Key Proofs

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Contents

Introduction

Online proof project description.

Student performance is generally quite poor on Ohio's end-of-course exams for Algebra 1, Geometry, Math 1, and Math 2, especially on items involving proof. In response to concerns that the items are too difficult, the following pages provide examples of more accessible computer-scorable proof items, mostly in Geometry. The proofs are written to focus on the most important steps and reasons in the argument.

For the convenience of teachers using an integrated curriculum, items are separated into two groups: those appropriate for Math 1, and those appropriate for Math 2, according to Ohio's assessments.

Fix note: This is work in progress. Draft items are first written as complete proofs in order to consider which expressions, words, or phrases students might be prompted to enter.

Questions to reviewers are written in red. Please send comments to Brad Findell, `findell.2@osu.edu`, Department of Mathematics, The Ohio State University.

The Ximera Environment

Students complete the proofs by filling in blanks, pulling down menus, and selecting correct answers. In Ximera, some answers are checked automatically when they are chosen. Others answers require pressing Enter, clicking the blue question mark, or clicking the blue "Check Work" button. Give it a try!

Example 1. *Some problems are multiple-choice:*

Multiple Choice:

(a) *Don't pick me.*

(b) *Not me either.*

(c) *Pick me!*

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(d) *Also an incorrect choice*

Example 2. *Some problems are select-all that are correct:*

Select All Correct Answers:

(a) *Don't pick me.*

(b) *Pick me!*

(c) *Pick me too!*

(d) *I'm a correct choice too.*

Example 3. *Some problems use (purple haze/ purple rain/ pull-down menus).*

Example 4. *Some problems are fill in the blank:* $3 \times 2 = \boxed{?}$

Part I

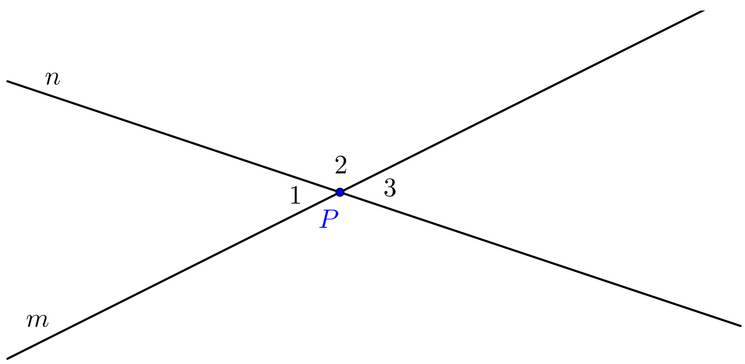
Math 1

Vertical Angles

Proofs updated.

Below are three different proofs that vertical angles are congruent. Please consider them separately.

Problem 1 *Point P is the intersection of lines m and n. Prove that $\angle 1 \cong \angle 3$.*



Fix note: When students write equations about linear pairs, they often write two equations for non-overlapping linear pairs—which doesn't help. The figure above is intended to help avoid that dead end, but it would be worthwhile to discuss that dead end anyway.

Proof *Using adjacent angles, $\angle 1 \cong \angle 3$ because they are both (complementary / supplementary / opposite / congruent) to $\angle 2$.* ■

Problem 2 *Point P is the intersection of lines m and n. Prove that $\angle 1 \cong \angle 3$.*

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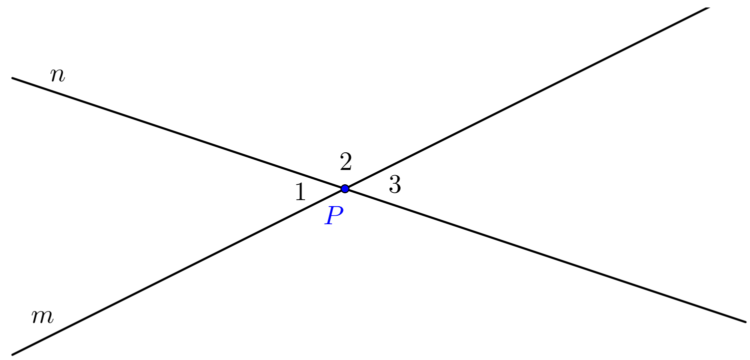
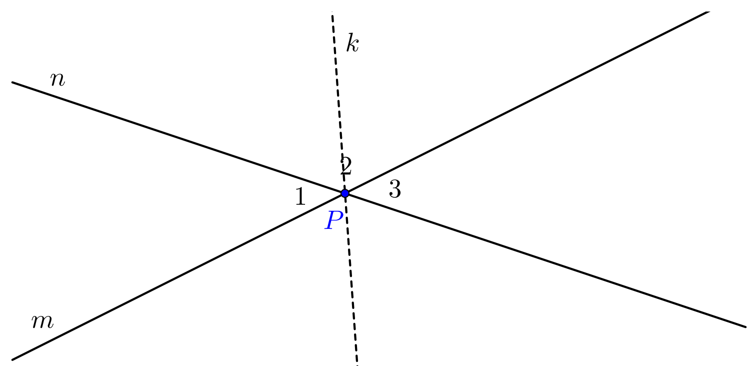


Proof *A rotation of (90° / 180° / 360°) about P maps m onto itself, maps n onto itself, and swaps $\angle 1$ and ($\angle 1$/ $\angle 2$/ $\angle 3$). Because rotations preserve angle measures, it must be that $\angle 1 \cong \angle 3$.* ■

Problem 3 *Point P is the intersection of lines m and n. Prove that $\angle 1 \cong \angle 3$.*



Proof *Reflecting about the (bisector/ supplement/ opposite) of $\angle 2$ swaps the sides of $\angle 2$ and therefore lines m and n. Thus, that reflection swaps $\angle 1$ and ($\angle 1$/ $\angle 2$/ $\angle 3$). Because reflections preserve angle measures, it follows that $\angle 1 \cong \angle 3$.* ■

Parallel Lines

Proofs updated.

