

Operations and Algorithms

Problems about operations and algorithms.

Problem 1 Explain what it means for an operation \star to be associative. Give some relevant and revealing examples and non-examples.

Free Response: **Hint:** An operation \star is associative if $(a \star b) \star c = a \star (b \star c)$ for all values of a , b , and c . Addition of numbers is associative, as is multiplication. Subtraction and division are not.

Problem 2 Consider the following pictures:



Jesse claims that these pictures represent $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$.

(a) Is Jesse's claim correct?

Multiple Choice:

- (i) Yes. ✓
- (ii) No.
- (iii) Not enough information.

(b) Explain your reasoning.

Free Response: **Hint:** Jesse is correct. The picture on the left is 4 copies of $2 \cdot 3$, and the picture on the right is 2 copies of $3 \cdot 4$.

(c) Do Jesse's pictures show the associativity of multiplication?

Multiple Choice:

- (i) Yes.

- (ii) No. ✓
- (iii) Not enough information.
- (d) If so, explain why. If not, draw new pictures representing $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$ that do show the associativity of multiplication.

Free Response: **Hint:** We can compute that in both cases the total area is 24, but that does not explain *why* they are the same. For that, imagine the volume of a box measuring 2 by 3 by 4. Slicing the layers different ways can explain associativity.

Problem 3 Explain what it means for an operation \star to be commutative. Give some relevant and revealing examples and non-examples.

Free Response: **Hint:** An operation \star is commutative if $a \star b = b \star a$ for all values of a and b . Addition of numbers is commutative, as is multiplication. Subtraction and division are not.

Problem 4 Explain what it means for an operation \star to distribute over another operation \dagger . Give some relevant and revealing examples and non-examples.

Free Response: **Hint:** An operation \star distributes over an operation \dagger if $a \star (b \dagger c) = (a \star b) \dagger (a \star c)$ and $(b \dagger c) \star a = (b \star a) \dagger (c \star a)$. Multiplication distributes over addition because $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$. But exponentiation does not distribute over addition because $(b + c) \wedge a \neq (b \wedge a) + (c \wedge a)$.

Problem 5 Explain what it means for an operation \star to be closed on a set of numbers.

Free Response: **Hint:** A set is closed under an operation \star if for all a and b in the set, $a \star b$ and $b \star a$ are also in the set.

Problem 5.1 Give some relevant and revealing examples and non-examples.

Free Response: **Hint:** The counting numbers are closed under addition and also under multiplication but not under subtraction or division. The set of even numbers (including 0 and negatives) are closed under addition, subtraction, and multiplication. The odd numbers, in contrast, are closed under multiplication but under neither addition nor subtraction.

Problem 6 Sometimes multiplication is described as *repeated addition*. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.

Free Response: **Hint:** Repeated addition by itself does not explain *why* multiplication is commutative. Instead, use an array model, interpreting $a \cdot b$ as, say, a rows by b columns. Rotate the array 90° , and it is clear that b rows by a columns, or $b \cdot a$, must be the same number of objects. Essentially the same reasoning works with an area model.

Problem 7 In beginning algebra, simplifying expressions often involves *collecting like terms*. But why does this work? Well, the expression $3x + 4x$ is equivalent to $(3 + 4)x$ by the (commutative/ associative/ distributive \checkmark) property. And then it is clear that $(3 + 4)x = \boxed{7x}$.

Problem 8 In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?

Solution: Build a solution in four steps:

- Use a specific starting price.
- Generalize the process of computing a price after a discount (assuming no tax).
- Generalize the process of computing a price with tax (assuming no discount).
- Generalize the two together.

To get started, try a specific starting price, say \$200. After the discount, the price would be \$ $\boxed{160}$. After the tax, the cost is \$ $\boxed{184}$.

Now try the other order. The original price with tax would be \$ $\boxed{230}$. Then with the discount, the cost would be \$ $\boxed{184}$. This specific example suggests that the order doesn't matter. But did we just get lucky?

Problem 8.1 To compute a price after a 20% discount, suppose the price is $\$p$. The discount will be $\$0.2p$. So the price after the discount would be $p - 0.2p$. And by the distributive property, this is equal to $0.8p$. In other words, rather than computing the discount and subtracting, we can directly compute the new price by multiplying the original price by 0.8 . This makes sense because with a discount of 20 percent, the price we pay will be 80 percent of the original price.

Problem 8.1.1 To compute a price with 15% tax, suppose the price is $\$p$. The tax will be $\$0.15p$. So the price with tax would be $p + 0.15p$. And by the distributive property, this is equal to $1.15p$. In other words, rather than computing the tax and adding, we can directly compute the new price by multiplying the original price by 1.15 . This makes sense because with a tax of 15 percent, the price we pay will be 115 percent of the original price.

Problem 8.1.1.1 Again, let the starting price be $\$p$.

If we apply the discount and then the tax, we multiply p first by 0.8 and then by 1.15 , resulting in the expression $1.15 \cdot 0.8p$.

If, on the other hand, we apply the tax and then the discount, we multiply p first by 1.15 and then by 0.8 , resulting in the expression $0.8 \cdot 1.15p$.

These expressions are equal because of the (commutative/ associative ✓/ distributive) property of (addition/ subtraction/ multiplication ✓/ division). Thus, the final cost is the same either way.

Problem 9 Money Bags Jon likes to give a tip of 20% when he is at restaurants. He does this by dividing his bill by 10 and then doubling it. Explain why this works.

Free Response: **Hint:** Because 10% is the same as $1/10$ of the bill, and 20% is twice that.

Problem 10 Regular Reggie likes to give a tip of 15% when he is at restaurants. He does this by dividing his bill by 10 and then adding half more to this number. Explain why this works.

Free Response: **Hint:** Because 10% is the same as $1/10$ of the bill, 5% is half that, and 15% is the same as $10\% + 5\%$.

Problem 11 Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.

Free Response: **Hint:** If the bill is already a multiple of 7, then $1/7$ of the bill is about 14.3%, which is slightly less than a standard tip of 15%. By rounding up to the nearest multiple of 7, the tip will always be at least 14.3% and usually slightly more.
