triangle. Now construct the medians. Describe what you notice. Does this work for every triangle?

A.13.6) The **circumcircle** of a triangle contains all three vertices of the triangle. The center of the circumcircle is called the **circumcenter**. Find the circumcenter on your sketch with the three perpendicular bisectors, and construct the circumcircle.

A.13.7) The **incircle** of a triangle is tangent to all three sides of the triangle. The center of the incircle is called the **incenter**. Find the incenter on your sketch with three angle bisectors. Construct the incircle. (Hint: To find the radius of the incircle, you will need to find the distance from the incenter to one of the sides of the triangle.)

A.13.8) The other "centers" of a triangle are called the **centroid** and the **orthocenter**. Make a thoughtful guess about how these correspond to the medians and the lines containing the altitudes.

A.13.9) Fill in the following handy chart summarizing what you found above.

	Associated point?	Always inside the triangle?	Meaning?
perpendicular bisectors			
angle bisectors			
lines containing altitudes			
lines containing the medians			

Be sure to put this in a safe place like in a safe, or under your bed.

A.14 The Euler Line and the Nine-Point Circle

A.14.1) Use *GeoGebra* to make the following constructions on an arbitrary triangle.

- Construct the circumcenter of the triangle. Hide all extraneous lines and points. Label this point C.
- Construct the centroid of the same triangle. Hide all extraneous lines and points. Label this point N.
- Construct the orthocenter of the same triangle. Hide all extraneous lines and points. Label this point O.
- Connect C and O with a segment.

Did a miracle happen? Describe what you notice about the segments $\overline{CN}$ and $\overline{ON}$.

A.14.2) Keeping the same triangle as used in the previous problem, use *GeoGebra* to make the following construction:

- Mark the midpoint of the segment that connects C and O. Label this point M.
- Mark the midpoints of each side. (Hint: Try to "unhide" those you have used already.)
- Mark where the altitudes meet the lines containing the sides of the triangle. Hide all extraneous lines and points.
- Mark the midpoints of the segments joining the orthocenter and the vertices. Hide all extraneous lines and points.
- Draw a circle centered at M that goes through one of the midpoints of the triangle.

Did a miracle happen? Describe what you notice.

A.14.3) Complete the following sentences:

(a) The *Euler line* contains the following points:

(b) The *nine-point circle* contains the following points:

A.15 Verifying our Constructions

When we do our compass and straightedge constructions, we should take care to verify that they actually work as advertised. We'll walk you through this process. To start, remember what a circle is:

Definition A **circle** is the set of points that are a fixed distance from a given point.

A.15.1) Is the center of a circle part of the circle?

A.15.2) Construct an equilateral triangle. Why does this construction work?

Now recall the SSS Theorem:

Theorem A.15.1 (SSS) *Specifying three sides uniquely determines a triangle.*

A.15.3) Now we'll analyze the construction for copying angles.

(a) Use a compass and straightedge construction to duplicate an angle. Explain how you are really just "measuring" the sides of some triangle.

(b) In light of the SSS Theorem, can you explain why the construction used to duplicate an angle works?

A.15.4) Now we'll analyze the construction for bisecting angles.

(a) Use compass and straightedge construction to bisect an angle. Explain how you are really just constructing (two) isosceles triangles. Draw these isosceles triangles in your figure.

(b) Find two more triangles on either side of your angle bisector where you may use the SSS Theorem to argue that they have equal side lengths and therefore equal angle measures.

(c) Can you explain why the construction used to bisect angles works?

Recall the SAS Theorem:

Theorem A.15.2 (SAS) *Specifying two sides and the angle between them uniquely determines a triangle.*

A.15.5) Now we'll analyze the construction for bisecting segments.

(a) Use a compass and straightedge construction to bisect a segment. Explain how you are really just constructing two isosceles triangles.

(b) Note that the bisector divides each of the above isosceles triangles in half. Find two triangles on either side of your bisector where you may use the SAS Theorem to argue that they have equal side lengths and angle measures.

(c) Can you explain why the construction used to bisect segments works?

A.15.6) Now we'll analyze the construction of a perpendicular line through a point not on the line.

(a) Use a compass and straightedge construction to construct a perpendicular through a point. Explain how you are really just constructing an isosceles triangle.

(b) Find two triangles in your construction where you may use the SAS Theorem to argue that they have equal side lengths and angle measures.

(c) Can you explain why the construction used to construct a perpendicular through a point works?

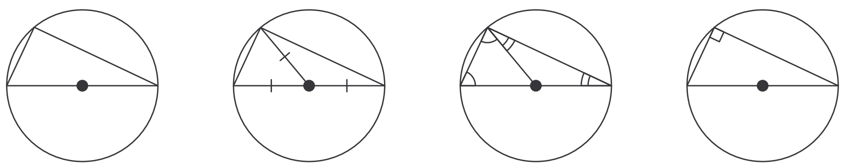
A.16 Of Angles and Circles

In this handout we are going to look at pictures and see if we can explain how they "prove" theorems.

Theorem A.16.1 *Any triangle inscribed in a circle having the diameter as a leg is a right triangle.*

A.16.1) Can you tell me in English what this theorem says? Provide some examples of this theorem in action.

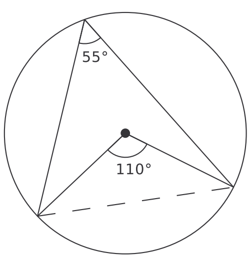
A.16.2) Here is a series of pictures, designed to be read from left to right.



Explain how these pictures "prove" the above theorem. In the process of your explanation, you may need to label parts of the pictures and do some algebra.

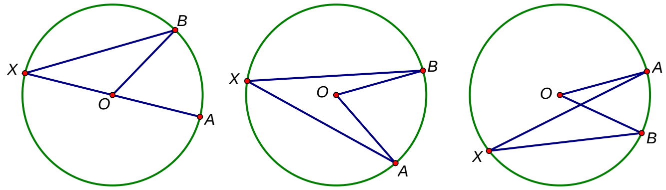
Theorem A.16.2 *Given a chord of a circle, the central angle defined by this chord is twice any inscribed angle defined by this chord.*

I'll play nice here and give you a picture of this theorem in action:

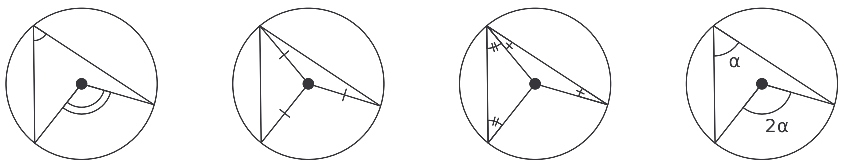


A.16.3) Can you tell me in English what this theorem says? For one thing, what do the fancy words mean? Specifically, what is meant by *chord*, *central angle*, and *inscribed angle*?

A.16.4) Sometimes it helps to organize the desired result into separate cases, beginning with the easiest:



For one possible line of reasoning, consider this series of pictures, designed to be read from left to right.



Explain how these pictures "prove" the above theorem. In the process of your explanation, you may need to label parts of the pictures and do some algebra.

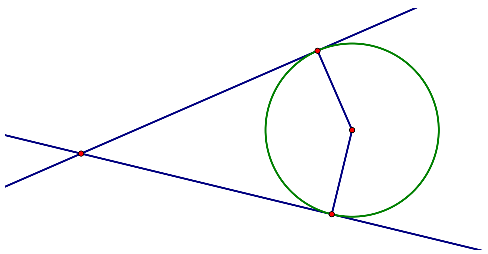
Corollary Given a chord of a circle, all inscribed angles defined by this chord are equal.

A.16.5) Firstly—what the heck is a corollary? Secondly—what is it saying? Thirdly—why is it true?

A.17 More Circles

A.17.1) Prove: The radius of a circle is perpendicular to the tangent where the radius intersects the circle. Hint: Suppose not.

A.17.2) Draw an angle that circumscribes a circle. Find a relationship between the measure of the angle and the measure of the central angle intercepted by the same chord.



A.17.3) Prove: A radius that is perpendicular to a chord bisects the chord.

A.17.4) Prove: A radius that bisects a chord is perpendicular to the chord.

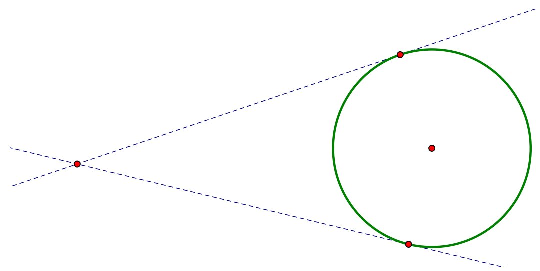
A.17.5) Show that, given any three non-collinear points in the Euclidean plane, there is a unique circle passing through the three points.

But how about four points in the plane, no three of which are collinear?

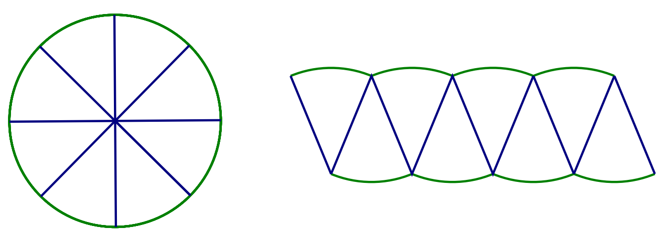
A.17.6) Draw four points in the Euclidean plane, no three of which are collinear, that cannot lie on a single circle.

A.17.7) Using a compass, draw a large circle, and inscribe a quadrilateral in the circle. Measure the four angles. Repeat with another circle and quadrilateral. What do you notice? Identify a condition on any quadrilateral that is inscribed in a circle.

A.17.8) Construct a tangent line to a circle from a point outside the given circle.



A.17.9) Give an informal derivation of the relationship between the circumference and area of a circle. Imagine cutting a circle into “pie pieces” and rearranging the pieces into a shape like the one below. As the circle is cut into more and more equal-sized “pie pieces,” what does the rearranged shape begin to resemble? Can you find the area of this shape?



A.17.10) Derive a formula for the length of the arc intercepted by an cenral angle of a circle.

A.17.11) Derive a formula for the area of a sector of a circle.

A.18 Quadrilateral Diagonals

Imagine you are working at a kite factory and you have been asked to design a new kite. The kite will be a quadrilateral made of synthetic cloth, and it will be formed by two intersecting rods that serve as the diagonals of the quadrilateral and provide structure for the kite.

A.18.1) To get started, review the definitions of all special quadrilaterals. Be sure to include *kite* on your list.

A.18.2) To consider the possible kite shapes, your first task is to describe how conditions on the diagonals determine the quadrilateral. Use spaghetti to model the intersecting rods, and use paper and pencil to draw the rod configurations and resulting kite shapes. Explore diagonals of various lengths, of the same length, and of different lengths. Explore various places at which to attach the diagonals to each other, including at one or both of their midpoints. Explore various angles that the diagonals might make with each other at their intersection, including the possibility of being perpendicular.

A.18.3) Summarize your findings in a table organized like the one below.

Diagonal Conditions	Quadrilateral	Definition	Other Key Properties

A.19 I'm Into Triangles

In this activity, we're going to see if we can discover a simple method for breaking *every* polygon into triangles.

A.19.1) Draw yourself a polygon with at least 8 sides. Show how to break this polygon into triangles.

A.19.2) See if you can figure out exactly **what** your method was for breaking the polygon into triangles. Write it down.

A.19.3) Find a casual acquaintance and declare "I challenge you to present me with a polygon that cannot be broken into triangles." Can you use your method to break their polygon into triangles?

A.19.4) Draw a polygon that would be really difficult to break into triangles.

A.19.5) Come up with a simple method that will **always** work for breaking a polygon into triangles. As a hint, draw a stick person in your polygon, and try to imagine what they see. . .

A.20 Morley's Miracle

Here is a construction that wasn't discovered until 1899. To make life easier, I'm going to allow you to use the following (somewhat imprecise) method for trisecting angles:

(a) Fold the paper so that the crease leads up to the angle, with the edge of the flap being folded-over bisecting the new angle of the crease and the edge that was not moved.

(b) Now fold the edge that was not moved on top of the flap that was just made. It should fit perfectly near the angle. If done correctly, the steps above should trisect the angle.

A.20.1) What does that say above? I know, I know, it sounds complicated. See if you can figure it out anyhow.

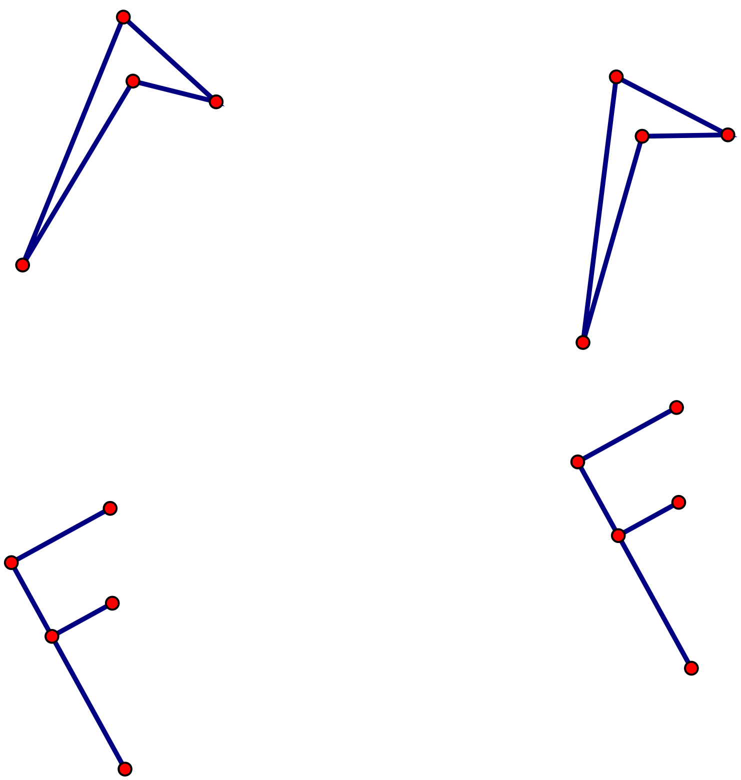
So now get your tracing paper out, make a big scalene triangle, and trisect all three angles.

A.20.2) Connect adjacent trisectors. Do you see a miracle happening? (I know, I know, if *anybody* can follow these directions than we truly will have a miracle on our hands!)

A.21 Congruence via Transformations

Two figures are said to be congruent if there exists a basic rigid motion (translation, rotation, or reflection) or a sequence of basic rigid motions that maps one figure onto the other. (Note that typical definitions of congruence rely on measures of angles and sides, which works for polygons, but not more general figures.)

A.21.1) One of the pairs of figures below shows a translation, and the other pair does not. To identify which is which, draw segments between each point and its image. Use those segments to explain your reasoning.