This page develops important results regarding parallel lines and transversals. **Read carefully, and complete the proofs.**

Axiom 1. *Parallel postulate (uniqueness of parallels): Given a line and a point not on the line, there is exactly one line through the given point parallel to the given line.*

Problem 4

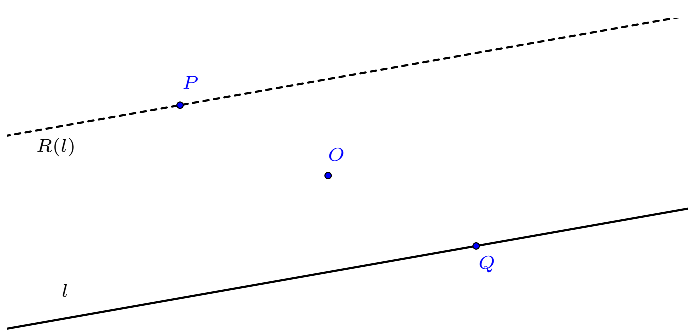
Theorem 1. *A 180° rotation about a point on a line takes the line to itself.*

Proof *Suppose point P is on line k. The point cuts the line into two opposite* $\boxed{?}$. *A 180° rotation about P swaps the two opposite rays, thereby mapping the line onto itself.* ■

Problem 5

Theorem 2. *A 180° rotation about a point not on a line takes the line to a parallel line.*

Proof *Let O be a point not on line l. Let P be an arbitrary point on $R(l)$, the rotated image of l. To show that $R(l)$ is parallel to l, it is sufficient to show that P cannot lie also on l.*



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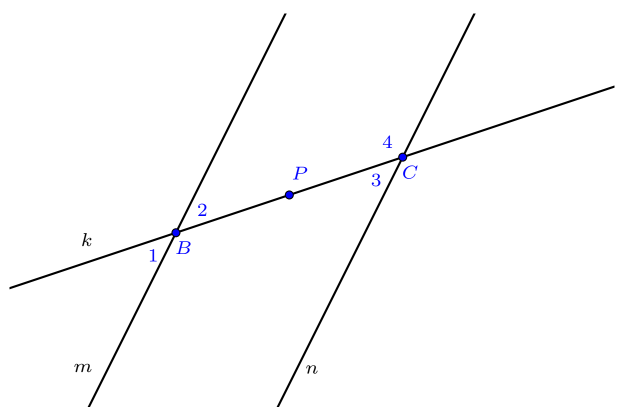
Because P is on $R(l)$, there is a point Q on l such that $P = R(Q)$. The rotated image of $\overrightarrow{OQ}$ is ($\overrightarrow{QO}$ / $\overrightarrow{OP}$ / $\overrightarrow{QP}$), and because $\angle QOP$ is $180°$, it follows that Q, O, and P are $\boxed{?}$. Call that line k. We know line k is distinct from l because point $\boxed{?}$ is on k but not on l. Now, if P were on l, then points P and Q would be on two distinct lines, k and l, contradicting the assumption that on two points there is a unique line. The theorem is proved. ■

Problem 6

Theorem 3. *If two parallel lines are cut by a transversal, alternate interior angles and corresponding angles are congruent.*

Proof *Given that parallel lines m and n are cut by transversal k, prove that alternate interior angles are congruent.*



Let B and C be the intersections of transversal k with lines m and n, respectively. Let P be the midpoint of $\overline{BC}$.

(a) *Rotate $180°$ about P, which takes k to (itself/ m / n).*

(b) *The rotation maps B to $\boxed{?}$ because $PB = PC$ and the rotation preserves distances.*

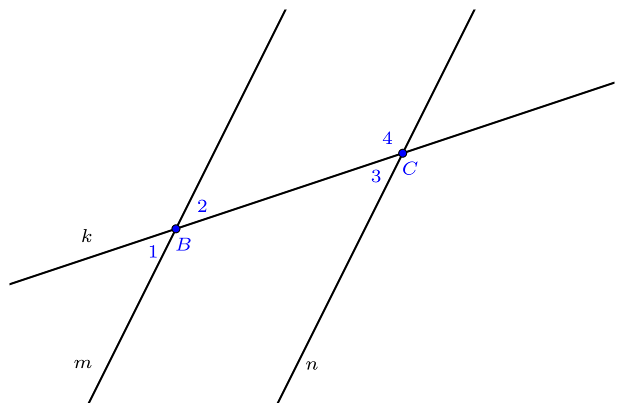
(c) *Because P is not on m, the rotation maps m to a parallel line through C, which must be (k / m / n) by the uniqueness of parallels.*

(d) *Thus, the rotation maps $\angle 2$ to ($\angle 1$/ $\angle 2$/ $\angle 3$/ $\angle 4$). These alternate interior angles must be congruent because the rotation preserves angle measures.*

■

Note: *The congruence of corresponding angles now follows from the congruence of vertical angles. But the next problem is another approach that uses a translation.*

Problem 7 ***Proof*** *Given that parallel lines m and n are cut by transversal k, prove that corresponding angles are congruent.*



Let B and C be the intersections of transversal k with lines m and n, respectively.

