Online HW 0: Bases

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Contents

Home Base

Problems about numbers in various bases.

Problem 1 *Explain why the following "joke" is "funny:" There are* 10 *types of people in the world. Those who understand base two and those who don't.*

Problem 2 *You meet some Tripod aliens, they tally by threes. Thankfully for everyone involved, they use the symbols* 0, 1, *and* 2.

(a) *Can you explain how a Tripod would count from* 11 *to* 201*? Be sure to carefully explain what number comes after* 22.

(b) *What number comes immediately before* 10*?* 210*?* 20110*? Explain your reasoning.*

Problem 3 *You meet some people who tally by sevens. They use the symbols* O, A, B, C, D, E, *and* F.

(a) *What do the individual symbols* O, A, B, C, D, E, *and* F *mean?*

(b) *Can you explain how they would count from* DD *to* AOC*? Be sure to carefully explain what number comes after* FF.

(c) *What number comes immediately before* AO*?* ABO*?* $EOFFA$*? Explain your reasoning.*

Problem 4 *Now, suppose that you meet a hermit who tallies by thirteens. Explain how he might count. Give some relevant and revealing examples.*

Problem 5 *While visiting Mos Eisley spaceport, you stop by Chalmun's Cantina. After you sit down, you notice that one of the other aliens is holding a discussion on fractions. Much to your surprise, they explain that* 1/6 *of* 36 *is* 7. *You are unhappy with this, knowing that* 1/6 *of* 36 *is in fact* 6, *yet their*

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audience seems to agree with it, not you. Next the alien challenges its audience by asking, "What is $1/4$ of 10?" What is the correct answer to this question, and how many fingers do the aliens have? Explain your reasoning.

Problem 6 *When the first Venusian to visit Earth attended a sixth grade class, it watched the teacher show that*

$$\frac{3}{12} = \frac{1}{4}.$$

"How strange," thought the Venusian. "On Venus, $\frac{4}{12} = \frac{1}{4}$." What base do Venusians use? Explain your reasoning.

Problem 7 *When the first Martian to visit Earth attended a high school algebra class, it watched the teacher show that the only solution of the equation*

$$5x^2 - 50x + 125 = 0$$

is $x = 5$.

"How strange," thought the Martian. "On Mars, $x = 5$ is a solution of this equation, but there also is another solution." If Martians have more fingers than humans, how many fingers do Martians have on both hands? Explain your reasoning.

Problem 8 *In one of your many space-time adventures, you see the equation*

$$\frac{3}{10} + \frac{4}{13} = \frac{21}{20}$$

written on a napkin. How many fingers did the beast who wrote this have? Explain your reasoning.

Problem 9 *What is the smallest number of weights needed to produce every integer-valued mass from 0 grams to say n grams? Explain your reasoning.*

Problem 10 *Starting at zero, how high can you count using just your fingers?*

(a) *Explain how to count to 10.*

(b) *Explain how to count to* 35.

(c) *Explain how to count to* 1023.

(d) *Explain how to count to* 59048.

(e) *Can you count even higher?*

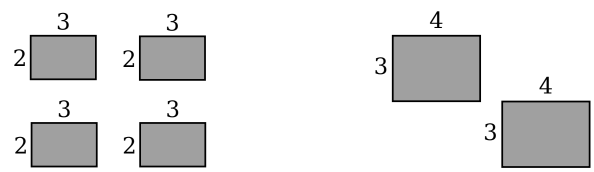
Explain your reasoning.

Algorithms

Problems about algorithms.

Problem 11 *Explain what it means for an operation $\star$ to be associative. Give some relevant and revealing examples and non-examples.*

Problem 12 *Consider the following pictures:*



Jesse claims that these pictures represent $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$.

