PARALLELS IN GEOMETRY (SUPPLEMENTS)

MATH 1166: SPRING 2015

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Contents

4 Toward Congruence and Similarity

4.4 Dilations, Scaling, and Similarity (cont.)

These pages are to follow p. 94 in the 2015 version of the notes for Math 1166.

4.4.1 Theorems for Similar Triangles

Recall the following:

Definition A geometric figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations.

We need to show that this general definition of similarity fits with our previous ideas about similar triangles. Here is one way of thinking about similar triangles:

$$\triangle ABC \sim \triangle A'B'C' \qquad \Leftrightarrow \qquad \begin{aligned} \angle A &\simeq \angle A' \\ \angle B &\simeq \angle B' \\ \angle C &\simeq \angle C' \end{aligned}$$

Question What does this mean?

?

Here is another way of thinking about similar triangles:

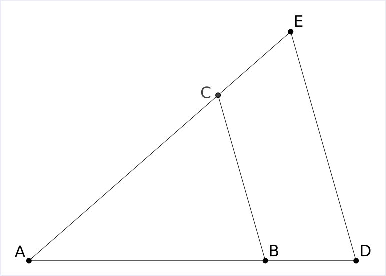
$$\triangle ABC \sim \triangle A'B'C' \qquad \Leftrightarrow \qquad \begin{aligned} AB &= k \cdot A'B' \\ BC &= k \cdot B'C' \\ CA &= k \cdot C'A' \end{aligned}$$

Question What does this mean?

?

Using merely the formula for the area of a triangle, we (meaning you) will explain why the following important theorem is true. Throughout this discussion we will use the convention that when we write AB we mean the *length* of the segment AB.

Theorem 4.4.1 (Parallel-Side) *Given:*



If side BC is parallel to side DE, then

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

Question Can you tell me in English what this theorem says? How does it relate to the definition of similarity in terms of rigid motions and dilations?

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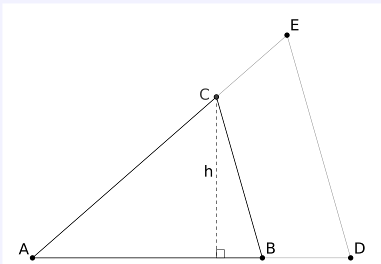
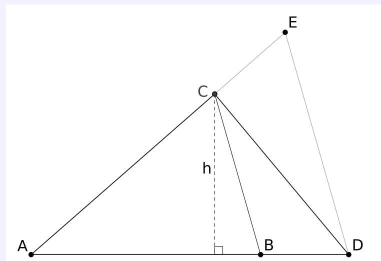
Now we (meaning you) are going to explore a bit. See if answering these questions sheds light on this.

Question If h is the height of $\triangle ABC$, find formulas for the areas of $\triangle ABC$ and $\triangle ADC$.





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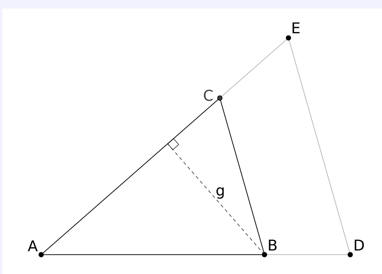
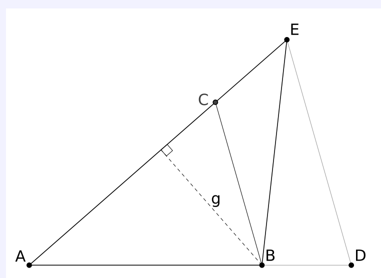
Question If g is the height of $\triangle ACB$, find formulas for the areas of $\triangle ACB$ and $\triangle AEB$.





?

Question Explain why

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle ACB).$$

?

Question Explain why

$$\text{Area}(\triangle CBE) = \text{Area}(\triangle CBD).$$

Big hint: Use the fact that you have two parallel sides! Draw a picture to help clarify your explanation.

?

Question Explain why

$$\text{Area}(\triangle ADC) = \text{Area}(\triangle AEB).$$

?

Question Explain why

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle ADC)} = \frac{\text{Area}(\triangle ACB)}{\text{Area}(\triangle AEB)}$$

?

Question Compute and simplify both of the following expressions:

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle ADC)} \quad \text{and} \quad \frac{\text{Area}(\triangle ACB)}{\text{Area}(\triangle AEB)}$$

?

Question How can you conclude that:

$$\frac{AB}{AD} = \frac{AC}{AE}$$

?

Question Why is it important that line DE is parallel to line CB?

?

Question Can you sketch out (in words) how the questions above prove the Parallel-Side Theorem?

?

Now comes the moment of truth.

Question Can you use the Parallel-Side Theorem to explain why if you know that if you have two triangles, $\triangle ABC$ and $\triangle A'B'C'$ with:

$$\angle A \simeq \angle A'$$
$$\angle B \simeq \angle B'$$
$$\angle C \simeq \angle C'$$

then we must have that

$$AB = k \cdot A'B'$$
$$BC = k \cdot B'C'$$
$$CA = k \cdot C'A'$$

These notes do not describe why side CA is also scaled by k. You address that question in the Side-Splitter Theorem activity.

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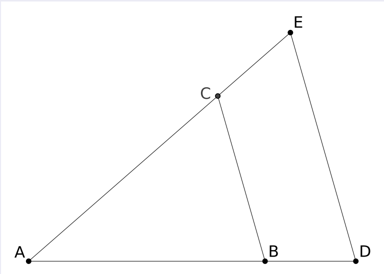
The Converse The converse of the Parallel-Side Theorem states:

Theorem 4.4.2 (Split-Side) *Given:*



If side BC intersects (splits) the sides of $\triangle ADE$ so that

$$\frac{AB}{AD} = \frac{AC}{AE},$$

then side BC is parallel to side DE.

Now we (meaning you) will answer questions in the hope that they will help us see why the above theorem is true.

Question Suppose that you **doubt** that side BC is parallel to side DE. Explain how to place a point C' on side AE so that side BC' is parallel to line DE. Be sure to sketch the situation(s).

?

Question You now have a triangle $\triangle ADE$ whose sides are split by a line BC' such that the line BC' is parallel to line DE. What does the Parallel-Side Theorem have to say about this?

?

Question What can you conclude about points C and C'?

?

Question What does this tell you about the Split-Side Theorem?

?

Let's see if you can put this all together:

Question Can you use the Split-Side Theorem to explain why you know that if you have two triangles, $\triangle ABC$ and $\triangle A'B'C'$ with:

$$AB = k \cdot A'B'$$
$$BC = k \cdot B'C'$$
$$CA = k \cdot C'A'$$

then we must have that

$$\angle A \simeq \angle A'$$
$$\angle B \simeq \angle B'$$
$$\angle C \simeq \angle C'$$

?

Putting all of our work above together, we may now say the following:

Theorem 4.4.3 *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are* ***similar*** *if either equivalent condition holds:*

$$\begin{matrix} \angle A \simeq \angle A' & & AB = k \cdot A'B' \\ \angle B \simeq \angle B' & \text{or} & BC = k \cdot B'C' \\ \angle C \simeq \angle C' & & CA = k \cdot C'A' \end{matrix}$$

Question How does this theorem connect back to the definition of similarity in terms of rigid motions and dilations?

