Fundamental Theorem

Problems about unique factorization.

Problem 1 *Explain what the GCD of two integers is. Give some relevant and revealing examples/nonexamples.*

Free Response: ***Hint:*** *The GCD of a and b is the greatest common divisor of the two integers. Imagine the following procedure:*

(a) *List the divisors of a.*
(b) *List the divisors of b.*
(c) *Compare the two lists to create a new list of divisors they have in common.*
(d) *From the new list, identify the greatest of these common divisors.*

Problem 2 *Explain what the LCM of two integers is. Give some relevant and revealing examples/nonexamples.*

Free Response: ***Hint:*** *Note: Use ellipses (i.e., three dots) to indicate a continuing pattern.*

The LCM of a and b is the least common multiple of the two integers. Imagine the following procedure:

(a) *List the multiples of a.*
(b) *List the multiples of b.*
(c) *Compare the two lists to create a new list of multiples they have in common.*
(d) *From the new list, identify the least of these common multiples.*

Problem 3 *How many zeros are at the end of the following numbers:*

(a) $2^2 \cdot 5^8 \cdot 7^3 \cdot 11^5$. *There are* $\boxed{2}$ *zeros.*

Hint: *Consider the 2s and 5s. (Why?)*

(b) $11!$. *There are* $\boxed{2}$ *zeros.*

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Hint: $11! = 11 \cdot 10 \cdot 9 \ldots 2 \cdot 1$, *and imagine its prime factorization. There will be plenty of 2s. Count the 5s.*

(c) $27!$. *There are* $\boxed{6}$ *zeros.*

Hint: *If you were to write out the 27 factors, 5s are contributed by the following factors: 5, 10, 15, 20, 25. And the 25 contributes a second 5.*

(d) $99!$. *There are* $\boxed{19+3}$ *zeros.*

Hint: *Among the 99 factors, there are 19 multiples of 5 and 3 multiples of 25.*

(e) $1001!$. *There are* $\boxed{200+40+8+1}$ *zeros.*

Hint: *Among the 1001 factors, there are 200 multiples of 5, 40 multiples of 25, 8 multiples of 125, and 625.*

In each case, explain your reasoning.

Problem 4 *Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.*

(a) $7|56$. *(True ✓/ False)*

Hint: $56 = 7 \cdot 8$.

(b) $55|11$. *(True/ False ✓)*

Hint: *But* $11|55$.

(c) $3|40$. *(True/ False ✓)*

Hint: $40 = 3 \cdot 39 + 1$. *Division by 3 gives remainder 1.*

(d) $100|(2^4 \cdot 3^{17} \cdot 5^2 \cdot 7)$ *(True ✓/ False)*

Hint: $100 = 2^2 5^2$.

(e) $5555|(5^{20} \cdot 7^9 \cdot 11^{11} \cdot 13^{23})$ *(True/ False ✓)*

Hint: $5555 = 5 \cdot 11 \cdot 101$.

(f) $3|(3+6+9+\cdots+300+303)$ *(True ✓/ False)*

Hint: *3 divides each of the terms.*

Problem 5 *Suppose that*

$$(3^5 \cdot 7^9 \cdot 11^x \cdot 13^y)|(3^a \cdot 7^b \cdot 11^{19} \cdot 13^7)$$

What values of a, b, x and y, make true statements? Explain your reasoning.

- a *($\geqslant$ ✓/ $=$/ $\leqslant$)* $\boxed{5}$.
- b *($\geqslant$ ✓/ $=$/ $\leqslant$)* $\boxed{9}$.
- x *($\geqslant$/ $=$/ $\leqslant$ ✓)* $\boxed{19}$.
- y *($\geqslant$/ $=$/ $\leqslant$ ✓)* $\boxed{7}$.

