Anatomy of Figures

Short-answer problems about centers of triangles.

Question 1 *What is your name?*

Free Response:

Problem 2 *Compare and contrast the idea of "intersecting sets" with the idea of "intersecting lines."*

Free Response: **Hint:** *Geometric figures are sets of points. The intersection of two geometric figures is the set of point(s) that the figures have in common. Two lines either intersect in a single point, say A, or they do not intersect. As sets, we would say the intersection is $\{A\}$ or $\{\}$, respectively.*

Problem 3 *Place three points in the plane. Give a detailed discussion explaining how they may or may not be on a line.*

Free Response: **Hint:** *If the three points are distinct, then there are two possibilities:*

- *The points are collinear (i.e., they all lie on the same line).*
- *The points are not collinear. Any two of the points determine a line that does not contain the third point. For arbitrary points, this is the more likely situation.*

Problem 4 *Place three lines in the plane. Give a detailed discussion explaining how they may or may not intersect.*

Free Response: **Hint:** *If the three lines are distinct, then there are several possibilities:*

- *The three lines are all parallel.*
- *Two of the three lines are parallel and the third line intersects the first two.*

- *The three lines are concurrent (i.e., they all lie on the same point).*
- *The three lines are not parallel and not concurrent. Any two of the lines intersect in a point that is not on the third line. For arbitrary lines, this is the most likely situation.*

Problem 5 *Explain how a perpendicular bisector is different from an altitude. Draw an example to illustrate the difference.*

Free Response: **Hint:** *A perpendicular bisector goes through the midpoint of a segment and is perpendicular to it. An altitude to a segment typically does not go through the midpoint of the segment—and it might not intersect the segment at all. Instead, the altitude goes through another vertex of the figure and is perpendicular to the (extended) line containing the segment.*

Problem 6 *Explain how a median is different from an angle bisector. Draw an example to illustrate the difference.*

Free Response: **Hint:** *A median extends from a vertex of a triangle to the midpoint of the opposite side (thereby bisecting that side). The angle bisector, um, bisects the angle.*

Problem 7 *What is the name of the point that is the same distance from all three sides of a triangle? Explain your reasoning.*

Free Response: **Hint:** *The points on an angle bisector are equidistant from the sides of the angle. So the point of concurrency of the angle bisectors of a triangle is equidistant from all three sides of the triangle. That point of concurrency is called the incenter, as it is the center of the incircle.*

Problem 8 *What is the name of the point that is the same distance from all three vertexes of a triangle? Explain your reasoning.*

Free Response: **Hint:** *The points on an perpendicular bisector are equidistant from the endpoints of the segment. So the point of concurrency of the perpendicular bisectors of a triangle is equidistant from all three vertices of the triangle. That point of concurrency is called the circumcenter, as it is the center of the circumcircle.*

Problem 9 *Could the circumcenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: **Hint:**   *Yes. Try an obtuse triangle.*

Problem 10 *Could the orthocenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: **Hint:**   *Yes. Try an obtuse triangle.*

Problem 11 *Could the incenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: **Hint:**   *No. The incenter is the center of the incircle, which is entirely inside the triangle.*

Problem 12 *Could the centroid be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: **Hint:**   *No. The centroid is the center of gravity of the triangle, which must lie inside the triangle.*

Problem 13 *Are there shapes that do not contain their centroid? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: **Hint:**   *Think of figures with holes in the "middle," where the center of gravity might be.*

Problem 14 *Draw an equilateral triangle. Now draw the lines containing the altitudes of this triangle. How many orthocenters do you have as intersections of lines in your drawing? Hints:*

(a) *More than one.*

(b) *How many triangles are in the picture you drew?*

Free Response: **Hint:** *Consider four points: the three original points and their orthocenter. Any of these points is the orthocenter of the triangle created by the other three points. (This is quite subtle. Examine your figure closely.)*

Problem 15 *Given a triangle, construct the circumcenter. Explain the steps in your construction.*