?

4.4.2 A New Meaning of Multiplication

School mathematics makes sense when concepts have *meaning*.

Question What can multiplication mean? Can you give multiplication meaning involving groups of groups or something of the sort?

?

Question Can you give multiplication meaning involving areas or something of the sort?

?

Question Can you somehow give meaning to multiplication using similarity? Use “scale factor” or “scaling” in your explanation.

?

4.4.3 Problem Solving with Similarity

We now have several ways of thinking more deeply about the naive "same shape" notion of similarity (imagined as zooming in and out): same angles; proportional sides; sequence of basic rigid motions and dilation(s).

Key issue is being able to distinguish situations in which things are similar from those in which when they are not: (e.g., baby from PowerPoint, types of televisions). Multiplication is not necessarily scaling–unless it is dilation. Direct proportion versus not.

Many real-world problems can be solved using similar triangles or other similar figures. For example, you can use shadows to compute the height of a flagpole. And maps, scale drawings, and scale models all involve similarity.

Students use proportional relationships between corresponding parts of similar figures, distinguishing "within figure" ratios from "across figure" ratios, relating the latter to the scale factor.[G-SRT.2] When the figures overlap, one challenge is being consistent about part-part versus part-whole ratios.

Students use the definition of similarity to show that any two circles are similar.[G-C.1] They can also see the more surprising result that any two parabolas are similar.

Similarity turns out to be very useful in right triangles. First, the altitude to the hypotenuse creates two triangles similar to the first. Second, among right triangles, similarity requires only one more angle, which leads to right triangle trigonometry.

CCSS G-SRT.2: Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

CCSS G-C.1: Prove that all circles are similar.

4.5 Length, Area, and Volume under Scaling

When two objects are similar, then lengths are related by a scale factor. What does this mean for other measurements? Perimeters, areas, volumes, weights, etc.

- General considerations of measurement and dimension.
- Reason about length, area, and volume in similarity situations. Rep-tiles.
- Use shearing and Cavalieri's principle to reason about area and volume.
- Using a grid (and scaling the grid) to reason about areas of general shapes under scaling.

- Volume as area of base times height: Imagine layers of cubic units covering the base.
- Volume of pyramid: Three pyramids make a cube.
- Volume of cylinder, cone, and sphere.
- If you know the area of a rectangle, what can you say about its perimeter? What about more general figures?
- If you know the perimeter of a rectangle, what can you say about its area? What about more general figures?
- LeBron problems
- Fractals?

A Supplemental Activities

A.52 Suitable Precision in Language

A.52.1) Improve upon the following imprecise statements.

Statement	Improved Version	Comments
A triangle has 180°.		
A line measures 180°.		
A circle is (or has) 360°.		

A.52.2) Explain the geometric distinction between a segment and its length. How are the two usually denoted differently?

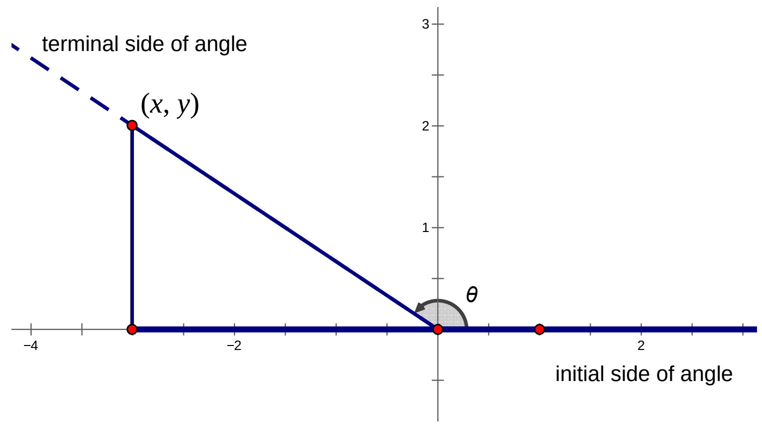
A.52.3) Explain the geometric distinction between an angle and its measure. How are the two usually denoted differently?

A.52.4) There are (at least) two ways of thinking about angles: (1) as a set of points, and (2) as an amount of turning. Describe how you have used both of these ways of thinking in this activity.

A.53 Circular Trigonometry

As we have seen, right triangle trigonometry is restricted to acute angles. But angles are often obtuse, so it is quite useful to extend trigonometry to angles greater than 90°. Here is one approach: Place the angle with the vertex at the origin in the coordinate and with one side of the angle (the initial side) along the positive x-axis. Measure to the other side of the angle (the terminal side) as a counter-clockwise rotation about the origin.

If we choose a point on the terminal side of this angle, we can draw what is called *reference triangle* by dropping a perpendicular to the x-axis. Then we can use the values of x, y, and r from this triangle, just as before. What is different this time is that x is negative, as well be the case for any angles with a terminal side in the second quadrant.



A.53.1) Draw a picture to demonstrate the following:

(a) $\sin 135^\circ =$

(b) $\cos 135^\circ =$

(c) $\tan 135^\circ =$

A.53.2) Draw spicture to demonstrate the following:

(a) $\sin 150^\circ =$

(b) $\cos 150^\circ =$

(c) $\tan 150^\circ =$

A.53.3) For some angles, the reference triangle is not actually a 'triangle,' but that's okay. Draw picture(s) to demonstrate the following:

(a) $\sin 90^\circ =$

(b) $\cos 90^\circ =$

(c) $\tan 90^\circ =$

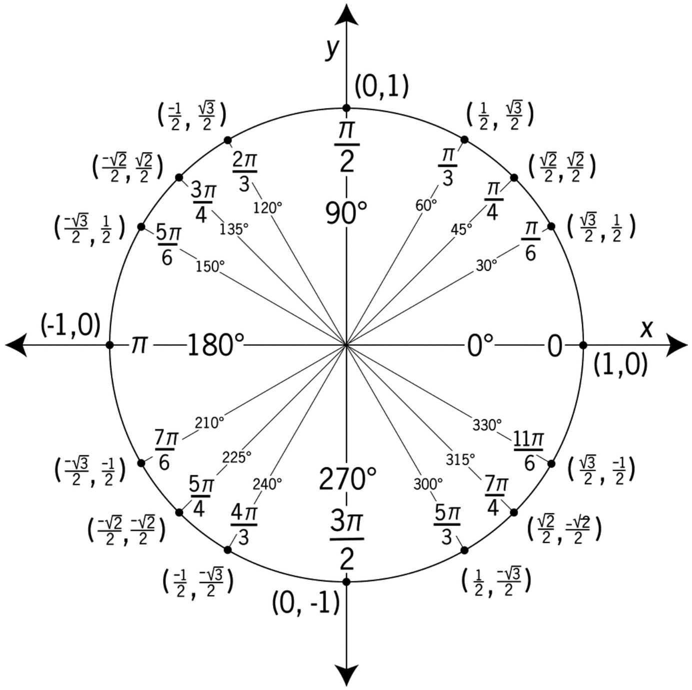
(d) $\sin 180^\circ =$

(e) $\cos 180^\circ =$

(f) $\tan 180^\circ =$

Because angles are often about rotation, angles greater than 180° can make sense, too. And negative angles can describe rotation in the opposite direction. If we consider the angle to change continuously, then rotation about the origin creates a situation that repeats every 360°. This repetition provides the foundation for modeling lots of repetitive (periodic) contexts in the real world. For this modeling, we need *circular trigonometry*, which turns out to be much cleaner if (1) angles are measured not in degrees but in a more "natural" unit, called radians; and (2) we use *the unit circle*, which is a circle of radius 1 centered at the origin.