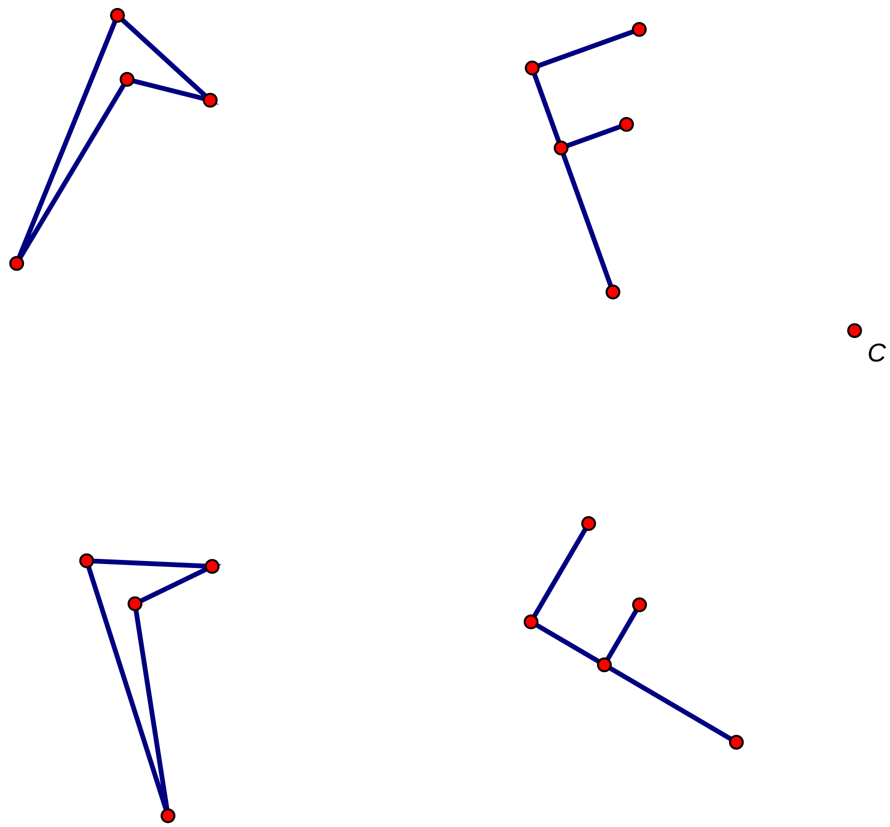


A.21.2) One of the pairs figures below shows a rotation about point C, and the other pair does not.

(a) Identify which pair of figures show a rotation about C, and explain how you know.

(b) Find the angle of rotation.

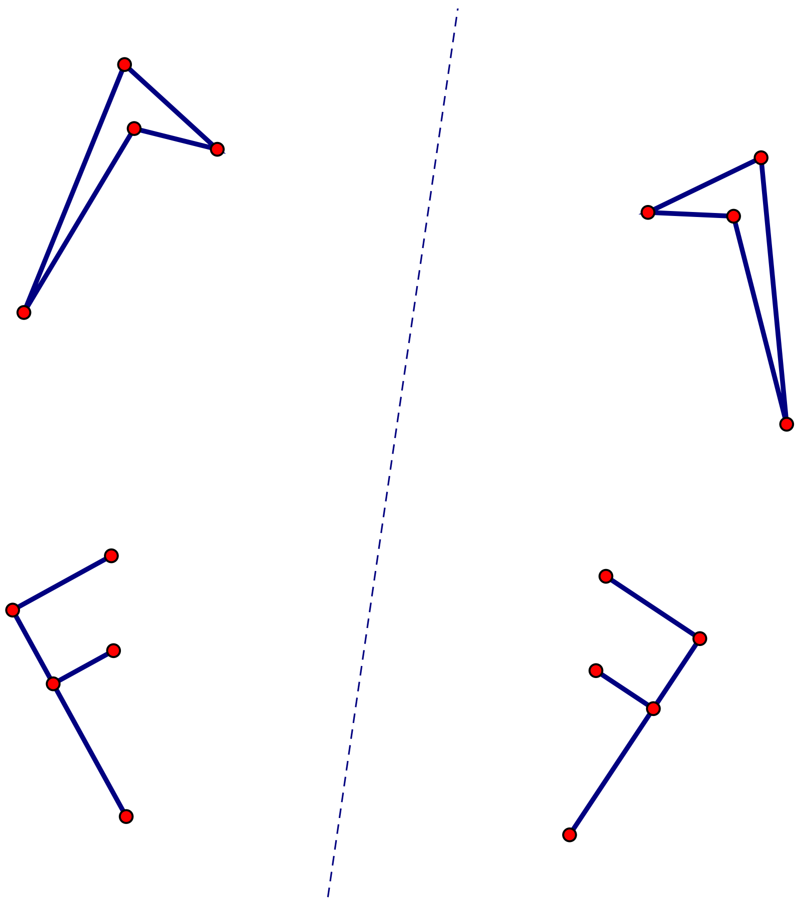
(c) Find the center of and angle of rotation for the other pair of figures. Explain your reasoning.



A.21.3) One of the pairs of figures shows a reflection about the given line, and the other pair does not.

(a) Identify which pair figures show a reflection about the given line, and explain how you know.

(b) Find the line of reflection for the other pair of figures, and explain your reasoning.

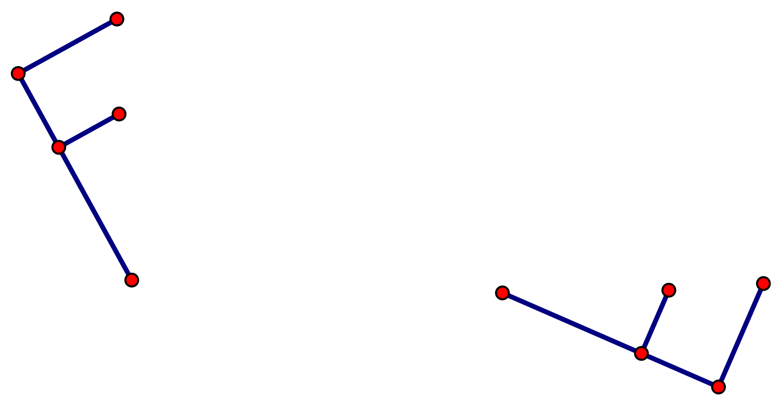


A.21.4) For the figures below, we want to identify a sequence of two or three basic rigid motions that takes one F onto the other. Possibilities include a rotation followed by a translation or a reflection followed by a rotation.

(a) Explain briefly why, for this pair of figures, we can quickly exclude sequences of the following types:

- a rotation followed by a rotation
- a translation followed by a translation
- a reflection followed by a reflection

(b) Using tracing paper to illustrate intermediate images, identify a sequence of basic rigid motions that takes one F onto the other. Explain your reasoning.



A.22 More Transformations

Transformations of the plane are considered to be functions that take points as inputs and produce points as outputs. Given a point as input, the corresponding output value is often called the *image* of the point under the transformation.[G-CO.2]

CCSS G-CO.2: Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

A.22.1) Based on your experience with the basic rigid motions, write definitions of translation, rotation, and reflection.[G-CO.4] For each definition, be sure to indicate (1) what it takes to specify the transformation, and (2) how to produce the image of a given point.

CCSS G-CO.4: Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

(a) Translation:

(b) Rotation:

(c) Reflection:

A.22.2) Now explore sequences of basic rigid motions. Here are some suggestions to support your explorations:

- Use a non-symmetric figure (such as an F).
- Use one sheet of tracing paper as the original plane, and use a second sheet of paper to carry out the sequence of transformations.
- Trace intermediate figures on both sheets of paper, to keep track of the work.
- For reflections, trace the line of reflection on both sheets.
- For rotations, use a protractor to help you keep track of angles.
- Consider special cases, such as reflections about the same line or rotations about the same point.
- Try to predict the result before you actually carry out the sequence of transformations.

Describe briefly what you can say about each of the following sequences of basic rigid motions. Include special cases in your descriptions.

(a) Translation followed by translation

(b) Rotation followed by rotation

(c) Reflection followed by reflection

(d) Translation followed by rotation

(e) Translation followed by reflection

(f) Rotation followed by reflection

A.23 Symmetries

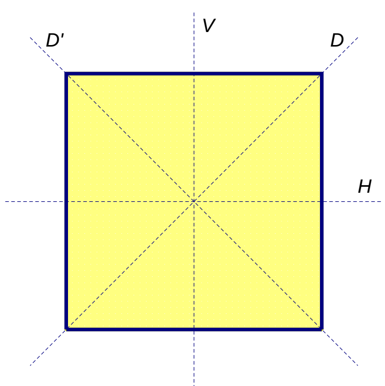
Definition A symmetry is a transformation that takes a figure onto itself.

A.23.1) List the symmetries of an equilateral triangle. Explain how you know you have them all.

A.23.2) Flip through these notes and describe the symmetries you notice. Try to find reflection symmetry, rotation symmetry, and translation symmetry.

Teaching Note: To augment these, bring some pictures from Web. Tesselations and Frieze patterns are necessary for translation symmetry.

A.23.3) Suppose the symmetries of a square are called R_0, R_{90}, R_{180}, R_{270}, V, H, D, D', based upon the figure below.



Hint: To identify a single transformation that accomplishes a sequence of transformations, do the transformations physically with a square piece of paper marked with “FRONT” on the side that starts facing you. Or mark the corners of the square with A, B, C, and D.

(a) Complete the following table, where the entry at (row, column) is the symmetry that results from the sequence of symmetries given by the row heading followed by the column heading.

(b) What patterns and not-quite-patterns do you notice in the table? For example, which elements "commute" with which other elements?

(c) What facts about isometries can you observe in the table? For example, what can you say generally about sequences of rotations and reflections?

	R_0	R_{90}	R_{180}	R_{270}	V	H	D	D'
R_0								
R_{90}								
R_{180}								
R_{270}								
V								
H								
D								
D'								

Teaching Note:* *Even blurring one's eyes, it is possible to notice (1) the composition of a reflection and a rotation (in either order) is a reflection; (2) the composition of two rotations is a rotation; and (3) the composition of two reflections is a rotation. Looking a bit closer, one can see that

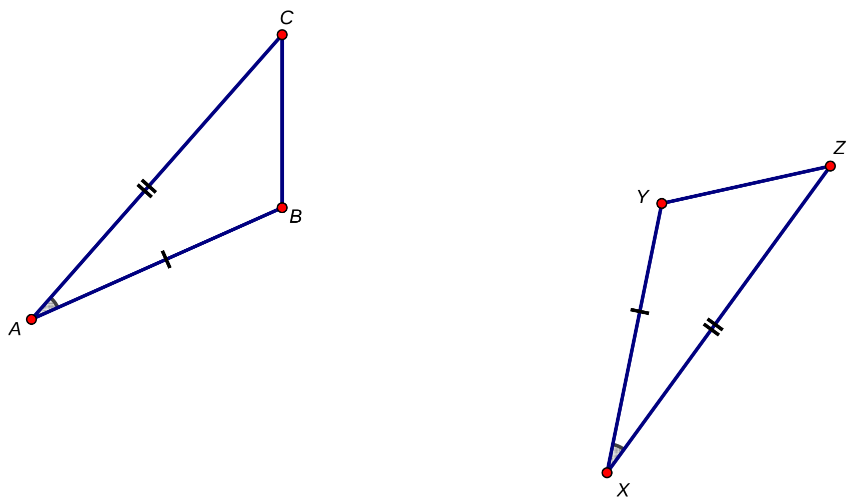
some elements commute with one another and others do not.

A.24 Congruence Criteria

In this activity, we show how the common triangle congruence criteria follow from what we now know about isometries.[G-CO.8] Recall that two figures are said to be congruent if there exists an isometry (translation, rotation, or reflection) or a sequence of isometries that maps one figure onto the other.

CCSS G-CO.8: Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

A.24.1) Proof of Side-Angle-Side (SAS) congruence. Suppose $\triangle ABC$ and $\triangle XYZ$ are such that $AB = XY$, $AC = XZ$, and $\angle A \cong \angle X$. Prove, using basic rigid motions, that $\triangle ABC \cong \triangle XYZ$. Consider the figure below.



Fill in the details of the following proof.

(a) Translate $\triangle ABC$ through the vector $\overrightarrow{AX}$. Call the image $\triangle A'B'C'$. Explain why A' and X coincide.

(b) Rotate $\triangle A'B'C'$ about $X = A'$ through $\angle B'XY$ so that ray $\overrightarrow{A'B'}$ is along ray $\overrightarrow{XY}$. Call the image $\triangle A''B''C''$ Explain how you know the segments $\overline{A''B''}$ and $\overline{XY}$ coincide.

(c) Reflect $\triangle A''B''C''$ about the line $\overleftrightarrow{A''B''} = \overleftrightarrow{XY}$. Call the image $\triangle A'''B'''C'''$. Explain why $\overline{A'''C'''}$ and $\overline{XZ}$ coincide.

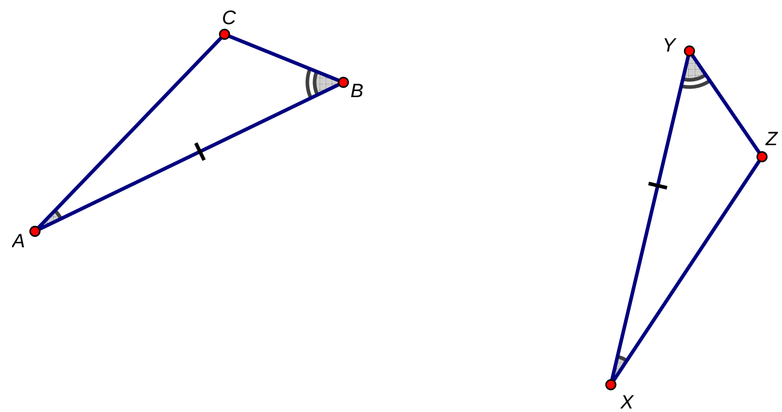
(d) Explain how you now know that all sides and angles of $\triangle A'''B'''C'''$ are congruent to the corresponding sides and angles of $\triangle XYZ$.

(e) Explain how to modify the above steps to handle the following different cases:

- Initially $X = A$.
- After the translation, $\overline{A'B'}$ and $\overline{XY}$ coincide.
- After the rotation, $\overline{A''C''}$ and $\overline{XZ}$ coincide. (Hint: Consider whether C'' and Z are on the same side or on opposite sides of $\overleftrightarrow{XZ}$.)

A.24.2) Proof of Angle-Side-Angle (ASA) congruence. Suppose $\triangle ABC$ and $\triangle XYZ$ are such that $AB = XY$, $\angle A \cong \angle X$, and $\angle B \cong \angle Y$. Prove, using basic rigid motions, that $\triangle ABC \cong \triangle XYZ$.

(a) Outline a general proof for the figure below.

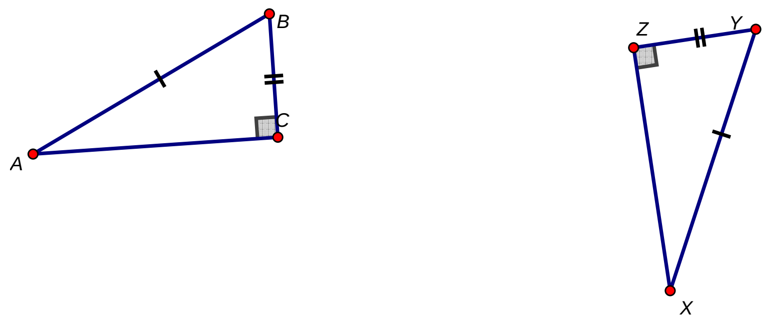


(b) Explain carefully how you know, after the sequence of rigid motions, that the "final image" of C coincides with Z.

(c) Describe how to modify the outline to handle other cases.

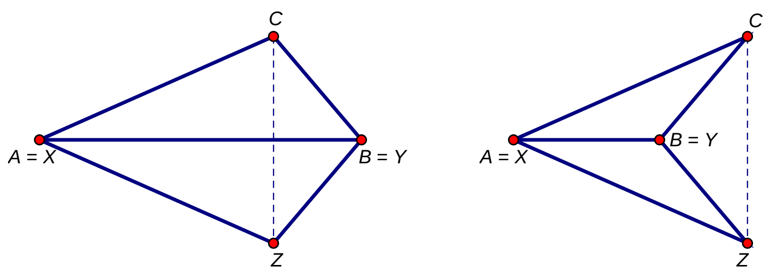
A.24.3) Proof of Hypotenuse-Leg (HL) congruence. Suppose $\triangle ABC$ and $\triangle XYZ$ are such that $\angle C$ and $\angle Z$ are right angles, $AB = XY$, and $BC = YZ$. Prove that

$\triangle ABC \cong \triangle XYZ$. (Hint: First extend side $\overrightarrow{AC}$ to a point A' so that $CA' = XZ$, and argue that $\triangle A'BC \cong \triangle XYZ$.)



