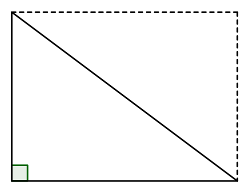
Proof by Picture

Short-answer proofs by pictures.

Problem 1 *Explain how the following picture "proves" that the area of a right triangle is half the base times the height.*



Free Response: ***Hint:*** *The area of the rectangle is base times height. The rectangle is made up of two congruent right triangles. Because congruent triangles have the same area, the area of each right triangle is (equal to/ half ✓/ double) the area of the rectangle.*

Problem 2 *Suppose you know that the area of a* ***right*** *triangle is half the base times the height. Explain how the following picture "proves" that the area of* ***every*** *triangle is half the base times the height.*

Free Response: ***Hint:*** *The whole triangle is made up of two right triangles. Call the bases of the small right triangles b_1 and b_2, and call the base of the large (combined) triangle b. Then $b = b_1 + b_2$, and the three triangles all have the same height, h. The area of the whole triangle is the sum of the areas of the two right triangles:*

$$\text{area} = \frac{hb_1}{2} + \frac{hb_2}{2} = \frac{h(b_1 + b_2)}{2} = \boxed{bh/2}$$

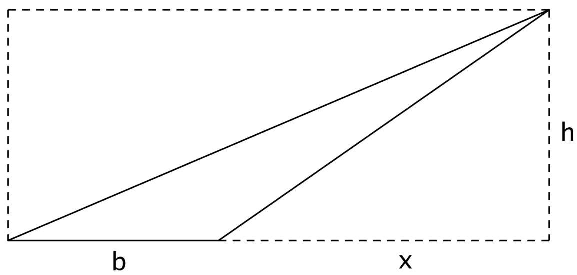
because $b = b_1 + b_2$.

Problem 3 *Now suppose that a student, say Geometry Giorgio attempts to solve a similar problem. Again knowing that the area of a right triangle is half the base times the height, he draws the following picture: Geometry Giorgio states that the diagonal line cuts the rectangle in half, and thus the area of*

Learning outcomes:
Author(s): Bart Snapp and Brad Findell

the triangle is half the base times the height. Is this correct reasoning? If so, give a complete explanation. If not, give correct reasoning based on Geometry Giorgio's picture.

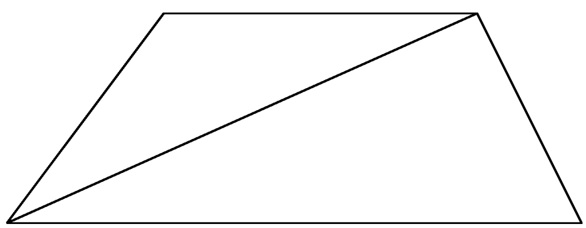
Free Response: ***Hint:*** *The triangle is not half the rectangle. Furthermore, the rectangle does not have the same base as the triangle, so "base times height" is unclear. The following picture allows us to distinguish these bases:*



One way to compute the area of the solid triangle is to (1) compute the area right triangle that is the lower half of the rectangle (with base $b+x$) and then (2) subtract the area of the small right triangle (with base x):

$$\text{area} = \boxed{\frac{h(b+x)}{2}} - \boxed{\frac{hx}{2}} = \frac{hb}{2} + \frac{hx}{2} - \frac{hx}{2} = \boxed{\frac{hb}{2}}$$

Problem 4 *Recall that a trapezoid is a quadrilateral with two parallel sides. Consider the following picture:*



How does the above picture prove that the area of a trapezoid is

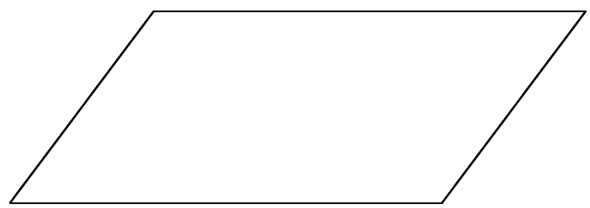
$$\text{area} = \frac{h(b_1 + b_2)}{2}$$

where h is the height of the trapezoid and b_1, b_2, are the lengths of the parallel sides?

Free Response: ***Hint:*** *Using the parallel sides as bases, the two triangles have the same height as the trapezoid. The area of the trapezoid is the sum of the areas of the triangles:*

$$\text{area} = \frac{hb_1}{2} + \frac{hb_2}{2} = \frac{h(b_1 + b_2)}{2}$$

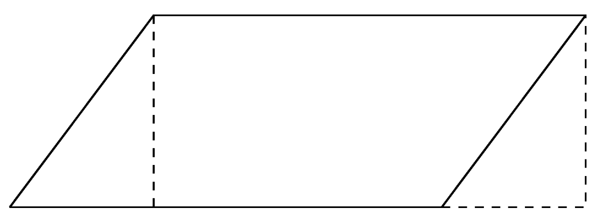
Problem 5 *Look at the previous problem. Can you use a similar idea to prove that the area of a parallelogram*



is the length of the base times the height?

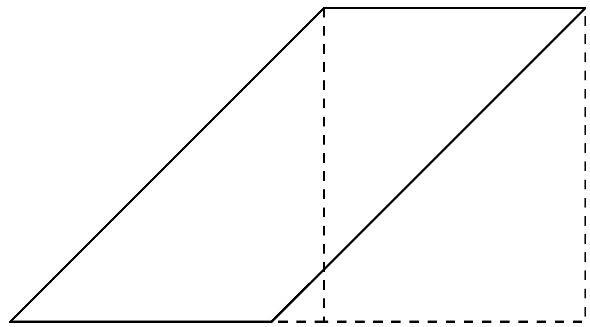
Free Response: ***Hint:*** *Hint: Draw a diagonal. Then there are two triangles with the same base and the same height.*

Problem 6 *Explain how the following picture "proves" that the area of a parallelogram is base times height.*



Free Response: ***Hint:*** *Imagine cutting off the triangle on the left and placing it in the (congruent) dotted region on the right. Then we have a rectangle with the same base and height as the parallelogram. The cutting and rearranging doesn't change the area of the figure, so the area of the parallelogram is the same as the area of the rectangle, which is bh.*

Problem 7 *Now suppose that a student, say Geometry Giorgio attempts to solve a similar problem. In an attempt to prove the formula for the area of a parallelogram, Geometry Giorgio draws the following picture:*



At this point Geometry Giorgio says that he has proved the formula for area of a parallelogram. What do you think of his picture? Give a complete argument based on his picture.

Free Response: ***Hint:*** *Hint: This parallelogram problem is much like Giorgio's triangle problem above. Use the same idea.*