A.53.4) The unit circle and special angles.



(a) Explain what the various numbers mean in this unit circle.

(b) Use the unit circle to make a table showing (1) angle in degrees, (2) angle in radians, (3) sine of the angle, and (4) cosine of the angle.

(c) Use your table to draw a graph of $\sin\vartheta$ versus ϑ.

(d) Use your table to draw a graph of $\cos\vartheta$ versus ϑ.

(e) Explain why it makes sense to connect the dots.

(f) Extend your graphs to angles greater than 360°, and use the unit circle to explain why your extension makes sense.

(g) Extend your graphs to angles less than 0°, and use the unit circle to explain why your extension makes sense.

A.54 Quadrilateral Diagonals (revision)

Imagine you are working at a kite factory and you have been asked to design a new kite. The kite will be a quadrilateral made of synthetic cloth, and it will be formed by two intersecting rods that serve as the diagonals of the quadrilateral and provide structure for the kite.

A.54.1) To get started, review the definitions of all special quadrilaterals. Be sure to include *kite* on your list.

A.54.2) To consider the possible kite shapes, your task is to describe how conditions on the diagonals determine the quadrilateral. Use fettuccine to model the intersecting rods, and use paper and pencil to draw the rod configurations and resulting kite shapes.

Here are some hints:

- Explore diagonals of various lengths, of the same length, and of different lengths.
- Explore various places at which to attach the diagonals to each other, including at one or both of their midpoints.
- Explore various angles that the diagonals might make with each other at their intersection, including the possibility of being perpendicular.

A.54.3) Summarize your findings in a table organized like the one on the next page.

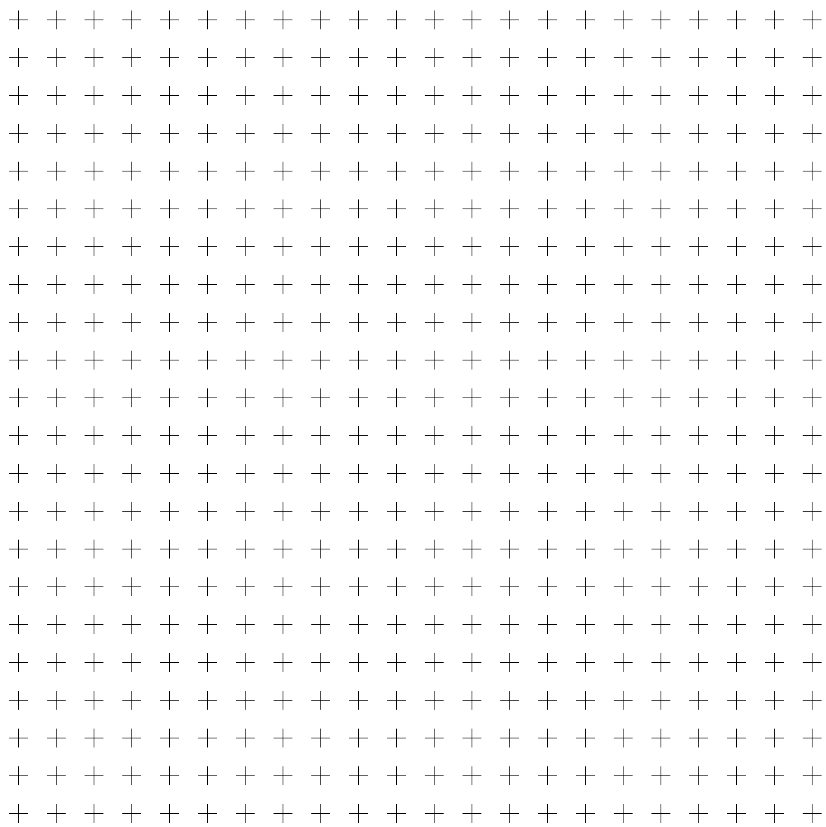
Quadrilateral	Definition	Diagonals: Cong.	Diagonals: Bisect	Diagonals: Perp.	Comments (including other key properties)
Square					
Rectangle					
Rhombus					
Parallelogram					
Kite					
Trapezoid					

A.55 Tenacity Paracity (revision, replaces p. 273 in 2015 notes)

In this activity we are going to investigate city geometry parabolas.

A.55.1) Remind me again, what is the definition of a *parabola*?

A.55.2) Use the definition of a parabola and taxicab distance to sketch the city geometry parabola when the focus is the point $(2, 1)$ and the directrix is $y = -3$.



A.55.3) Comparing geometries with algebra.

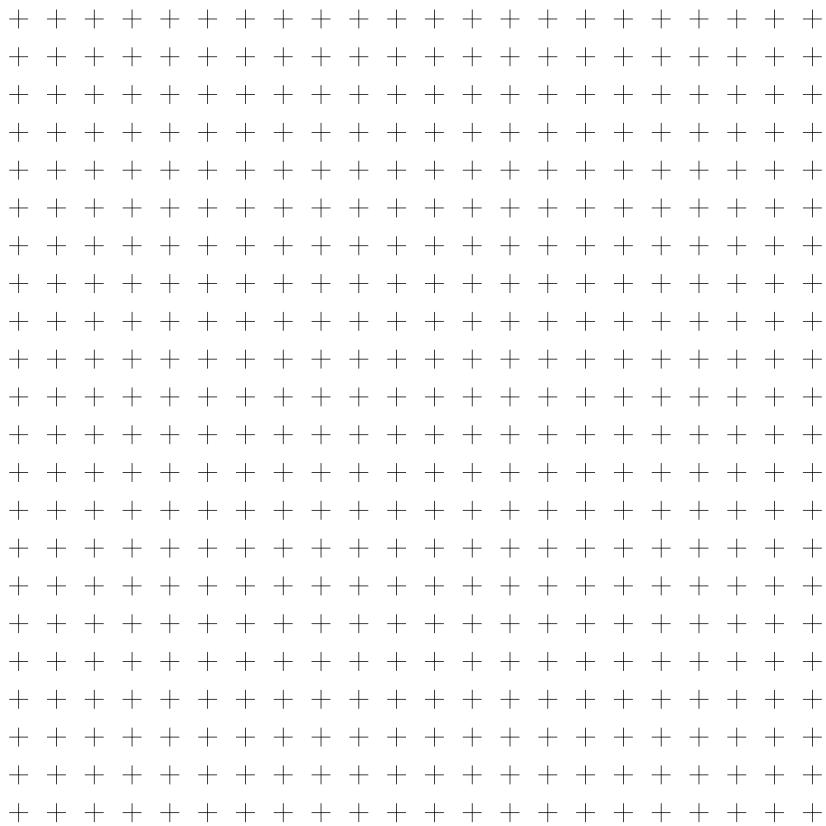
(a) Use coordinate constructions to write an equation for the Euclidean geometry parabola with its focus at $(2, 1)$ and its directrix being the line $y = -3$. (Hint: No need to simplify. Just use the definition and set the distances equal to one another.)

