Online HW 3

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Contents

Anatomy of Figures

Short-answer problems about centers of triangles.

Problem 1 *Compare and contrast the idea of "intersecting sets" with the idea of "intersecting lines."*

Free Response: ***Hint:*** *Geometric figures are sets of points. The intersection of two geometric figures is the set of point(s) that the figures have in common. Two lines either intersect in a single point, say A, or they do not intersect. As sets, we would say the intersection is $\{A\}$ or $\{\}$, respectively.*

Problem 2 *Place three points in the plane. Give a detailed discussion explaining how they may or may not be on a line.*

Free Response: ***Hint:*** *If the three points are distinct, then there are two possibilities:*

- *The points are collinear (i.e., they all lie on the same line).*
- *The points are not collinear. Any two of the points determine a line that does not contain the third point. For arbitrary points, this is the more likely situation.*

Problem 3 *Place three lines in the plane. Give a detailed discussion explaining how they may or may not intersect.*

Free Response: ***Hint:*** *If the three lines are distinct, then there are several possibilities:*

- *The three lines are all parallel.*
- *Two of the three lines are parallel and the third line intersects the first two.*
- *The three lines are concurrent (i.e., they all lie on the same point).*
- *The three lines are not parallel and not concurrent. Any two of the lines intersect in a point that is not on the third line. For arbitrary lines, this is the most likely situation.*

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Problem 4 *Explain how a perpendicular bisector is different from an altitude. Draw an example to illustrate the difference.*

Free Response: ***Hint:*** *A perpendicular bisector goes through the midpoint of a segment and is perpendicular to it. An altitude to a segment typically does not go through the midpoint of the segment—and it might not intersect the segment at all. Instead, the altitude goes through another vertex of the figure and is perpendicular to the (extended) line containing the segment.*

Problem 5 *Explain how a median is different from an angle bisector. Draw an example to illustrate the difference.*

Free Response: ***Hint:*** *A median extends from a vertex of a triangle to the midpoint of the opposite side (thereby bisecting that side). The angle bisector, um, bisects the angle.*

Problem 6 *What is the name of the point that is the same distance from all three sides of a triangle? Explain your reasoning.*

Free Response: ***Hint:*** *The points on an angle bisector are equidistant from the sides of the angle. So the point of concurrency of the angle bisectors of a triangle is equidistant from all three sides of the triangle. That point of concurrency is called the incenter, as it is the center of the incircle.*

Problem 7 *What is the name of the point that is the same distance from all three vertexes of a triangle? Explain your reasoning.*

Free Response: ***Hint:*** *The points on an perpendicular bisector are equidistant from the endpoints of the segment. So the point of concurrency of the perpendicular bisectors of a triangle is equidistant from all three vertices of the triangle. That point of concurrency is called the circumcenter, as it is the center of the circumcircle.*

Problem 8 *Could the circumcenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: ***Hint:*** *Yes. Try an obtuse triangle.*

Problem 9 *Could the orthocenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: ***Hint:*** *Yes. Try an obtuse triangle.*

Problem 10 *Could the incenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: ***Hint:*** *No. The incenter is the center of the incircle, which is entirely inside the triangle.*

Problem 11 *Could the centroid be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: ***Hint:*** *No. The centroid is the center of gravity of the triangle, which must lie inside the triangle.*

Problem 12 *Are there shapes that do not contain their centroid? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: ***Hint:*** *Think of figures with holes in the "middle," where the center of gravity might be.*

Problem 13 *Draw an equilateral triangle. Now draw the lines containing the altitudes of this triangle. How many orthocenters do you have as intersections of lines in your drawing? Hints:*

(a) *More than one.*

(b) *How many triangles are in the picture you drew?*

Free Response: ***Hint:*** *Consider four points: the three original points and their orthocenter. Any of these points is the orthocenter of the triangle created by the other three points. (This is quite subtle. Examine your figure closely.)*

Problem 14 *Given a triangle, construct the circumcenter. Explain the steps in your construction.*

Free Response: ***Hint:*** *The circumcenter is the point of concurrency of the perpendicular bisectors. Because the perpendicular bisectors are concurrent, it is enough to find the intersection of two of them.*

Problem 15 *Given a triangle, construct the orthocenter. Explain the steps in your construction.*

Free Response: ***Hint:*** *The orthocenter is the point of concurrency of the altitudes. Because the altitudes are concurrent, it is enough to find the intersection of two of them.*