A.24.4) Proof of Side-Side-Side (SSS) congruence. Suppose $\triangle ABC$ and $\triangle XYZ$ are such that $AB = XY$, $AC = XZ$, and $BC = YZ$. Prove, using basic rigid motions, that $\triangle ABC \cong \triangle XYZ$. Build toward the general case through the following steps:

(a) Case 1a: $A = X$, $B = Y$, and C and Z lie on opposite sides of $\overleftrightarrow{AB}$. (Hint: Explain why the situation must be like one of the figures below, argue that $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{CZ}$, and then use a reflection.)



(b) Case 1b: $A = X$, $B = Y$, and C and Z lie on the same side of $\overleftrightarrow{AB} = \overleftrightarrow{XY}$. (Hint: Consider a reflection of one of the triangles and use the previous case.)

(c) Case 2: $A = X$ but $B \neq Y$.

(d) Case 3: The general case.

A.25 Parallels

In the following problems, you may assume the following:

Postulate (Parallel Postulate) Given a line and a point not on the line, there is exactly one line passing through the point which is parallel to the given line.

You may also use previously-established results, such as the following:

- The measures of adjacent angles add as they should.
- A straight angle measures 180°.
- A 180° rotation about a point on a line takes the line to itself.
- A 180° rotation about a point off a line takes the line to a parallel line.

Now you may get started!

A.25.1) Use adjacent angles to prove that vertical angles are equal.

A.25.2) Now use rotations to prove that vertical angles are equal.

A.25.3) Prove: Alternate interior angles and corresponding angles of a transversal with respect to a pair of parallel lines are equal.

A.25.4) Prove: If a pair of alternate interior angles or a pair of corresponding angles of a transversal with respect to two lines are equal, then the lines are parallel.

A.25.5) The previous two problems seem almost identical to one another. How are they different?

A.25.6) Prove: The angle sum of a triangle is 180°.

A.26 Midsegments

Definition In a triangle, a *midsegment* is a line joining the midpoints of two sides.

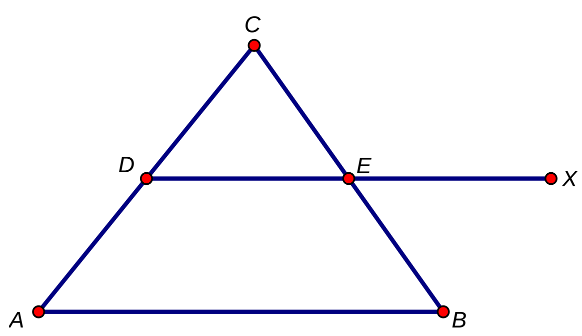
Theorem A.26.1 *Midsegment Theorem: A midsegment in a triangle is parallel to and half the length of the corresponding side.*

In this activity, we prove the midsegment theorem. First, we need some results about parallelograms.

A.26.1) Prove the following theorem: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

A.26.2) Prove the following theorem: If one pair of sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

A.26.3) Prove the midsegment theorem. (Hint: Extend the midsegment $\overline{DE}$ to a point X such that $EX = DE$, and then find quadrilaterals that must be parallelograms by the previous results.)



A.27 Similarities

A.27.1) Based on your experience with the stretching activity, write a definition of dilation. Be sure to indicate (1) what it takes to specify the transformation, and (2) how to produce the image of a given point.

A.27.2) Based on your experience with the stretching activity, describe for a dilation:

(a) What happens to line segments?

(b) What happens to angles?

(c) What happens to lines passing through the center of the dilation?

(d) What happens to lines not passing through the center of the dilation?

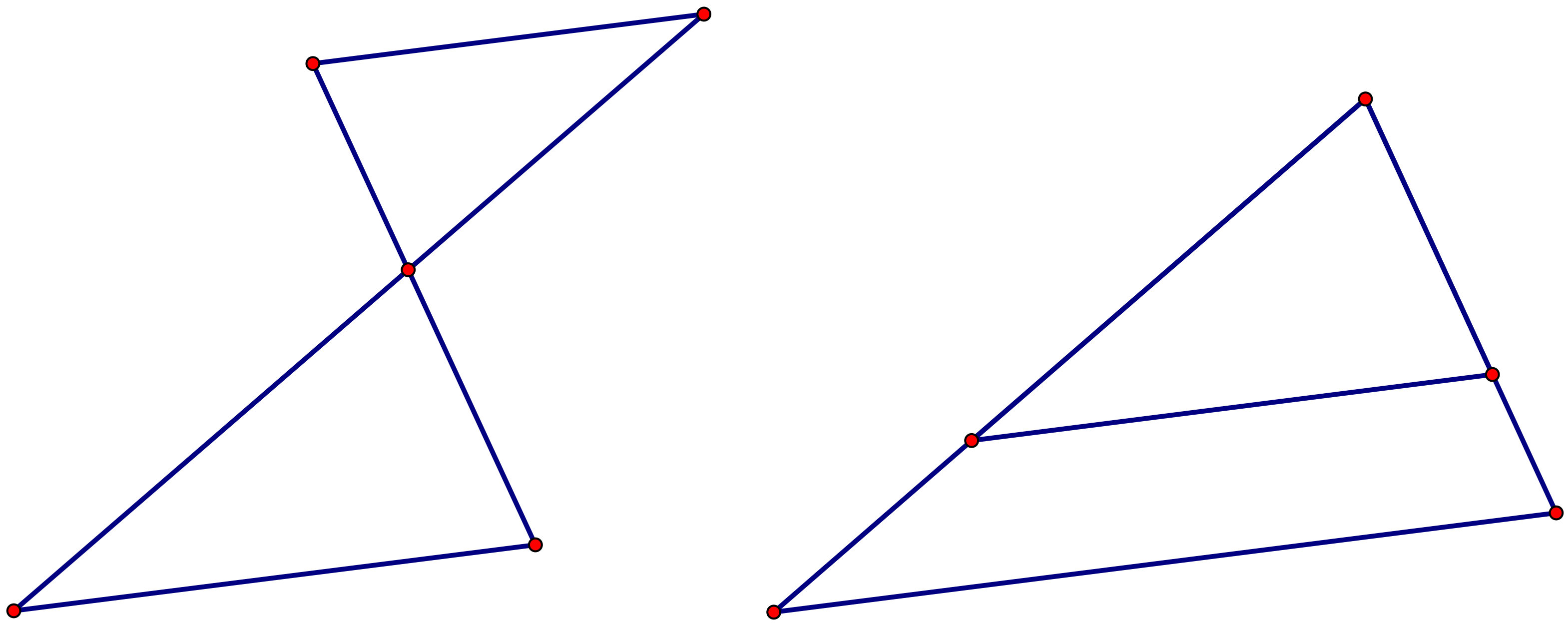
Definition A geometric figure is *simliar* to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations.

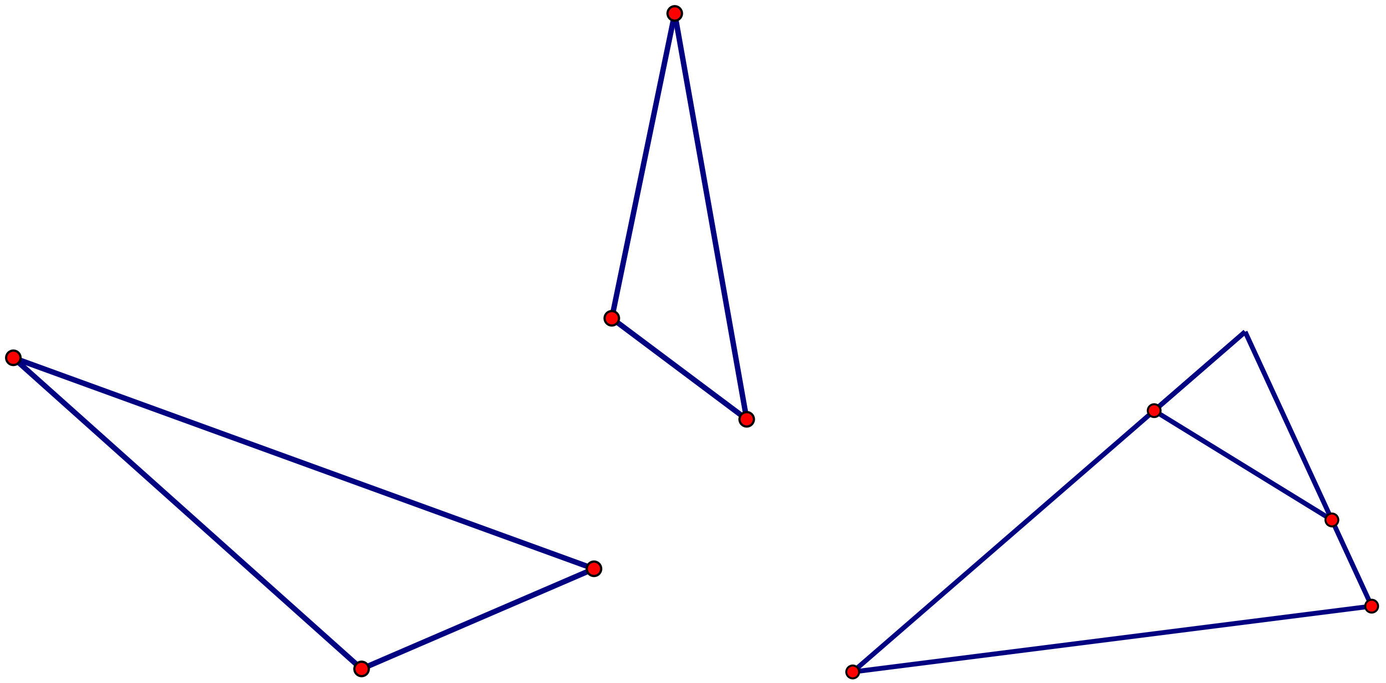
A.27.3) For each of the pairs of objects on the following pages, do the following:

(a) Trace the smaller figure on plastic. Then close one eye and try to hold the plastic between your eye and the paper so that the tracing "exactly" covers the larger figure. Be sure that the plane of the paper and the plane of the plastic are parallel. (Why does this matter?)

(b) If the objects are similar, find a sequence of rotations, reflections, translations, and dilations that takes one figure onto the other.

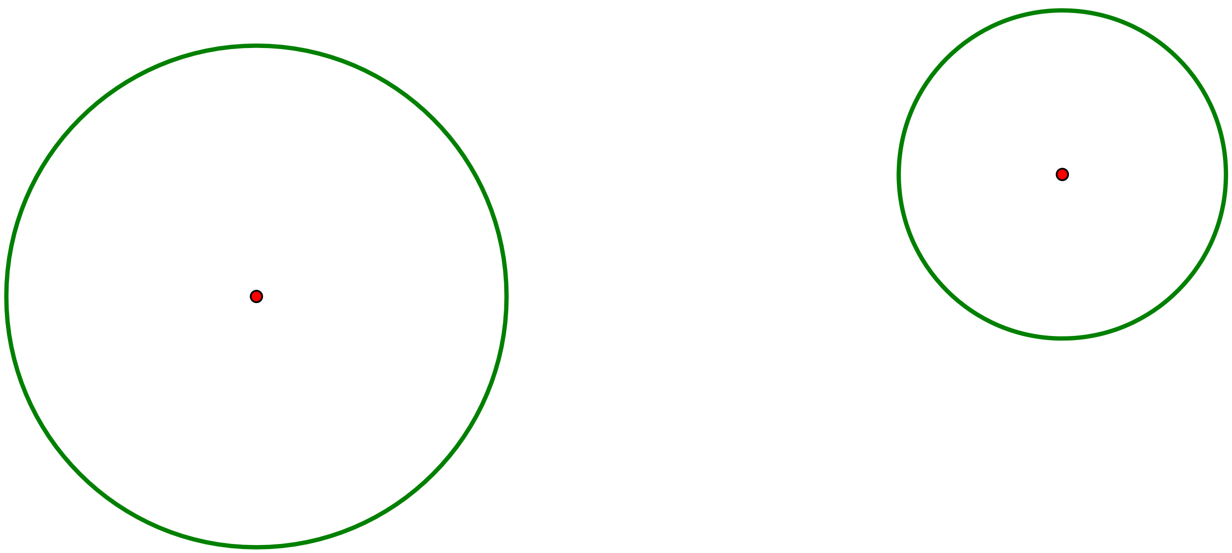
(c) If the objects are similar, try to find a single dilation that demonstrates the similarity. If you cannot find such a dilation, explain how you know you cannot.



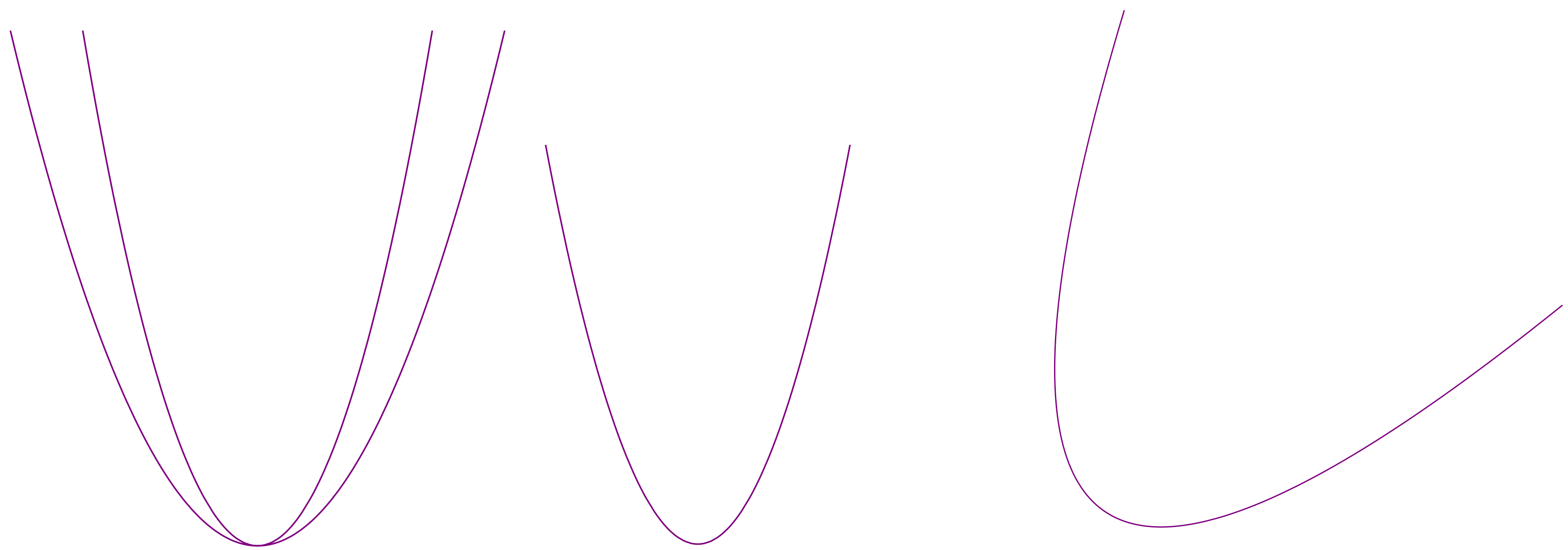


A.27.4) Describe a general (and foolproof) way of demonstrating that any two circles are similar.[G-C.1]

CCSS G-C.1: Prove that all circles are similar.



A.27.5) Describe a general (and foolproof) way of demonstrating that any two parabolas are similar.

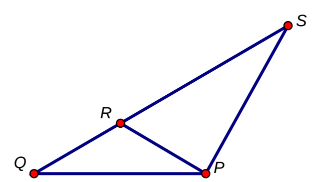


A.28 Side-Splitter Theorems

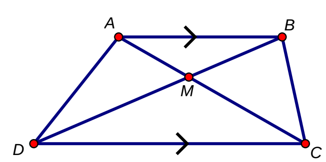
In this activity, we will show that the properties of dilations, which you noticed in a previous activity, can be proven *without* using facts about transversals and parallel lines. Instead, we use the area formulas for rectangles, triangles, and parallelograms.

Question What must be true about the base and height measurements for these area formulas to be valid?

