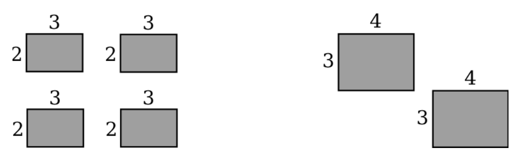
Operations and Algorithms

Problems about operations and algorithms.

Problem 1 *Explain what it means for an operation $\star$ to be associative. Give some relevant and revealing examples and non-examples.*

Free Response: ***Hint:*** *An operation $\star$ is associative if $(a \star b) \star c = a \star (b \star c)$ for all values of a, b, and c. Addition of numbers is associative, as is multiplication. Subtraction and division are not.*

Problem 2 *Consider the following pictures:*



Jesse claims that these pictures represent $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$.

(a) *Is Jesse's claim correct?*

Multiple Choice:

(i) *Yes.* ✓
(ii) *No.*
(iii) *Not enough information.*

(b) *Explain your reasoning.*

Free Response: ***Hint:*** *Jesse is correct. The picture on the left is 4 copies of $2 \cdot 3$, and the picture on the right is 2 copies of $3\dot{4}$.*

(c) *Do Jesse's pictures show the associativity of multiplication?*

Multiple Choice:

(i) *Yes.*

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(ii) *No.* ✓

(iii) *Not enough information.*

(d) *If so, explain why. If not, draw new pictures representing* $(2 \cdot 3) \cdot 4$ *and* $2 \cdot (3 \cdot 4)$ *that do show the associativity of multiplication.*

Free Response: ***Hint:*** *We can compute that in both cases the total area is 24, but that does not explain why they are the same. For that, imagine the volume of a box measuring 2 by 3 by 4. Slicing the layers different ways can explain associativity.*

Problem 3 *Explain what it means for an operation* $\star$ *to be commutative. Give some relevant and revealing examples and non-examples.*

Free Response: ***Hint:*** *An operation* $\star$ *is commutative if* $a \star b = b \star a$ *for all values of* a *and* b*. Addition of numbers is commutative, as is multiplication. Subtraction and division are not.*

Problem 4 *Explain what it means for an operation* $\star$ *to distribute over another operation* $\dagger$*. Give some relevant and revealing examples and non-examples.*

Free Response: ***Hint:*** *An operation* $\star$ *distributes over an operation* $\dagger$ *if* $a \star (b \dagger c) = (a \star b) \dagger (a \star c)$ *and* $(b \dagger c) \star a = (b \star a) \dagger (c \star a)$*. Multiplication distributes over addition because* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ *and* $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$*. But exponentiation does not distribute over addition because* $(b + c) \wedge a \neq (b \wedge a) + (c \wedge a)$*.*

Problem 5 *Explain what it means for an operation* $\star$ *to be closed on a set of numbers.*

Free Response: ***Hint:*** *A set is closed under an operation* $\star$ *if for all* a *and* b *in the set,* $a \star b$ *and* $b \star a$ *are also in the set.*

Problem 5.1 *Give some relevant and revealing examples and non-examples. The counting numbers are closed under addition and also under multiplication but not under subtraction or division. The set of even numbers (including* 0 *and negatives) are closed under addition, subtraction, and multiplication. The odd numbers, in contrast, are closed under multiplication but under neither addition nor subtraction.*

Problem 6 *Sometimes multiplication is described as repeated addition. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.*

Free Response: ***Hint:*** *Repeated addition by itself does not explain why multiplication is commutative. Instead, use an array model, interpreting $a \cdot b$ as, say, a rows by b columns. Rotate the array 90°, and it is clear that b rows by a columns, or $b \cdot a$, must be the same number of objects. Essentially the same reasoning works with an area model.*

Problem 7 *In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?*

Solution: *We build up a solution in four steps:*

(a) *Use a specific starting price.*

(b) *Generalize the process of computing a price after a discount (assuming no tax).*

(c) *Generalize the process of computing a price with tax (assuming no discount).*

(d) *Generalize the two together.*

To get started, try a specific starting price, say \$200. After the discount, the price would be \$$\boxed{160}$. After the tax, the cost is \$$\boxed{184}$.

Now try the other order. The original price with tax would be \$$\boxed{230}$. Then with the discount, the cost would be \$$\boxed{184}$. This specific example suggests that the order doesn't matter. But did we just get lucky?

Problem 7.1 *To compute a price after a 20%, discount, suppose the price is \$$p$. The discount will be \$$\boxed{0.2p}$. So the price after the discount would $\boxed{p} - \boxed{0.2p}$. And by the distributive property, this is equal to $\boxed{0.8p}$. In other words, rather than computing the discount and subtracting, we can directly compute the new price by multiplying the original price by $\boxed{0.8}$. This makes sense because with a discount of 20 percent, the price we pay will be $\boxed{80}$ percent of the original price.*

Problem 7.1.1 *To compute a price with* 15% *tax, suppose the price is* $\$p$. *The tax will be* $\$\boxed{0.15p}$. *So the price with tax would* $\boxed{p} + \boxed{0.15p}$. *And by the distributive property, this is equal to* $\boxed{1.15p}$. *In other words, rather than computing the tax and adding, we can directly compute the new price by multiplying the original price by* $\boxed{1.15}$. *This makes sense because with a tax of* 15 *percent, the price we pay will be* $\boxed{115}$ *percent of the original price.*

Problem 7.1.1.1 *Again, let the starting price be* $\$p$.

If we apply the discount and then the tax, we multiply p *first by* $\boxed{0.8}$ *and then by* $\boxed{1.15}$, *resulting in the expression* $\boxed{1.15 \cdot 0.8p}$.

If, on the other hand, we apply the tax and then the discount, we multiply p *first by* $\boxed{1.15}$ *and then by* $\boxed{0.8}$, *resulting in the expression* $\boxed{0.8 \cdot 1.15p}$.

These expressions are equal because of the (commutative/ associative ✓/ distributive) property of (addition/ subtraction/ multiplication ✓/ division). Thus, the final cost is the same either way.

Problem 8 *Money Bags Jon likes to give a tip of* 20% *when he is at restaurants. He does this by dividing his bill by* 10 *and then doubling it. Explain why this works.*

Free Response: ***Hint:*** *Because* 10% *is the same as* $1/10$ *of the bill, and* 20% *is twice that.*

Problem 9 *Regular Reggie likes to give a tip of* 15% *when he is at restaurants. He does this by dividing his bill by* 10 *and then adding half more to this number. Explain why this works.*

Free Response: ***Hint:*** *Because* 10% *is the same as* $1/10$ *of the bill,* 5% *is half that, and* 15% *is the same as* $10\% + 5\%$.

Problem 10 *Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.*

Free Response: ***Hint:*** *If the bill is already a multiple of 7, then 1/7 of the bill is about 14.3%, which is slightly less than a standard tip of 15%. By rounding up to the nearest multiple of 7, the tip will alway be at least 14.3% and usually slightly more.*