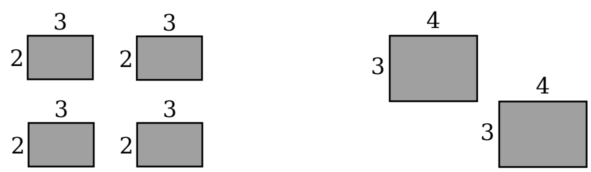
Algorithms

Problems about algorithms.

Problem 1 *Explain what it means for an operation $\star$ to be associative. Give some relevant and revealing examples and non-examples.*

Problem 2 *Consider the following pictures:*



Jesse claims that these pictures represent $(2 \cdot 3) \cdot 4$ *and* $2 \cdot (3 \cdot 4)$.

(a) *Is Jesse's claim correct? Explain your reasoning.*

(b) *Do Jesse's pictures show the associativity of multiplication? If so, explain why. If not, draw new pictures representing* $(2 \cdot 3) \cdot 4$ *and* $2 \cdot (3 \cdot 4)$ *that do show the associativity of multiplication.*

Problem 3 *Explain what it means for an operation $\star$ to be commutative. Give some relevant and revealing examples and non-examples.*

Problem 4 *Explain what it means for an operation $\star$ to distribute over another operation $\dagger$. Give some relevant and revealing examples and non-examples.*

Problem 5 *Explain what it means for an operation $\star$ to be closed on a set of numbers. Give some relevant and revealing examples and non-examples.*

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Problem 6 *Sometimes multiplication is described as* repeated addition. *Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.*

Problem 7 *In a warehouse you obtain* 20% *discount but you must pay a* 15% *sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?*

Problem 8 *Money Bags Jon likes to give a tip of* 20% *when he is at restaurants. He does this by dividing his bill by* 10 *and then doubling it. Explain why this works.*

Problem 9 *Regular Reggie likes to give a tip of* 15% *when he is at restaurants. He does this by dividing his bill by* 10 *and then adding half more to this number. Explain why this works.*

Problem 10 *Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of* 7 *and then taking the quotient (when that new number is divided by* 7*). Explain why this isn't as wacky as it might sound.*

Problem 11 *Cheap Carl likes to give a tip of* $13\frac{1}{3}\%$ *when he is at restaurants. He does this by dividing his bill by* 10 *and then adding one-third more to this number. Explain why this works.*

Problem 12 *Reasonable Rebbecca likes to give a tip of* 18% *when she is at restaurants. She does this by dividing her bill by* 5 *and then removing one-tenth of this number. Explain why this works.*

Problem 13 *Can you think of and justify any other schemes for computing the tip?*

Problem 14 *Here is an example of a standard addition algorithm:*

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 15 *Here is an example of the column addition algorithm:*

$$
\begin{array}{r}
892 \\
+398 \\
\hline
10 \\
18 \\
11 \\
\hline
1290
\end{array}
$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 16 *If you check out Problems and , you will learn about "partial" algorithms.*

(a) *Develop a "partial" algorithm for addition, give it a name, and describe how to perform this algorithm.*

(b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 17 *Here is an example of the banker's addition algorithm:*

$$
\begin{array}{r}
892 \\
+398 \\
\hline
1\boldsymbol{0} \\
1\boldsymbol{9} \\
\boldsymbol{12} \\
\hline
1290
\end{array}
$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 18 *Here is an example of a standard subtraction algorithm:*

$$\begin{array}{rccc}
 & & 8 & \\
 & 8 & \cancel{9} & {}^{1}2 \\
- & 3 & 7 & 8 \\
\hline
 & 5 & 1 & 4
\end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 19 *Here is an example of the subtraction by addition algorithm:*

$$\begin{array}{r} 892 \\ -378 \\ \hline 514 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} 8+\boldsymbol{4}=12 & \textit{add 1 to 7 to get 8} \\ 8+\boldsymbol{1}=9 & \\ 3+\boldsymbol{5}=8 & \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 20 *Here is an example of the Austrian subtraction algorithm:*

$$\begin{array}{rccc} & 8 & 9 & {}^{1}2 \\ - & 3 & {}^{8}\cancel{7} & 8 \\ \hline & 5 & 1 & 4 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 21 *If you check out Problems and , you will learn about "partial" algorithms.*

(a) *Develop a "partial" algorithm for subtraction, give it a name, and describe how to perform this algorithm.*