(a) *Translate to the right along line k by distance BC, which takes k to (itself / m/ n).*

(b) *The translation maps B to $\boxed{?}$, and it maps m to (k / m / n) because the translation maintains parallels, and there is a unique parallel to m through C.*

(c) *The translation maps $\angle 1$ to ($\angle 1$/ $\angle 2$/ $\angle 3$/ $\angle 4$). These corresponding angles must be congruent because the translation preserves angle measures.*

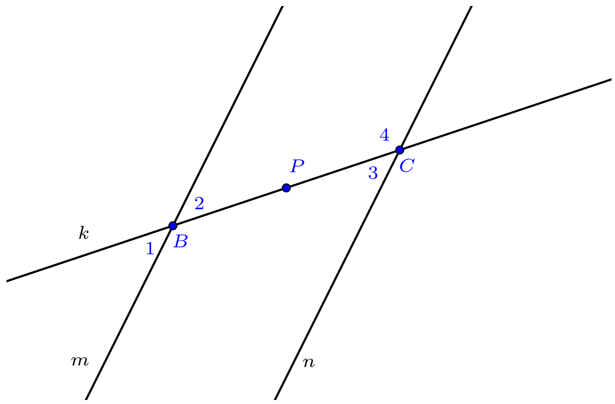
■

Problem 8

Theorem 4. *If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.*

Note: *This theorem is the $\boxed{?}$ of the previous theorem about alternate interior angles.*

Proof *Given that m and n are cut by transversal k with alternate interior angles congruent, prove that lines m and n are parallel.*



Let B and C be the intersections of transversal k with lines m and n, respectively. Let P be the midpoint of $\overline{BC}$.

(a) *Rotate 180° about P, which takes k to (itself / m / n), and which swaps B and $\boxed{?}$ because distances are preserved.*

(b) *Because $\angle 2 \cong \angle 3$ and because a side of $\angle 2$ (i.e., $\overrightarrow{BP}$) is mapped to a side of $\angle 3$ (i.e., ($\overrightarrow{CP}$ / $\overrightarrow{PC}$ / $\overrightarrow{BP}$)), it must be that the other side of $\angle 2$ (which lies on m) is mapped to the other side of $\angle 3$ (which lies on line $\boxed{?}$). Thus, n is the image of m.*

(c) *Because P is not on m, the 180° rotation maps m to a parallel line through C. Thus, n must be parallel to m.*

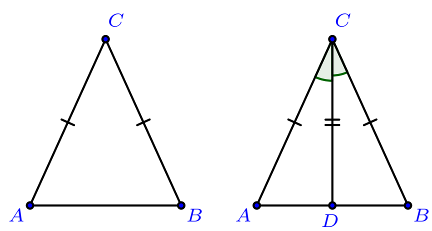
■

Isosceles Triangle Theorem

Proofs updated.

Fix note: Below are several different proofs, along with one that is not a proof. Please consider them separately.
Any (or all) of the proofs might be extended to conclude that, in the case of an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude all lie on the same line.

Problem 9 *Prove that the base angles of an isosceles triangle are congruent.*

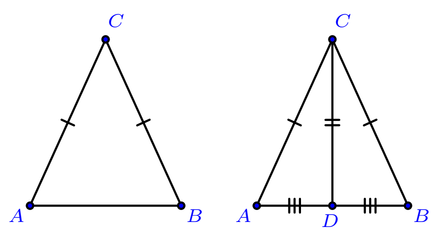


(a) *Beginning with the given figure on the left, Morgan draws $\overline{CD}$ and marks the figure intending that this new segment is a(n) (median/ angle bisector / perpendicular bisector/ altitude).*

(b) *Based on the marked figure, Morgan claims that the $\triangle ACD \cong \triangle\boxed{?}$ by (SAS/ SSS/ SSA/ ASA/ HL).*

(c) *Finally, Morgan concludes that $\angle A \cong \angle\boxed{?}$, as they are corresponding parts of congruent triangles.*

Problem 10 *Prove that the base angles of an isosceles triangle are congruent.*



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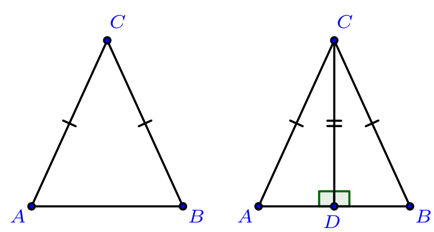
(a) *Beginning with the given figure on the left, Deja draws $\overline{CD}$ and marks the figure intending that this new segment is a(n) (median/ angle bisector/ perpendicular bisector/ altitude).*

(b) *Based on the marked figure, Deja claims that the $\triangle ACD \cong \triangle\boxed{?}$ by (SAS / SSS/ SSA/ ASA/ HL).*

(c) *Finally, Deja concludes that $\angle A \cong \angle\boxed{?}$, as they are corresponding parts of congruent triangles.*

Problem 11 *Prove that the base angles of an isosceles triangle are congruent.*

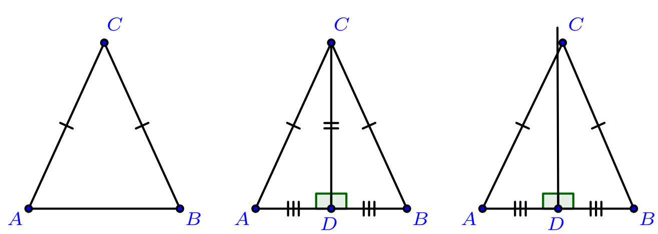


(a) *Beginning with the given figure on the left, Elle draws $\overline{CD}$ and marks the figure intending that this new segment is a(n) (median/ angle bisector/ perpendicular bisector/ altitude).*

(b) *Based on the marked figure, Elle claims that the $\triangle ACD \cong \triangle\boxed{?}$ by (SAS / SSS/ SSA/ ASA/ HL).*

(c) *Finally, Elle concludes that $\angle A \cong \angle\boxed{?}$, as they are corresponding parts of congruent triangles.*

Problem 12 *Simon and Taylor are trying to prove that the base angles of an isosceles triangle are congruent.*



Given figure *Simon's figure* *Taylor's figure*

Beginning with the given figure on the left, Simon draws $\overline{CD}$ and marks the second figure intending that this new segment is a perpendicular bisector of $\overline{AB}$.

