

# Algorithms

*Problems about algorithms.*

**Problem 1** Explain what it means for an operation  $\star$  to be *associative*. Give some relevant and revealing examples and non-examples.

**Problem 2** Consider the following pictures: Jesse claims that these pictures represent  $(2 \cdot 3) \cdot 4$  and  $2 \cdot (3 \cdot 4)$ .

- (a) Is Jesse's claim correct? Explain your reasoning.
- (b) Do Jesse's pictures show the associativity of multiplication? If so, explain why. If not, draw new pictures representing  $(2 \cdot 3) \cdot 4$  and  $2 \cdot (3 \cdot 4)$  that do show the associativity of multiplication.

**Problem 3** Explain what it means for an operation  $\star$  to be *commutative*. Give some relevant and revealing examples and non-examples.

**Problem 4** Explain what it means for an operation  $\star$  to *distribute* over another operation  $\dagger$ . Give some relevant and revealing examples and non-examples.

**Problem 5** Explain what it means for an operation  $\star$  to be *closed* on a set of numbers. Give some relevant and revealing examples and non-examples.

**Problem 6** Sometimes multiplication is described as *repeated addition*. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.

Learning outcomes:

Author(s): Bart Snapp and Brad Findell

**Problem 7** In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?

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**Problem 8** Money Bags Jon likes to give a tip of 20% when he is at restaurants. He does this by dividing his bill by 10 and then doubling it. Explain why this works.

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**Problem 9** Regular Reggie likes to give a tip of 15% when he is at restaurants. He does this by dividing his bill by 10 and then adding half more to this number. Explain why this works.

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**Problem 10** Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.

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**Problem 11** Cheap Carl likes to give a tip of  $13\frac{1}{3}\%$  when he is at restaurants. He does this by dividing his bill by 10 and then adding one-third more to this number. Explain why this works.

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**Problem 12** Reasonable Rebbecca likes to give a tip of 18% when she is at restaurants. She does this by dividing her bill by 5 and then removing one-tenth of this number. Explain why this works.

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**Problem 13** Can you think of and justify any other schemes for computing the tip?

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**Problem 14** Here is an example of a standard addition algorithm:

$$\begin{array}{r} 11 \\ 892 \\ + 398 \\ \hline 1290 \end{array}$$

- (a) *Describe how to perform this algorithm.*
- (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
- (c) *Show the “behind-the-scenes” algebra that is going on here.*

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**Problem 15** *Here is an example of the column addition algorithm:*

$$\begin{array}{r} 892 \\ +398 \\ \hline 10 \\ 18 \\ 11 \\ \hline 1290 \end{array}$$

- (a) *Describe how to perform this algorithm.*
  - (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 16** *If you check out Problems and , you will learn about “partial” algorithms.*

- (a) *Develop a “partial” algorithm for addition, give it a name, and describe how to perform this algorithm.*
  - (b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 17** *Here is an example of the banker’s addition algorithm:*

$$\begin{array}{r} 892 \\ + 398 \\ \hline 10 \\ 19 \\ 12 \\ \hline 1290 \end{array}$$

- (a) *Describe how to perform this algorithm.*
  - (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 18** *Here is an example of a standard subtraction algorithm:*

$$\begin{array}{r}
 8 \\
 8 \cancel{9}^1 2 \\
 - 378 \\
 \hline
 514
 \end{array}$$

- (a) *Describe how to perform this algorithm.*
  - (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 19** Here is an example of the subtraction by addition algorithm:

$$\begin{array}{r} 892 \\ -378 \\ \hline 514 \end{array} \quad \rightsquigarrow \quad \begin{array}{l} 8 + \mathbf{4} = 12 \quad \text{add 1 to 7 to get 8} \\ 8 + \mathbf{1} = 9 \\ 3 + \mathbf{5} = 8 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.

**Problem 20** Here is an example of the Austrian subtraction algorithm:

$$\begin{array}{r} 8 \ 9^1 2 \\ - 3 \ 8 \cancel{7} \ 8 \\ \hline 5 \ 1 \ 4 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.



**Problem 21** *If you check out Problems and , you will learn about “partial” algorithms.*

- (a) *Develop a “partial” algorithm for subtraction, give it a name, and describe how to perform this algorithm.*
  - (b) *Provide a relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 22** *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} 23 \\ 634 \\ \times \quad 8 \\ \hline 5072 \end{array}$$

- (a) *Describe how to perform this algorithm.*
- (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
- (c) *Show the “behind-the-scenes” algebra that is going on here.*

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**Problem 23** *Here is an example of the partial-products algorithm:*

$$\begin{array}{r} 634 \\ \times 8 \\ \hline 4800 \\ 240 \\ 32 \\ \hline 5072 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.

**Problem 24** Here is an example of a standard division algorithm:

$$\begin{array}{r}
 97 \text{ R } 1 \\
 \hline
 8 \overline{) 777} \\
 \underline{72} \phantom{0} \\
 57 \\
 \underline{56} \\
 1
 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.