(a) *Is Jesse's claim correct? Explain your reasoning.*

(b) *Do Jesse's pictures show the associativity of multiplication? If so, explain why. If not, draw new pictures representing $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$ that do show the associativity of multiplication.*

Problem 13 *Explain what it means for an operation $\star$ to be commutative. Give some relevant and revealing examples and non-examples.*

Problem 14 *Explain what it means for an operation $\star$ to distribute over another operation $\dagger$. Give some relevant and revealing examples and non-examples.*

Problem 15 *Explain what it means for an operation $\star$ to be closed on a set of numbers. Give some relevant and revealing examples and non-examples.*

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Problem 16 *Sometimes multiplication is described as* repeated addition. *Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.*

Problem 17 *In a warehouse you obtain* 20% *discount but you must pay a* 15% *sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?*

Problem 18 *Money Bags Jon likes to give a tip of* 20% *when he is at restaurants. He does this by dividing his bill by* 10 *and then doubling it. Explain why this works.*

Problem 19 *Regular Reggie likes to give a tip of* 15% *when he is at restaurants. He does this by dividing his bill by* 10 *and then adding half more to this number. Explain why this works.*

Problem 20 *Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of* 7 *and then taking the quotient (when that new number is divided by* 7*). Explain why this isn't as wacky as it might sound.*

Problem 21 *Cheap Carl likes to give a tip of* $13\frac{1}{3}\%$ *when he is at restaurants. He does this by dividing his bill by* 10 *and then adding one-third more to this number. Explain why this works.*

Problem 22 *Reasonable Rebbecca likes to give a tip of* 18% *when she is at restaurants. She does this by dividing her bill by* 5 *and then removing one-tenth of this number. Explain why this works.*

Problem 23 *Can you think of and justify any other schemes for computing the tip?*

Problem 24 *Here is an example of a standard addition algorithm:*

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 25 *Here is an example of the column addition algorithm:*

$$\begin{array}{r} 892 \\ +398 \\ \hline 10 \\ 18 \\ 11 \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 26 *If you check out Problems* **??** *and* **??***, you will learn about "partial" algorithms.*

(a) *Develop a "partial" algorithm for addition, give it a name, and describe how to perform this algorithm.*

(b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 27 *Here is an example of the banker's addition algorithm:*

$$\begin{array}{r} 892 \\ +398 \\ \hline 1\boldsymbol{0} \\ 1\boldsymbol{9} \\ \boldsymbol{12} \\ \hline 1290 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 28 *Here is an example of a standard subtraction algorithm:*

$$\begin{array}{r} 8\phantom{\,{}^{1}2} \\ 8\cancel{9}\,{}^{1}2 \\ -3\,7\;\,8 \\ \hline 5\,1\;\,4 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 29 *Here is an example of the subtraction by addition algorithm:*

$$\begin{array}{r} 892 \\ -378 \\ \hline 514 \end{array} \qquad \leftrightsquigarrow \qquad \begin{array}{ll} 8 + \boldsymbol{4} = 12 & \textit{add 1 to 7 to get 8} \\ 8 + \boldsymbol{1} = 9 & \\ 3 + \boldsymbol{5} = 8 & \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 30 *Here is an example of the Austrian subtraction algorithm:*

$$\begin{array}{r} 8\ \ 9\ {}^{1}2 \\ -3\ {}^{8}\cancel{7}\ \ 8 \\ \hline 5\ \ 1\ \ 4 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 31 *If you check out Problems* **??** *and* **??**, *you will learn about "partial" algorithms.*

(a) *Develop a "partial" algorithm for subtraction, give it a name, and describe how to perform this algorithm.*