Taylor claims that a perpendicular bisector of a side of a triangle usually misses the opposite vertex, so the figure should allow for that possibility.

Fix note: Taylor's claim, the prompt, and the choices below need attention. Simon's figure suggests congruence by "SSAS," which might indicate that too much is being assumed. Would that make sense as a distractor?

Without using other facts about isosceles triangles or perpendicular bisectors, choose the best assessment of their disagreement:

Multiple Choice:

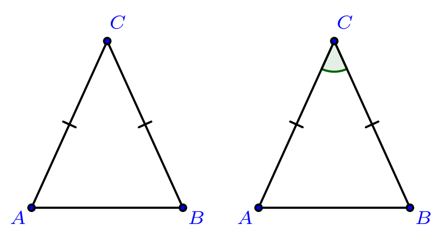
(a) *Simon is correct, and $\triangle ACD \cong \triangle BCD$ by SAS.*

(b) *Simon is correct, and $\triangle ACD \cong \triangle BCD$ by SSS*

(c) *Taylor is correct, and the perpendicular bisector should not be used to complete this proof.*

(d) *Neither of them are correct.*

Problem 13 *Prove that the base angles of an isosceles triangle are congruent.*



Given figure

(a) *Examining the given figure on the left, Lissy notices symmetry in the triangle and claims that the triangle is congruent to itself by a (translation / reflection/ rotation).*

(b) *Based on the marked figure, Lissy claims that the $\triangle ACB \cong \triangle\boxed{?}$ by (SAS/ SSS/ SSA/ ASA/ HL).*

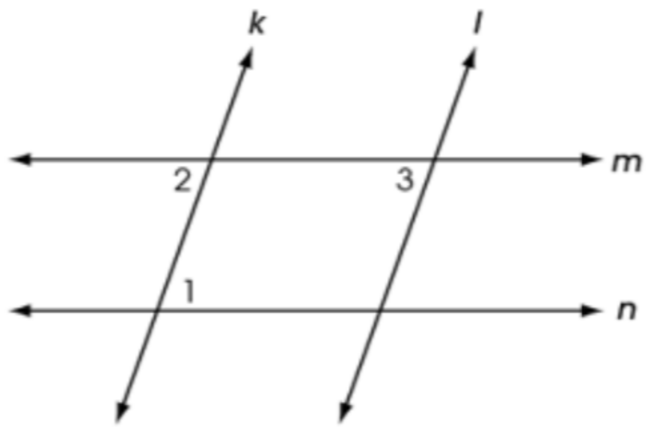
(c) *Finally, Lissy concludes that $\angle A \cong \angle\boxed{?}$, as they are corresponding parts of congruent triangles.*

Quadrilaterals

Proof.

Problem 14 *Adapted from Ohio's 2017 Geometry released item 13.*

Two pairs of parallel lines intersect to form a parallelogram as shown.



Complete the following proof that opposite angles of a parallelogram are congruent:

(a) $\angle 1 \cong \angle 2$ as *(opposite angles / alternate interior angles / corresponding angles) for parallel lines* (m and n / k and l).

(b) $\angle 3 \cong \angle 2$ as *(opposite angles / alternate interior angles / corresponding angles)for parallel lines* (m and n / k and l).

(c) *Then* $\angle 1 \cong \angle 3$ *because they are both congruent to* $\angle 2$.

Problem 15 *Adapted from Ohio's 2018 Geometry released item 21.*

Given the parallelogram $WXYZ$, *prove that* $\overline{WX} \cong \overline{YZ}$.

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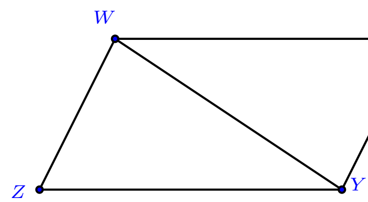


> *Fix note: It really would help to have an online environment that allows students to mark diagrams.*

Complete the proof below:

(a) $\angle ZWY \cong \angle XYW$ as (alternate interior angles/ corresponding angles/ opposite angles) for parallel segments ($\overline{WZ}$ and $\overline{XY}$ / $\overline{WX}$ and $\overline{YZ}$).

(b) $\angle ZYW \cong \angle XWY$ for the same reason, this time for parallel segments ($\overline{WZ}$ and $\overline{XY}$ / $\overline{WX}$ and $\overline{YZ}$).

