

# Algorithms

*Problems about algorithms.*

**Problem 1** Explain what it means for an operation  $\star$  to be *associative*. Give some relevant and revealing examples and non-examples.

**Problem 2** Consider the following pictures:



Jesse claims that these pictures represent  $(2 \cdot 3) \cdot 4$  and  $2 \cdot (3 \cdot 4)$ .

- Is Jesse's claim correct? Explain your reasoning.
- Do Jesse's pictures show the associativity of multiplication? If so, explain why. If not, draw new pictures representing  $(2 \cdot 3) \cdot 4$  and  $2 \cdot (3 \cdot 4)$  that do show the associativity of multiplication.

**Problem 3** Explain what it means for an operation  $\star$  to be *commutative*. Give some relevant and revealing examples and non-examples.

**Problem 4** Explain what it means for an operation  $\star$  to *distribute* over another operation  $\dagger$ . Give some relevant and revealing examples and non-examples.

**Problem 5** Explain what it means for an operation  $\star$  to be *closed* on a set of numbers. Give some relevant and revealing examples and non-examples.

**Problem 6** Sometimes multiplication is described as *repeated addition*. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.

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**Problem 7** In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?

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**Problem 8** Money Bags Jon likes to give a tip of 20% when he is at restaurants. He does this by dividing his bill by 10 and then doubling it. Explain why this works.

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**Problem 9** Regular Reggie likes to give a tip of 15% when he is at restaurants. He does this by dividing his bill by 10 and then adding half more to this number. Explain why this works.

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**Problem 10** Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.

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**Problem 11** Cheap Carl likes to give a tip of  $13\frac{1}{3}\%$  when he is at restaurants. He does this by dividing his bill by 10 and then adding one-third more to this number. Explain why this works.

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**Problem 12** Reasonable Rebbecca likes to give a tip of 18% when she is at restaurants. She does this by dividing her bill by 5 and then removing one-tenth of this number. Explain why this works.

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**Problem 13** Can you think of and justify any other schemes for computing the tip?

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**Problem 14** Here is an example of a standard addition algorithm:

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

- (a) Describe how to perform this algorithm.
  - (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 15** Here is an example of the column addition algorithm:

$$\begin{array}{r} 892 \\ +398 \\ \hline 10 \\ 18 \\ 11 \\ \hline 1290 \end{array}$$

- (a) Describe how to perform this algorithm.
  - (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 16** If you check out Problems 14 and 15, you will learn about “partial” algorithms.

- (a) Develop a “partial” algorithm for addition, give it a name, and describe how to perform this algorithm.
  - (b) Provide a relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 17** Here is an example of the banker's addition algorithm:

$$\begin{array}{r}
 892 \\
 +398 \\
 \hline
 10 \\
 19 \\
 \hline
 12 \\
 \hline
 1290
 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.

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**Problem 18** Here is an example of a standard subtraction algorithm:

$$\begin{array}{r}
 8 \\
 8 \cancel{9}^1 2 \\
 - 378 \\
 \hline
 514
 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.

**Problem 19** Here is an example of the subtraction by addition algorithm:

$$\begin{array}{r}
 892 \\
 -378 \\
 \hline
 514
 \end{array}
 \rightsquigarrow
 \begin{array}{ll}
 8 + \mathbf{4} = 12 & \text{add 1 to 7 to get 8} \\
 8 + \mathbf{1} = 9 & \\
 3 + \mathbf{5} = 8 &
 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.

**Problem 20** Here is an example of the Austrian subtraction algorithm:

$$\begin{array}{r}
 89^12 \\
 -3^8\cancel{7}8 \\
 \hline
 514
 \end{array}$$

- (a) Describe how to perform this algorithm.
  - (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 21** If you check out Problems and , you will learn about “partial” algorithms.

- (a) Develop a “partial” algorithm for subtraction, give it a name, and describe how to perform this algorithm.
  - (b) Provide a relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 22** Here is an example of a standard multiplication algorithm:

$$\begin{array}{r} 23 \\ 634 \\ \times 8 \\ \hline 5072 \end{array}$$

- (a) Describe how to perform this algorithm.
  - (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 23** Here is an example of the partial-products algorithm:

$$\begin{array}{r} 634 \\ \times 8 \\ \hline 4800 \\ 240 \\ 32 \\ \hline 5072 \end{array}$$

- (a) Describe how to perform this algorithm.