(b) Use your taxicab distance formula to write an equation for the city geometry parabola with its focus at $(2, 1)$ and its directrix being the line $y = -3$.

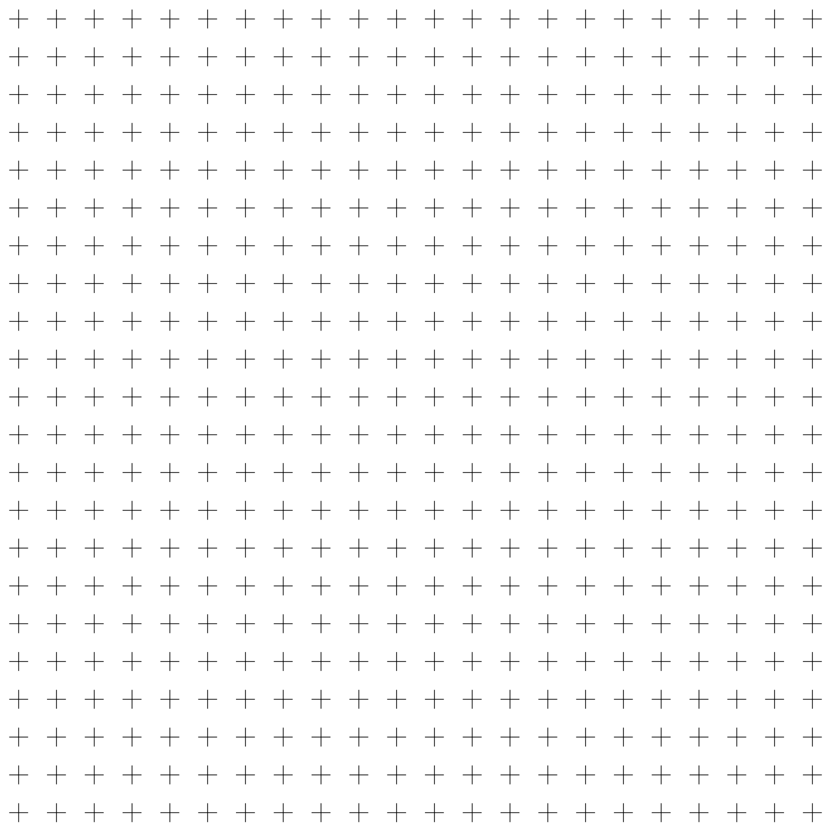
(c) Compare and contrast the two equations.

(d) Use algebra of absolute value to show that the graph in the previous problem is the correct graph. (Hint: Consider three cases: $y > 1$, $-3 \leq y \leq 1$, and $y < -3$.)

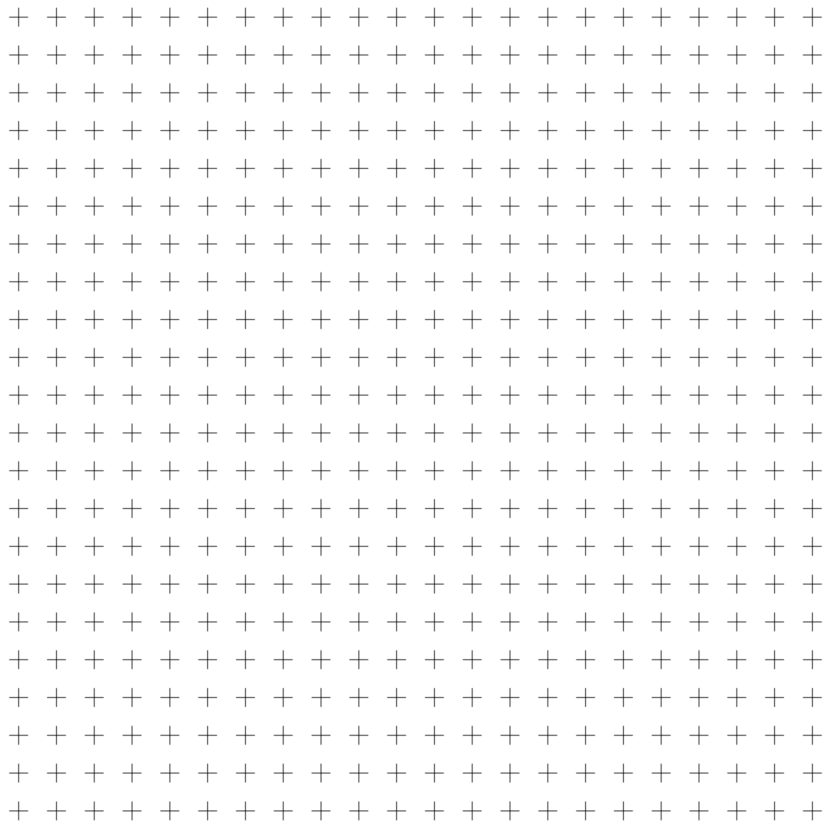
A.55.4) Sketch the city geometry parabola when the focus is the point $(4, 4)$ and the directrix is $y = -x$



A.55.5) Sketch the city geometry parabola when the focus is the point $(0, 4)$ and the directrix is $y = x/3$