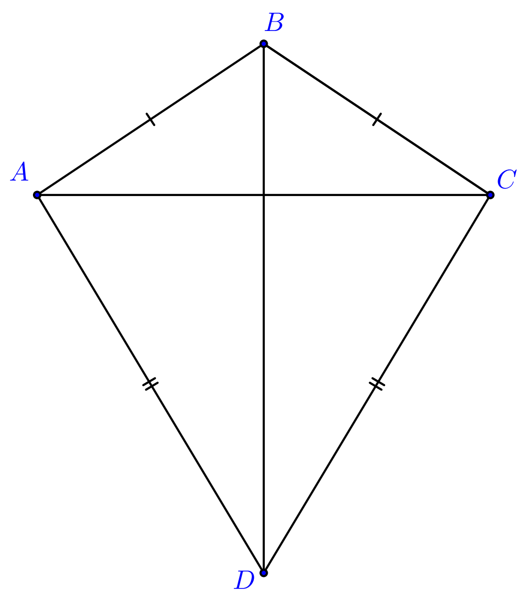
(c) $\overline{WY} \cong \overline{YW}$ because a segment is congruent to itself.

(d) $\triangle WYZ \cong \triangle YWX$ by (SAS/ ASA/ SSS).

(e) Then $\overline{YZ} \cong \overline{WX}$ as corresponding parts of congruent triangles.

> *Fix note: Maybe number the angles.*

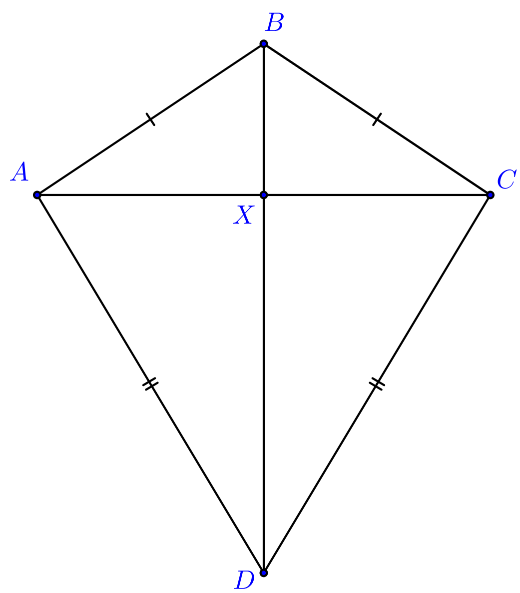
Problem 16 *Quadrilateral $ABCD$ is a kite as marked. Prove that $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$.*



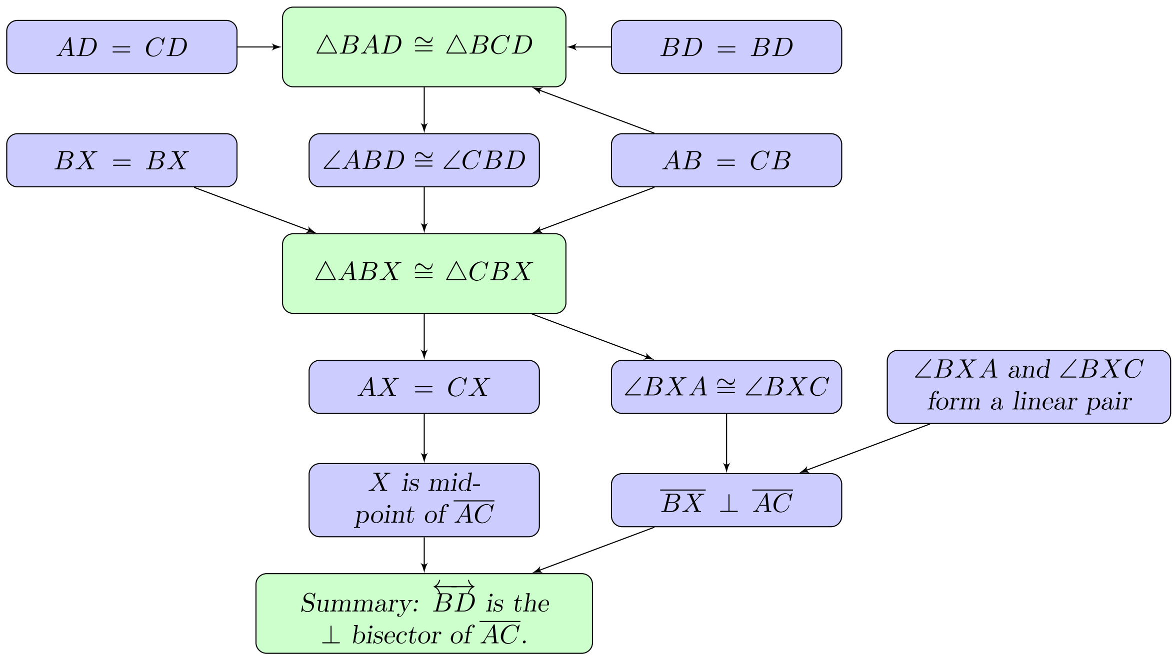
Key theorem: The points on a perpendicular bisector are exactly those that are equidistant from the endpoints of a segment.

Proof: Because B and D are each $\boxed{?}$ from A and C, they each must lie on the perpendicular bisector of segment $\boxed{?}$, which implies that $\overleftrightarrow{BD}$ is its perpendicular bisector.

Problem 17 *Quadrilateral $ABCD$ is a kite as marked. Prove that $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$.*



A proof that makes use of triangle congruence:



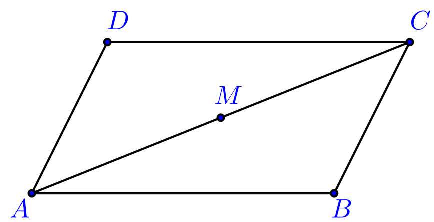
Fix note: Do we need a step about $\overleftrightarrow{BX}$ and $\overleftrightarrow{BD}$ being the same line?

