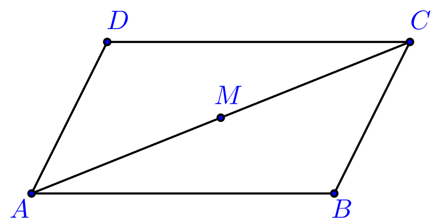
Quadrilateral Symmetry

Proof.

Problem 1 *Use symmetry to prove properties of parallelograms.*



Consider a 180° *rotation about* M*, the midpoint of diagonal* $\overline{AC}$*. Show that this rotation maps the parallelogram onto itself.*

> *Fix note: The following proof is quite elegant, but some of the details are subtle, especially distinguishing between mapping the sides (i.e., segments) and the lines containing the sides. Can any of this be omitted or abbreviated? Which parts might students supply?*

(a) *The rotation maps* A *to* C *and* C *to* A *because a* 180° *rotation about a point on a line takes the line to itself and preserves lengths.*

(b) *The rotation maps* $\overleftrightarrow{AB}$ *to a parallel line through* C *(the image of* A*), which by the uniqueness of parallels must be* $\overleftrightarrow{CD}$*. Similarly, the rotation maps* $\overleftrightarrow{CD}$ *to* $\overleftrightarrow{AB}$*,* $\overleftrightarrow{AD}$ *to* $\overleftrightarrow{CB}$*, and* $\overleftrightarrow{CB}$ *to* $\overleftrightarrow{AD}$*.*

(c) *Furthermore, the intersection of* $\overleftrightarrow{AB}$ *and* $\overleftrightarrow{CB}$*, which is* B*, must map to the intersection of their images,* $\overleftrightarrow{CD}$ *and* $\overleftrightarrow{AD}$*, and that intersection is* D*. And likewise,* D *must map to* B*.*

(d) *Because vertices are mapped to vertices, sides are mapped to opposite sides, angles are mapped to opposite angles, and thus the parallelogram is mapped onto itself.*

Now this symmetry proves the following properties for free*:*

- *opposite sides are congruent (sides are mapped to opposite sides),*

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- *opposite angles are congruent (angles are mapped to opposite angles), and*
- *the diagonals bisect each other.*

  Detail: The $180°$ *rotation about* M *swaps* $\overrightarrow{MB}$ *and* $\overrightarrow{MD}$*, so they must be opposite rays, and thus* B*,* M*, and* D *are collinear.*
  Because the rotation preserves lengths, $MB = MD$*, so that* M *is also the midpoint of* $\overline{BD}$*, which means that the diagonals bisect each other.*