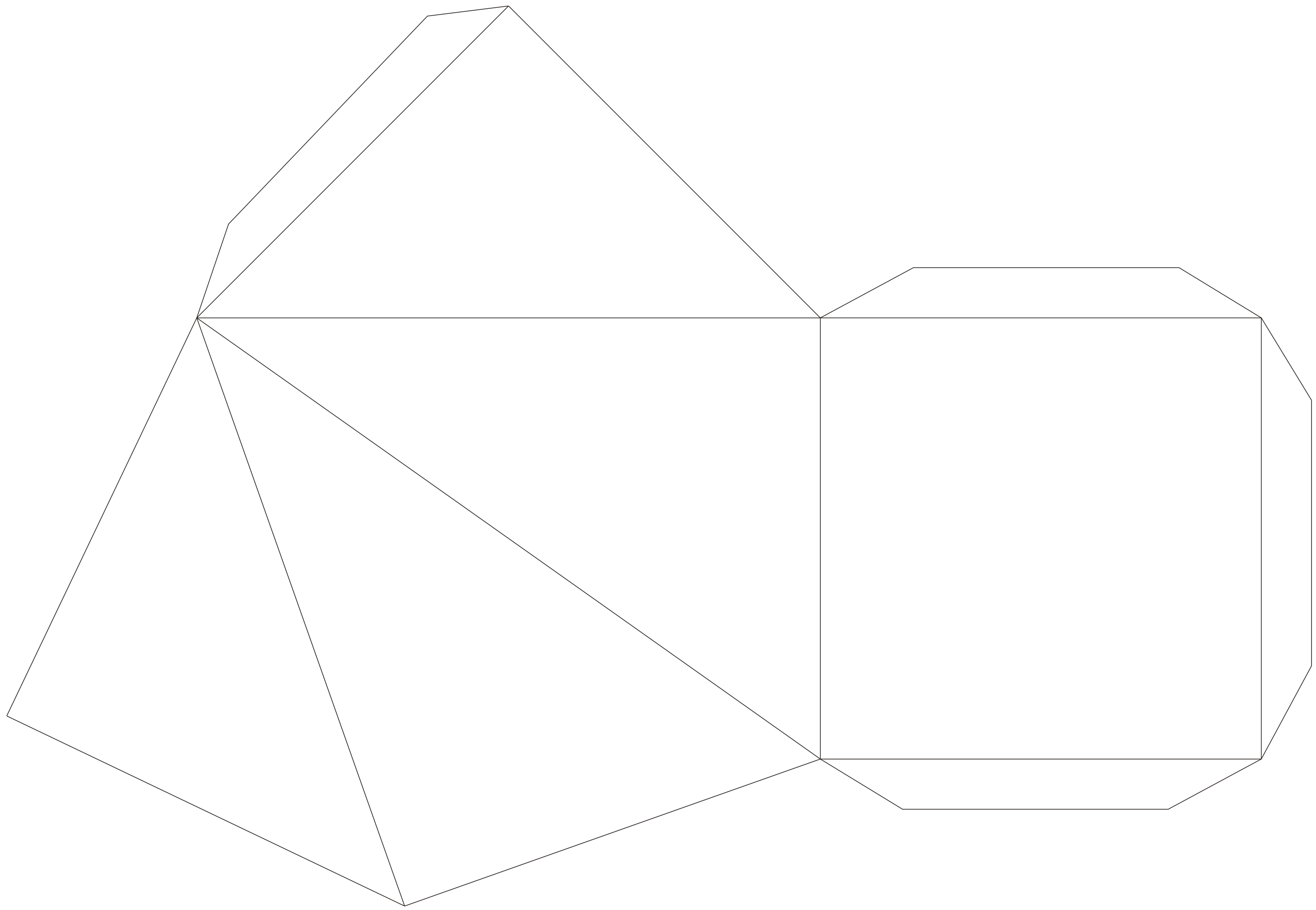


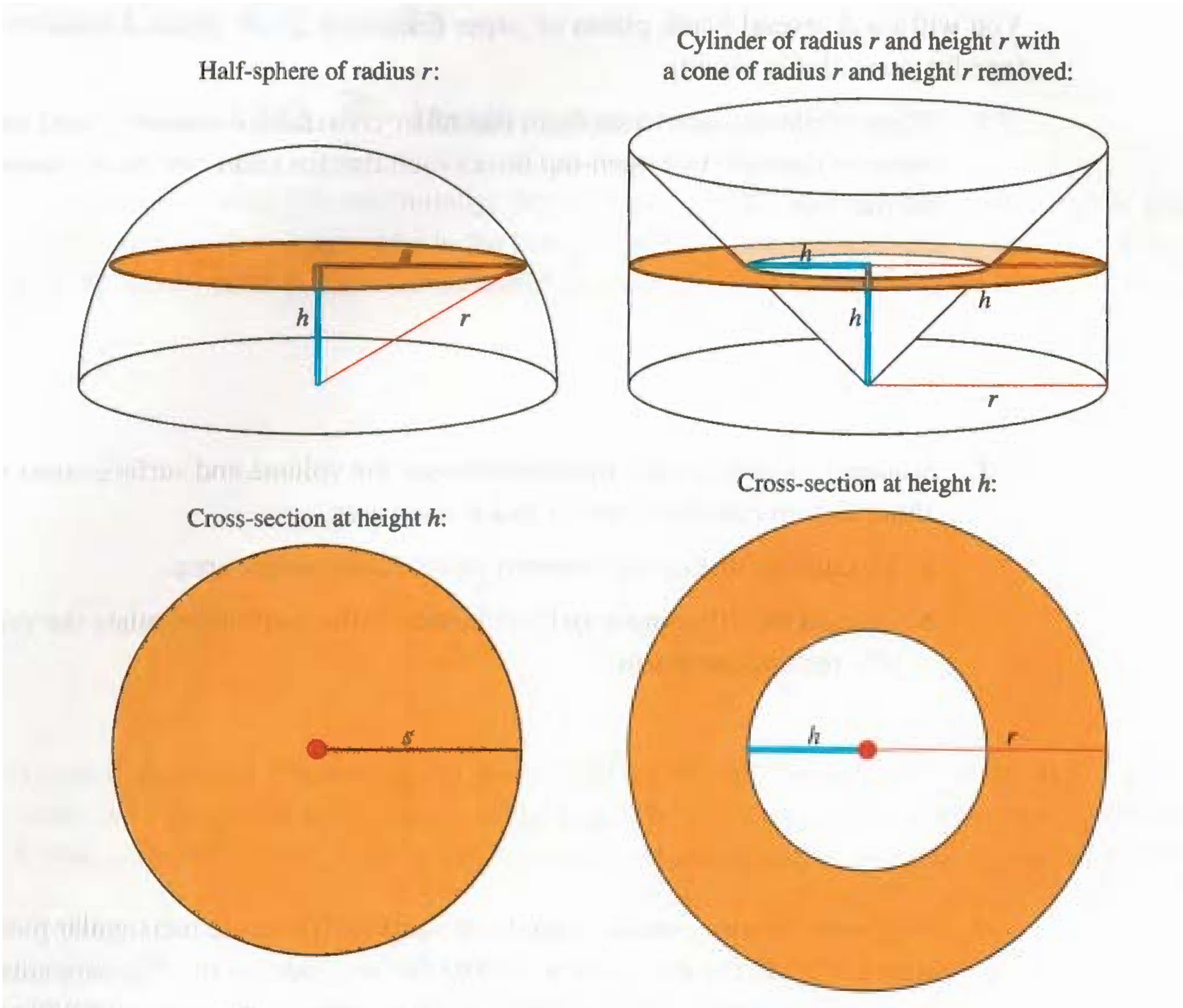
A.55.6) Sketch the city geometry parabola when the focus is the point $(4, 1)$ and the directrix is $y = 3x/2$



A.55.7) Explain how to find the distance between a point and a line in city geometry.

A.55.8) Give instructions for sketching city geometry parabolas.




Half-sphere of radius r:
Cylinder of radius r and height r with
a cone of radius r and height r removed:
h
r
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h
h
r
Cross-section at height h:
Cross-section at height h:
s
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B Supplemental Problems

B.1 Midterm 1 Review

Midterm Exam 1 will cover Chapters 1 – 3 (except section 1.2 and the angle trisection part of 3.1) and activities A.1 – A.18 (except A.14). We did not complete all of these activities in class so you will need to fill in some gaps. In particular, we did not do A.15 at all and we did only a few problems from A.17.