Problem 16 *Given a triangle, construct the incenter. Explain the steps in your construction.*

Free Response: ***Hint:*** *The incenter is the point of concurrency of the angle bisectors. Because the angle bisectors are concurrent, it is enough to find the intersection of two of them.*

Problem 17 *Given a triangle, construct the centroid. Explain the steps in your construction.*

Free Response: ***Hint:*** *The centroid is the point of concurrency of the medians. Because the medians are concurrent, it is enough to find the intersection of two of them.*

Problem 18 *Given a triangle, construct the incircle. Explain the steps in your construction.*

Free Response: ***Hint:*** *The center of the incircle is the point of concurrency of the angle bisectors. As stated above, the incenter is equidistant from the sides of the triangle. To find that distance, which is the radius of the incircle, construct a perpendicular from the incenter to one side of the triangle, and then "measure" along that perpendicular bisector.*

Problem 19 *Given a triangle, construct the circumcircle. Explain the steps in your construction.*

Free Response: ***Hint:*** *The center of the circumcircle is the point of concurrency of the perpendicular bisectors. As stated above, the circumcenter is equidistant from the vertices of the triangle. So use that distance as the radius of the circumcircle.*

Problem 20 *Given a circle, give a construction that finds its center.*

Free Response: ***Hint:*** *Any three points on the circle form a triangle. The circumcenter of that triangle will be the center of the circle.*

Problem 21 *Where is the circumcenter of a right triangle? Explain your reasoning.*

Free Response: ***Hint:*** *The circumcenter is at the midpoint of the hypotenuse. The perpendicular bisector of the hypotenuse clearly contains this midpoint. And it is just a tad harder to see that the perpendicular bisectors of the legs also contains this point. (To see this, note that a midsegment to a leg lies on the perpendicular bisector of that leg.)*

Problem 22 *Where is the orthocenter of a right triangle? Explain your reasoning.*

Free Response: ***Hint:*** *The orthocenter of a right triangle is the vertex of the right angle. The altitude to the hypotenuse goes through the vertex of the right angle. Both legs of the right triangle are also altitudes of that right triangle, so they also go through the vertex of the right angle.*

Problem 23 *Can you draw a triangle where the circumcenter, orthocenter, incenter, and centroid are all the same point? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: ***Hint:*** *Try an equilateral triangle.*

Problem 24 *True or False: Explain your conclusions.*

(a) *An altitude of a triangle is always perpendicular to a line containing some side of the triangle.*

(b) *An altitude of a triangle always bisects some side of the triangle.*

(c) *The incenter is always inside the triangle.*

(d) *The circumcenter, the centroid, and the orthocenter always lie in a line.*

(e) *The circumcenter can be outside the triangle.*

(f) *The orthocenter is always inside the triangle.*

(g) *The centroid is always inside the incircle.*

Free Response: ***Hint:*** *Don't worry about (d) and (g):*

(a) *True. This is part of the definition of an altitude.*
(b) *False. Any scalene triangle will do.*
(c) *True. The incenter is the center of the incircle, which lies entirely inside the triangle.*
(d) *True. An amazing fact. Try it.*
(e) *True. Try an obtuse triangle.*
(f) *False. Try an obtuse triangle.*
(g) *False. Try a thin triangle.*

Problem 25 *Given 3 distinct points not all in a line, construct a circle that passes through all three points. Explain the steps in your construction.*

Free Response: ***Hint:*** *The three points form a triangle. Construct the circumcircle of that triangle.*

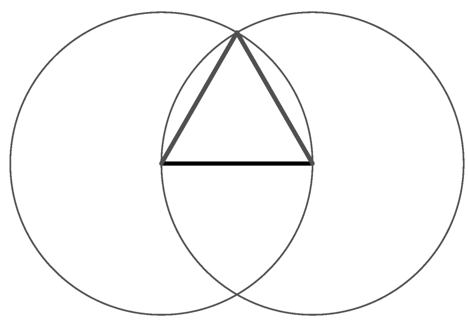
Constructions Problems

Short-answer problems about constructions.

