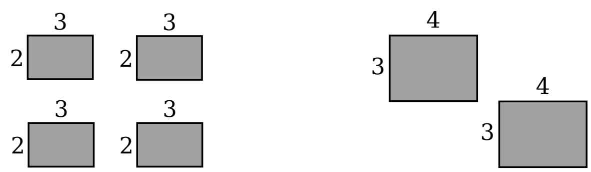
Algorithms

Problems about algorithms.

Problem 1 *Explain what it means for an operation $\star$ to be associative. Give some relevant and revealing examples and non-examples.*

Problem 2 *Consider the following pictures:*



Jesse claims that these pictures represent $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$.

(a) *Is Jesse's claim correct? Explain your reasoning.*

(b) *Do Jesse's pictures show the associativity of multiplication? If so, explain why. If not, draw new pictures representing $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$ that do show the associativity of multiplication.*

Problem 3 *Explain what it means for an operation $\star$ to be commutative. Give some relevant and revealing examples and non-examples.*

Problem 4 *Explain what it means for an operation $\star$ to distribute over another operation $\dagger$. Give some relevant and revealing examples and non-examples.*

Problem 5 *Explain what it means for an operation $\star$ to be closed on a set of numbers. Give some relevant and revealing examples and non-examples.*

Learning outcomes:
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Problem 6 *Sometimes multiplication is described as repeated addition. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.*

Problem 7 *In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?*

Problem 8 *Money Bags Jon likes to give a tip of 20% when he is at restaurants. He does this by dividing his bill by 10 and then doubling it. Explain why this works.*

Problem 9 *Regular Reggie likes to give a tip of 15% when he is at restaurants. He does this by dividing his bill by 10 and then adding half more to this number. Explain why this works.*

Problem 10 *Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.*

Problem 11 *Cheap Carl likes to give a tip of* $13\frac{1}{3}\%$ *when he is at restaurants. He does this by dividing his bill by 10 and then adding one-third more to this number. Explain why this works.*

Problem 12 *Reasonable Rebbecca likes to give a tip of 18% when she is at restaurants. She does this by dividing her bill by 5 and then removing one-tenth of this number. Explain why this works.*

Problem 13 *Can you think of and justify any other schemes for computing the tip?*

Problem 14 *Here is an example of a standard addition algorithm:*

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 15 *Here is an example of the column addition algorithm:*

$$\begin{array}{r} 892 \\ +398 \\ \hline 10 \\ 18 \\ 11 \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 16 *If you check out Problems and , you will learn about "partial" algorithms.*

(a) *Develop a "partial" algorithm for addition, give it a name, and describe how to perform this algorithm.*

(b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 17 *Here is an example of the banker's addition algorithm:*

$$\begin{array}{r} 892 \\ +398 \\ \hline 1\boldsymbol{0} \\ 1\boldsymbol{9} \\ \boldsymbol{12} \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 18 *Here is an example of a standard subtraction algorithm:*

$$\begin{array}{rccc} & & 8 & \\ & 8 & \cancel{9} & {}^{1}2 \\ - & 3 & 7 & 8 \\ \hline & 5 & 1 & 4 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 19 *Here is an example of the subtraction by addition algorithm:*

$$\begin{array}{r} 892 \\ -378 \\ \hline 514 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} 8 + \boldsymbol{4} = 12 & \textit{add 1 to 7 to get 8} \\ 8 + \boldsymbol{1} = 9 & \\ 3 + \boldsymbol{5} = 8 & \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 20 *Here is an example of the Austrian subtraction algorithm:*

$$\begin{array}{r} 8\;\;9\;{}^{1}2 \\ -3\;{}^{8}\cancel{7}\;\;8 \\ \hline 5\;\;1\;\;4 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 21 *If you check out Problems and , you will learn about "partial" algorithms.*

(a) *Develop a "partial" algorithm for subtraction, give it a name, and describe how to perform this algorithm.*

(b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 22 *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} 23 \\ 634 \\ \times\quad 8 \\ \hline 5072 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 23 *Here is an example of the partial-products algorithm:*

$$\begin{array}{r} 634 \\ \times \quad 8 \\ \hline 4800 \\ 240 \\ 32 \\ \hline 5072 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 24 *Here is an example of a standard division algorithm:*

$$\begin{array}{r@{}l} & 97\ R1 \\ \cline{2-2} 8 \big) & 777 \\ & 72 \\ \cline{2-2} & 57 \\ & 56 \\ \cline{2-2} & 1 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 25 *Here is an example of the partial quotients algorithm:*

$$\begin{array}{r@{}r} & 7 \\ & 90 \\ \cline{2-2} 8 \big) & 777 \\ & 720 \\ \cline{2-2} & 57 \\ & 56 \\ \cline{2-2} & 1 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 26 *Here is another example of the partial-quotients division algorithm:*

$$\begin{array}{r} 4 \\ 10 \\ 10 \\ 10 \\ 8\,\overline{)\,277} \\ \underline{80} \\ 197 \\ \underline{80} \\ 117 \\ \underline{80} \\ 37 \\ \underline{32} \\ 5 \end{array}$$

(a) *Describe how to perform this algorithm—be sure to explain how this is different from the scaffolding division algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 27 *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} 634 \\ \underline{\times 216} \\ 3804 \\ 6340 \\ \underline{126800} \\ 136944 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here—you may assume that you already know the algebra behind the standard multiplication algorithm.*

Problem 28 *Here is an example of the addition algorithm with decimals:*

$$\begin{array}{r} {}^{1} \\ 37.2 \\ +8.74 \\ \hline 45.94 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 29 *Here is an example of the multiplication algorithm with decimals:*

$$\begin{array}{r} 3.40 \\ \times\ \ .21 \\ \hline 340 \\ 6800 \\ \hline .7140 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 30 *Here is an example of the division algorithm without remainder:*

$$\begin{array}{r} 0.75 \\ 4\overline{)3.00} \\ \underline{2\ 8} \\ 20 \\ \underline{20} \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 31 *In the following addition problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{MOON} \\ +\ \texttt{SUN} \\ \hline \texttt{PLUTO} \end{array}$$

Recover the original problem and solution. Explain your reasoning. Hint: $\texttt{S} = 6$ *and* $\texttt{U} = 5$.

Problem 32 *In the following addition problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{SEND} \\ +\texttt{MORE} \\ \hline \texttt{MONEY} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 33 *In the following subtraction problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{DEFER} \\ -\texttt{DU7Y} \\ \hline \texttt{N2G2} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 34 *In the following two subtraction problems, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{NINE} \\ -\texttt{TEN} \\ \hline \texttt{TWO} \end{array} \qquad \begin{array}{r} \texttt{NINE} \\ -\texttt{ONE} \\ \hline \texttt{ALL} \end{array}$$

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

Problem 35 *In the following multiplication problem, every digit has been replaced with a letter.*

```
   LET
×  NO
------
   SOT
  NOT
------
  FRET
```

Recover the original problem and solution. Explain your reasoning.

Problem 36 *The following is a long division problem where every digit except 7 was replaced by X.*

```
       X7X
    ------
XX ) XXXXX
     X77
     ---
      X7X
      X7X
      ---
        XX
        XX
        ==
```

Recover the digits from this long division problem. Explain your reasoning.

Problem 37 *The following is a long division problem where the various digits were replaced by X except for a single 8. The double bar indicates that the remainder is 0.*

```
          XX8XX
      ---------
XXX ) XXXXXXXX
      XXX
      ---
       XXXX
        XXX
        ---
         XXXX
         XXXX
         ====
```

Recover the digits from this long division problem. Explain your reasoning.