B.1.1 Review Ideas

- Be able to state definitions of commonly used terms.
- Be able to perform standard constructions and explain why they work.
- Be able to explain key concepts such as area and angle.
- Be able to state precisely the triangle congruence criteria.
- Know the properties of various special quadrilaterals and be able to prove them.
- Know the various centers of a triangle, how to construct them, and whether they can lie outside the triangle.
- Be able to state key theorems and prove them in at least two (2) ways, especially:
  - Isosceles triangle theorem
  - Pythagorean theorem
  - The angle sum of a triangle

Review Problems

B.1.1) Describe Euclid's (compass and straightedge) construction for an equilateral triangle, and explain why it works.

B.1.2) Construct a 30-60-90 right triangle. Explain the steps in your construction and how you know it works.

B.1.3) Construct a 45-45-90 right triangle. Explain the steps in your construction and how you know it works.

B.1.4) Where is the orthocenter of a right triangle? Explain your reasoning. What about the circumcenter? Again, explain your reasoning.

B.1.5) Show that, given any three non-collinear points in the Euclidean plane, there is a unique circle passing through the three points.

B.1.6) Given a circle, give a construction that finds its center.

B.1.7) State and prove a condition on any quadrilateral that is inscribed in a circle.

B.1.8) Construct a tangent line from a point outside a given circle to the circle.

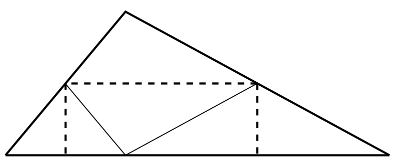
B.1.9) Give an informal derivation of the relationship between the circumference and area of a circle.

B.1.10) Prove: If a quadrilateral is a parallelogram, then opposite sides are congruent.

B.1.11) Prove: If opposite sides of a quadrilateral are congruent, then it is a parallelogram.

B.1.12) Draw an arbitrary convex quadrilateral. Form a second quadrilateral by connecting the midpoints of the sides of the first quadrilateral. Make a conjecture about the second quadrilateral and prove it.

B.1.13) The following picture shows a triangle that has been folded along the dotted lines:



Explain how the picture "proves" the following statements:

(a) The interior angles of a triangle sum to 180°.

(b) The area of a triangle is given by $bh/2$.

B.2 Midterm 2 Review (final version)

Midterm 2 will cover Chapter 4, Sections 5.1 and 5.2, Activities A.21–A.36, and the beginning of A.39. Here are some specific reminders:

- Know the distinction between synthetic and analytic geometry.
- Know the basic rigid motions, what is required to specify them, and their definitions.
- Know what it means to say that transformations of the plane are functions.
- Know how to define congruence in terms of basic rigid motions.
- Know how to define similarity in terms of dilations and basic rigid motions.
- Know and be able to use criteria for congruence and similarity of triangles.
- Know how to use transformations to describe symmetries of figures, including tessellations.
- Know the focus and directrix definition of parabola and be able to use it in both synthetic and analytic geometry.
- Know the definition of circle and be able to use it in both synthetic and analytic geometry.
- Be aware of assumptions underlying Euclidean geometry and how those assumptions can be different in other geometries (such as spherical geometry).
- Use similarity to find missing lengths by reasoning from the scale factor or from within-figure comparisons.
- Reason about length, area, and volume in similarity situations. How are rep-tiles related to this question?
- Use shearing and Cavalieri's principle to reason about area and volume.
- If you know the area of a rectangle, what can you say about its perimeter? What about more general figures?
- If you know the perimeter of a rectangle, what can you say about its area? What about more general figures?
- In coordinate geometry, what is a point?
- In coordinate geometry, what is a line?
- Know how to find an equation of the line containing two given points.
- Know how to derive the distance formula quickly.
- Know how to derive the midpoint formula quickly.

B.2.1 Midterm 2 Review Problems

B.2.1) During a solar eclipse we see that the apparent diameter of the Sun and Moon are nearly equal. If the Moon is around 240,000 miles from Earth, the Moon's diameter is about 2000 miles, and the Sun's diameter is about 865,000 miles how far is the Sun from the Earth?

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Write an expression that gives the solution to this problem—show all work.

B.2.2) A typical adult male gorilla is about 5.5 feet tall and weighs about 400 pounds. Suppose King Kong was about 22 feet tall and proportioned like a typical adult male gorilla.

(a) Write an expression that approximates King Kong's weight. Briefly explain your reasoning.

(b) The circumference of the neck of a typical adult male gorilla is 36 inches. Approximately what would be the circumference of King Kong's neck? Briefly explain your reasoning.

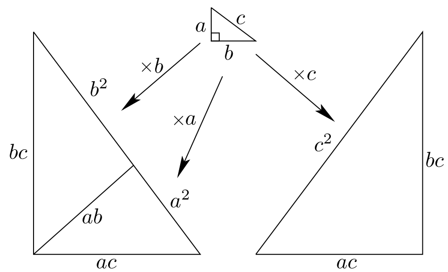
(c) Suppose an Ohio State sweatshirt for a typical adult male gorilla requires 3 square yards of fabric. Approximately how much fabric would be required for an Ohio State sweatshirt for King Kong? Briefly explain your reasoning.

B.2.3) Brenah is drinking fruit punch from a glass shaped like an inverted cone. Suppose the glass has a height 5 in. and a base of radius 2 in. What is the volume of the glass? What is the height of the fruit punch when the glass is half full? Generalize your result for any glass shaped like an inverted cone.

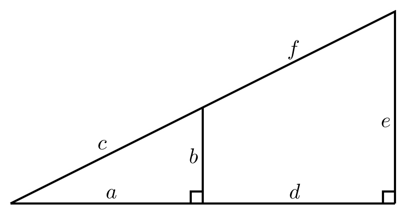
B.2.4) A cup has a circular opening, a circular base, and circular cross sections at every height parallel to the base. The opening has a diameter of 9 cm, the base has a diameter of 6 cm, and the cup is 12 cm high. What is the volume of the cup? Explain your reasoning. If the cup is filled to half its height, what fraction of the cup's volume is filled? Explain your reasoning.

B.2.5) Suppose you use a photocopier to enlarge a figure to 125% of its original size. What is the scale factor of the enlargement? What happens to areas under the enlargement? If you lost the original figure, what reduction percentage would you use on the enlargment to create a figure congruent to the original? What is the scale factor for the reduction?

B.2.6) Explain how the following picture "proves" the Pythagorean Theorem.



B.2.7) Here is a right triangle, note it is **not** drawn to scale:



Solve for all unknowns in the following cases.

(a) $a = 3$, $b = ?$, $c = ?$, $d = 12$, $e = 5$, $f = ?$

(b) $a = ?$, $b = 3$, $c = ?$, $d = 8$, $e = 13$, $f = ?$

(c) $a = 7$, $b = 4$, $c = ?$, $d = ?$, $e = 11$, $f = ?$

(d) $a = 5$, $b = 2$, $c = ?$, $d = 6$, $e = ?$, $f = ?$

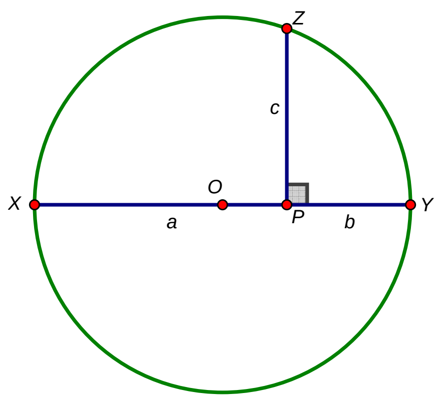
In each case explain your reasoning.