In the proof above, $\triangle BAD \cong \triangle BCD$ *by* $\boxed{?}$, *and* $\triangle ABX \cong \triangle CBX$ *by* $\boxed{?}$.

Quadrilateral Symmetry

Proof.

Problem 18 *Use symmetry to prove properties of parallelograms.*



Consider a 180° rotation about M, the midpoint of diagonal $\overline{AC}$. Show that this rotation maps the parallelogram onto itself.

Fix note: The following proof is quite elegant, but some of the details are subtle, especially distinguishing between mapping the sides (i.e., segments) and the lines containing the sides. Can any of this be omitted or abbreviated? Which parts might students supply?

(a) *The rotation maps A to C and C to A because a 180° rotation about a point on a line takes the line to itself and preserves lengths.*

(b) *The rotation maps $\overleftrightarrow{AB}$ to a parallel line through C (the image of A), which by the uniqueness of parallels must be $\overleftrightarrow{CD}$. Similarly, the rotation maps $\overleftrightarrow{CD}$ to $\overleftrightarrow{AB}$, $\overleftrightarrow{AD}$ to $\overleftrightarrow{CB}$, and $\overleftrightarrow{CB}$ to $\overleftrightarrow{AD}$.*

(c) *Furthermore, the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$, which is B, must map to the intersection of their images, $\overleftrightarrow{CD}$ and $\overleftrightarrow{AD}$, and that intersection is D. And likewise, D must map to B.*

(d) *Because vertices are mapped to vertices, sides are mapped to opposite sides, angles are mapped to opposite angles, and thus the parallelogram is mapped onto itself.*

Now this symmetry proves the following properties for free*:*

- *opposite sides are congruent (sides are mapped to opposite sides),*

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- *opposite angles are congruent (angles are mapped to opposite angles), and*
- *the diagonals bisect each other.*

  Detail: The $180°$ rotation about M swaps $\overrightarrow{MB}$ and $\overrightarrow{MD}$, so they must be opposite rays, and thus B, M, and D are collinear.
  Because the rotation preserves lengths, $MB = MD$, so that M is also the midpoint of $\overline{BD}$, which means that the diagonals bisect each other.

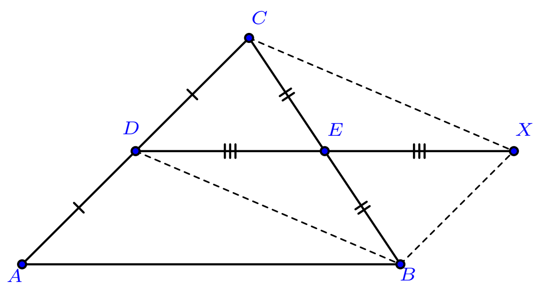
Midsegment Theorem

Proofs updated.

Theorem 5. *Midsegment Theorem: The segment joining the midpoints of two sides of a triangle is parallel to and half the length of the third side.*

In preparation for the midsegment theorem, the class proved several useful theorems about parallelograms.

Problem 19 *To prove the midsegment theorem for $\triangle ABC$ with midpoints D and E of sides AC and BC, respectively, Mitch extended $\overline{DE}$ to a point X such that $EX = DE$, as shown in the marked figure. Then he added dotted lines to the figure to show parallelograms.*



Mitch organized his reasoning in the following flow chart:

Fix note: The flowchart omits reasons to reduce clutter. The most significant steps are green whereas the details are blue.

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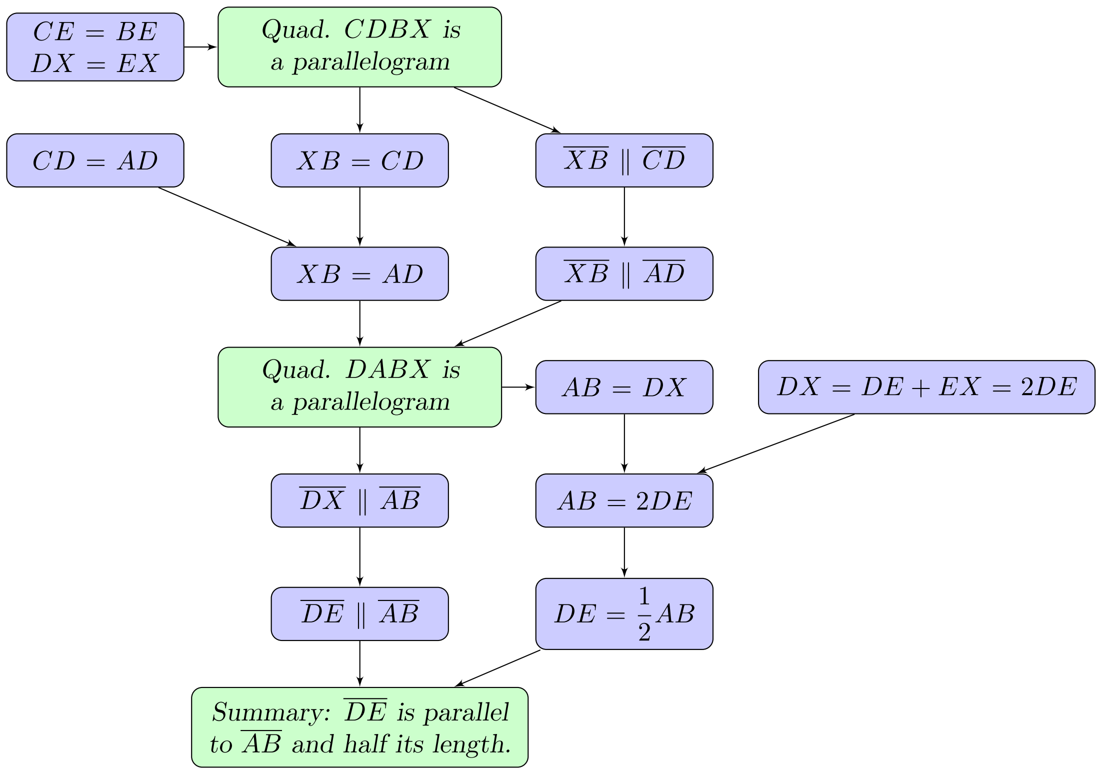