A.28.1) If the area of $\triangle SPR = 5$ square inches and the area of $\triangle QPR = 8$ square inches, then what can you say about $\dfrac{SR}{RQ}$? What about $\dfrac{SR}{SQ}$? What can you say generally about how these ratios depend upon the areas of the triangles?

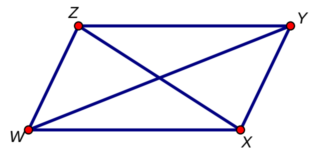


A.28.2) For the trapezoid below, explain why the area of $\triangle BAD$ is equal to the area of $\triangle BAC$. Name two other triangles that have the same area.

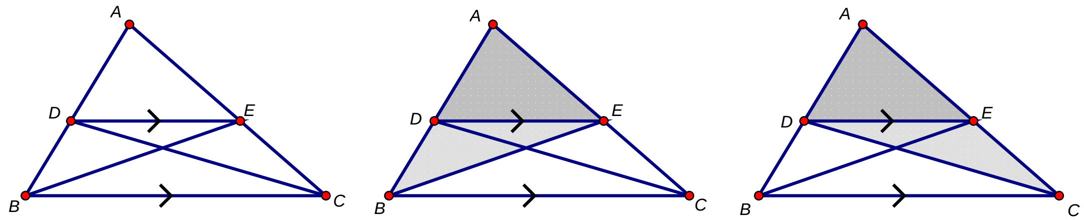


A.28.3) For the parallelogram below, which triangle has the greatest area: $\triangle XYZ$,

$\triangle WXY$, $\triangle ZWX$, or $\triangle YZW$? Explain.



A.28.4) Prove: If a line in a triangle is parallel to a side of a triangle, then it splits the other sides of the triangle proportionally.



(a) How do the areas of $\triangle ADE$ and $\triangle DBE$ relate to AD and DB? Explain.

(b) How do the areas of $\triangle ADE$ and $\triangle CED$ relate to AE and CE? Explain.

(c) How do the areas of $\triangle BDE$ and $\triangle CED$ compare? Explain.

(d) Use the previous results to show that $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.

(e) Where in the proof did you use the fact that $\overline{DE} \parallel \overline{BC}$?

A.28.5) Prove: In the previous figure, $\dfrac{AB}{AD} = \dfrac{AC}{AE} = \dfrac{BC}{DE}$.

(a) Use the results of the previous problem and some algebra to show that $\dfrac{AB}{AD} = \dfrac{AC}{AE}$

(b) How do we know that $\angle ADE \cong \angle ABC$?

(c) Translate $\triangle ADE$ by the vector $\overrightarrow{DB}$ so that the image $\angle A'D'E'$ of $\angle ADE$ coincides with $\angle ABC$. Draw a picture of the result.

(d) What segments are parallel now? How do you know?

(e) Now explain why $\dfrac{BC}{DE}$ is equal to the common ratio in part (a).

A.28.6) Prove: If a line in a triangle splits two sides proportionally, then it is parallel to the third side of the triangle. (Hint: Using the previous figures, draw a line through D and parallel to $\overline{BC}$, and let X be the point where the new line intersects $\overline{AC}$. By the previous results, $\overline{DX}$ divides the sides proportionally. Then argue that E and X must be the same point.)

A.29 Trigonometry Checkup

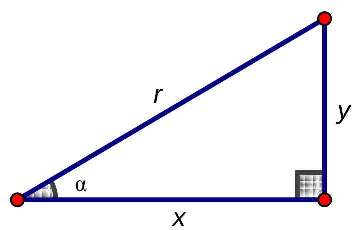
This activity is intended to reminder you of key ideas from high school trigonometry.

A.29.1 Right Triangle Trigonometry

A.29.1) What are the ratios of side lengths in a 45°-45°-90° triangle? Explain where the ratios come from, including why they work for any such triangle, no matter what size. (Hint: Use the Pythagorean Theorem.)

A.29.2) What are the ratios of side lengths in a 30°-60°-90° triangle? Explain where those the come from. (Hint: Think of half of an equilateral triangle.)

A.29.3) Consider the right triangle below with an angle of a, sides of length x and y, and hypotenuse of length r, as labeled.



(a) If we imagine angle a is fixed, why are ratios of pairs of side lengths the same, no matter the size of the triangle?[G-SRT.6]

CCSS G-SRT.6: Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

(b) Using the triangle above (and your memory of Precalculus), write down the side-length ratios for sine, cosine, and tangent:

$\sin a =$ $\qquad\qquad$ $\cos a =$ $\qquad\qquad$ $\tan a =$

(c) What values of a make sense in *right triangle trigonometry*? (We overcome these bounds later in circular trigonometry.)

(d) What does it mean to say that these ratios depend upon the angle α?

(e) Why is only one of the triangle's three angles necessary in defining these ratios?

A.29.4) Use your work so far to find the following trigonometric ratios:

(a) $\sin 30^\circ =$ $\quad$ $\cos 30^\circ =$ $\quad$ $\tan 30^\circ =$

(b) $\sin 45^\circ =$ $\quad$ $\cos 45^\circ =$ $\quad$ $\tan 45^\circ =$

(c) $\sin 60^\circ =$ $\quad$ $\cos 60^\circ =$ $\quad$ $\tan 60^\circ =$

(d) $\sin 0^\circ =$ $\quad$ $\cos 0^\circ =$ $\quad$ $\tan 0^\circ =$

A.29.5) You may recall the identity $\sin^2 \vartheta + \cos^2 \vartheta = 1$.[F-TF.8]

CCSS F-TF.8: Prove the Pythagorean identity $\sin^2(\vartheta) + \cos^2(\vartheta) = 1$ and use it to find $\sin(\vartheta)$, $\cos(\vartheta)$, or $\tan(\vartheta)$ given $\sin(\vartheta)$, $\cos(\vartheta)$, or $\tan(\vartheta)$ and the quadrant of the angle.

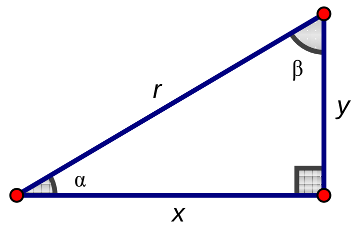
(a) Explain why the equation is true.

(b) Why is it called an identity?

(c) Why is it called a Pythagorean identity?

A.29.6) In right triangle trigonometry, there are indeed two acute angles, as shown in the figure below.G-SRT.7

CCSS G-SRT.7: Explain and use the relationship between the sine and cosine of complementary angles.



(a) How are the angles α and β related? Explain why.

(b) Using lengths in the above triangle, find the following ratios:

$\sin\alpha =$ $\cos\alpha =$

$\sin\beta =$ $\cos\beta =$

(c) What do you notice about the sine and cosine of complementary angles?

(d) Explain why the result makes sense.

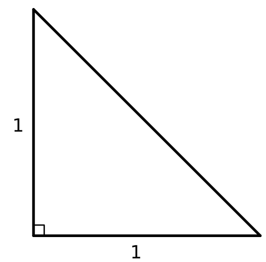
Given an angle and a side length of a right triangle, you can find the missing side lengths.G-SRT.8 This is called "solving the right triangle." And given the sine, cosine, or tangent of an angle, you can find the other two ratios. (Hint: In either case, draw a triangle.)

CCSS G-SRT.8: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

A.29.7) Suppose $\sin\alpha = \frac{3}{5}$. Then $\cos\alpha =$, $\tan\alpha =$.

A.30 Please be Rational

Let's see if we can give yet another proof that the square root of two is not rational. Consider the following isosceles right triangle:

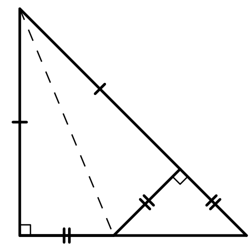


A.30.1) Using the most famous theorem of all, how long is the unmarked side?

A.30.2) Suppose that the unmarked side has a rational length. In that case how could we express it?

A.30.3) Explain why there would then be a *smallest* isosceles right triangle with integer sides. Considering the problem above, how long would the sides be? Draw and label a picture.

A.30.4) Now fold your smallest isosceles right triangle with integer sides along the dotted line like so:



Explain why the segments we have marked above as “equal” are in fact equal.

A.30.5) Explain how we have now found an isosceles right triangle with integer sides that is now smaller than the smallest isosceles right triangle with integer sides. Is this possible? What must we now conclude?

A.31 Rep-Tiles

A **rep-tile** is a polygon where several copies of a given rep-tile fit together to make a larger, similar, version of itself. If 2 copies are used, we call it a *rep-2-tile*, if 3 copies are used, we call it a *rep-3-tile*, and if n copies are used, we call it a *rep-n-tile*.

A.31.1) With a separate sheet of paper, draw and cut out:

(a) An isosceles right triangle whose sides have lengths $1''$, $1''$, and $\sqrt{2}''$.

(b) A rectangle whose sides have lengths $1''$ and $\sqrt{2}''$.

Working with a partner, show that each of these polygons is a rep-2-tile.

A.31.2) For each rep-tile above, compute the perimeter and area. In each case, how does this relate to the perimeter and area of the larger polygon?

Teaching Note: This is at least a two-day activity. Bring printed versions of the figures so that they can cut out already drawn ones. Bring a printed version of the table. Some time working with the figures, computing areas and perimeters, and practicing arithmetic of radicals. Move the summary to the second day.

A.31.3) With a fresh sheet of paper, start a table to keep track of your work:

rep-tile	scale factor (new:old)	perimeter (new:old)	area (new:old)
description			
⋮	⋮	⋮	⋮

A.31.4) Geometry Giorgio suggests that a rectangle whose sides have lengths $1''$ and $4''$ is also a rep-2-tile. Is he right? If you should happen to search the internet for other examples of rep-2-tiles, you might find a surprise.

A.31.5) With a separate sheet of paper, draw and cut-out:

(a) A 30-60-90 right triangle whose shortest side has length $1''$.

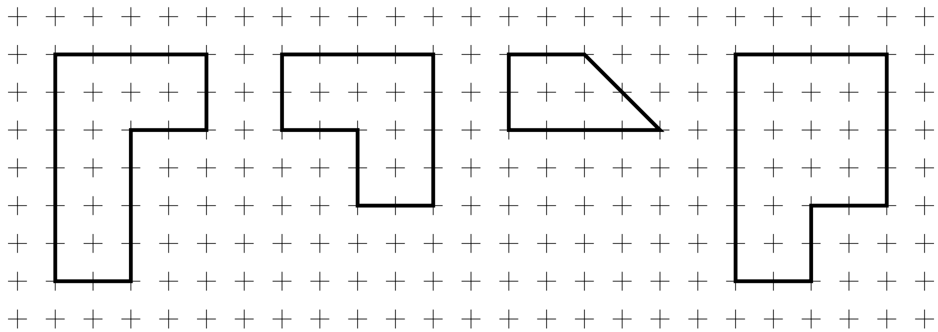
(b) A rectangle whose sides have lengths $1''$ and $\sqrt{3}''$.

Working with a partner, show that each of these polygons is a rep-3-tile.

A.31.6) For each rep-tile above, compute the perimeter and area. In each case, how does this relate to the perimeter and area of the larger polygon? Add this information to your table.

A.31.7) Explain why every triangle and every parallelogram is a rep-4-tile. Give an example of each, and compute the perimeter and area. In both cases, compare the perimeter and area to that of the larger polygons.

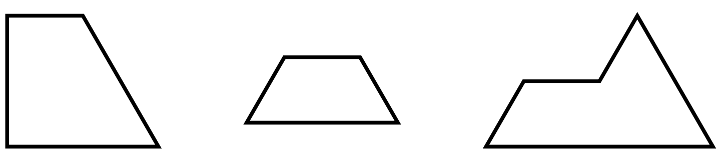
A.31.8) With a separate sheet of graph paper, draw and cut out the following polygons:



Working with a partner, show that each of these polygons is a rep-4-tile.

A.31.9) For each rep-tile above, compute the perimeter and area. In each case, how does this relate to the perimeter and area of the larger polygon? Add this information to your table.

A.31.10) With a separate sheet of paper, trace and cut out the following polygons:



Working with a partner, show that each of these polygons is a rep-4-tile.

A.31.11) Explain why every rectangle whose sides have ratio $1 : \sqrt{n}$ is a rep-n-tile.

A.31.12) Explain how you know that any rep-tile will tessellate the plane.

A.31.13) Give an example of a polygon that tessellates the plane that is not a rep-tile.

A.31.14) Every tessellation made by rep-tiles will have **symmetry of scale**. What does it mean to have *symmetry of scale*?

A.31.15) Consider the tessellations made by rep-tiles you've seen so far. What other symmetries do they have?

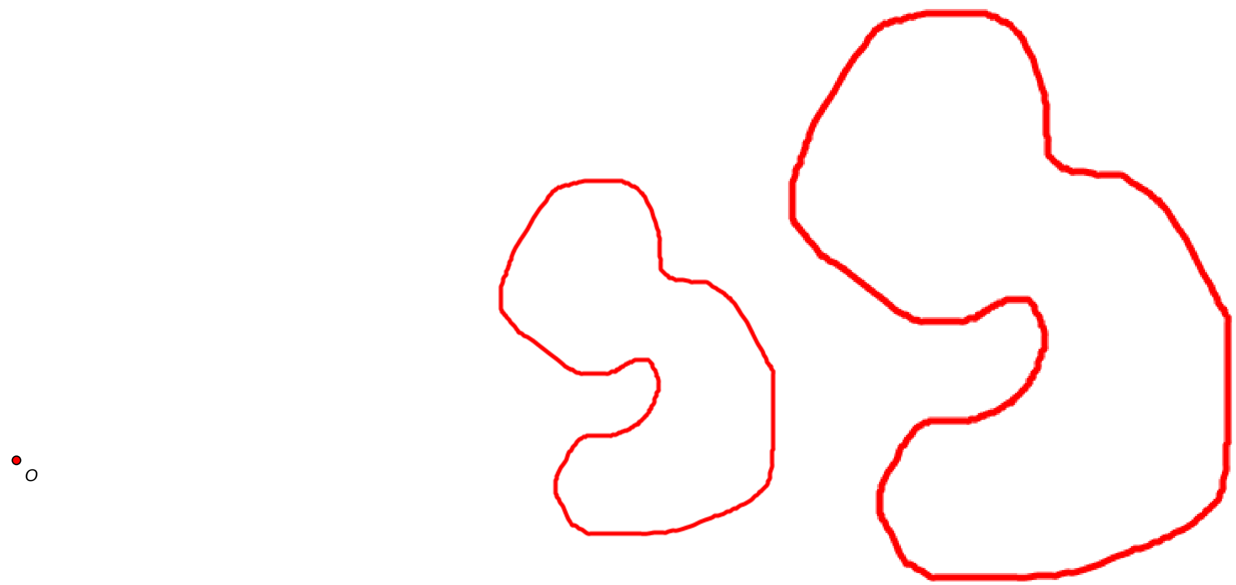
A.31.16) Do you think you can have a tessellation that has symmetry of scale but no other symmetries?

A.32 Scaling Area

A.32.1) Is a 3×5 rectangle similar to a 4×6 rectangle? Explain your reasoning. Now come up with another explanation.

A.32.2) Use area formulas to explain what happens to the area of a rectangle under scaling by a factor of k? What about a triangle? What about a circle?

A.32.3) Below is a figure and a dilation of that figure about point O.



(a) Find the scale factor of the dilation. Explain your reasoning.

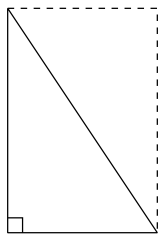
(b) What can you say about the areas of the two figures? Explain your reasoning.

Fixnote: Need a scaling volume activity. See comments for a start.

A.33 Turn Up the Volume!

In this activity, we will investigate formulas for area and volume.

A.33.1) Explain how the following picture "proves" that the area of a right triangle is one half of the base times the height.