Problem 26 *Given a line segment, construct an equilateral triangle whose edge has the length of the given segment. Explain the steps in your construction and how you know it works.*

Free Response: ***Hint:*** (a) *Draw two circles, one with each end point as the center and with the other as a point on the circle.*

(b) *The circles intersect at two points. Choose one and connect it to both of the line segment's endpoints.*



Problem 27 *Use a compass and straightedge to bisect a given line segment. Explain the steps in your construction and how you know it works.*

Free Response: ***Hint:*** (a) *Draw two circles, one with each end point as the center and with the other as a point on the circle.*

(b) *The circles intersect at two points. Draw a line through these two points.*

(c) *The new line bisects the original line segment.*

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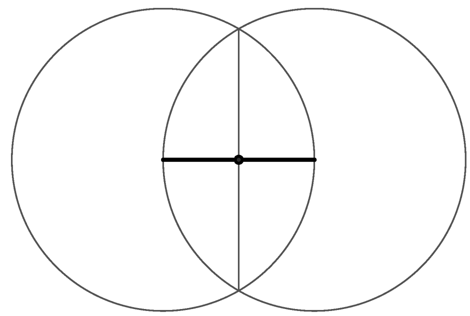
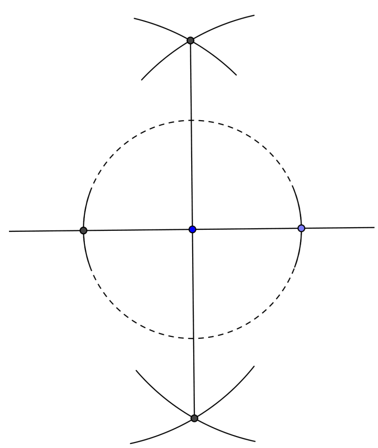


Problem 28 *Given a line segment with a point on it, construct a line perpendicular to the segment that passes through the given point. Explain the steps in your construction and how you know it works.*

Free Response: ***Hint:***

(a) *With an arbitrary radius, draw a circle to identify two points on the given line equidistant from the given point.*

(b) *Now (as above) bisect the segment defined by those two new points.*



Problem 29 *Use a compass and straightedge to bisect a given angle. Explain the steps in your construction and how you know it works.*

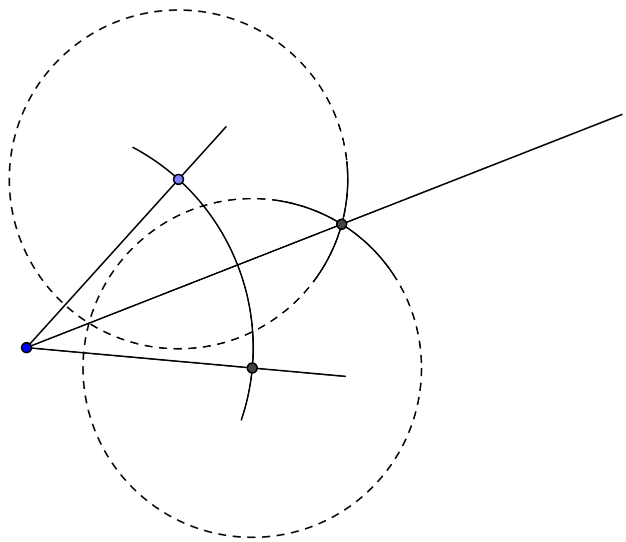
Free Response: ***Hint:***

(a) *Draw a circle with its center being the vertex of the angle.*

(b) *At each of the points where that circle intersects the sides of the angle, draw a circle with the same radius.*

(c) *The two circles intersect in two points. Draw a ray from the vertex of the angle through one of those points.*

(d) *The line bisects the angle.*



Problem 30 *Given an angle and some point [or a ray], use a compass and straightedge to copy the angle so that the new angle has as its vertex the given point [or a ray as one side of the angle]. Explain the steps in your construction and how you know it works.*

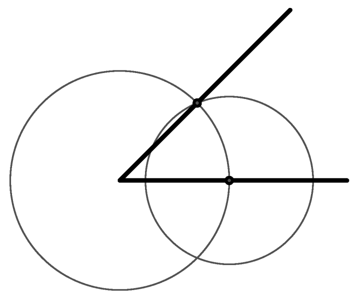
Free Response: ***Hint:*** (a) *Open the compass to a fixed width and make a circle centered at the vertex of the angle.*

(b) *Make a circle of the same radius on the line with the point [or on the ray].*

(c) *Open the compass so that one end touches the first circle where it hits one side of the original angle, with the other end of the compass extended to where the first circle hits the other side of the original angle.*

(d) *Draw a circle with the radius found above with its center where the second circle hits the line.*

(e) *Connect the point to where the circles meet. This is the other side of the angle we are constructing.*