In the proof above, which theorem may Mitch use to conclude that quadrilateral $CDBX$ *a parallelogram?*

Multiple Choice:

(a) *If a pair of sides of a quadrilateral are congruent and parallel, then it is a parallelogram.*

(b) *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

(c) *If opposite sides of a quadrilateral are congruent, then it is a parallelogram.*

(d) *If opposite angles of a quadrilateral are congruent, then it is a parallelogram.*

(e) *The Pythagorean Theorem.*

(f) *None of these.*

In the proof above, which theorem may Mitch use to conclude that quadrilateral $DABX$ *a parallelogram?*

Multiple Choice:

(a) *If one pair of sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.*

(b) *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

(c) *If opposite sides of a quadrilateral are congruent, then it is a parallelogram.*

(d) *If opposite angles of a quadrilateral are congruent, then it is a parallelogram.*

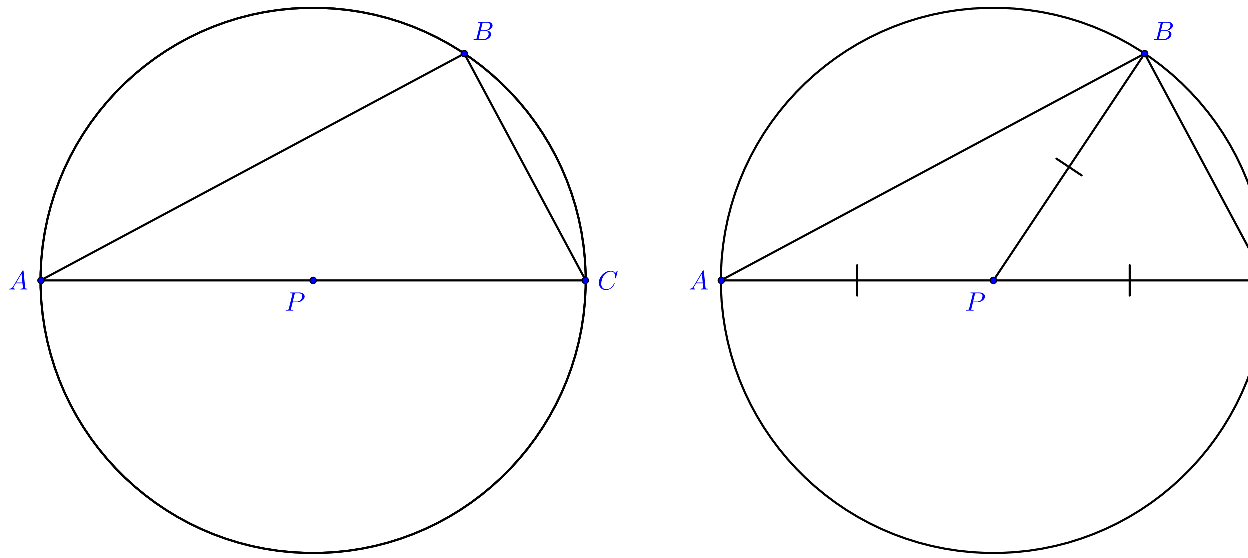
(e) *The Pythagorean Theorem.*

(f) *None of these.*

Inscribed Angles

Proofs updated.

Problem 20 *In the figure below, $\overline{AC}$ is a diameter of a circle with center P. Prove that $\angle ABC$ is a right angle.*



(a) *Beginning with the diagram on the left, Natalia draws $\overline{PB}$ and marks the diagram to show segments that she knows to be congruent because each one is a $\boxed{?}$ of the circle.*

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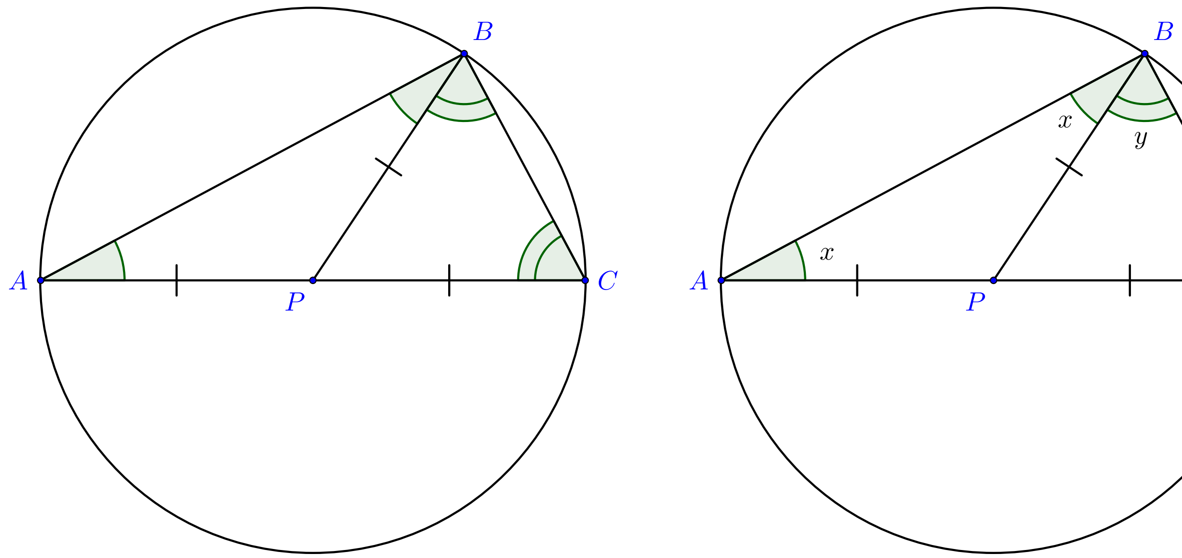