Problem 6 *Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.*

(a) *If $7|13a$, then $7|a$. (True ✓/ False)*

Hint: *Follows from Euclid's Lemma because 7 is prime.*

(b) *If $6|49a$, then $6|a$. (True ✓/ False)*

Hint: *Because $6 = 2 \cdot 3$ is not prime, we handle its prime factors separately.*
Because $6|49a$, it must be that $2|49a$. Then $2|a$ by Euclid's Lemma.
Similarly, because $6|49a$, it must be that $3|49a$. Then $3|a$ by Euclid's Lemma.
Because both $2|a$ and $3|a$, it follows that $6|a$.

(c) *If $10|65a$, then $10|a$. (True/ False ✓)*

Hint: *Counterexample: $a = 2$.*

(d) *If $14|22a$, then $14|a$. (True/ False ✓)*

Hint: *Counterexample: $a = 7$.*

(e) $54|931^{21}$. *(True/ False ✓)*

Hint: *54 is even (i.e., it has 2 as a factor), but 931^{21} is not.*

(f) $54|810^{33}$. *(True ✓/ False)*

Hint: *From $54 = 2 \cdot 3^3$ and $810 = 2 \cdot 5 \cdot 3^4$, we can see that $54|810$.*

Problem 7 *Joanna thinks she can see if a number is divisible by 24 by checking to see if it's divisible by 4 and divisible by 6. She claims that if the number is divisible by 4 and by 6, then it must be divisible by 24.*

Lindsay has a similar divisibility test for 24: She claims that if a number is divisible by 3 and by 8, then it must be divisible by 24.

Are either correct? Explain your reasoning.

Joanna is (correct/ incorrect ✓). Lindsay is (correct ✓/ incorrect).

Free Response: ***Hint:*** *For Joanna, the least (positive) counterexample is* $\boxed{12}$. *Her method doesn't work because the* $gcd(6,4) = \boxed{2}$, *so there are common multiples* before $6 \cdot 4$.

Lindsay's method, works because $gcd(3,8) = \boxed{1}$, *so every common multiple of* 3 *and* 8 *is also a multiple of* $3 \cdot 8$.

Problem 8 *Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.*

(a) *If* $a^2 | b^2$, *then* $a|b$. *(True ✓/ False)*

Hint: *However many times a prime appears in the prime factorizations of* a, *it will appear twice as many times in the prime factorization of* a^2. *Same idea for* b *and* b^2. *Because* $a^2|b^2$, *we know that* $b^2 = ka^2$ *for some integer* k. *Both* b^2 *and* a^2 *must have an even number of factors any prime, which implies that* k *must also have an even number of factors of that prime. This means that* k *is a perfect square, which is to say it is the square of some integer* c. *Substituting* $k = c^2$, *we find that* $b^2 = c^2a^2 = (ca)^2$. *Assuming* $a, b, c > 0$, *we have* $b = ca$, *which means that* $a|b$.

(b) *If* $a|b^2$, *then* $a|b$. *(True ✓/ False)*

Hint: *Counterexample:* $a = 12$, $b = 6$.

(c) *If* $a|b$ and $\gcd(a, b) = 1$, *then* $a = 1$. *(True ✓/ False)*

Hint: *Because* $a|b$ *there is an integer* k *such that* $b = ak$. *Because* $\gcd(a, b) = 1$, *every prime in the factorization of* b *is not a factor of* a *and therefore must be in* k. *This implies that* k *has the same prime factorization of* b. *Assuming* $a, b, k > 0$, *this means that* $b = k$ *and therefore* $a = 1$.

Problem 9 *Suppose x and y are integers. If $x^2 = 11 \cdot y$, what can you say about y? Explain your reasoning.*

Free Response: ***Hint:*** *Imagine comparing the prime factorizations of x, x^2, and y. However many times a prime appears in the prime factorization of x, it will appear twice as many times in the prime factorization of x^2. Because $11|(11y)$, it follows that $11|x^2$. Because x^2 must have an even number of factors of 11, y will have an odd number of factors of 11. And y will have an even number of each of its other prime factors.*

Problem 10 *Suppose x and y are integers. If $x^2 = 25 \cdot y$, what can you say about y? Explain your reasoning.*

Free Response: ***Hint:*** *Because x^2, $25y^2$, and 25 are all perfect squares, y must also be a perfect square*