(b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 32 *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} 23 \\ 634 \\ \times \quad 8 \\ \hline 5072 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 33 *Here is an example of the partial-products algorithm:*

$$\begin{array}{r} 634 \\ \times \quad 8 \\ \hline 4800 \\ 240 \\ 32 \\ \hline 5072 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 34 *Here is an example of a standard division algorithm:*

$$\begin{array}{r} 97\,R1 \\ 8\overline{)777} \\ \underline{72} \\ 57 \\ \underline{56} \\ 1 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 35 *Here is an example of the partial quotients algorithm:*

$$\begin{array}{r} 7 \\ 90 \\ 8\overline{)777} \\ \underline{720} \\ 57 \\ \underline{56} \\ 1 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 36 *Here is another example of the partial-quotients division algorithm:*

$$
\begin{array}{r@{}r}
 & 4\\
 & 10\\
 & 10\\
 & 10\\ \cline{2-2}
8\,\big) & 277\\
 & 80\\ \cline{2-2}
 & 197\\
 & 80\\ \cline{2-2}
 & 117\\
 & 80\\ \cline{2-2}
 & 37\\
 & 32\\ \cline{2-2}
 & 5
\end{array}
$$

(a) *Describe how to perform this algorithm—be sure to explain how this is different from the scaffolding division algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 37 *Here is an example of a standard multiplication algorithm:*

$$
\begin{array}{r}
634\\
\times 216\\ \hline
3804\\
6340\\
126800\\ \hline
136944
\end{array}
$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here—you may assume that you already know the algebra behind the standard multiplication algorithm.*

Problem 38 *Here is an example of the addition algorithm with decimals:*

$$\begin{array}{r} 1 \\ 37.2 \\ +8.74 \\ \hline 45.94 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 39 *Here is an example of the multiplication algorithm with decimals:*

$$\begin{array}{r} 3.40 \\ \times\ \ .21 \\ \hline 340 \\ 6800 \\ \hline .7140 \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 40 *Here is an example of the division algorithm without remainder:*

$$\begin{array}{r} 0.75 \\ 4\,\overline{)\,3.00} \\ \underline{2\ 8} \\ 20 \\ \underline{\underline{20}} \end{array}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 41 *In the following addition problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{MOON} \\ + \ \texttt{SUN} \\ \hline \texttt{PLUTO} \end{array}$$

Recover the original problem and solution. Explain your reasoning. Hint: $\texttt{S} = 6$ *and* $\texttt{U} = 5$.

Problem 42 *In the following addition problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{SEND} \\ +\texttt{MORE} \\ \hline \texttt{MONEY} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 43 *In the following subtraction problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{DEFER} \\ -\texttt{DU7Y} \\ \hline \texttt{N2G2} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 44 *In the following two subtraction problems, every digit has been replaced with a letter.*

$$\begin{array}{r} \texttt{NINE} \\ -\texttt{TEN} \\ \hline \texttt{TWO} \end{array} \qquad \begin{array}{r} \texttt{NINE} \\ -\texttt{ONE} \\ \hline \texttt{ALL} \end{array}$$

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

Problem 45 *In the following multiplication problem, every digit has been replaced with a letter.*

$$\begin{array}{r} \mathtt{LET} \\ \times\ \ \mathtt{NO} \\ \hline \mathtt{SOT} \\ \mathtt{NOT} \\ \hline \mathtt{FRET} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

Problem 46 *The following is a long division problem where every digit except 7 was replaced by X.*

$$\begin{array}{r} X7X \\ XX\,\big)\overline{XXXXX} \\ \underline{X77} \\ X7X \\ \underline{X7X} \\ XX \\ \underline{\underline{XX}} \end{array}$$

Recover the digits from this long division problem. Explain your reasoning.

Problem 47 *The following is a long division problem where the various digits were replaced by X except for a single 8. The double bar indicates that the remainder is 0.*

$$\begin{array}{r} XX8XX \\ XXX\,\big)\overline{XXXXXXXX} \\ \underline{XXX} \\ XXXX \\ \underline{XXX} \\ XXXX \\ \underline{\underline{XXXX}} \end{array}$$

Recover the digits from this long division problem. Explain your reasoning.

Polynomials

Problems about polynomials

Problem 48 *Explain what is meant by a polynomial in a variable x.*

Problem 49 *Given:*

$$3x^7 - x^5 + x^4 - 16x^3 + 27 = a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0$$

Find a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7.

