Online HW 1: Operations and Algorithms

Bart Snapp and Brad Findell

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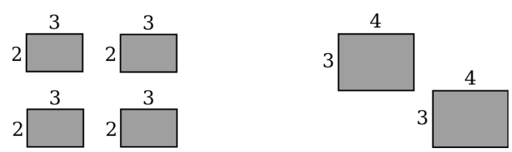
Contents

Operations and Algorithms

Problems about operations and algorithms.

Problem 1 *Explain what it means for an operation $\star$ to be associative. Give some relevant and revealing examples and non-examples.*

Problem 2 *Consider the following pictures:*



Jesse claims that these pictures represent $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$.

(a) *Is Jesse's claim correct?*

Multiple Choice:

(i) *Yes.*
(ii) *No.*
(iii) *Not enough information.*

(b) *Explain your reasoning.*

(c) *Do Jesse's pictures show the associativity of multiplication?*

Multiple Choice:

(i) *Yes.*
(ii) *No.*
(iii) *Not enough information.*

(d) *If so, explain why. If not, draw new pictures representing $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$ that do show the associativity of multiplication.*

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Problem 3 *Explain what it means for an operation $\star$ to be commutative. Give some relevant and revealing examples and non-examples.*

Problem 4 *Explain what it means for an operation $\star$ to distribute over another operation $\dagger$. Give some relevant and revealing examples and non-examples.*

Problem 5 *Explain what it means for an operation $\star$ to be closed on a set of numbers.*

Problem 5.1 *Give some relevant and revealing examples and non-examples.*

Problem 6 *Sometimes multiplication is described as repeated addition. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.*

Problem 7 *In beginning algebra, simplifying expressions often involves collecting like terms. But why does this work? Well, the expression $3x + 4x$ is equivalent to $(3+4)x$ by the (commutative/ associative/ distributive) property. And then it is clear that $(3 + 4)x = \boxed{?}$.*

Problem 8 *In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?*

Outline: *Build a solution in four steps:*

(a) *Use a specific starting price.*

(b) *Generalize the process of computing a price after a discount (assuming no tax).*

(c) *Generalize the process of computing a price with tax (assuming no discount).*

(d) *Generalize the two together.*

Try a Specific Price. *To get started, try a specific starting price, say \$200. Applying the discount first, the price would be \$$\boxed{?}$. After the tax, the cost is \$$\boxed{?}$.*

Now try applying the tax first. The original price with tax would be \$$\boxed{?}$. Then with the discount, the cost would be \$$\boxed{?}$, which is (greater than/ equal to/ less than) the cost when applying the discount first.

Will this work for any starting price? We need to generalize.

Problem 8.1 ***Apply a Discount.*** *Suppose the starting price is \$$p$. A 20% discount, in terms of p, will be \$$\boxed{?}$. So the price after the discount would $p - 0.2p$. And by the (commutative/ inverse/ distributive) property, this is equal to $(1 - 0.2)p$, or $\boxed{?}$. In other words, rather than computing the discount and subtracting, we can directly compute the new price by multiplying the original price by $\boxed{?}$. This makes sense because with a discount of 20 percent, the price we pay will be $\boxed{?}$ percent of the original price.*

Problem 8.1.1 ***Apply a Tax.*** *Again, suppose the starting price is \$$p$. A 15% tax, in terms of p, will be \$$\boxed{?}$. So the price with tax would $p + 0.15p$. And by the (commutative/ identity/ distributive) property, this is equal to $\boxed{?}$. In other words, rather than computing the tax and adding, we can directly compute the new price by multiplying the original price by $\boxed{?}$. This makes sense because with a tax of 15 percent, the price we pay after tax will be $\boxed{?}$ percent of the original price.*

Problem 8.1.1.1 ***Apply both.*** *Again, let the starting price be \$$p$.*

If we apply the discount and then the tax, we multiply p first by $\boxed{?}$ and then by $\boxed{?}$, resulting in the expression $\boxed{?}$.

If, on the other hand, we apply the tax and then the discount, we multiply p first by $\boxed{?}$ and then by $\boxed{?}$, resulting in the expression $\boxed{?}$.

These expressions are equal because of the (identity/ associative/ distributive) and (commutative/ inverse/ distributive) properties of (addition/ subtraction / multiplication/ division). Thus, the final cost is the same either way.

Problem 9 *Money Bags Jon likes to give a tip of* 20% *when he is at restaurants. He does this by dividing his bill by* 10 *and then doubling it. Explain why this works.*

Problem 10 *Regular Reggie likes to give a tip of* 15% *when he is at restaurants. He does this by dividing his bill by* 10 *and then adding half more to this number. Explain why this works.*

Problem 11 *Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of* 7 *and then taking the quotient (when that new number is divided by* 7*). Explain why this isn't as wacky as it might sound.*

Algorithms

More problems about algorithms.