(b) *Natalia sees that* $\triangle APB$ *and* $\triangle BPC$ *are* $\boxed{?}$ *triangles, so she marks the figure to show angles that must be congruent.*

(c) *In order to do some algebra with these congruent angles, Natalia labels their measures* x *and* y*, as shown in the picture on the right.*

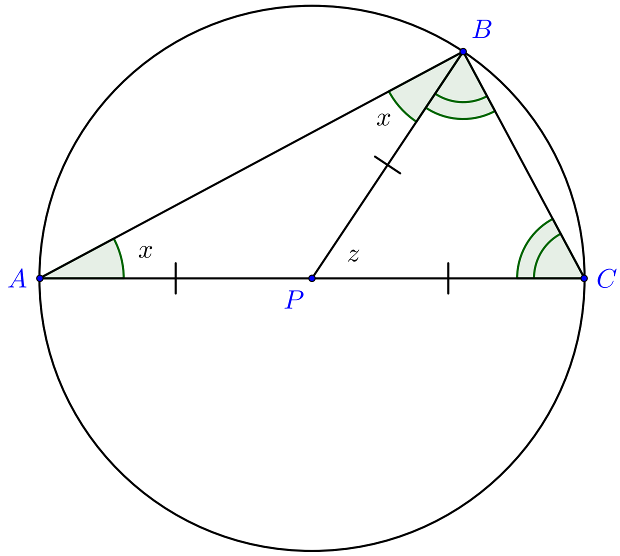
(d) *She writes an equation for the sum of the angles of* $\triangle ABC$*:*

$$\boxed{?} = 180^\circ$$

(e) *She divides that equation by* $\boxed{?}$ *to conclude that* $m\angle ABC = x + y = \boxed{?}$ *degrees.*

Problem 21 *A special case of the relationship between an inscribed angle and the corresponding central angle.*

In the figure below, $\overline{AC}$ *is a diameter of a circle with center* P*. Prove that* $z = 2x$*.*



Because z is the measure of an angle exterior to $\triangle\boxed{?}$, it is equal to the sum of the measures of the (opposite/ adjacent/ remote interior/ alternate interior) angles. In other words $z = 2x$.

Alternatively, without using the exterior angle theorem, one might proceed as follows:

(a) $\angle APB + x + x = 180^\circ$ *because of the angle sum in* $\triangle\boxed{?}$.

(b) $\angle APB + z = 180^\circ$ *because they form a linear pair.*

(c) *Then $z = 2x$ by comparing the two equations.*

Note: *This handles the special case in which the center of the circle lies on one side of the inscribed angle. For the general result, consider two cases: (1) When the center of the circle is in the interior of the inscribed angle; and (2) When the center of the circle is not in the interior of the inscribed angle.*

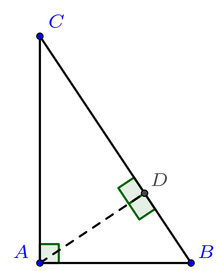
Part II

Math 2

Similar Right Triangles

Proofs.

Problem 22 *Adapted from Ohio's 2017 Geometry released item 17.*



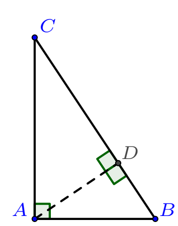
Complete the following proof that $\triangle DAC$ is similar to $\triangle DBA$:

(a) $\triangle DBA \sim \triangle\boxed{?}$ *by (AA similarity / CPCTC / right triangle similarity) because they share $\angle B$ and they each have a right angle.*

(b) $\triangle DAC \sim \triangle\boxed{?}$ *for the same reason because they share ($\angle A$ / $\angle B$ / $\angle C$) and they each have a right angle.*

(c) $\triangle DAC \sim \triangle DBA$ *because (CPCTC / right triangle similarity / they are both similar to $\triangle ABC$).*

Problem 23 *A different proof, also adapted from Ohio's 2017 Geometry released item 17.*



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Complete the following proof that $\triangle DAC$ is similar to $\triangle DBA$:

(a) $\angle B$ *and* $\angle BAD$ *are* $\boxed{?}$ *because they are acute angles in a right triangle.*

(b) $\angle DAC$ *and* $\angle BAD$ *are complementary because they are adjacent angles that form* $\angle BAC$, *which is (right/ acute/ obtuse).*

(c) $\angle B \cong \angle DAC$ *because they are both complementary to* $\angle BAD$.

(d) $\triangle DAC \sim \triangle DBA$ *by (AA similarity/ CPCTC/ right triangle similarity) because* $\angle B \cong \angle DAC$ *and they each have a right angle.*