Polynomials

Problems about polynomials.

Problem 1 *Explain what is meant by a polynomial in a variable x.*

Free Response: ***Hint:*** *Informally: A polynomial in x is an algebraic expression that can be written as a sum of terms, each of which is a whole-number power of x multiplied by some real number.*

Formally: For a whole number n, a polynomial (in x) of degree n *is an expression that can be written in the form*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are real numbers and $a_n \neq 0$. A polynomial function of degree n *is a function that can be defined by a polynomial.*

Problem 2 *Indicate the degree of the following polynomials. For expressions that are not polynomials, type NP.*

$3x - 3$	$\boxed{1}$
$\sqrt{x}$	$\boxed{NP}$
x^{15}	$\boxed{15}$
$1 - x - x^2$	$\boxed{2}$
$2^x + x^2$	$\boxed{NP}$
$(1+x)(2+x)x$	$\boxed{3}$
56	$\boxed{0}$
$2/x$	$\boxed{NP}$

Problem 3 *Given:*

$$3x^7 - x^5 + x^4 - 16x^3 + 27 = a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0$$

Find a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7.

Answer: $a_0 = \boxed{27}$, $a_1 = \boxed{0}$, $a_2 = \boxed{0}$, $a_3 = \boxed{-16}$, $a_4 = \boxed{1}$, $a_5 = \boxed{-1}$, $a_6 = \boxed{0}$, $a_7 = \boxed{3}$.

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Problem 4 *Given:*

$$6x^5 + a_4x^4 - x^2 + a_0 = a_5x^5 - 24x^4 + a_3x^3 + a_2x^2 - 5$$

Find a_0, a_1, a_2, a_3, a_4, a_5.

Answers: $a_0 = \boxed{-5}$, $a_1 = \boxed{0}$, $a_2 = \boxed{-1}$, $a_3 = \boxed{0}$, $a_4 = \boxed{-24}$, $a_5 = \boxed{6}$.

Problem 5 *Is it true that polynomials are equal if and only if their coefficients are equal? Explain your reasoning.*

Free Response: ***Hint:*** *Yes. This is the fact we used to complete the previous two problems.*

Problem 6 *Is it true that numbers are equal if and only if their digits are equal? Explain your reasoning.*

Free Response: ***Hint:*** *For whole numbers written in the same base, the answer is (yes ✓/ no).*

For decimals, the answer is (yes/ no ✓). [Note: We will learn why and how later in the course.]

For fractions, of course, the answer is (yes/ no ✓). For example, $\frac{2}{3} = \frac{4}{6}$*, and the "digits" of the fraction on the left are clearly different from those of the fraction on the right.*

Problem 7 *Explain how to add two polynomials. Explain, in particular, how "collecting like terms" is an application of the properties of arithmetic.*

Free Response: ***Hint:*** *Use the associative and commutative properties to rearrange the sum so that like terms are consecutive. Then use the distributive property to "collect" the like terms. For example,* $3x^2 + 4x^2 = (3 + 4)x^2 = 7x^2$.

Problem 8 *Explain how to multiply two polynomials.*

Free Response: ***Hint:*** *Use the distributive property: Multiply each term in the first polynomial by each term in the second polynomial. Then collect like terms.*

Problem 9 *Here is an example of the polynomial division algorithm:*

$$
\begin{array}{rl}
 & \phantom{x^3 + 0x^2 + {}} x - 3 \quad R\; 9x+4 \\
x^2+3x+1 \big) & \overline{x^3 + 0x^2 + x + 1} \\
 & \underline{x^3 + 3x^2 + x} \\
 & -3x^2 + 0x + 1 \\
 & \underline{-3x^2 - 9x - 3} \\
 & \phantom{x^3 + 3x^2 + {}} 9x + 4
\end{array}
$$

Describe how to perform this algorithm:

Free Response: ***Hint:*** *This is very much like the long division algorithm for counting numbers:*

(a) *Write part of the quotient;*

(b) *Multiply that part of the quotient by the divisor;*

(c) *Subtract that product from the dividend;*

(d) *Repeat until the remaining part of the dividend is has a degree less than the degree of the divisor.*

Note that, in the last step, what counts as "less than" is different for dividing polynomials than for dividing counting numbers.

Problem 10 *Find the quotient and divisor when dividing* $x^3 + 4x^2 - 1$ *by* $x + 2$. *Quotient:* $\boxed{x^2 + 2x - 4}$; *remainder:* $\boxed{7}$.

Problem 11 *Find the quotient and divisor when dividing* $x^3 + 4x^2 - 1$ *by* $x^2 + 1$. *Quotient:* $\boxed{x + 4}$; *remainder:* $\boxed{-x - 5}$.

Problem 12 *State the Division Theorem for polynomials. Give some relevant and revealing examples of this theorem in action.*

Free Response: ***Hint:*** *Informally, when dividing a polynomial by a (non-zero) polynomial, we can always find a sensible quotient and remainder (both polynomials). More formally, given* $n(x)$ *and a non-zero divisor* $d(x)$, *we can find* $q(x)$ *and* $r(x)$ *such that* $n(x) = \boxed{d(x)q(x) + r(x)}$ *with the degree of* $r(x)$ *(greater than/ equal to/ less than ✓) the degree of* $d(x)$.

Problem 13 *Write 35_{ten} in base two.*

$$35_{ten} = \underbrace{\boxed{100011}}_{\text{given}}{}_{two}$$

Problem 13.1 *Find a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ such that the a_i's are integers greater than or equal to 0 and less than 2 such that $p(2) = 35$.*

$$p(x) = \underbrace{\boxed{1}}_{\text{given}} x^5 + \underbrace{\boxed{0}}_{\text{given}} x^4 + \underbrace{\boxed{0}}_{\text{given}} x^3 + \underbrace{\boxed{0}}_{\text{given}} x^2 + \underbrace{\boxed{1}}_{\text{given}} x + \underbrace{\boxed{1}}_{\text{given}}$$

Problem 13.1.1 *How are the previous two problems related?*

Free Response: ***Hint:*** *First, we think of our polynomial $p(x)$ as an object in base x. Since the coefficients are 0 or 1 only, when we plug in $x = 2$, we won't have to do any rearranging between the places. So, once we plug in $x = 2$, the coefficients form an element in base $x = 2$. Plugging in $x = 2$ is the process we would go through to convert from base two to base ten. So, the value of the polynomial at $x = 2$ will be the base ten value of 100011_{two}, or 35_{ten}.*

Problem 14 *Consider $x^2 + 2x + 3$. This can be thought of as a "number" in base x. Express this number in base $x + 1$, that is, find b_0, b_1, b_2 such that*

$$b_2(x+1)^2 + b_1(x+1) + b_0 = x^2 + x + 1.$$

$$x^2 + x + 1 = \underbrace{\boxed{1}}_{\text{given}} (x+1)^2 + \underbrace{\boxed{-1}}_{\text{given}}(x+1) + \underbrace{\boxed{1}}_{\text{given}}.$$