Problem 12 *Here is an example of a standard addition algorithm:*

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 13 *Here is an example of the column addition algorithm:*

$$\begin{array}{r} 892 \\ +398 \\ \hline 10 \\ 18 \\ 11 \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 14 *If you check out Problems* **??** *and* **??***, you will learn about "partial" algorithms.*

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(a) *Develop a "partial" algorithm for addition, give it a name, and describe how to perform this algorithm.*

(b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 15 *Here is an example of the banker's addition algorithm:*

$$\begin{array}{r} 892 \\ +398 \\ \hline 1\boldsymbol{0} \\ 1\boldsymbol{9} \\ \boldsymbol{12} \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 16 *Here is an example of a standard subtraction algorithm:*

$$\begin{array}{r} 8\phantom{\,{}^{1}2} \\ 8\cancel{9}\,{}^{1}2 \\ -3\,7\,\;\;8 \\ \hline 5\,1\,\;\;4 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 17 *Here is an example of the subtraction by addition algorithm:*

$$\begin{array}{r} 892 \\ -378 \\ \hline 514 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} 8 + \boldsymbol{4} = 12 & \textit{add 1 to 7 to get 8} \\ 8 + \boldsymbol{1} = 9 & \\ 3 + \boldsymbol{5} = 8 & \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 18 *Here is an example of the Austrian subtraction algorithm:*

$$\begin{array}{rrr} 8 & 9 & {}^{1}2 \\ -3 & {}^{8}\cancel{7} & 8 \\ \hline 5 & 1 & 4 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 19 *If you check out Problems* **??** *and* **??**, *you will learn about "partial" algorithms.*

(a) *Develop a "partial" algorithm for subtraction, give it a name, and describe how to perform this algorithm.*

(b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 20 *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} 23 \\ 634 \\ \times \quad 8 \\ \hline 5072 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 21 *Here is an example of the partial-products algorithm:*

$$\begin{array}{r} 634 \\ \times \quad 8 \\ \hline 4800 \\ 240 \\ 32 \\ \hline 5072 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 22 *Here is an example of a standard division algorithm:*

$$\begin{array}{r@{}l} & 97\,R\,1 \\ 8\,\big) & 777 \\ & \underline{72} \\ & 57 \\ & \underline{56} \\ & 1 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 23 *Here is an example of the partial quotients algorithm:*

$$\begin{array}{r@{}r} & 7 \\ & 90 \\ \cline{2-2} 8\big) & 777 \\ & 720 \\ \cline{2-2} & 57 \\ & 56 \\ \cline{2-2} & 1 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 24 *Here is another example of the partial-quotients division algorithm:*

$$\begin{array}{r@{}r} & 4 \\ & 10 \\ & 10 \\ & 10 \\ \cline{2-2} 8\big) & 277 \\ & 80 \\ \cline{2-2} & 197 \\ & 80 \\ \cline{2-2} & 117 \\ & 80 \\ \cline{2-2} & 37 \\ & 32 \\ \cline{2-2} & 5 \end{array}$$

(a) *Describe how to perform this algorithm—be sure to explain how this is different from the scaffolding division algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 25 *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} 634 \\ \times 216 \\ \hline 3804 \\ 6340 \\ 126800 \\ \hline 136944 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here—you may assume that you already know the algebra behind the standard multiplication algorithm.*

Problem 26 *Here is an example of the addition algorithm with decimals:*

$$\begin{array}{l} \;1 \\ \;37.2 \\ +8.74 \\ \hline \;45.94 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 27 *Here is an example of the multiplication algorithm with decimals:*

$$\begin{array}{r} 3.40 \\ \times\ \ .21 \\ \hline 340 \\ 6800 \\ \hline .7140 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 28 *Here is an example of the division algorithm without remainder:*

$$\begin{array}{r} 0.75 \\ 4\overline{)3.00} \\ 2\ 8 \\ \hline 20 \\ 20 \\ \hline\hline \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 29 *In the following addition problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \mathtt{MOON} \\ +\ \ \mathtt{SUN} \\ \hline \mathtt{PLUTO} \end{array}$$

Recover the original problem and solution. Explain your reasoning. Hint: $\mathtt{S} = 6$ *and* $\mathtt{U} = 5$.

Problem 30 *In the following addition problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \mathit{SEND} \\ +\mathit{MORE} \\ \hline \mathit{MONEY} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 31 *In the following subtraction problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \mathit{DEFER} \\ -\mathit{DU7Y} \\ \hline \mathit{N2G2} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 32 *In the following two subtraction problems, every digit has been replaced with a letter.*

$$\begin{array}{r} \mathit{NINE} \\ -\mathit{TEN} \\ \hline \mathit{TWO} \end{array} \qquad \begin{array}{r} \mathit{NINE} \\ -\mathit{ONE} \\ \hline \mathit{ALL} \end{array}$$

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

Problem 33 *In the following multiplication problem, every digit has been replaced with a letter.*

$$\begin{array}{rrrr} & L & E & T \\ \times & & N & O \\ \hline & S & O & T \\ N & O & T & \\ \hline F & R & E & T \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 34 *The following is a long division problem where every digit except 7 was replaced by X.*

$$
\begin{array}{r}
X7X \\
XX\,\big)\overline{XXXXX} \\
\underline{X77} \\
X7X \\
\underline{X7X} \\
XX \\
\underline{\underline{XX}}
\end{array}
$$

Recover the digits from this long division problem. Explain your reasoning.

Problem 35 *The following is a long division problem where the various digits were replaced by X except for a single 8. The double bar indicates that the remainder is 0.*

$$
\begin{array}{r}
XX8XX \\
XXX\,\big)\overline{XXXXXXXXX} \\
\underline{XXX} \\
XXXX \\
\underline{XXX} \\
XXXX \\
\underline{\underline{XXXX}}
\end{array}
$$

Recover the digits from this long division problem. Explain your reasoning.