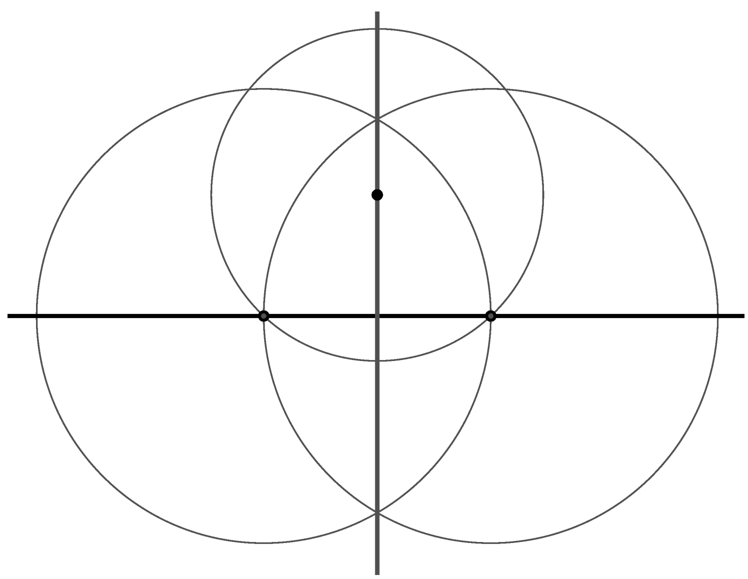


Problem 31 *Given a point and line, construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction and how you know it works.*

Free Response: ***Hint:*** *the original line that passes through the given point.*

(a) *Draw a circle centered at the point large enough to intersect the line in two distinct points.*

(b) *Bisect the line segment. The line used to do this will be the desired line.*



Problem 32 *Given a point and line, construct a line parallel to the given line that passes through the given point. Explain the steps in your construction and how you know it works.*

Free Response: ***Hint:*** *Through the given point, construct a perpendicular to the given line. Then through the same point, construct a perpendicular to the new line.*

Problem 33 *Construct a* 30-60-90 *right triangle. Explain the steps in your construction and how you know it works.*

Free Response: ***Hint:*** *Construct an equilateral triangle and cut it in half.*

Problem 34 *Construct an isosceles right triangle. Explain the steps in your construction and how you know it works.*

Free Response: ***Hint:*** *Construct a square and draw a diagonal.*

Trickier Constuctions

Short-answer questions about tricky constructions.

Problem 35 *Construct a square. Explain the steps in your construction.*

Problem 36 *Construct a regular hexagon. Explain the steps in your construction.*

Problem 37 *Your friend Margy is building a clock. She needs to know how to align the twelve numbers on her clock so that they are equally spaced on a circle. Explain how to use a compass and straightedge construction to help her out. Illustrate your answer with a construction and explain the steps in your construction.*

Problem 38 *Construct a triangle given two sides of a triangle and the angle between them. Explain the steps in your construction.*

Problem 39 *State the SAS Theorem.*

Problem 40 *Construct a triangle given three sides of a triangle. Explain the steps in your construction.*

Problem 41 *State the SSS Theorem.*

Problem 42 *Construct a triangle given a side and two angles where one of the angles does not touch the given side. Explain the steps in your construction.*

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Problem 43 *State the SAA Theorem.*

Problem 44 *Construct a triangle given a side between two given angles. Explain the steps in your construction.*

Problem 45 *State the ASA Theorem.*

Problem 46 *Explain why when given an isosceles triangle, that two of its angles have equal measure. Hint: Use the SAS Theorem.*

Problem 47 *Construct a figure showing that a triangle cannot always be uniquely determined when given an angle, a side adjacent to that angle, and the side opposite the angle. Explain the steps in your construction and explain how your figure shows what is desired. Explain what this says about the possibility of a SSA theorem. Hint: Draw many pictures to help yourself out.*

Problem 48 *Give a construction showing that a triangle is uniquely determined if you are given a right-angle, a side touching that angle, and another side not touching the angle. Explain the steps in your construction and explain how your figure shows what is desired.*

Problem 49 *Construct a triangle given two adjacent sides of a triangle and a median to one of the given sides. Explain the steps in your construction.*

Problem 50 *Construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.*

Problem 51 *Construct a triangle given a side, the median to the side, and the angle opposite to the side. Explain the steps in your construction.*

Problem 52 *Construct a triangle given an altitude, and two angles not touching the altitude. Explain the steps in your construction.*