B.2.8) Describe a general (and foolproof) way of demonstrating that any two parabolas are similar.

B.2.9) Prove that the angle sum of a triangle is 180°.

B.2.10) Given a figure and a rotation of that figure, find the center and angle of rotation.

B.2.11) In the figure below O is the center of the circle, $\overline{XY}$ is a diameter, $a = PX$, $b = PY$, and $c = PZ$.



(a) Show that $c = \sqrt{ab}$.

(b) Use the figure to explain the Arithmetic-Geometric Mean Inequality: $\dfrac{a+b}{2} \geqslant \sqrt{ab}$.

B.2.12) If the perimeter of a rectangle is 20 feet, what is the most one can say about the rectangle's area? If the perimeter of any simple closed 2-dimensional shape is 20 feet, what is the most anyone can say about its area?

B.2.13) If the surface area of a rectangular prism is 20 square feet, what is the most one can say about the prism's volume? If the surface area of any simple closed 3-dimensional shape is 20 square feet, what is the most one can say about its volume?

B.2.14) Why do cute furry animals curl up to stay warm in the winter? Why are most ugly desert reptiles long and skinny?

B.2.15) Simple closed curve A is contained entirely inside simple closed curve B.

(a) True or False: The area enclosed by A is less than the area enclosed by B. Explain

(b) True or False: The perimeter of A is less than the perimeter of B. Explain.

B.2.16) Is it correct to say that “area is length times width”? Think about what these three quantities mean. When would it be correct in the numerical sense and why? (Make sure you use the meaning of multiplication.)

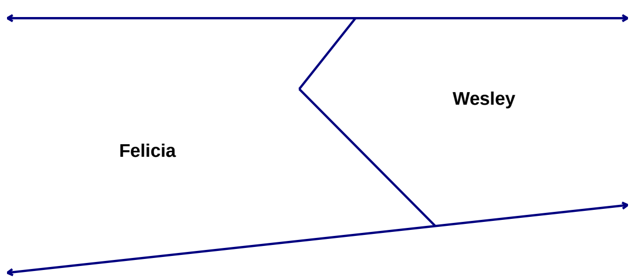
B.2.17) Is it correct to say that “volume is length times width times height”? What must be true about a figure so that the numerical volume can be more easily measured by “area times height”?

B.2.18) Are there figures for which there is no formula for measuring length, area, and volume? Explain. What does your answer to this question imply about the teaching of geometric measurement?

B.2.19) Convert 25 yards to meters (and 25 meters to yards) using “2.54 cm in each inch” as the only Metric-English unit conversion. Now convert 25 square yards to square meters and 25 square meters to square yards. Do the same with cubic yards and cubic meters.

B.2.20) In track and field, 1600 meters is often called the “one mile,” but this is not exactly correct. Is 1600 meters longer or shorter than one mile? By how much?

B.2.21) Felicia and Wesley are neighbors. The common boundary between their properties consists of two line segments, as shown below.



They would prefer their common boundary to be a single straight segment. How might they change their boundary so that they each have the same area as they have now?

B.2.22) Write an equation of the line through $(2, 4)$ parallel to $5x - 3y = 1$. Now write an equation of the line through (x_1, y_1) parallel to $ax + by = c$.

B.2.23) Write an equation of the line through $(2, 4)$ perpendicular to $5x - 3y = 1$. Now write an equation of the line through (x_1, y_1) perpendicular to $ax + by = c$.

B.2.24) Intersections of lines.

(a) Find the intersection of the lines $2x - 3y = 4$ and $3x - 5y = 3$.

(b) Find the intersection of the lines $2x - 3y = 4$ and $-4x + 6y = -8$.

(c) Find the intersection of the lines $2x - 3y = 4$ and $-4x + 6y = 5$.

(d) How might you have predicted in advance how many solutions to expect for each previous system of equations?

(e) Use algebra to help explain why lines intersect in zero, one, or infinitely many points. (You know this geometrically, of course. Here you demonstrate how algebra gives the same result.) Indicate clearly the algebraic conditions for which you get zero, one, or infinitely many points.

B.2.25) Suppose you have a rectangle with vertices at $(0, 0)$, $(a, 0)$, (a, b) and $(0, b)$. Use algebra to prove that the diagonals have the same length.

B.3 Final Exam Review (draft, 4/28/15)

Note: The final exam is cumulative, so be sure to include the **previous midterm review documents** as part of your review. Since the second midterm, we have covered Sections 5.3, 5.4, 6.1 and 6.2 from your notes as well as Activities A.38–A.48 and A.53 (a handout). Below is a summary of that content.

Coordinate Geometry

- Find an equation of a line through a given point parallel to a given line.
- Find an equation of a line through a given point perpendicular to a given line.
- Given two lines, two circles, or a line and a circle, find their intersection(s) if any. Describe what is happening geometrically when the algebra yields no solutions, exactly one solution, exactly two solution(s) or infinitely many solutions.
- Given an equation of a circle or a parabola, complete the square to find the center of the circle or the vertex of the parabola.
- What are constructible numbers? What are some numbers that are not constructible?

City Geometry

- Know the distance formula in city geometry, be able to use it, and explain its meaning.
- Given a center and a radius, graph a city-geometry circle and write its equation.
- Given two points in city geometry, graph their midset and write an equation of the midset.
- Given a focus and a directrix, graph the city-geometry parabola and write its equation.
- Explain the absolute value function in several ways.
- Graph and analyze absolute value equations (such as equations of city-geometry circles, midsets, or parabolas) by checking cases.

Functions

- What is a function? What do domain and range mean?

- Why is “Is this a function?” a poor question. What is a better question?
- Graph and analyze parametric equations describing a path in the plane
- In what sense are transformations of the plane functions? What are the input and output values? What are the domain and range of an isometry or dilation of the plane?
- Describe and analyze functions involving several related variables, such as length, width, area, and perimeter of rectangles. When fixing one of these quantities, what kinds of functions can you find among the other quantities?

B.3.1 Supplemental Review Problems

The problems below target content since the second midterm exam.

B.3.1) Consider a nonzero vector defined by the ordered pair (a, b). If $\|(a, b)\|$ is the magnitude of this vector, **use algebra** to explain why

$$\frac{(a, b)}{\|(a, b)\|}$$

is a new vector whose magnitude is 1 and whose direction is the same as (a, b).

B.3.2) Suppose you have a parametric plot defined by $x(t)$ and $y(t)$.

(a) Compare and contrast the plots of

$$\Big(x(t), y(t)\Big) \qquad \text{and} \qquad \Big(x(t-6), y(t-6)\Big).$$

(b) Suppose that there are two bugs whose positions are given by:

$$\text{bug}_1(t) = \Big(x(t), y(t)\Big) \qquad \text{and} \qquad \text{bug}_2 = \Big(x(t-6), y(t-6)\Big).$$

where t represents time in seconds. Describe what happens as t runs from 0 seconds to 36 seconds.