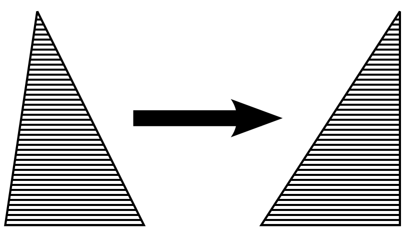


A.33.2) "Shearing" is a process where you take a shape, cut it into thin strips, then push the strips around in one direction to make a new shape. Cavalieri's principle states:

> Shearing parallel to a fixed direction does not change the n-dimensional measure of an object.

What is this saying?

A.33.3) Building on the first two problems, explain how the following picture "proves" that the area of any triangle is one half of the base times the height.

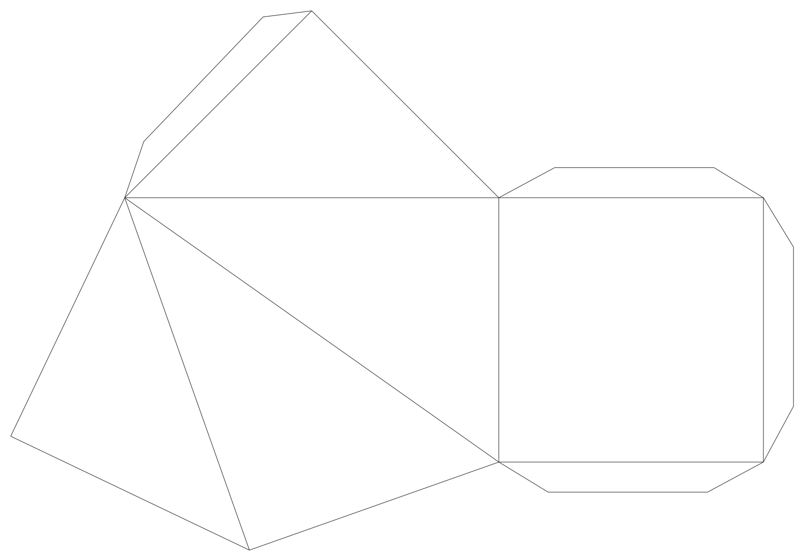


A.33.4) Give an intuitive argument explaining why Cavalieri's principle is true.

A.33.5) Explain how to use a picture to "prove" that a triangle of a given area could have an arbitrarily large perimeter.

A.33.6) Cut out the provided net. Then fold it and tape it to create a square-based pyramid. With your neighbors, show that three such square-based pyramids can form a cube.

Teaching Note: Here is the net



A.33.7) Use your work above to derive a formula for the volume of a right pyramid with a square base. The formula should be in terms of the side length of the square base.

A.33.8) Use Cavalieri's principle to explain the formula for **every** pyramid with an $s \times s$ square base of height s in terms of s. Be sure to describe how this formula is different from the previous one.

A.33.9) Provide an informal explanation of a volume formula for any pyramid-like object with a base of area B and height h. Be sure to describe what you mean by "pyramid-like" and whether your formula works for a cone.

A.33.10) Answer the following question using the provided figure showing cross-sections at height h of a half-sphere and and of a cylinder with a cone inside it.[8.G.9G-GMD.1G-GMD.2]

CCSS 8.G.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

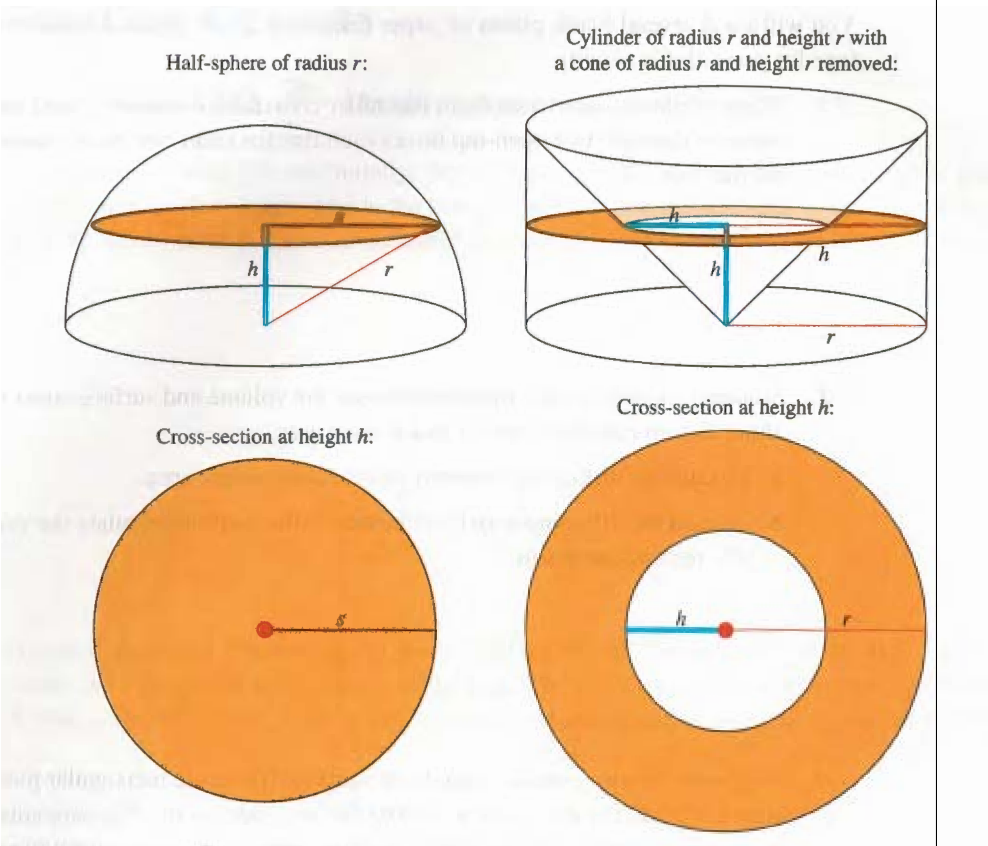
CCSS G-GMD.1: Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.

CCSS G-GMD.2: Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

(a) Explain why the two colored cross-sections at height h have the same area.

(b) Use the formula for the volume of a cone and Cavalieri's principle to derive a formula for the radius of a sphere of radius r.

Teaching Note: Here is the figure:



A.34 Coordinate Constructions

In synthetic geometry, point, line and plane are taken to be undefined terms. In analytic (coordinate) geometry, in contrast, we make the following definitions.

Definition A *point* is an ordered pair (x, y) of real numbers. A *line* is the set of ordered pairs (x, y) that satisfy an equation of the form $ax + by = c$, where a, b, and c are real numbers and a and b are not both 0.

Many of the problems below are expressed generally. You may find it useful to try some specific examples before the general case.

A.34.1) Given points (x_1, y_1) and (x_2, y_2), find the distance between them in the coordinate plane.[8.G.8]

CCSS 8.G.8: Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

A.34.2) Find the midpoint of the segment from (x_1, y_1) and (x_2, y_2). Explain why your formula makes sense.

A.34.3) Recall that in synthetic geometry, a circle is defined as the set of points that are equidistant from a center. Use this definition to determine the equation of circle with center (h, k) and radius r.[G-GPE.1]

CCSS G-GPE.1: Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

A.34.4) For each pair of points below, find an equation of the line containing the two points.

(a) Points $(2, 3)$ and $(5, 7)$.

(b) Points $(2, 3)$ and $(2, 7)$.

(c) Points $(2, 3)$ and $(5, 3)$.

(d) Points (x_1, y_1) and (x_2, y_2).

A.34.5) Express each of your previous equations in the form $ax + by = c$ and also in the form $y = mx + b$. What are the advantages and disadvantages of these forms?

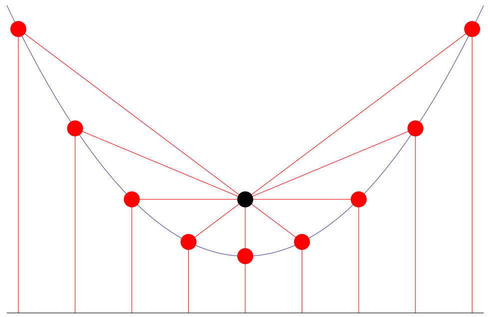
A.34.6) In the above definition of a line in coordinate geometry, why is it important to require that a and b are not both 0?

A.34.7) In school mathematics, lines are usually of the form $y = mx + b$. Why is it unambiguous to talk about *the slope* of such a line? In other words, given a non-vertical line in the plane, explain why any two points on the line will yield the same slope.[8.EE.6]

CCSS 8.EE.6: Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx+b$ for a line intercepting the vertical axis at b.

A.35 Bola, Para Bola

We've mentioned several times that a parabola is the set of points that are equidistant from a given point (the focus) and a given line (the directrix):



In this activity we are going to try an reconcile the definition given above with the equation that you know and love (admit it!):

$$y = ax^2 + bx + c$$

A.35.1) How do we compute the distance between two points? Be explicit!

A.35.2) Let's see if we can derive the formula for a parabola with its focus at $(0, 1)$ and its directrix being the line $y = 0$.

(a) Given a point (x, y), write an expression for the distance from this point to the focus.

(b) Write an expression for the distance from (x, y) to the directrix.

(c) Use these two expressions and some algebra to find the formula for the parabola.

A.35.3) Let's see if we can derive the formula for a parabola with its focus at $(0, 1)$ and its directrix being the line $y = -1$.

(a) Given a point (x, y), write an expression for the distance from this point to the focus.

(b) Write an expression for the distance from (x, y) to the directrix.

(c) Use these two expressions and some algebra to find the formula for the parabola.

A.35.4) Let's see if we can derive the formula for a parabola with its focus at $(1, 1)$ and its directrix being the line $y = -2$.

(a) Given a point (x, y), write an expression for the distance from this point to the focus.

(b) Write an expression for the distance from (x, y) to the directrix.

(c) Use these two expressions and some algebra to find the formula for the parabola.

A.36 More Medians

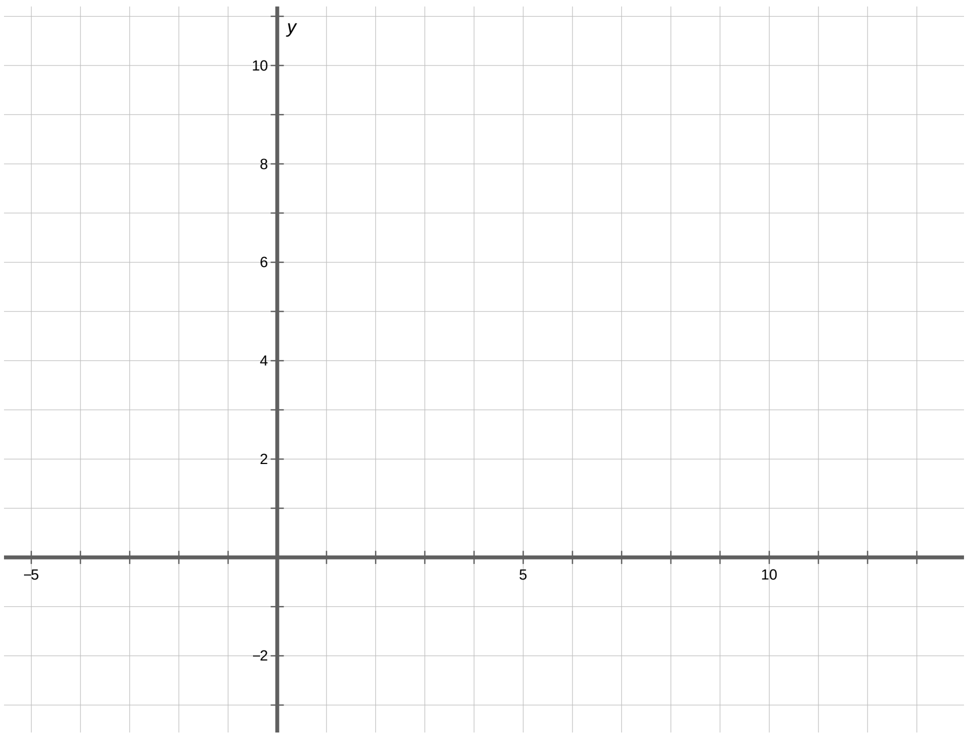
Here we use coordinates to explore several ways of thinking about the medians of triangles.

A.36.1) For each set of points below, plot the points in the coordinate plane, and use a ruler to draw the triangle. Locate the midpoint of each side, and use a ruler to draw the medians. Check that the medians are concurrent, and find the coordinates of the centroid.

(a) $A = (2, 1)$, $B = (10, 2)$, $C = (3, 6)$. Centroid: ________.

(b) $D = (6, 6)$, $E = (9, 10)$, $F = (4, 8)$. Centroid: ________.

(c) $G = (-1, 1)$, $H = (1, 6)$, $I = (-3, 4)$. Centroid: ________.



A.36.2) What do you notice about how the coordinates of the centroid depend upon the coordinates of the vertices? Make a conjecture about the centroid of a triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3). Check that your formula works for all of the triangles above.

A.36.3) Imagine a triangle made of nearly weightless material with one-pound weights placed at each of the vertices, $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$.

(a) Explain why the triangle will balance on a ruler along the median to side $\overline{AB}$.

(b) Explain why the triangle will continue to balance along the median when the masses at A and B are both moved to the midpoint of $\overline{AB}$.

(c) Now imagine trying to balance the triangle at a single point along the median. Where will it balance? Use the phrase "weighted average" to explain your reasoning.

(d) Use weighted-average reasoning to compute the coordinates of this balance point, assuming the vertices are $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$.

A.36.4) Consider a triangle with vertices at $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$.

(a) Explain why the equation of the line containing the median from C to the midpoint of $\overline{AB}$ can be written as follows:

$$\frac{y - y_3}{x - x_3} = \frac{y_1 + y_2 - 2y_3}{x_1 + x_2 - 2x_3}$$

(b) From reasoning alone (i.e., without doing additional calculations) write down analogous equations for the lines containing the other two medians.

(c) Use algebra and reasoning to show that the previously-conjectured coordinates of the centroid satisfy all three equations of lines containing medians.

(d) Have you now proven that the medians are concurrent? Explain.

A.37 Perpendicular Bisector: Reasoning with Algebra

Prove: If $C = (x, y)$ is on the perpendicular bisector of the segment from $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then C is equidistant from A and B.

(a) Why is it sufficient to show the following?

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$$

(b) We are given that C is on the perpendicular bisector of $\overline{AB}$. Explain why we can write:

$$y - \frac{y_1 + y_2}{2} = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)\left(x - \frac{x_1 + x_2}{2}\right)$$

(c) In the spaces that follow each equation, provide a supporting explanation for each step.

$$2y - y_1 - y_2 = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)(2x - x_1 - x_2)$$

$$(y - y_1) + (y - y_2) = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)((x - x_1) + (x - x_2))$$

$$(y_2 - y_1)\,[(y - y_1) + (y - y_2)] = (x_1 - x_2)\,[(x - x_1) + (x - x_2)]$$

$$[(y - y_1) - (y - y_2)]\,[(y - y_1) + (y - y_2)] = [(x - x_2) - (x - x_1)]\,[(x - x_2) + (x - x_1)]$$

$$(y - y_1)^2 - (y - y_2)^2 = (x - x_2)^2 - (x - x_1)^2$$

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$$

(d) How does this algebraic argument prove the statement generally, for any points A, B, and C?

A.38 Constructible Numbers

Compass and straightedge constructions involve drawing and finding intersections of two fundamental geometric objects: lines and circles. All more complicated constructions are combinations of pieces of these.

In this activity, we explore what numbers are constructible (as lengths or distances) with compass and straightedge, assuming only that we begin with a segment of length 1. We call such numbers *constructible numbers*. First we must establish how to do arithmetic with compass and straightedge.

A.38.1) Suppose you are given a compass and a straightedge and segments of lengths a, b, and 1.

(a) How would you construct a segment of length $a + b$?

(b) How would you construct a segment of length $a - b$?

(c) How would you construct a segment of length ab? (Hint: Use similar triangles.)

(d) How would you construct a segment of length $a \div b$?