(b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 22 *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} {}^{2}\,{}^{3} \\ 634 \\ \times \quad 8 \\ \hline 5072 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 23 *Here is an example of the partial-products algorithm:*

$$
\begin{array}{r}
634 \\
\times \quad 8 \\
\hline
4800 \\
240 \\
32 \\
\hline
5072
\end{array}
$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 24 *Here is an example of a standard division algorithm:*

$$
\begin{array}{r@{\,}l}
 & 97\ R\ 1 \\
8 \,\big) & 777 \\
 & 72 \\
\cline{2-2}
 & 57 \\
 & 56 \\
\cline{2-2}
 & 1
\end{array}
$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 25 *Here is an example of the partial quotients algorithm:*

$$
\begin{array}{r}
7 \\
90 \\
8\,\overline{)\,777} \\
\underline{720} \\
57 \\
\underline{56} \\
1
\end{array}
$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 26 *Here is another example of the partial-quotients division algorithm:*

$$
\begin{array}{rr}
 & 4\\
 & 10\\
 & 10\\
 & 10\\
\cline{2-2}
8 & \big)\,277\\
 & 80\\
\cline{2-2}
 & 197\\
 & 80\\
\cline{2-2}
\end{array}
$$

(a) *Describe how to perform this algorithm—be sure to explain how this is different from the scaffolding division algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 27 *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} 634 \\ \times 216 \\ \hline 3804 \\ 6340 \\ 126800 \\ \hline 136944 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here—you may* *assume that you already know the algebra behind the standard multiplication algorithm.*

Problem 28 *Here is an example of the addition algorithm with decimals:*

$$\begin{array}{r} {}^{1} \\ 37.2 \\ +8.74 \\ \hline 45.94 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 29 *Here is an example of the multiplication algorithm with decimals:*

$$
\begin{array}{r}
3.40 \\
\times \quad .21 \\
\hline
340 \\
6800 \\
\hline
.7140
\end{array}
$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 30 *Here is an example of the division algorithm without remainder:*

$$\begin{array}{r} 0.75 \\ 4\overline{)3.00} \\ \underline{2\,8} \\ 20 \\ \underline{\underline{20}} \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 31 *In the following addition problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{MOON} \\ +\ \ \texttt{SUN} \\ \hline \texttt{PLUTO} \end{array}$$

Recover the original problem and solution. Explain your reasoning. Hint: $\texttt{S} = 6$ *and* $\texttt{U} = 5$.

Problem 32 *In the following addition problem, every digit has been replaced with a letter.*

$$
\begin{array}{r}
SEND \\
+MORE \\
\hline
MONEY
\end{array}
$$

Recover the original problem and solution. Explain your reasoning.

Problem 33 *In the following subtraction problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \mathtt{DEFER} \\ -\mathtt{DU7Y} \\ \hline \mathtt{N2G2} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 34 *In the following two subtraction problems, every digit has been replaced with a letter.*

$$\begin{array}{r} \mathtt{NINE} \\ -\mathtt{TEN} \\ \hline \mathtt{TWO} \end{array} \qquad \begin{array}{r} \mathtt{NINE} \\ -\mathtt{ONE} \\ \hline \mathtt{ALL} \end{array}$$

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

Problem 35 *In the following multiplication problem, every digit has been replaced with a letter.*

$$
\begin{array}{r}
LET \\
\times \quad NO \\
\hline
SOT \\
NOT \\
\hline
FRET
\end{array}
$$

Recover the original problem and solution. Explain your reasoning.

Problem 36 *The following is a long division problem where every digit except*

7 was replaced by X.

$$
\begin{array}{r}
X7X \\
XX\overline{)XXXXX} \\
\underline{X77} \\
X7X \\
\underline{X7X} \\
XX \\
\underline{\underline{XX}}
\end{array}
$$

Recover the digits from this long division problem. Explain your reasoning.

Problem 37 *The following is a long division problem where the various digits were replaced by X except for a single 8. The double bar indicates that the remainder is 0.*

$$
\begin{array}{r}
XX8XX \\
XXX\,\overline{)\,XXXXXXXX} \\
\underline{XXX} \\
XXXX \\
\underline{XXX} \\
XXXX \\
\underline{\underline{XXXX}}
\end{array}
$$

Recover the digits from this long division problem. Explain your reasoning.