**Problem 25** *Here is an example of the partial quotients algorithm:*

$$\begin{array}{r} 7 \\ 90 \\ \hline 8 \overline{) 777} \\ 720 \\ \hline 57 \\ 56 \\ \hline 1 \end{array}$$

- (a) *Describe how to perform this algorithm.*
  - (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 26** *Here is another example of the partial-quotients division algorithm:*

$$\begin{array}{r}
 4 \\
 10 \\
 10 \\
 10 \\
 \hline
 8 \overline{) 277} \\
 \phantom{0} 80 \\
 \hline
 \phantom{0} 197 \\
 \phantom{0} 80 \\
 \hline
 \phantom{0} 117
 \end{array}$$



## Algorithms

- (a) *Describe how to perform this algorithm—be sure to explain how this is different from the scaffolding division algorithm.*
  - (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
  - (c) *Show the “behind-the-scenes” algebra that is going on here.*
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**Problem 27** *Here is an example of a standard multiplication algorithm:*

$$\begin{array}{r} 634 \\ \times 216 \\ \hline 3804 \\ 6340 \\ 126800 \\ \hline 136944 \end{array}$$

- (a) *Describe how to perform this algorithm.*
- (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
- (c) *Show the “behind-the-scenes” algebra that is going on here—you may*

assume that you already know the algebra behind the standard multiplication algorithm.

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**Problem 28** Here is an example of the addition algorithm with decimals:

$$\begin{array}{r} 1 \\ 37.2 \\ + 8.74 \\ \hline 45.94 \end{array}$$

- (a) Describe how to perform this algorithm.
- (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
- (c) Show the “behind-the-scenes” algebra that is going on here.

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**Problem 29** Here is an example of the multiplication algorithm with decimals:

$$\begin{array}{r} 3.40 \\ \times .21 \\ \hline 340 \\ 6800 \\ \hline .7140 \end{array}$$

(a) Describe how to perform this algorithm.

- (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*
- (c) *Show the “behind-the-scenes” algebra that is going on here.*

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**Problem 30** *Here is an example of the division algorithm without remainder:*

$$\begin{array}{r}
 0.75 \\
 \hline
 4 \overline{) 3.00} \\
 \underline{28} \phantom{0} \\
 20 \\
 \underline{20} \\
 \hline
 \hline
 \end{array}$$

- (a) *Describe how to perform this algorithm.*
- (b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

(c) Show the “behind-the-scenes” algebra that is going on here.

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**Problem 31** In the following addition problem, every digit has been replaced with a letter.

$$\begin{array}{r}
 MOON \\
 + \quad SUN \\
 \hline
 PLUTO
 \end{array}$$

Recover the original problem and solution. Explain your reasoning. Hint:  $S = 6$  and  $U = 5$ .

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**Problem 32** In the following addition problem, every digit has been replaced with a letter.

$$\begin{array}{r} SEND \\ + MORE \\ \hline MONEY \end{array}$$

Recover the original problem and solution. Explain your reasoning.

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**Problem 33** In the following subtraction problem, every digit has been replaced with a letter.

$$\begin{array}{r}
 DEFER \\
 - DU7Y \\
 \hline
 N2G2
 \end{array}$$

Recover the original problem and solution. Explain your reasoning.

**Problem 34** In the following two subtraction problems, every digit has been replaced with a letter.

$$\begin{array}{r}
 NINE \\
 - TEN \\
 \hline
 TWO
 \end{array}$$

$$\begin{array}{r}
 NINE \\
 - ONE \\
 \hline
 ALL
 \end{array}$$

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

**Problem 35** In the following multiplication problem, every digit has been replaced with a letter.



$$\begin{array}{r}
 \text{LET} \\
 \times \text{ NO} \\
 \hline
 \text{SOT} \\
 \text{NOT} \\
 \hline
 \text{FRET}
 \end{array}$$

Recover the original problem and solution. Explain your reasoning.

**Problem 36** The following is a long division problem where every digit except

7 was replaced by X.

$$\begin{array}{r}
 \phantom{XX}X7X \\
 \hline
 XX \overline{) XXXXX} \\
 \phantom{XX}X77 \\
 \hline
 \phantom{XX}X7X \\
 \phantom{XX}X7X \\
 \hline
 \phantom{XX}XX \\
 \phantom{XX}\underline{\underline{XX}}
 \end{array}$$

Recover the digits from this long division problem. Explain your reasoning.

**Problem 37** The following is a long division problem where the various digits were replaced by X except for a single 8. The double bar indicates that the remainder is 0.

$$\begin{array}{r}
 \phantom{XXX} \overline{XX8XX} \\
 XXX \overline{) XXXXXXXXX} \\
 \phantom{XXXX} \underline{XXX} \\
 \phantom{XXXX} XXXX \\
 \phantom{XXXX} \underline{XXX} \\
 \phantom{XXXX} XXXX \\
 \phantom{XXXX} \underline{XXXX} \\
 \phantom{XXXX} \underline{\underline{XXXX}}
 \end{array}$$

Recover the digits from this long division problem. Explain your reasoning.

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