(e) How would you construct a segment of length $\sqrt{a}$? (Hint: Recall how to construct a geometric mean.)

A.38.2) Beginning with a segment of length 1, describe briefly how you might construct segments of the following lengths.

(a) $\dfrac{7}{5}$

(b) $3 + 2\sqrt{5}$

(c) $\dfrac{4 + \sqrt{5}}{3}$

(d) $\dfrac{3 + \sqrt{2 - \sqrt{3}}}{1 + \sqrt{5}}$

A.38.3) Based on the previous problems, if you begin with a segment of length 1, how would you describe the set of all numbers constructible with methods used so far?

Note that with the methods so far, we can construct neither $\sqrt[3]{2}$ nor π. The question now is whether we have described the whole set of constructible numbers or whether there are additional constructions that will broaden our arithmetic and thereby enlarge the set.

For this question, we turn to coordinate constructions, which allow us to use the methods of algebra to solve geometric problems. A key habit here will be *imagining the algebra without actually doing it*—based on your extensive algebra experience with these kinds of problems.

A.38.4) Suppose you are given points (p, q), and (r, s) with integer coordinates.

(a) What arithmetic operations are involved in finding an equation $ax + by = c$ of the line containing these points?

(b) What can you conclude about the numbers a, b, and c?

(c) What if you begin with points that have coordinates that are rational numbers?

A.38.5) Suppose you are given equations of the form

$$ax + by = c$$

$$dx + ey = f$$

where a, b, c, d, e, and f are all integers.

(a) What kind of geometric objects do these equations describe in the xy-plane?

(b) What arithmetic operations would you use to solve the equations simultaneously?

(c) What can you conclude about the numbers x and y that are the (simultaneous) solutions of these equations?

(d) How will your answers change if a, b, c, d, e, and f are all rational numbers?

A.38.6) Suppose you are given points (h, k), and (p, q) with integer coordinates?

(a) Write an equation of the circle with center (h, k) and containing the point (p, q)?

(b) What arithmetic operations were involved in writing your equation of the circle?

(c) What can you conclude about the numbers that are coefficients in your equation?

A.38.7) Suppose you are given equations of the form

$$x^2 + ax + y^2 + by = c$$

$$x^2 + dx + y^2 + ey = f$$

where a, b, c, d, e, and f are all integers.

(a) What kind of geometric objects do these equations describe in the xy-plane?

(b) What arithmetic operations would you use to solve the equations simultaneously?

(c) What can you conclude about the numbers x and y that are the (simultaneous) solutions of these equations?

(d) How will your answers change if a, b, c, d, e, and f are all rational numbers?

A.38.8) Based on the previous problems, if you begin with a coordinate system with only integer coordinates, how would you describe the set of all numbers (coordinates) that are constructible via lines and circles?

A.38.9) Considering that all compass and straightedge constructions are about lines, circles, and their intersections, what do your results about coordinate constructions imply about compass and straightedge constructions?

A.38.10) Name some numbers that are **not constructible** with compass and straightedge.

The idea that some numbers are not constructible is exactly what was needed to address several problems first posed by the Greeks in antiquity, such as doubling the cube and trisecting an angle. In a paper published in 1987, Pierre Wantzel used algebraic methods to prove the impossibility of these geometric constructions.

A.38.11) Suppose you have a square of side length s and you want to “double the square.” In other words, you want to construct a square with **twice the area**.

(a) What is the side length of the desired square? Explain your reasoning.

(b) Is this side length constructible? Explain.

A.38.12) Suppose you have a cube of side length s and you want to "double the cube." In other words, you want to construct a cube with **twice the volume**.

(a) What is the side length of the desired cube? Explain your reasoning.

(b) Is this side length constructible? Explain.

A.38.13) You may remember some double angle formulas from trigonometry. There are also triple angle formulas. For example, for any angle ϑ, $\cos 3\vartheta = 4\cos^3\vartheta - 3\cos\vartheta$.

(a) What is $\cos 60^\circ$?

(b) Write the above triple angle formula for $\vartheta = 20^\circ$.

(c) Explain why $x = \cos 20^\circ$ must be a root of the polynomial $8x^3 - 6x - 1$.

(d) Explain how the rational root theorem implies that this polynomial has no linear factors.

(e) Explain why this polynomial must therefore be irreducible over the rational numbers.

(f) You may recall from Math 1165 that some methods of solving cubic equations involve extracting cube roots. What does this imply about trisecting angles?

(g) You may recall, from earlier this semester, discussing a method for trisecting an angle with paper folding. What does that method imply about the relationship between the numbers that are constructible by paper folding and those that are constructible by compass and straightedge? Explain.

A.39 Area and Perimeter

A.39.1) You have been asked to put together the dance floor for your sister's wedding. The dance floor is made up of 24 square tiles that measure one meter on each side.

(a) Experiment with different rectangles that could be made using all of these tiles, and record your data in a table.

(b) Draw a graph of your data. Describe patterns in the data, as seen in the table or graph.

A.39.2) Suppose the dance floor is held together by a border made of edge pieces one meter long.

(a) What determines how many edge pieces are needed: area or perimeter? Explain.

(b) Make a graph showing the perimeter vs. length for various rectangles with an area of 24 square meters.

(c) Describe the graph. How do patterns that you observed in the table show up in the graph?

(d) Which design would require the most edge pieces? Explain.

(e) Which design would require the fewest edge pieces? Explain.

A.39.3) Suppose you had begun with a different number of floor tiles, such as 30, 21, or 19, or 36.

(a) In general, describe the rectangle with whole-number dimensions that has the greatest perimeter for a fixed area.

(b) Which rectangle has the least perimeter for a fixed area?

A.39.4) Consider the graphs you drew in the previous problems.

(a) Can we connect the dots in the graphs? Explain.

(b) How might we change the context so that the dimensions can be other than whole numbers? In the new context, how would the previous answers change?

A.39.5) The previous problems were about rectangles with constant area and changing perimeter.

(a) Make up a problem about rectangles with whole-number dimensions, constant perimeter, and changing area.

(b) Make a table of length, width, perimeter, and area for these rectangles.

(c) Draw a graph of width versus length for your rectangles.

(d) Draw a graph of area versus length for your rectangles.

(e) Now modify the context and your graphs to allow dimensions that are not whole numbers.

(f) Which rectangle will have a maximum area?

(g) Which rectangle will have a minimum area?

A.39.6) So far we have considered rectangles with fixed area and those with fixed perimeter. What about fixing the width or the length? Since they behave in much the same way, let's fix the width.

(a) Make up a problem about rectangles with constant width and changing area and perimeter.

(b) Make a table of length, width, perimeter, and area for these rectangles.

(c) Draw a graph of area versus length for your rectangles.

(d) Draw a graph of perimeter versus length for your rectangles.

(e) What kinds of functions do you see?

A.39.7) Explain how and where you saw the following advanced algebra ideas in the above problems:

(a) Domain, range and "limiting cases"

(b) Rates of change, maxima, minima, and asymptotic behavior

(c) Generalizing from a specific to a generic fixed quantity

(d) Equation solving with several variables

(e) Distinguishing among various types of functions

A.40 Reading Information From a Graph

On the next page is the graph of a function called $h(t)$, which represents the distance (in miles) and direction (east = positive, west = negative) Johnny is from home t hours after noon. It does not have a simple formula, so don't try to find one. Answer the following questions about h, briefly explaining how you obtained your answer(s):

A.40.1) On the given graph of h, what are the least and greatest values of t? What are the least and greatest values of $h(t)$? What do these answers say about Johnny?

A.40.2) Evaluate the following expressions: $h(0)$, $h(3)$, and $h(-3)$. What do each of these say about Johnny?

A.40.3) For each of the following, solve for t (i.e., find all the values of t that make the statement true). Describe what you did with the graph to determine the solutions. Where possible, interpret the statement and its solutions in terms of Johnny.

(a) $h(t) = 0$

(b) $h(t) = 3$

(c) $h(t) \leq 3$

(d) $h(t) = h(4.5)$

(e) $h(t) = t$

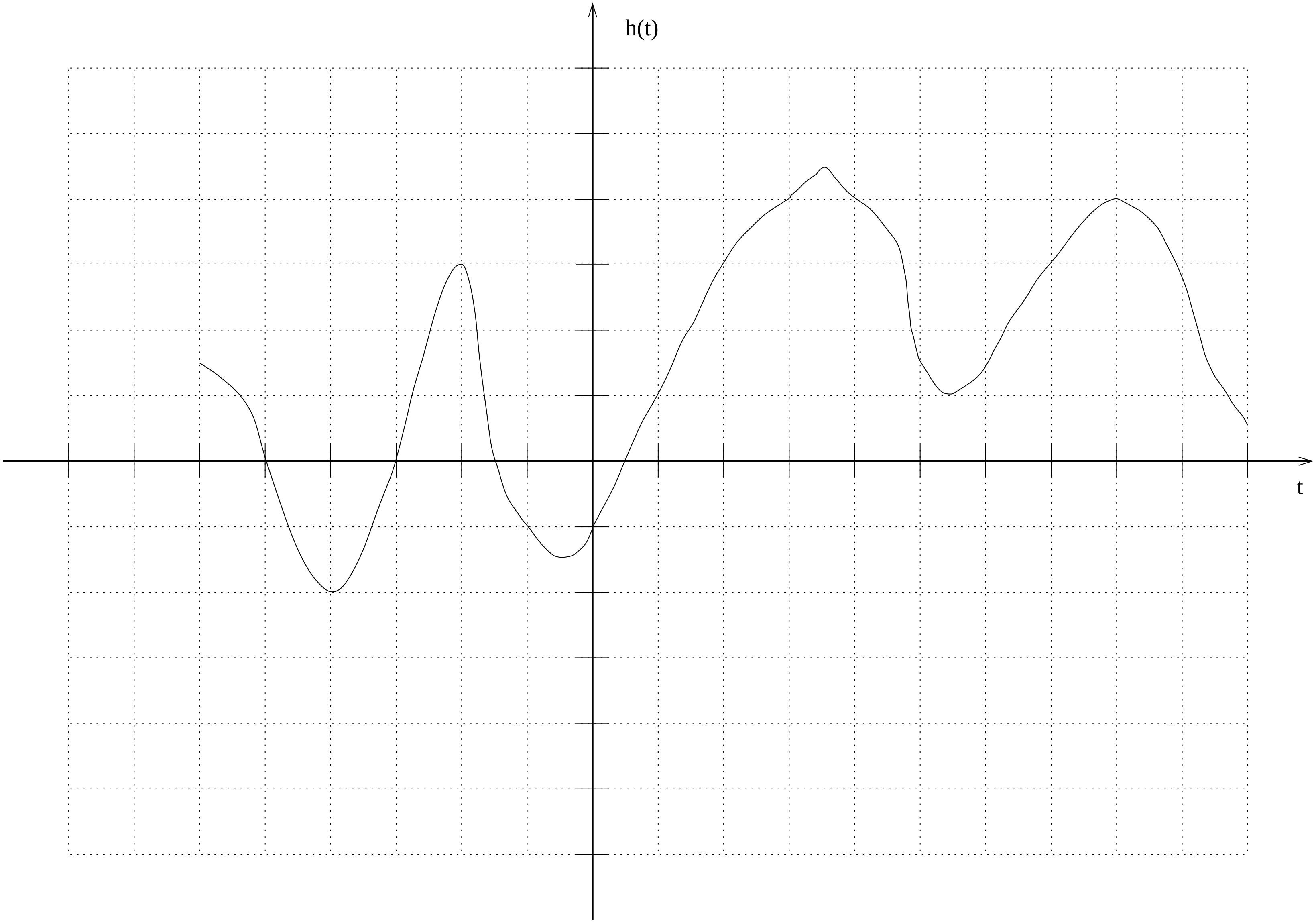
(f) $h(t) = -t$

(g) $h(t) = h(-t)$

(h) $h(t) = -h(-t)$

(i) $h(t + 1) = h(t)$

(j) $h(t) + 1 = h(t)$



h(t)
t

A.41 Parametric Equations

Fixnote: Perhaps design a preliminary activity about connecting the dots.

When graphs are given by *parametric equations*, the coordinates x and y may be given as functions of t, often thought of as "time." To begin graphing parametric equations, make a table of values for t, x, and y, and then plot the order pairs (x, y).

A.41.1) Consider the parametric equation

$$(x, y) = (2t + 3, -t - 4).$$

This is an equation about points that vary with t. To see the individual coordinates as functions of time, this equation can also be written as a pair of equations about coordinates, as follows:

$$x(t) = 2t + 3y(t) = -t - 4 \tag{A.1}$$

(a) Graph the equation. It might help to note various values of t on your graph.

(b) Describe the graph and explain why it looks the way it does.

(c) Locate the points corresponding to $t = \frac{2}{3}, \frac{5}{4}, 3.1$, and π.

(d) Why is it okay to connect the dots? Consider what happens to the x and y coordinates near and between points you have already plotted.

(e) What are the input values in the graph of a parametric equation?

(f) What are the output values in the graph of a parametric equation?

A *vector* has both direction and magnitude (i.e., length). In this course, vectors will often be given as ordered pairs, and they may be drawn or imagined as arrows from the origin to the given point. For example, the vector $(3, 2)$ can be represented as an arrow from $(0, 0)$ to $(3, 2)$. But the position of the vector is unimportant.

A.41.2) Explain why an arrow from $(1, 6)$ to $(4, 8)$ describes the vector $(3, 2)$.

A.41.3) What vector may be represented by an arrow from $(6, 4)$ to $(2, 1)$?

A.41.4) Consider the equation $(x, y) = (2, 1) + t(-1, 3)$.

(a) Graph the equation.

(b) Use the ideas of a starting point and a direction vector to explain why the graph looks the way it does.

(c) Pick an arbitrary point on your graph and describe how to arrive at that point using the starting point and scaling the direction vector.

A.41.5) Graph the equation $(x, y) = (2, 1) + t(2, -6)$. Compare and contrast this problem with the previous problem.

A.41.6) Write a parametric equation for the line containing $(-3, 2)$ and $(2, 1)$.

A.41.7) Write a parametric equation for the line containing the points (a, b) and (c, d).

A.41.8) Consider the line containing the points $A = (2, 4)$ and $B = (-1, 8)$.

(a) Find the coordinates of the point $2/3$ of the way from A to B.

(b) Find the coordinates of the point $5/4$ of the way from A to B.

(c) Find the coordinates of the point p/q of the way from A to B.

(d) What would it mean for p/q to be greater than 1? Explain

(e) What would it mean for p/q to be negative? Explain.

(f) What geometric object will result if p/q varies through all possible rational numbers? Explain.

(g) Find the coordinates of the point p/q of the way between (a, b) and (c, d).

A.42 Parametric Plots of Circles

In this activity we'll investigate parametric plots of circles.

A.42.1) One problem with the standard form for a circle, even the form for the unit circle

$$x^2 + y^2 = 1,$$

is that it is somewhat difficult to find points on the circle. We claim that for any value of t,

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$

will be a point on the unit circle. Can you give me some explanation as to why this is true? Two hints, for two answers: The unit circle; The Pythagorean identity.

A.42.2) Another way to think about parametric formulas for circles is to imagine

$$x(\vartheta) = \cos(\vartheta)$$
$$y(\vartheta) = \sin(\vartheta)$$

where ϑ is an angle. What is the connection between value of ϑ and the point $(x(\vartheta), y(\vartheta))$?

A.42.3) One way to think about parametric formulas for circles is to imagine

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$

as "drawing" the circle as t changes. Starting with $t = 0$, describe how the circle is "drawn." Make a table of values of t, x, and y. Use values of t that are special angles. Includes values of t that are negative as well as some values of t that are greater than 2π.

A.42.4) One day you accidentally write down

$$x(t) = \sin(t)$$
$$y(t) = \cos(t)$$

Again, make a table of values of t, x, and y What happens now? Do you still get a circle? How is this different from what we did in the previous question?

A.42.5) Do the formulas

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$

define a function? Discuss. Clearly identify the domain and range as part of your discussion. Remember, the domain is the set of input values and the range is the set of output values.

