

# Operations and Algorithms

*Problems about operations and algorithms.*

**Problem 1** Explain what it means for an operation  $\star$  to be associative. Give some relevant and revealing examples and non-examples.

**Free Response:** **Hint:** An operation  $\star$  is associative if  $(a \star b) \star c = a \star (b \star c)$  for all values of  $a$ ,  $b$ , and  $c$ . Addition of numbers is associative, as is multiplication. Subtraction and division are not.

**Problem 2** Consider the following pictures:



Jesse claims that these pictures represent  $(2 \cdot 3) \cdot 4$  and  $2 \cdot (3 \cdot 4)$ .

(a) Is Jesse's claim correct?

**Multiple Choice:**

- (i) Yes. ✓
- (ii) No.
- (iii) Not enough information.

(b) Explain your reasoning.

**Free Response:** **Hint:** Jesse is correct. The picture on the left is 4 copies of  $2 \cdot 3$ , and the picture on the right is 2 copies of  $3 \cdot 4$ .

(c) Do Jesse's pictures show the associativity of multiplication?

**Multiple Choice:**

- (i) Yes.

- (ii) No. ✓
- (iii) Not enough information.
- (d) If so, explain why. If not, draw new pictures representing  $(2 \cdot 3) \cdot 4$  and  $2 \cdot (3 \cdot 4)$  that do show the associativity of multiplication.

**Free Response:** **Hint:** We can compute that in both cases the total area is 24, but that does not explain *why* they are the same. For that, imagine the volume of a box measuring 2 by 3 by 4. Slicing the layers different ways can explain associativity.

**Problem 3** Explain what it means for an operation  $\star$  to be commutative. Give some relevant and revealing examples and non-examples.

**Free Response:** **Hint:** An operation  $\star$  is commutative if  $a \star b = b \star a$  for all values of  $a$  and  $b$ . Addition of numbers is commutative, as is multiplication. Subtraction and division are not.

**Problem 4** Explain what it means for an operation  $\star$  to distribute over another operation  $\dagger$ . Give some relevant and revealing examples and non-examples.

**Free Response:** **Hint:** An operation  $\star$  distributes over an operation  $\dagger$  if  $a \star (b \dagger c) = (a \star b) \dagger (a \star c)$  and  $(b \dagger c) \star a = (b \star a) \dagger (c \star a)$ . Multiplication distributes over addition because  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  and  $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ . But exponentiation does not distribute over addition because  $(b + c) \wedge a \neq (b \wedge a) + (c \wedge a)$ .

**Problem 5** Explain what it means for an operation  $\star$  to be closed on a set of numbers.

**Free Response:** **Hint:** A set is closed under an operation  $\star$  if for all  $a$  and  $b$  in the set,  $a \star b$  and  $b \star a$  are also in the set.

**Problem 5.1** Give some relevant and revealing examples and non-examples.

**Free Response:** **Hint:** The counting numbers are closed under addition and also under multiplication but not under subtraction or division. The set of even numbers (including 0 and negatives) are closed under addition, subtraction, and multiplication. The odd numbers, in contrast, are closed under multiplication but under neither addition nor subtraction.

**Problem 6** Sometimes multiplication is described as *repeated addition*. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.

**Free Response:** **Hint:** Repeated addition by itself does not explain *why* multiplication is commutative. Instead, use an array model, interpreting  $a \cdot b$  as, say,  $a$  rows by  $b$  columns. Rotate the array  $90^\circ$ , and it is clear that  $b$  rows by  $a$  columns, or  $b \cdot a$ , must be the same number of objects. Essentially the same reasoning works with an area model.

**Problem 7** In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?

**Solution:** Build a solution in four steps:

- Use a specific starting price.
- Generalize the process of computing a price after a discount (assuming no tax).
- Generalize the process of computing a price with tax (assuming no discount).
- Generalize the two together.

To get started, try a specific starting price, say \$200. After the discount, the price would be \$160. After the tax, the cost is \$184.

Now try the other order. The original price with tax would be \$230. Then with the discount, the cost would be \$184. This specific example suggests that the order doesn't matter. But did we just get lucky?

**Problem 7.1** To compute a price after a 20% discount, suppose the price is  $p$ . The discount will be  $0.2p$ . So the price after the discount would be  $p - 0.2p$ . And by the distributive property, this is equal to  $0.8p$ . In other words, rather than computing the discount and subtracting, we can directly compute the new price by multiplying the original price by 0.8. This makes sense because with a discount of 20 percent, the price we pay will be 80 percent of the original price.

**Problem 7.1.1** To compute a price with 15% tax, suppose the price is  $\$p$ . The tax will be  $\$0.15p$ . So the price with tax would  $p + 0.15p$ . And by the distributive property, this is equal to  $1.15p$ . In other words, rather than computing the tax and adding, we can directly compute the new price by multiplying the original price by  $1.15$ . This makes sense because with a tax of 15 percent, the price we pay will be  $115$  percent of the original price.

**Problem 7.1.1.1** Again, let the starting price be  $\$p$ .

If we apply the discount and then the tax, we multiply  $p$  first by  $0.8$  and then by  $1.15$ , resulting in the expression  $1.15 \cdot 0.8p$ .

If, on the other hand, we apply the tax and then the discount, we multiply  $p$  first by  $1.15$  and then by  $0.8$ , resulting in the expression  $0.8 \cdot 1.15p$ .

These expressions are equal because of the (commutative/ associative ✓/ distributive) property of (addition/ subtraction/ multiplication ✓/ division). Thus, the final cost is the same either way.

**Problem 8** Money Bags Jon likes to give a tip of 20% when he is at restaurants. He does this by dividing his bill by 10 and then doubling it. Explain why this works.

**Free Response:** **Hint:** Because 10% is the same as  $1/10$  of the bill, and 20% is twice that.

**Problem 9** Regular Reggie likes to give a tip of 15% when he is at restaurants. He does this by dividing his bill by 10 and then adding half more to this number. Explain why this works.

**Free Response:** **Hint:** Because 10% is the same as  $1/10$  of the bill, 5% is half that, and 15% is the same as  $10\% + 5\%$ .

**Problem 10** Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.

**Free Response:** **Hint:** If the bill is already a multiple of 7, then  $1/7$  of the bill is about 14.3%, which is slightly less than a standard tip of 15%. By rounding up to the nearest multiple of 7, the tip will always be at least 14.3% and usually slightly more.

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