Parallel Lines

Proofs updated.

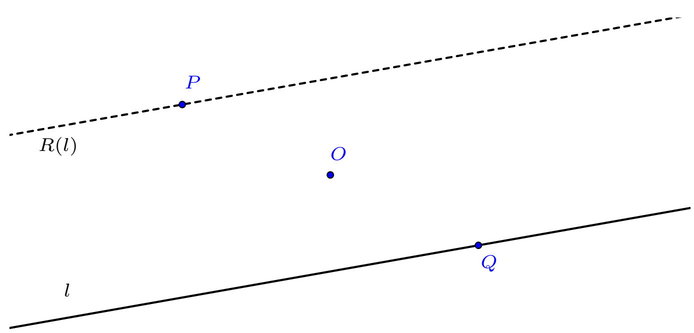
Axiom 1. *Parallel postulate (uniqueness of parallels): Given a line and a point not on the line, there is exactly one line through the given point parallel to the given line.*

Theorem 1. *A 180° rotation about a point on a line takes the line to itself.*

Proof Suppose point P is on line k. The point cuts the line into two opposite rays. A 180° rotation about P swaps the two opposite rays, thereby mapping the line onto itself. ■

Theorem 2. *A 180° rotation about a point not on a line takes the line to a parallel line.*

Proof Let O be a point not on line l. Let P be an arbitrary point on $R(l)$, the rotated image of l. To show that $R(l)$ is parallel to l, it is sufficient to show that P cannot lie also on l.



Because P is on $R(l)$, there is a point Q on l such that $P = R(Q)$. The rotated image of $\overrightarrow{OQ}$ is $\overrightarrow{OP}$, and because $\angle QOP$ is 180°, it follows that Q, O, and P are $\boxed{\textit{collinear}}$. Call that line k. We know line k is distinct from l because O is on k but not on l. Now, if P were on l, then points P and Q would be on two distinct lines, k and l, contradicting the assumption that on two points there is a unique line. The theorem is proved. ■

Theorem 3. *If two parallel lines are cut by a transversal alternate interior angles and corresponding angles are congruent.*

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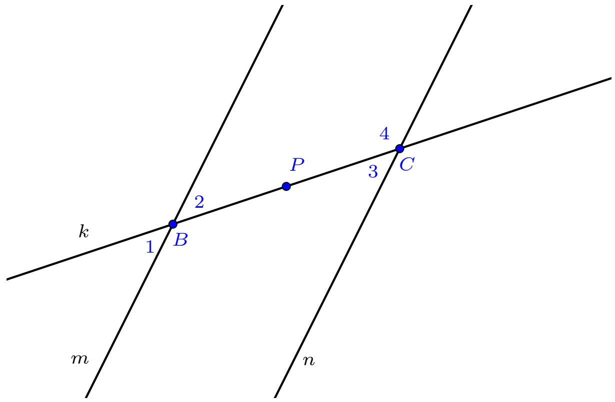
Proof Given that parallel lines m and n are cut by transversal k, prove that alternate interior angles are congruent.



(a) Let B and C be the intersections of transversal k with lines m and n, respectively. Let P be the midpoint of $\overline{BC}$

(b) Rotate 180° about P, which takes k to (itself ✓/ m/ n).

(c) The rotation maps B to C and C to B because distances are preserved.

(d) The rotation maps m to a parallel line through C, which must be (k/ m / n ✓) by the uniqueness of parallels.

(e) The rotation maps n to (k/ m ✓/ n) by the same reasoning.

(f) The rotation swaps $\angle 2$ and ($\angle 1$/ $\angle 2$/ $\angle 3$ ✓/ $\angle 4$). These alternate interior angles must be congruent because the rotation preserves angle measures.

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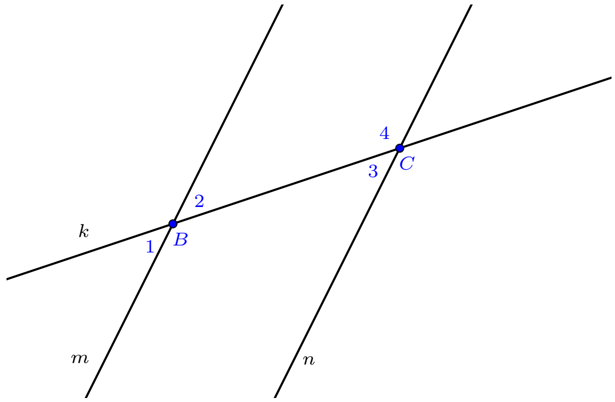
Note: The congruence of corresponding angles now follows from the congruence of vertical angles. But here is a direct approach that uses a translation.

Proof Given that parallel lines m and n are cut by transversal k, prove that corresponding angles are congruent.



(a) Let B and C be the intersections of transversal k with lines m and n, respectively.

(b) Translate to the right along line k by distance BC, which takes k to (itself ✓/ m/ n).

(c) The translation maps B to $\boxed{C}$, and it maps m to (k/ m/ n ✓) because the translation maintains parallels, and there is a unique parallel to m through C.

(d) The translation maps $\angle 1$ to ($\angle 1$/ $\angle 2$/ $\angle 3$ ✓/ $\angle 4$). These corresponding angles must be congruent because the translation preserves angle measures.

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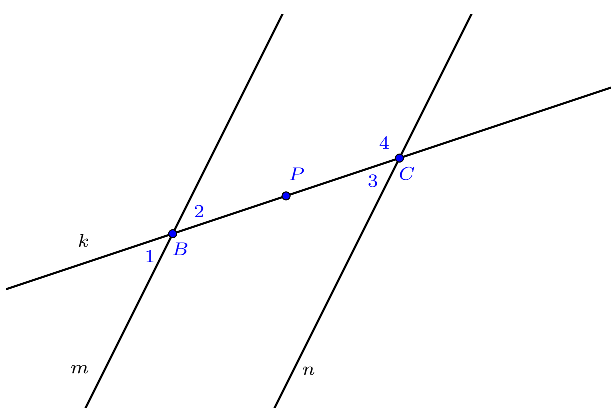
Theorem 4. *If two lines are cut by a transversal so that alternate interior (and corresponding) angles are congruent, then the lines are parallel.*

Proof Given that m and n are cut by transversal k with alternate interior angles congruent, prove that lines m and n are parallel.



(a) Let B and C be the intersections of transversal k with lines m and n, respectively. Let P be the midpoint of $\overline{BC}$

(b) Rotate 180° about P, which takes k to (itself ✓/ m/ n), and which swaps B and $\boxed{C}$ because distances are preserved.

(c) Because $\angle 2 \cong \angle 3$ and because a side of $\angle 2$ (i.e., $\overrightarrow{BP}$) is mapped to a side of $\angle 3$ (i.e., ($\overrightarrow{CP}$ ✓/ $\overrightarrow{PC}$/ $\overrightarrow{BP}$)), it must be that the other side of $\angle 2$ (which lies on m) is mapped to the other side of $\angle 3$ (which lies on line $\boxed{n}$). Thus, n is the image of m.

(d) Because P is not on m, the 180° rotation maps m to a parallel line through C. Thus, n must be parallel to m.

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