A.42.6) Reason with your previous tables of x- and y-values to determine the graph of the following parametric equations.

$$x(t) = 2\cos(t) + 3$$
$$y(t) = 2\sin(t) - 4$$

Explain your reasoning.

A.42.7) Now we will go backwards. The standard form for a circle centered at a point (a, b) with radius c is given by

$$(x - a)^2 + (y - b)^2 = r^2.$$

Explain why this makes perfect sense from the definition of a circle.

A.42.8) Here are three circles

$$(x - 1)^2 + (y + 2)^2 = 4^2 \qquad (x + 4)^2 + (y - 2)^2 = 8 \qquad x^2 + y^2 - 4x + 6y = 12.$$

Convert each of these circles to parametric form.

A.43 Eclipse the Ellipse

In this activity we'll investigate parametric plots of ellipses and other curves.

A.43.1) Recall that for $0 \leqslant t < 2\pi$

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \end{aligned}$$

gives a parametric plot of a unit circle. Describe the plot of

$$\begin{aligned} x(t) &= 3\cos(t) \\ y(t) &= \sin(t) \end{aligned}$$

for $0 \leqslant t < 2\pi$.

A.43.2) Now describe the plot of

$$\begin{aligned} x(t) &= 2\cos(t) \\ y(t) &= 5\sin(t) \end{aligned}$$

for $0 \leqslant t < 2\pi$.

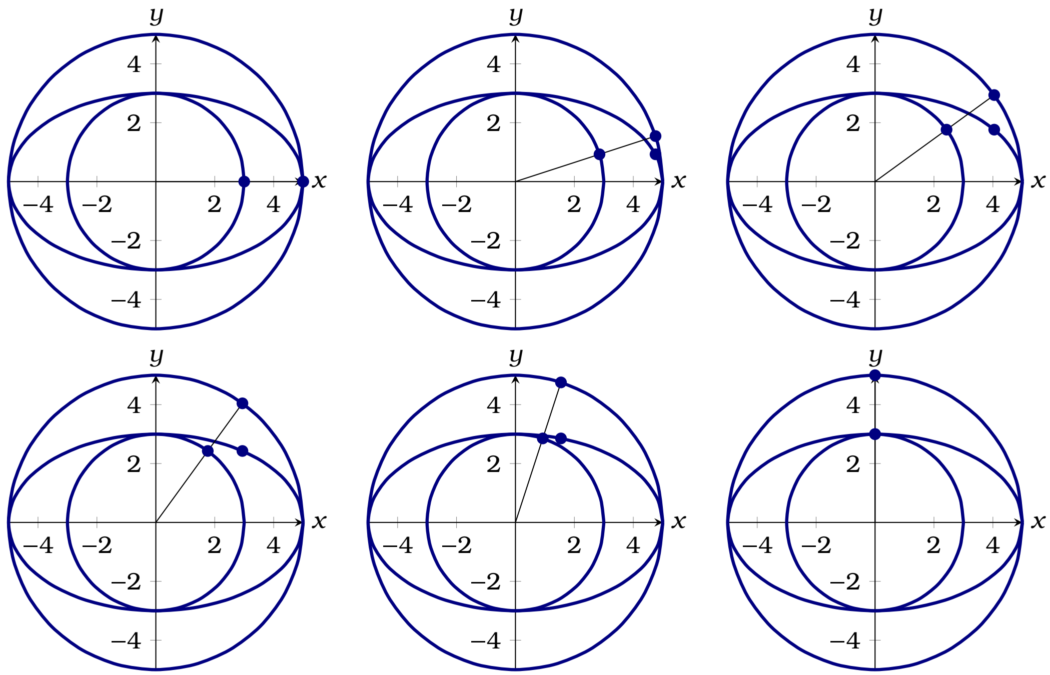
A.43.3) We claim that an ellipse centered at the origin is defined by points (x, y) satisfying

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Are the parametric curves we found above ellipses? Explain why or why not.

Fixnote: Use this for something.

A.43.4) Here we have some plots showing two concentric circles and an ellipse that touches both.

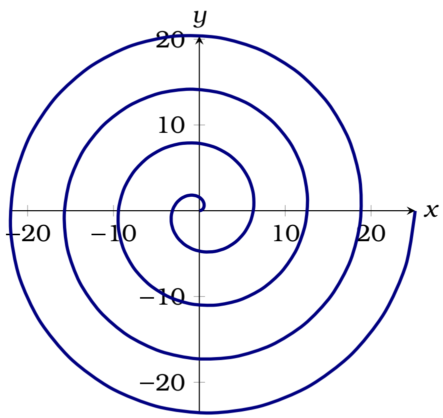


(a) Can you guess parametric formulas for the circles and for the ellipse?

(b) Do you notice anything about the dots in the pictures? Can you explain why this happens?

(c) Can you give a compass and straightedge construction that will give you as many points on a given ellipse as you desire? Give a detailed explanation.

A.43.5) Can you give a parametric formula for this cool spiral?

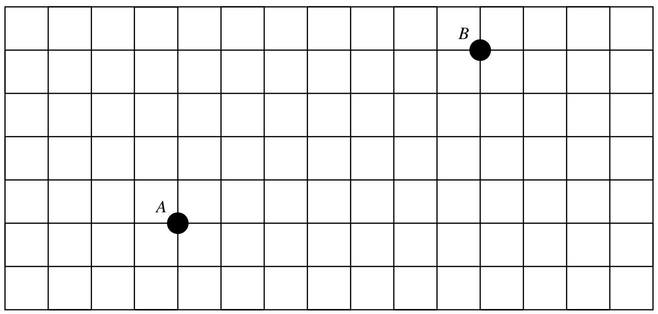


A.43.6) Remind me once more, do the formulas that produce these plots define functions? Discuss. Clearly identify the domain and range as part of your discussion.

A.44 Taxicab Distance

In this activity, we explore *City Geometry*, where points are Euclidean points, given with coordinates; lines are Euclidean lines, defined with equations or by two points, as in Euclidean coordinate geometry; and angles are Euclidean angles. Distance, however, is measured according the path a taxicab might travel. Let's get started.

A.44.1) Suppose we are in a city that is neatly laid out in blocks of two-way streets, with streets running north-south and east-west, and suppose we want to travel from point A to point B in the figure below.



(a) What is the *taxicab distance*, measured in city blocks, from point A to point B? (Do we mean the shortest distance, the longest distance, or something else?)

(b) Is there a single shortest path for the taxi to take? Explain.

(c) Let $A = (1, 2)$. What would be the coordinates of B?

(d) Describe a calculation that yields the taxicab distance between points A and B.

(e) Suppose the taxicab may travel on alleys also running north-south and east-west. Better yet, suppose the taxicab can create alleys wherever they would be most useful, except that they must still run north-south or east-west. What then would be the taxicab distance from A to B? Explain.

(f) Based on your reasoning, given points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, write a formula for, $d_T(P, Q)$, the *taxicab distance* between points P and Q. Check that it works for several pairs of points.

Continue in section 5.1.1.

A.45 City Geometry and Absolute Value

A.45.1) Let's consider circles in city geometry.•

(a) Use the taxicab distance formula to derive the equation of a city-geometry circle with radius r and center (a, b).

(b) Write the equation of a city-geometry circle with radius 1, centered at the origin, and draw a graph of this city-geometry circle.

• First, remind yourself how to use the definition of circle and the distance formula in Euclidean coordinate geometry to derive the equation of a Euclidean circle with radius r and center (a, b).

To better understand the equation of this city-geometry circle, we need to explore the idea of absolute value.

A.45.2) Consider the following attempts to characterize the absolute value function.

$$|x| \text{ is the “magnitude” of } x\text{—the size of } x\text{, ignoring its sign.} \tag{A.2}$$

$$|x| \text{ is the distance from the origin to } x. \tag{A.3}$$

$$|x| = \sqrt{x^2} \tag{A.4}$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{A.5}$$

(a) Which characterization is the definition of the absolute value function?

(b) Are the other characterizations of the absolute value function equivalent to the definition? Explain.

(c) What difficulties might students have with each one?

(d) Use one or more of these characterizations to explain the solution(s) to $|x - 5| = 8$.

(e) Use one or more of these characterizations to develop meanings for and $|x - a|$ and $|a - x|$ where where a is a constant.

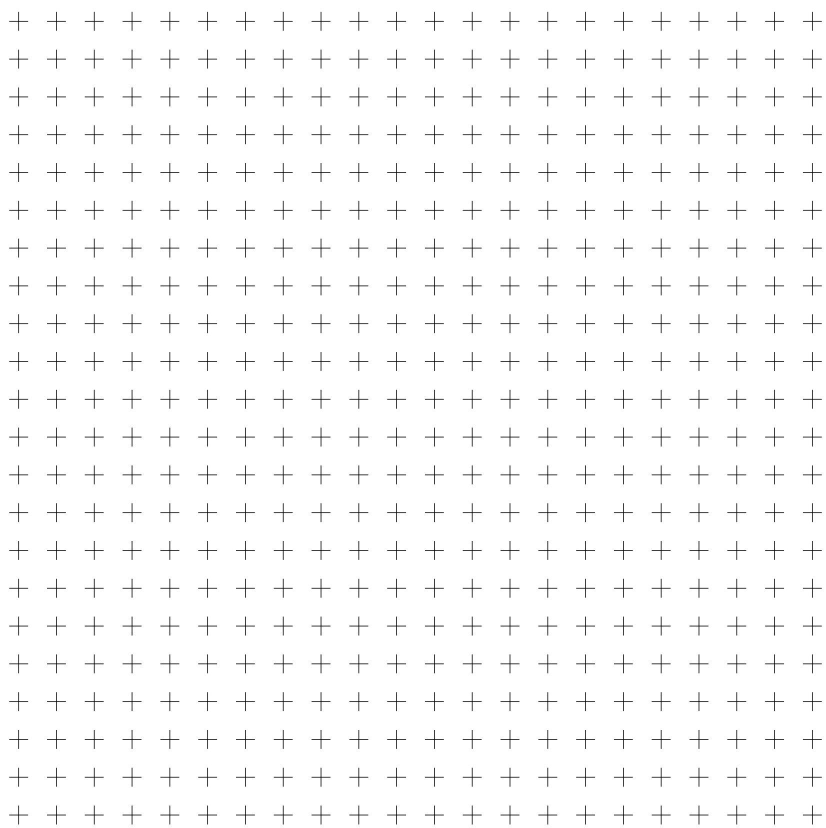
(f) What are the benefits of using more than one characterization of this idea?

A.45.3) Use the piecewise characterization of the absolute value function to explain why the equation $|x| + |y| = 1$ has the graph that it does. (Hint: Consider various cases, depending upon the sign of x and the sign of y.)

A.46 The Path Not Taken

In Euclidean geometry, there is a unique shortest path between two points. Not so in city geometry, here you have many different choices. Let's investigate this further.

A.46.1) Place two points 5 units apart on the grid below. How many paths are there that follow the grid lines? Note, if your answer is 1, then maybe you should pick another point!



Be sure to demand that your results are shared with the rest of the class.

A.46.2) Do the first problem again, except for points that are 4 units apart and then for points that are 6 units apart. What do you notice? Can you explain this?

A.46.3) Construct a chart showing your findings from your work above, and other findings that may be relevant.

A.46.4) Suppose you know how many paths there are to all points of distance n away from a given point. Can you easily figure out how many paths there are to all points of distance $n + 1$ away? Try to explain this in the context of paths in city geometry.

A.47 Midsets Abound

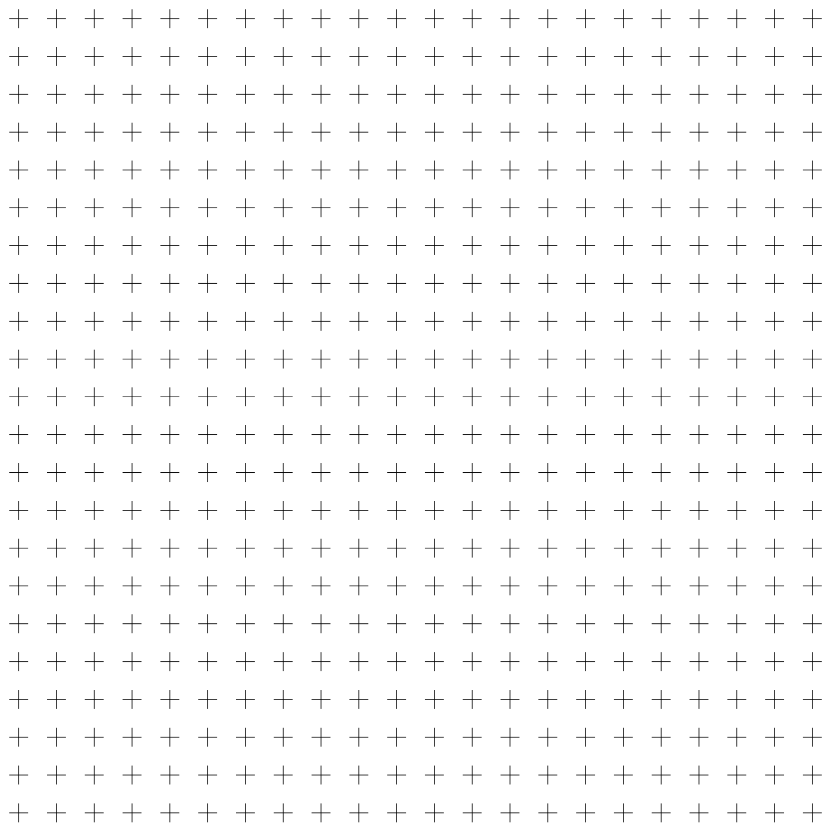
In this activity we are going to investigate *midsets*.

Definition Given two points A and B, their **midset** is the set of points that are an equal distance away from both A and B.

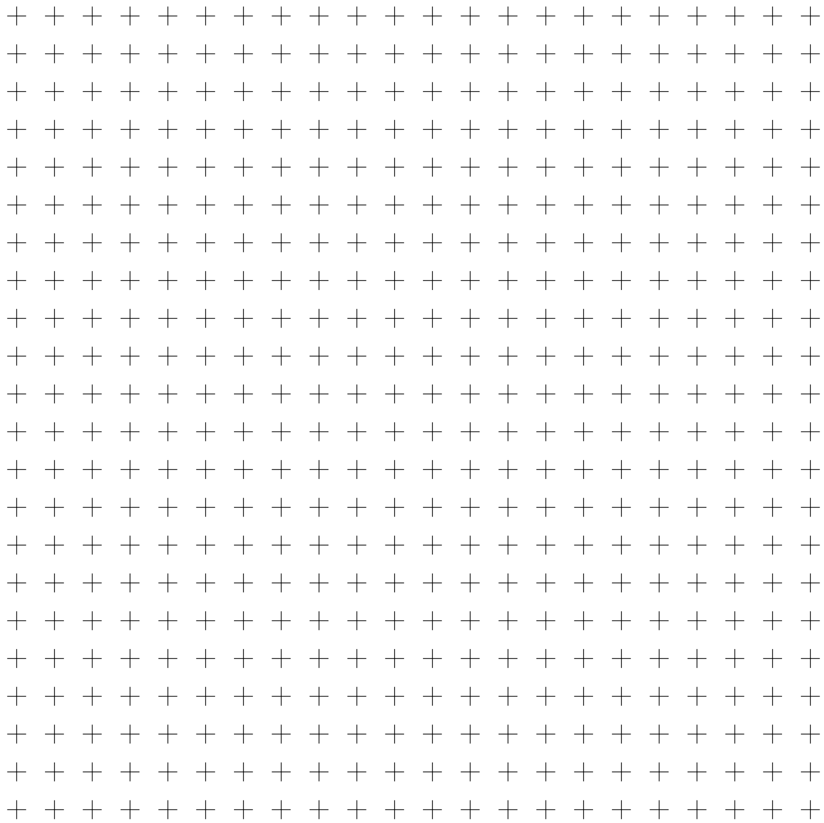
A.47.1) Draw two points in the plane A and B. See if you can sketch the Euclidean midset of these two points.