(c) Now suppose that there are two bugs whose positions are given by:

$$\text{bug}_1(t) = \Big(x(t), y(t)\Big) \qquad \text{and} \qquad \text{bug}_2 = \Big(x(t) - 6, y(t) - 6\Big).$$

where t represents time in seconds. Describe what happens as t runs from 0 seconds to 36 seconds.

B.3.3) Find the intersection of the lines

$$x_1(t) = -6 + 9t \qquad x_2(t) = 3 + t$$
$$y_1(t) = 3 - 2t \qquad y_2(t) = -4 - 2t$$

If $(x_1(t), y_1(t))$ gives the position of jogger$_1$ and $(x_2(t), y_2(t))$ gives the position of jogger$_2$, what is the significance of the point of intersection of these lines, from the perspective of the joggers?

B.3.4) A bug moves according to the following parametric equations, where t is measured in seconds and x and y are measured in centimeters: $x = 2t^2$, $y = t - 2$. (Suppose t can be any real number.)

(a) Describe the path of the bug.

(b) Is the bug's position a function of time?

(c) On the path, is y a function of x?

(d) Is x a function of y?

(e) If you know one of x, y, or t, can you determine the other two? How does this question relate to the previous two questions?

(f) In school mathematics, students are often given a graph and asked, "Is it a function." Explain why this is a poor question. What better questions could you ask?

B.3.5) Consider the following equations:

$$x^2 - y^2 = 0 \qquad x^2 = y^2 \qquad |y| = |x| \qquad y = \pm x$$
$$(x - y)(x + y) = 0 \qquad x = \pm y \qquad y = \pm|x| \qquad x = \pm|y|$$

(a) Which equations are equivalent to which other equations? Say how you know. (Be sure to state what it means for the equations to be equivalent.)

(b) For each set of equivalent equations, graph the solution set, and describe how each of the equations provides a different way about thinking about that solution set.

B.3.6) Distance formulas and circle equations across dimensions.

(a) What is the (Euclidean) distance formula in 2 dimensions, on the xy-plane?

(b) What is the distance formula in 3 dimensions?

(c) What is the distance formula in 1 dimension?

(d) Write an equation of the circle of radius r and center (a, b).

(e) Explain how a circle is a one-dimensional figure living in a two-dimensional "space."

(f) In three-dimensional space, write an equation of the two-dimensional "circle" of radius r and center (a, b, c).

(g) In one-dimensional space, write an equation of the zero-dimensional "circle" of radius r and center a.

B.3.7) Recall the method in Euclidean geometry of constructing an equilateral triangle on a given segment. Suppose a "city geometry compass" draws a city geometry circle. Imagine using such a "city geometry compass" below.

(a) Construct a "city geometry equilateral triangle" on the segment defined by the points $(0, 0)$ and $(4, 0)$. Explain your steps.

(b) Now construct a "city geometry equilateral triangle" on the segment defined by the points $(0, 0)$ and $(2, 2)$. Explain your steps.

(c) Will the construction always give a (unique!) equilateral triangle? What does "unique" mean in this context? Give a detailed discussion.

B.3.8) A fundamental feature of the basic rigid motions in Euclidean geometry is that they preserve distance and angle. In city geometry, some basic rigid motions preserve both distance and angle and others fail for various reasons. Explain. (Hint: Some but not all rotations preserve both distance and angle.)

B.3.9) Clearly identify which of the following numbers are constructible and which numbers are not constructible.

$$4 \qquad \sqrt[3]{2} \qquad 3.1415926 \qquad \sqrt[3]{125} \qquad \sqrt[6]{7} \qquad \frac{6}{1+\sqrt{5}}$$

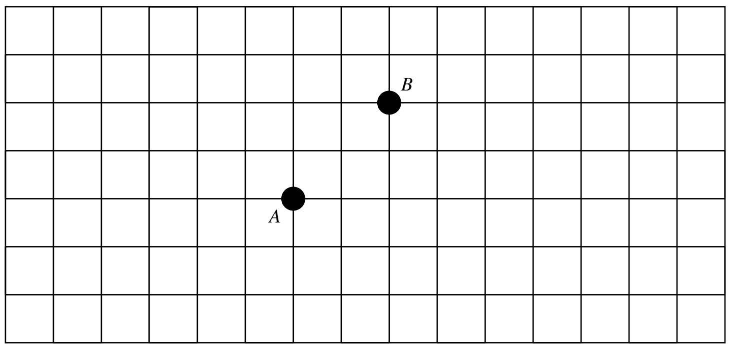
B.3.10) Consider $x = (y-2)^2 + 3$.

(a) Plot this curve.

(b) Explain how this might not be the plot of a function.

(c) Explain how this could be the plot of a function.

B.3.11) Below are two points in City Geometry.



(a) Sketch the City Geometry midset of these two points.

(b) Suppose that $A = (0,0)$ and $B = (2,2)$. Using taxicab distance in City Geometry, write an equation that must be satisfied by any point (x, y) that is in the midset of A and B.

(c) Explain the connection between the geometry in part (a) and the algebra in part (b) for the case $x > 2$ and $y < 0$.

B.3.12) Use coordinate constructions to derive the formula for the parabola whose focus is the point $(-3,-4)$ and whose directrix is the line $y = 2$. Show your work. Note, you should use the Euclidean distance formula, and your final answer should start with "$y =$".

B.3.13) Comparing Celsius and Fahrenheit.

(a) Water freezes at 0° C, which is 32° F. Water boils at 100° C, which is 212° F. Use this information to derive a Celsius to Fahrenheit conversion formula.

(b) Suppose the temperature of a model house increases from 24° C to 30° X, which seems to be a 25% increase. Convert these temperatures to Fahrenheit and compute the percent change in degrees Fahrenheit.

(c) Use your conversion formula to explain why the percent changes are not the same in Fahrenheit as in Celsius.

B.3.14) Surface area of a cylinder.

(a) Derive and explain a formula for the surface area of a right cylinder of radius of radius r and height h.

(b) Assuming the radius is fixed, how does the surface area vary with the height? In other words, what kind of function is it? Explain briefly.

(c) Assuming the height is fixed, how does the surface area vary with the radius? Explain briefly.

B.3.15) Volume of a cylinder.

(a) Derive and explain a formula for the volume of a right cylinder of radius of radius r and height h.

(b) Assuming the radius is fixed, how does the volume vary with the height? In other words, what kind of function is it? Explain briefly.

(c) Assuming the height is fixed, how does the volume vary with the radius? Explain briefly.

B.3.16) Standard televisions usually have an aspect ratio (width:length) of 4:3. Wide-screen televisions have an aspect ratio of 16:9. When Brad's first wide-screen television was a 36 inch (diagonal) model. Although the new television was clearly wider than the 27 inch (diagonal) standard television it replaced, he was surprised that it did not seem taller than the old television. Which television was actually taller? And by how much?