Problem 53 *Construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.*

Problem 54 *Construct a triangle, given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.*

Problem 55 *Construct a circle with a given radius tangent to two other given circles. Explain the steps in your construction.*

Problem 56 *Does a given angle and a given opposite side uniquely determine a triangle? Explain your answer.*

Problem 57 *You are on the bank of a river. There is a tree directly in front of you on the other side of the river. Directly left of you is a friend a known distance away. Your friend knows the angle starting with them, going to the tree, and ending with you. How wide is the river? Explain your work.*

Problem 58 *You are on a boat at night. You can see three lighthouses, and you know their position on a map. Also you know the angles of the light rays from the lighthouses. How do you figure out where you are? Explain your work.*

Problem 59 *Construct a triangle given an angle, the length of a side adjacent to the given angle, and the length of the angle's bisector to the opposite side. Explain the steps in your construction.*

Problem 60 *Construct a triangle given an angle, the length of the opposite side, and the length of the altitude of the given angle. Explain the steps in your construction.*

Problem 61 *Construct a triangle given one side, the length of the altitude of the opposite angle, and the radius of the circumcircle. Explain the steps in your construction.*

Problem 62 *Construct a triangle given one side, the length of the altitude of an adjacent angle, and the radius of the circumcircle. Explain the steps in your construction.*

Problem 63 *Construct a triangle given one side, the length of the median connecting that side to the opposite angle, and the radius of the circumcircle. Explain the steps in your construction.*

Problem 64 *Construct a triangle given one angle and the lengths of the altitudes to the two other angles. Explain the steps in your construction.*

Problem 65 *Construct a circle with a given radius tangent to two given intersecting lines. Explain the steps in your construction.*

Problem 66 *Given a circle and a line, construct another circle of a given radius that is tangent to both the original circle and line. Explain the steps in your construction.*

Problem 67 *Construct a circle with three smaller circles of equal size inside such that each smaller circle is tangent to the other two and the larger outside circle. Explain the steps in your construction.*

Folding and Tracing

Short-answer questions about folding and tracing.

Problem 68 *What are the rules for folding and tracing constructions?*

Problem 69 *Use folding and tracing to bisect a given line segment. Explain the steps in your construction.*

Hint: *Fold one endpoint of the segment onto the other. The midpoint is where the fold intersects the segment. (Note that the fold is the perpendicular bisector of the segment.)*

Problem 70 *Given a line segment with a point on it, use folding and tracing to construct a line perpendicular to the segment that passes through the given point. Explain the steps in your construction.*

Hint: *Fold the line onto itself so that the fold goes through the given point. (You may need to extend the segment to see enough of the line.)*

Problem 71 *Use folding and tracing to bisect a given angle. Explain the steps in your construction.*

Hint: *Fold one side of the angle onto the other so that the fold goes through the vertex of the angle.*

Problem 72 *Given a point and line, use folding and tracing to construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction.*

Hint: *Fold the line onto itself so that the fold goes through the given point.*

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Problem 73 *Given a point and line, use folding and tracing to construct a line parallel to the given line that passes through the given point. Explain the steps in your construction.*

Hint: *Construct a perpendicular to a perpendicular as follows: (1) Fold the line onto itself so that the fold goes through the given point. (2) Fold the new fold onto itself so that the fold goes through the given point.*

Problem 74 *Given a length of* 1*, construct a triangle whose perimeter is a multiple of* 6*. Explain the steps in your construction.*

Problem 75 *Construct a* 30-60-90 *right triangle. Explain the steps in your construction.*

Problem 76 *Given a length of* 1*, construct a triangle with a perimeter of* $3+\sqrt{5}$*. Explain the steps in your construction.*

Lines in Triangles

Short-answer questions about lines in triangles.

Problem 77 *Explain how a perpendicular bisector is different from an altitude. Use folding and tracing to illustrate the difference.*

Problem 78 *Explain how a median different from an angle bisector. Use folding and tracing to illustrate the difference.*

Problem 79 *Given a triangle, use folding and tracing to construct the circumcenter. Explain the steps in your construction.*

Problem 80 *Given a triangle, use folding and tracing to construct the orthocenter. Explain the steps in your construction.*

Problem 81 *Given a triangle, use folding and tracing to construct the incenter. Explain the steps in your construction.*

Problem 82 *Given a triangle, use folding and tracing to construct the centroid. Explain the steps in your construction.*

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