A.47.2) See if you can use coordinate constructions to find the equation of the midset of two points A and B. If necessary, set $A = (2, 3)$ and $B = (5, 7)$.

A.47.3) Now working in city geometry, place two points and see if you can find their midset.



A.47.4) Let's try to classify the various midsets in city geometry:



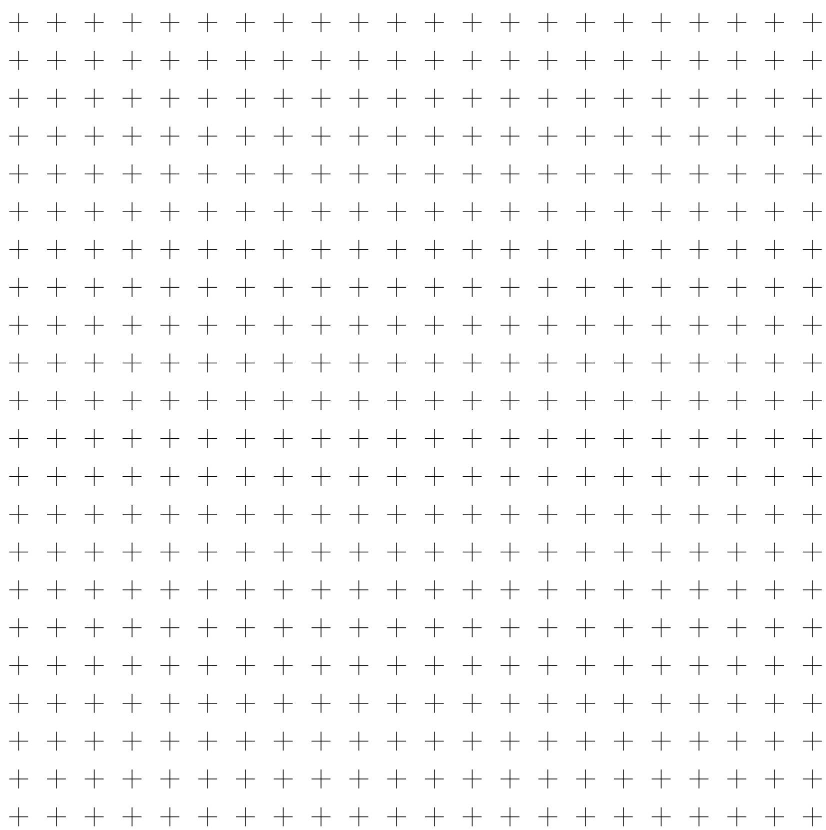
A.48 Tenacity Paracity

In this activity we are going to investigate city geometry parabolas.

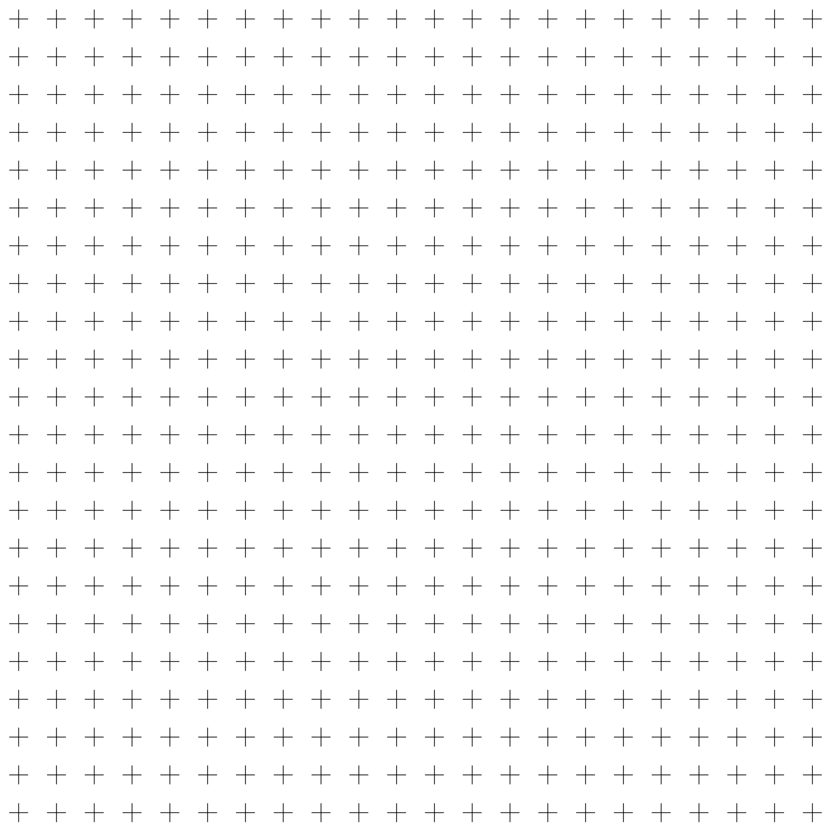
A.48.1) Remind me again, what is the definition of a *parabola*?

A.48.2) Use coordinate constructions to find the equation of the parabola with its focus at $(1, 2)$ and its directrix being the line $y = -3$.

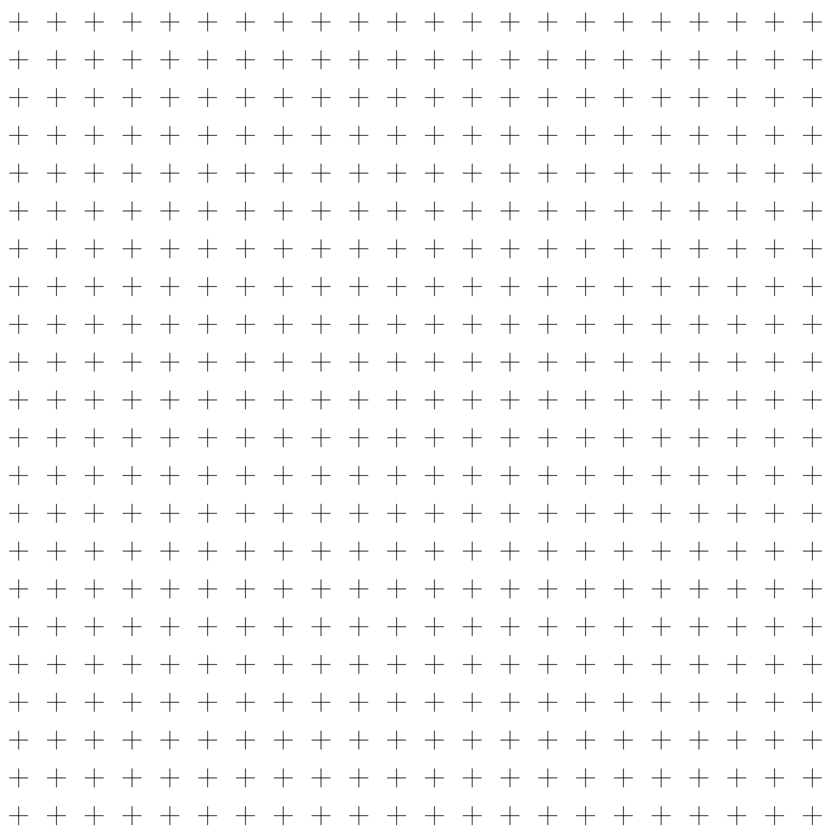
A.48.3) Sketch the city geometry parabola when the focus is the point $(0, 2)$ and the directrix is $y = 0$



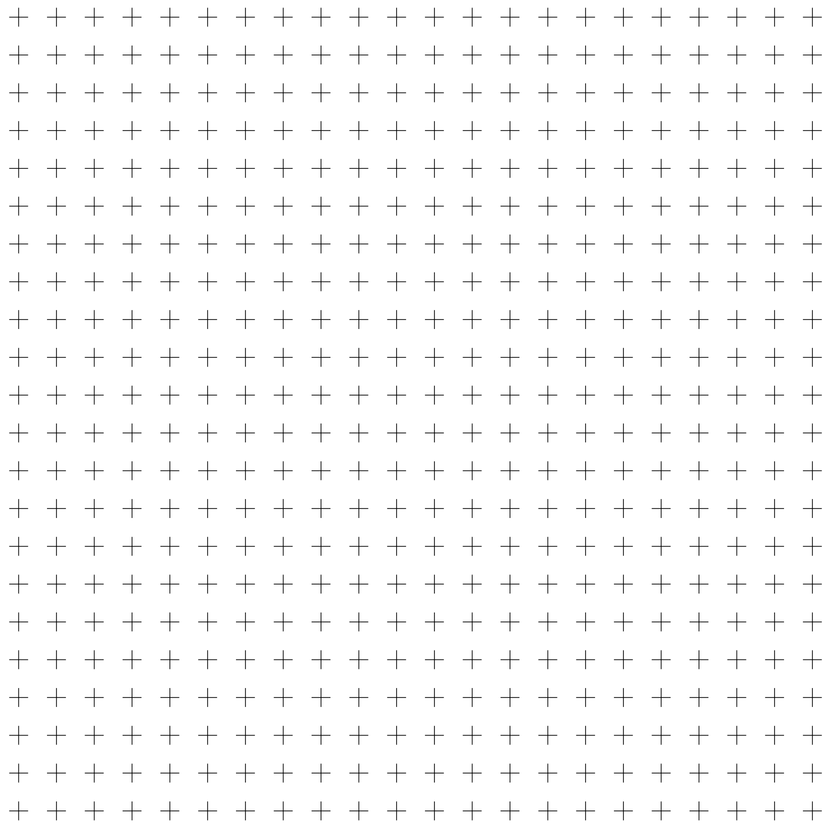
A.48.4) Sketch the city geometry parabola when the focus is the point $(4, 4)$ and the directrix is $y = -x$



A.48.5) Sketch the city geometry parabola when the focus is the point $(0, 4)$ and the directrix is $y = x/3$



A.48.6) Sketch the city geometry parabola when the focus is the point $(4, 1)$ and the directrix is $y = 3x/2$



A.48.7) Explain how to find the distance between a point and a line in city geometry.

A.48.8) Give instructions for sketching city geometry parabolas.

A.49 Who Mapped the What Where?

Let M represent some mysterious matrix that maps the plane to itself. So M is of the form:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

A.49.1) If I tell you that $\mathbf{p} = (3, 4)$ and

$$\mathsf{M}\mathbf{p} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix},$$

give 3 possible matrices for M.

A.49.2) Now suppose that in addition to the fact above, I tell that

$$\mathsf{M}\mathbf{o} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

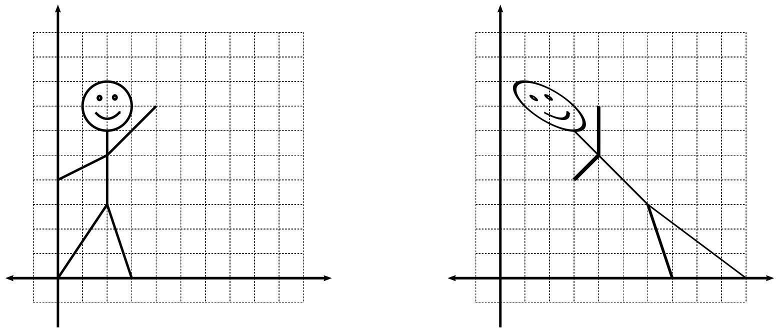
where $\mathbf{o}$ is the origin. Give 3 possible matrices for M.

A.49.3) Now suppose that in addition to the two facts above, I tell you that

$$\mathsf{M}\mathbf{q} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

where $\mathbf{q} = (1, 1)$. How many possibilities do you have for M now? What are they?

A.49.4) Here is a picture of my buddy *Sticky*:



As you can see, he's been dancing with some matrix M. Can you tell me which matrix it was? What are good points to pay attention to?

A.50 How Strange Could It Be?

In this activity, we are going to investigate just how strange a map given by

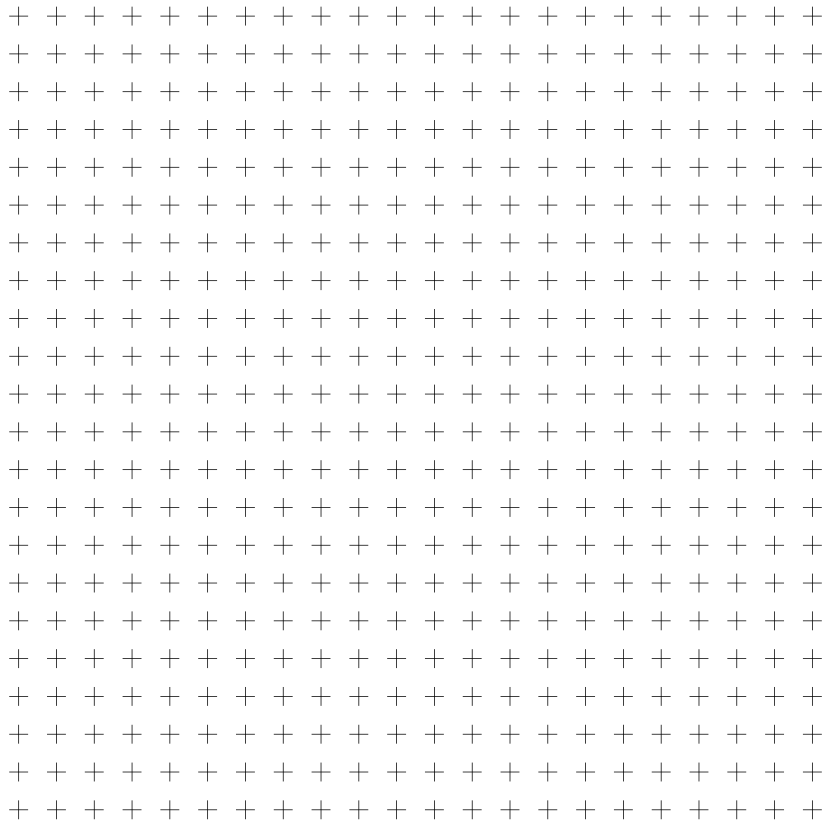
$$\mathsf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

could possibly be.

A.50.1) Let α, β, γ, and δ be real numbers, and let x be a variable. Consider the point:

$$(\alpha x + \beta, \gamma x + \delta)$$

Choose values for the Greek letters and plot this for varying values of x. What sort of curve do you get?



A.50.2) Now consider the line $y = mx + p$. Express its coordinates *without* using y.

A.50.3) Apply M to the coordinates you found above. What do you get? What does this tell you about what happens to lines after you apply a matrix to them?

A.50.4) Tell me some things about the line $y = mx + q$.

A.50.5) Apply M to the coordinates associated to $y = mx + q$. What does this tell you about what happens to parallel lines after you apply a matrix to them?

A.50.6) What's going to happen to a parallelogram after you apply a matrix to it?

A.51 Composing Transformations

In this activity, we use the following notation:

- R_ϑ denotes a counterclockwise rotation by ϑ about the origin.
- F_ℓ denotes a reflection about the line ℓ.
- $T_{(a,b)}$ denotes a translation by a vector (a, b).

A.51.1) Pick a specific point. Use matrices to find the image of your point under the following sequences of transformations:

(a) R_{90} followed by $F_{y=x}$

(b) $F_{y=x}$ followed by R_{90}

A.51.2) Repeat problem 1 for a general point.

A.51.3) Reason geometrically to identify a single transformation that accomplishes the sequences from problem 1. (Hint: Do the transformations physically with a square piece of paper marked with "FRONT" on the side that starts facing you.)

A.51.4) Use your answers to the previous problems to write a single matrix for each of the sequences of transformations in problem 1.

A.51.5) Compute the following products of matrices:

(a) $R_{90}F_{y=x}$

(b) $F_{y=x}R_{90}$

A.51.6) What do you notice about your previous answers? Explain why it works that way. Hint: Without actually doing the computations, write a matrix expression that represents the result of each of the sequences of transformations.

B References and Further Reading

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