

Polynomials

Problems about polynomials.

Problem 1 Explain what is meant by a *polynomial* in a variable x .

Free Response: **Hint:** Informally: A polynomial in x is an algebraic expression that can be written as a sum of terms, each of which is a whole-number power of x multiplied by some real number.

Formally: For a whole number n , a polynomial (in x) of degree n is an expression that can be written in the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i 's are real numbers and $a_n \neq 0$. A polynomial function of degree n is a function that can be defined by a polynomial.

Problem 2 Indicate the degree of the following polynomials. For expressions that are not polynomials, type NP.

$3x - 3$	<input type="text" value="1"/>
\sqrt{x}	<input type="text" value="NP"/>
x^{15}	<input type="text" value="15"/>
$1 - x - x^2$	<input type="text" value="2"/>
$2^x + x^2$	<input type="text" value="NP"/>
$(1 + x)(2 + x)x$	<input type="text" value="3"/>
56	<input type="text" value="0"/>
$2/x$	<input type="text" value="NP"/>

Problem 3 Given:

$$3x^7 - x^5 + x^4 - 16x^3 + 27 = a_7 x^7 + a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0$$

Find $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$.

Answer: $a_0 = \boxed{27}, a_1 = \boxed{0}, a_2 = \boxed{0}, a_3 = \boxed{-16}, a_4 = \boxed{1}, a_5 = \boxed{-1}, a_6 = \boxed{0}, a_7 = \boxed{3}$.

Problem 4 Given:

$$6x^5 + a_4x^4 - x^2 + a_0 = a_5x^5 - 24x^4 + a_3x^3 + a_2x^2 - 5$$

Find $a_0, a_1, a_2, a_3, a_4, a_5$.

Answers: $a_0 = \boxed{-5}, a_1 = \boxed{0}, a_2 = \boxed{-1}, a_3 = \boxed{0}, a_4 = \boxed{-24}, a_5 = \boxed{6}$.

Problem 5 Is it true that polynomials are equal if and only if their coefficients are equal? Explain your reasoning.

Free Response: **Hint:** Yes. This is the fact we used to complete the previous two problems.

Problem 6 Is it true that numbers are equal if and only if their digits are equal? Explain your reasoning.

Free Response: **Hint:** For whole numbers written in the same base, the answer is (yes ✓/ no).

For decimals, the answer is (yes/ no ✓). [Note: We will learn why and how later in the course.]

For fractions, of course, the answer is (yes/ no ✓). For example, $\frac{2}{3} = \frac{4}{6}$, and the “digits” of the fraction on the left are clearly different from those of the fraction on the right.

Problem 7 Explain how to add two polynomials. Explain, in particular, how “collecting like terms” is an application of the properties of arithmetic.

Free Response: **Hint:** Use the associative and commutative properties to rearrange the sum so that like terms are consecutive. Then use the distributive property to “collect” the like terms. For example, $3x^2 + 4x^2 = (3 + 4)x^2 = 7x^2$.

Problem 8 Explain how to multiply two polynomials.

Free Response: **Hint:** Use the distributive property: Multiply each term in the first polynomial by each term in the second polynomial. Then collect like terms.

Problem 9 Here is an example of the polynomial division algorithm:

$$\begin{array}{r}
 x - 3 \quad R \ 9x + 4 \\
 x^2 + 3x + 1 \overline{) x^3 + 0x^2 + x + 1} \\
 \underline{x^3 + 3x^2 + x} \\
 -3x^2 + 0x + 1 \\
 \underline{-3x^2 - 9x - 3} \\
 9x + 4
 \end{array}$$

Describe how to perform this algorithm:

Free Response: **Hint:** This is very much like the long division algorithm for counting numbers:

- Write part of the quotient;
- Multiply that part of the quotient by the divisor;
- Subtract that product from the dividend;
- Repeat until the remaining part of the dividend is has a degree less than the degree of the divisor.

Note that, in the last step, what counts as “less than” is different for dividing polynomials than for dividing counting numbers.

Problem 10 Find the quotient and divisor when dividing $x^3 + 4x^2 - 1$ by $x + 2$. Quotient: $x^2 + 2x - 4$; remainder: 7 .

Problem 11 Find the quotient and divisor when dividing $x^3 + 4x^2 - 1$ by $x^2 + 1$. Quotient: $x + 4$; remainder: $-x - 5$.

Problem 12 State the Division Theorem for polynomials. Give some relevant and revealing examples of this theorem in action.

Free Response: **Hint:** Informally, when dividing a polynomial by a (non-zero) polynomial, we can always find a sensible quotient and remainder (both polynomials). More formally, given $n(x)$ and a non-zero divisor $d(x)$, we can find $q(x)$ and $r(x)$ such that $n(x) = d(x)q(x) + r(x)$ with the degree of $r(x)$ (greater than / equal to / less than \checkmark) the degree of $d(x)$.