Problem 50 *Given:*

$$6x^5 + a_4x^4 - x^2 + a_0 = a_5x^5 - 24x^4 + a_3x^3 + a_2x^2 - 5$$

Find a_0, a_1, a_2, a_3, a_4, a_5.

Problem 51 *Is it true that polynomials are equal if and only if their coefficients are equal? Explain your reasoning.*

Problem 52 *Is it true that numbers are equal if and only if their digits are equal? Explain your reasoning.*

Problem 53 *Explain how to add two polynomials. Explain, in particular, how "collecting like terms" is an application of the properties of arithmetic.*

Problem 54 *Explain how to multiply two polynomials.*

Problem 55 *Here is an example of the polynomial division algorithm:*

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$$
\begin{array}{r|l}
 & \phantom{x^3+0x^2+{}} x - 3 \quad R\, 9x+4 \\
\cline{2-2}
x^2+3x+1 & x^3+0x^2+x+1 \\
 & \underline{x^3+3x^2+x} \\
 & -3x^2+0x+1 \\
 & \underline{-3x^2-9x-3} \\
 & \phantom{x^3+3x^2-{}} 9x+4
\end{array}
$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) *Show the "behind-the-scenes" algebra that is going on here.*

Problem 56 *State the Division Theorem for polynomials. Give some relevant and revealing examples of this theorem in action.*

Problem 57 *Given a polynomial*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

can you find two numbers L and U such that $L \leqslant p(x) \leqslant U$ for all x? If so, explain why. If not, explain why not.

Problem 58 *Consider all polynomials of the form*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are integers. If you substitute an integer for x will you always get an integer out? Explain your reasoning.

Problem 59 *Consider the following polynomial:*

$$p(x) = \frac{x^2}{2} + \frac{x}{2}$$

Will $p(x)$ always returns an integer when an integer is substituted for x? Explain your reasoning.

Problem 60 *Fix some integer value for x and consider all polynomials of the form*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0. Which numbers can be represented by such polynomials? Explain your reasoning.

Problem 61 *Find a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 2 such that $p(2) = 35$. Discuss how your answer compares to the representation of 35 in base two. Explain your reasoning.

Problem 62 *Find a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 7 such that $p(7) = 234$. Discuss how your answer compares to the representation of 234 in base seven. Explain your reasoning.

Problem 63 *Find a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 10 such that $p(10) = 18$. Discuss how your answer compares to the representation of 18 in base ten. Explain your reasoning.

Problem 64 *Find a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 15 such that $p(15) = 201$. Discuss how your answer compares to the representation of 201 in base fifteen. Explain your reasoning.

Problem 65 *Fix some integer value for x and consider all polynomials of the form*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0 and less than x. Which numbers can be represented by such polynomials? Explain your reasoning. Big hint: Base x.

Problem 66 *Fix some integer value for x and consider all polynomials of the form*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0 and less than 10. Which numbers can be represented by such polynomials? Explain your reasoning.

Problem 67 *Consider $x^2 + x + 1$. This can be thought of as a "number" in base x. Express this number in base $(x+1)$, that is, find b_0, b_1, b_2 such that*

$$b_2(x+1)^2 + b_1(x+1) + b_0 = x^2 + x + 1.$$

Explain your reasoning.

Problem 68 *Consider $x^2 + 2x + 3$. This can be thought of as a "number" in base x. Express this number in base $(x-1)$, that is, find b_0, b_1, b_2 such that*

$$b_2(x-1)^2 + b_1(x-1) + b_0 = x^2 + 2x + 3.$$

Explain your reasoning.

Problem 69 *Consider $x^3 + 2x + 1$. This can be thought of as a "number" in base x. Express this number in base $(x-1)$, that is, find b_0, b_1, b_2, b_3 such that*

$$b_3(x-1)^3 + b_2(x-1)^2 + b_1(x-1) + b_0 = x^3 + 2x + 1.$$

Explain your reasoning.

Problem 70 *If the polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is thought of as a "number" in base x, describe two different ways to find the base $(x-1)$ coefficients of $p(x)$.
