Parallel Lines

Proofs updated.

This page develops important results regarding parallel lines and transversals. **Read carefully, and complete the proofs.**

Axiom 1. *Parallel postulate (uniqueness of parallels): Given a line and a point not on the line, there is exactly one line through the given point parallel to the given line.*

Problem 1

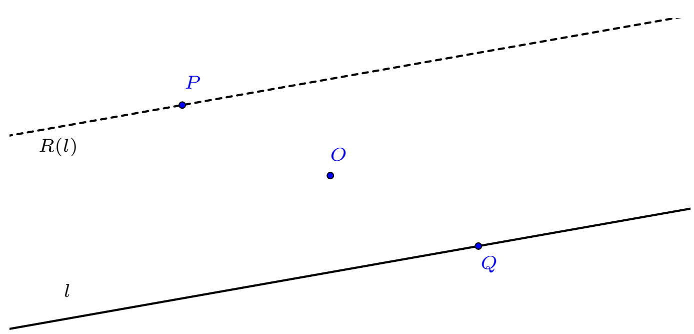
Theorem 1. *A 180° rotation about a point on a line takes the line to itself.*

Proof *Suppose point P is on line k. The point cuts the line into two opposite $\boxed{rays}$. A 180° rotation about P swaps the two opposite rays, thereby mapping the line onto itself.* ■

Problem 2

Theorem 2. *A 180° rotation about a point not on a line takes the line to a parallel line.*

Proof *Let O be a point not on line l. Let P be an arbitrary point on $R(l)$, the rotated image of l. To show that $R(l)$ is parallel to l, it is sufficient to show that P cannot lie also on l.*



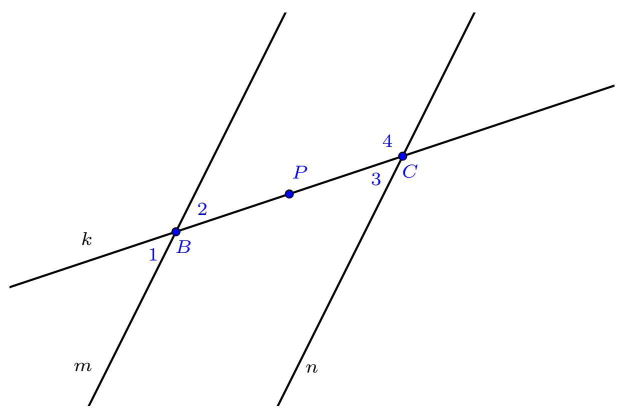
Author(s): Brad Findell

Because P is on $R(l)$, there is a point Q on l such that $P = R(Q)$. The rotated image of $\overrightarrow{OQ}$ is ($\overrightarrow{QO}$ / $\overrightarrow{OP}$ ✓/ $\overrightarrow{QP}$), and because $\angle QOP$ is $180°$, it follows that Q, O, and P are $\boxed{\textit{collinear}}$. Call that line k. We know line k is distinct from l because point $\boxed{O}$ is on k but not on l. Now, if P were on l, then points P and Q would be on two distinct lines, k and l, contradicting the assumption that on two points there is a unique line. The theorem is proved. ■

Problem 3

Theorem 3. *If two parallel lines are cut by a transversal, alternate interior angles and corresponding angles are congruent.*

Proof *Given that parallel lines m and n are cut by transversal k, prove that alternate interior angles are congruent.*



Let B and C be the intersections of transversal k with lines m and n, respectively. Let P be the midpoint of $\overline{BC}$.

(a) *Rotate $180°$ about P, which takes k to (itself ✓/ m / n).*

(b) *The rotation maps B to $\boxed{C}$ because $PB = PC$ and the rotation preserves distances.*

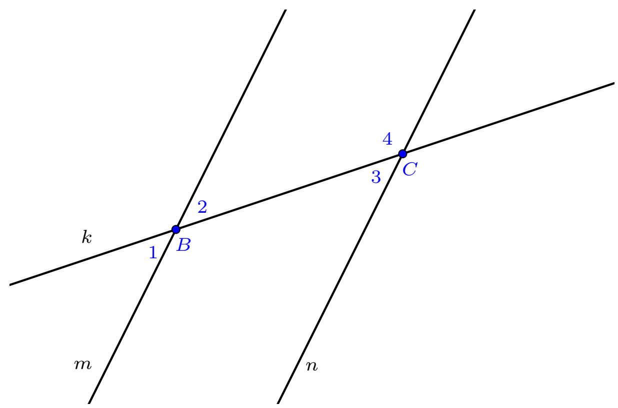
(c) *Because P is not on m, the rotation maps m to a parallel line through C, which must be (k / m / n ✓) by the uniqueness of parallels.*

(d) *Thus, the rotation maps $\angle 2$ to ($\angle 1$ / $\angle 2$ / $\angle 3$ ✓/ $\angle 4$). These alternate interior angles must be congruent because the rotation preserves angle measures.*

■

Note: *The congruence of corresponding angles now follows from the congruence of vertical angles. But the next problem is another approach that uses a translation.*

Problem 4 ***Proof*** *Given that parallel lines m and n are cut by transversal k, prove that corresponding angles are congruent.*



Let B and C be the intersections of transversal k with lines m and n, respectively.

(a) *Translate to the right along line k by distance BC, which takes k to (itself ✓/ m / n).*

(b) *The translation maps B to $\boxed{C}$, and it maps m to (k / m / n ✓) because the translation maintains parallels, and there is a unique parallel to m through C.*

(c) *The translation maps $\angle 1$ to ($\angle 1$/ $\angle 2$/ $\angle 3$ ✓/ $\angle 4$). These corresponding angles must be congruent because the translation preserves angle measures.*

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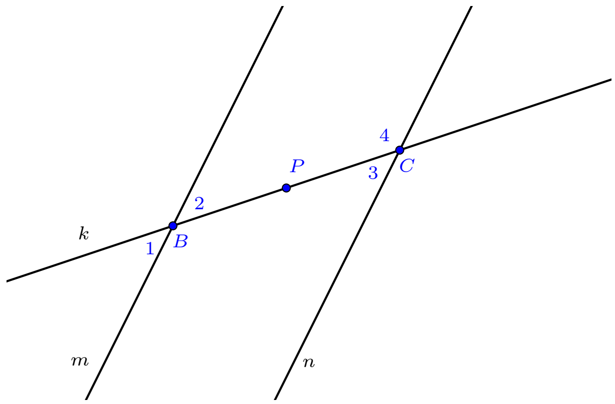
Problem 5

Theorem 4. *If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.*

Note: *This theorem is the* $\boxed{\textit{converse}}$ *of the previous theorem about alternate interior angles.*

Proof *Given that m and n are cut by transversal k with alternate interior angles congruent, prove that lines m and n are parallel.*



Let B and C be the intersections of transversal k with lines m and n, respectively. Let P be the midpoint of $\overline{BC}$.

(a) *Rotate 180° about P, which takes k to (itself ✓/ m/ n), and which swaps B and $\boxed{C}$ because distances are preserved.*

(b) *Because $\angle 2 \cong \angle 3$ and because a side of $\angle 2$ (i.e., $\overrightarrow{BP}$) is mapped to a side of $\angle 3$ (i.e., ($\overrightarrow{CP}$ ✓/ $\overrightarrow{PC}$/ $\overrightarrow{BP}$)), it must be that the other side of $\angle 2$ (which lies on m) is mapped to the other side of $\angle 3$ (which lies on line $\boxed{n}$). Thus, n is the image of m.*

(c) *Because P is not on m, the 180° rotation maps m to a parallel line through C. Thus, n must be parallel to m.*

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