Free Response: **Hint:** *The circumcenter is the point of concurrency of the perpendicular bisectors. Because the perpendicular bisectors are concurrent, it is enough to find the intersection of two of them.*

Problem 16 *Given a triangle, construct the orthocenter. Explain the steps in your construction.*

Free Response: **Hint:** *The orthocenter is the point of concurrency of the altitudes. Because the altitudes are concurrent, it is enough to find the intersection of two of them.*

Problem 17 *Given a triangle, construct the incenter. Explain the steps in your construction.*

Free Response: **Hint:** *The incenter is the point of concurrency of the angle bisectors. Because the angle bisectors are concurrent, it is enough to find the intersection of two of them.*

Problem 18 *Given a triangle, construct the centroid. Explain the steps in your construction.*

Free Response: **Hint:** *The centroid is the point of concurrency of the medians. Because the medians are concurrent, it is enough to find the intersection of two of them.*

Problem 19 *Given a triangle, construct the incircle. Explain the steps in your construction.*

Free Response: **Hint:** *The center of the incircle is the point of concurrency of the angle bisectors. As stated above, the incenter is equidistant from the sides of the triangle. To find that distance, which is the radius of the incircle, construct a perpendicular from the incenter to one side of the triangle, and then "measure" along that perpendicular bisector.*

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Problem 20 *Given a triangle, construct the circumcircle. Explain the steps in your construction.*

Free Response: **Hint:**   *The center of the circumcircle is the point of concurrency of the perpendicular bisectors. As stated above, the circumcenter is equidistant from the vertices of the triangle. So use that distance as the radius of the circumcircle.*

Problem 21 *Given a circle, give a construction that finds its center.*

Free Response: **Hint:**   *Any three points on the circle form a triangle. The circumcenter of that triangle will be the center of the circle.*

Problem 22 *Where is the circumcenter of a right triangle? Explain your reasoning.*

Free Response: **Hint:**   *The circumcenter is at the midpoint of the hypotenuse. The perpendicular bisector of the hypotenuse clearly contains this midpoint. And it is just a tad harder to see that the perpendicular bisectors of the legs also contains this point. (To see this, note that a midsegment to a leg lies on the perpendicular bisector of that leg.)*

Problem 23 *Where is the orthocenter of a right triangle? Explain your reasoning.*

Free Response: **Hint:**   *The orthocenter of a right triangle is the vertex of the right angle. The altitude to the hypotenuse goes through the vertex of the right angle. Both legs of the right triangle are also altitudes of that right triangle, so they also go through the vertex of the right angle.*

Problem 24 *Can you draw a triangle where the circumcenter, orthocenter, incenter, and centroid are all the same point? If so, draw a picture and explain. If not, explain why not using pictures as necessary.*

Free Response: **Hint:**   *Try an equilateral triangle.*

Problem 25 *True or False: Explain your conclusions.*

(a) *An altitude of a triangle is always perpendicular to a line containing some side of the triangle.*

(b) *An altitude of a triangle always bisects some side of the triangle.*

(c) *The incenter is always inside the triangle.*

(d) *The circumcenter, the centroid, and the orthocenter always lie in a line.*

(e) *The circumcenter can be outside the triangle.*

(f) *The orthocenter is always inside the triangle.*

(g) *The centroid is always inside the incircle.*

Free Response: **Hint:**   *Don't worry about (d) and (g):*

(a) *True. This is part of the definition of an altitude.*

(b) *False. Any scalene triangle will do.*

(c) *True. The incenter is the center of the incircle, which lies entirely inside the triangle.*

(d) *True. An amazing fact. Try it.*

(e) *True. Try an obtuse triangle.*

(f) *False. Try an obtuse triangle.*

(g) *False. Try a thin triangle.*

Problem 26 *Given 3 distinct points not all in a line, construct a circle that passes through all three points. Explain the steps in your construction.*

Free Response: **Hint:**   *The three points form a triangle. Construct the circumcircle of that triangle.*