

Fundamental Theorem

Problems about unique factorization.

Problem 1 Explain what the GCD of two integers is. Give some relevant and revealing examples/nonexamples.

Free Response: **Hint:** The GCD of a and b is the greatest common divisor of the two integers. Imagine the following procedure:

- (a) List the divisors of a .
- (b) List the divisors of b .
- (c) Compare the two lists to create a new list of divisors they have in common.
- (d) From the new list, identify the greatest of these common divisors.

Problem 2 Explain what the LCM of two integers is. Give some relevant and revealing examples/nonexamples.

Free Response: **Hint:** Note: Use ellipses (i.e., three dots) to indicate a continuing pattern.

The LCM of a and b is the least common multiple of the two integers. Imagine the following procedure:

- (a) List the multiples of a .
- (b) List the multiples of b .
- (c) Compare the two lists to create a new list of multiples they have in common.
- (d) From the new list, identify the least of these common multiples.

Problem 3 How many zeros are at the end of the following numbers:

- (a) $2^2 \cdot 5^8 \cdot 7^3 \cdot 11^5$. There are 2 zeros.

Hint: Consider the 2s and 5s. (Why?)

- (b) $11!$. There are 2 zeros.

Learning outcomes:
Author(s): Bart Snapp and Brad Findell

Hint: $11! = 11 \cdot 10 \cdot 9 \cdots 2 \cdot 1$, and imagine its prime factorization. There will be plenty of 2s. Count the 5s.

- (c) $27!$. There are $\boxed{6}$ zeros.

Hint: If you were to write out the 27 factors, 5s are contributed by the following factors: 5, 10, 15, 20, 25. And the 25 contributes a second 5.

- (d) $99!$. There are $\boxed{19 + 3}$ zeros.

Hint: Among the 99 factors, there are 19 multiples of 5 and 3 multiples of 25.

- (e) $1001!$. There are $\boxed{200 + 40 + 8 + 1}$ zeros.

Hint: Among the 1001 factors, there are 200 multiples of 5, 40 multiples of 25, 8 multiples of 125, and 625.

In each case, explain your reasoning.

Problem 4 Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

- (a) $7|56$. (True \checkmark / False)

Hint: $56 = 7 \cdot 8$.

- (b) $55|11$. (True/ False \checkmark)

Hint: But $11|55$.

- (c) $3|40$. (True/ False \checkmark)

Hint: $40 = 3 \cdot 39 + 1$. Division by 3 gives remainder 1.

- (d) $100|(2^4 \cdot 3^{17} \cdot 5^2 \cdot 7)$ (True \checkmark / False)

Hint: $100 = 2^2 5^2$.

- (e) $5555|(5^{20} \cdot 7^9 \cdot 11^{11} \cdot 13^{23})$ (True/ False \checkmark)

Hint: $5555 = 5 \cdot 11 \cdot 101$.

- (f) $3|(3 + 6 + 9 + \cdots + 300 + 303)$ (True \checkmark / False)

Hint: 3 divides each of the terms.

Problem 5 Suppose that

$$(3^5 \cdot 7^9 \cdot 11^x \cdot 13^y) | (3^a \cdot 7^b \cdot 11^{19} \cdot 13^7)$$

What values of a , b , x and y , make true statements? Explain your reasoning.

- a (\geq ✓ / $=$ / \leq) 5.
- b (\geq ✓ / $=$ / \leq) 9.
- x (\geq / $=$ / \leq ✓) 19.
- y (\geq / $=$ / \leq ✓) 7.

Problem 6 Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

- (a) If $7|13a$, then $7|a$. (True ✓ / False)

Hint: Follows from Euclid's Lemma because 7 is prime.

- (b) If $6|49a$, then $6|a$. (True ✓ / False)

Hint: Because $6 = 2 \cdot 3$ is not prime, we handle its prime factors separately. Because $6|49a$, it must be that $2|49a$. Then $2|a$ by Euclid's Lemma. Similarly, because $6|49a$, it must be that $3|49a$. Then $3|a$ by Euclid's Lemma. Because both $2|a$ and $3|a$, it follows that $6|a$.

- (c) If $10|65a$, then $10|a$. (True / False ✓)

Hint: Counterexample: $a = 2$.

- (d) If $14|22a$, then $14|a$. (True / False ✓)

Hint: Counterexample: $a = 7$.

- (e) $54|931^{21}$. (True / False ✓)

Hint: 54 is even (i.e., it has 2 as a factor), but 931^{21} is not.

- (f) $54|810^{33}$. (True ✓ / False)

Hint: From $54 = 2 \cdot 3^3$ and $810 = 2 \cdot 5 \cdot 3^4$, we can see that $54|810$.

Problem 7 Joanna thinks she can see if a number is divisible by 24 by checking to see if it's divisible by 4 and divisible by 6. She claims that if the number is divisible by 4 and by 6, then it must be divisible by 24.

Lindsay has a similar divisibility test for 24: She claims that if a number is divisible by 3 and by 8, then it must be divisible by 24.

Are either correct? Explain your reasoning.

Joanna is (correct / incorrect ✓). Lindsay is (correct ✓ / incorrect).

Free Response: **Hint:** For Joanna, the least (positive) counterexample is $\boxed{12}$. Her method doesn't work because the $\gcd(6, 4) = \boxed{2}$, so there are common multiples before $6 \cdot 4$.

Lindsay's method, works because $\gcd(3, 8) = \boxed{1}$, so every common multiple of 3 and 8 is also a multiple of $3 \cdot 8$.

Problem 8 Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

- (a) If $a^2 | b^2$, then $a | b$. (True ✓ / False)

Hint: However many times a prime appears in the prime factorizations of a , it will appear twice as many times in the prime factorization of a^2 . Same idea for b and b^2 . Because $a^2 | b^2$, we know that $b^2 = ka^2$ for some integer k . Both b^2 and a^2 must have an even number of factors any prime, which implies that k must also have an even number of factors of that prime. This means that k is a perfect square, which is to say it is the square of some integer c . Substituting $k = c^2$, we find that $b^2 = c^2 a^2 = (ca)^2$. Assuming $a, b, c > 0$, we have $b = ca$, which means that $a | b$.

- (b) If $a | b^2$, then $a | b$. (True ✓ / False)

Hint: Counterexample: $a = 12, b = 6$.

- (c) If $a | b$ and $\gcd(a, b) = 1$, then $a = 1$. (True ✓ / False)

Hint: Because $a | b$ there is an integer k such that $b = ak$. Because $\gcd(a, b) = 1$, every prime in the factorization of b is not a factor of a and therefore must be in k . This implies that k has the same prime factorization of b . Assuming $a, b, k > 0$, this means that $b = k$ and therefore $a = 1$.

Problem 9 Suppose x and y are integers. If $x^2 = 11 \cdot y$, what can you say about y ? Explain your reasoning.

Free Response: **Hint:** Imagine comparing the prime factorizations of x , x^2 , and y . However many times a prime appears in the prime factorization of x , it will appear twice as many times in the prime factorization of x^2 . Because $11|(11y)$, it follows that $11|x^2$. Because x^2 must have an even number of factors of 11, y will have an odd number of factors of 11. And y will have an even number of each of its other prime factors.

Problem 10 Suppose x and y are integers. If $x^2 = 25 \cdot y$, what can you say about y ? Explain your reasoning.

Free Response: **Hint:** Because x^2 , $25y^2$, and 25 are all perfect squares, y must also be a perfect square