- (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 24** *Here is an example of a standard division algorithm:*

$$\begin{array}{r}
 97 \text{ R } 1 \\
 8 \overline{) 777} \\
 \underline{72} \phantom{00} \\
 57 \phantom{00} \\
 \underline{56} \phantom{00} \\
 1
 \end{array}$$

- (a) *Describe how to perform this algorithm.*
  - (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 25** *Here is an example of the partial quotients algorithm:*

$$\begin{array}{r} 7 \\ 90 \\ \hline 8 \overline{) 777} \\ 720 \\ \hline 57 \\ 56 \\ \hline 1 \end{array}$$



- (a) *Describe how to perform this algorithm.*
  - (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 26** *Here is another example of the partial-quotients division algorithm:*

$$\begin{array}{r}
 4 \\
 10 \\
 10 \\
 10 \\
 \hline
 8 \overline{) 277} \\
 \phantom{0} 80 \\
 \hline
 \phantom{0} 197 \\
 \phantom{0} 80 \\
 \hline
 \phantom{0} 117
 \end{array}$$

- (a) Describe how to perform this algorithm—be sure to explain how this is different from the scaffolding division algorithm.
  - (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 27** Here is an example of a standard multiplication algorithm:

$$\begin{array}{r}
 634 \\
 \times 216 \\
 \hline
 3804 \\
 6340 \\
 126800 \\
 \hline
 136944
 \end{array}$$

- (a) Describe how to perform this algorithm.
  - (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here—you may assume that you already know the algebra behind the standard multiplication algorithm.
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**Problem 28** Here is an example of the addition algorithm with decimals:

$$\begin{array}{r}
 1 \\
 37.2 \\
 +8.74 \\
 \hline
 45.94
 \end{array}$$

- (a) Describe how to perform this algorithm.
  - (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 29** Here is an example of the multiplication algorithm with decimals:

$$\begin{array}{r} 3.40 \\ \times .21 \\ \hline 340 \\ 6800 \\ \hline .7140 \end{array}$$

- (a) Describe how to perform this algorithm.
  - (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
  - (c) Show the “behind-the-scenes” algebra that is going on here.
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**Problem 30** Here is an example of the division algorithm without remainder:

$$\begin{array}{r}
 0.75 \\
 \hline
 4 \overline{) 3.00} \\
 \underline{28} \phantom{0} \\
 20 \\
 \underline{20} \\
 \hline
 \hline
 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.

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**Problem 31** In the following addition problem, every digit has been replaced with a letter.

$$\begin{array}{r}
 MOON \\
 + SUN \\
 \hline
 PLUTO
 \end{array}$$

Recover the original problem and solution. Explain your reasoning. Hint:  $S = 6$  and  $U = 5$ .

**Problem 32** In the following addition problem, every digit has been replaced with a letter.

$$\begin{array}{r} \text{SEND} \\ + \text{MORE} \\ \hline \text{MONEY} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

**Problem 33** In the following subtraction problem, every digit has been replaced with a letter.

$$\begin{array}{r} \text{DEFER} \\ - \text{DU7Y} \\ \hline \text{N2G2} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

**Problem 34** In the following two subtraction problems, every digit has been

replaced with a letter.

$$\begin{array}{r} \text{NINE} \\ - \text{TEN} \\ \hline \text{TWO} \end{array} \qquad \begin{array}{r} \text{NINE} \\ - \text{ONE} \\ \hline \text{ALL} \end{array}$$

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

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**Problem 35** In the following multiplication problem, every digit has been replaced with a letter.

$$\begin{array}{r} \text{LET} \\ \times \text{NO} \\ \hline \text{SOT} \\ \text{NOT} \\ \hline \text{FRET} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

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**Problem 36** The following is a long division problem where every digit except 7 was replaced by X.

$$\begin{array}{r}
 \phantom{XX}X7X \\
 \hline
 XX \overline{) XXXXX} \\
 \phantom{XX}X77 \\
 \hline
 \phantom{XX}X7X \\
 \phantom{XX}X7X \\
 \hline
 \phantom{XX}XX \\
 \phantom{XX}\underline{\underline{XX}}
 \end{array}$$

Recover the digits from this long division problem. Explain your reasoning.

**Problem 37** The following is a long division problem where the various digits were replaced by X except for a single 8. The double bar indicates that the remainder is 0.



$$\begin{array}{r}
 \phantom{XXX} \overline{XX8XX} \\
 XXX \overline{) XXXXXXXXX} \\
 \phantom{XXX} \underline{XXX} \\
 \phantom{XXX} XXXX \\
 \phantom{XXX} \underline{XXX} \\
 \phantom{XXX} XXXX \\
 \phantom{XXX} \underline{XXX} \\
 \phantom{XXX} \underline{\underline{XXXX}}
 \end{array}$$

Recover the digits from this long division problem. Explain your reasoning.

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