Online HW 6: Symmetry

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Symmetry

Short-answer questions about symmetry.

Question 1 *Categorize the capital letters of the alphabet by their symmetries. Use the following font:*

ABCDEFGHIJKLMNOPQRSTUVWXYZ

Free Response: ***Hint:***

- *Vertical line symmetry: A, H, I, M, O, T, U, V, W, X, Y*
- *Horizontal line symmetry: B, C, D, E H, I, K, O, X*
- 180° *rotational symmetry: H, I, N, O, S, X, Z*
- *None: F, G, J, L, P, Q, R.*

Notes: (1) In many fonts that look much the same, the K has no symmetry. (2) In this font, the O is slightly taller than it is wide. If it were a circle, there would be more symmetry. (See later problem.)

Question 2 *Write the words COKE and PEPSI in capital letters so that they read vertically. Use a mirror to look at a reflection of the words. What is different about the reflections of the two words? Explain.*

Free Response: ***Hint:*** *If the K has horizontal line symmetry in the font, then all the letters in COKE have horizontal line symmetry, which becomes vertical line symmetry when the word is written vertically. PEPSI, on the other hand, has several letters without that symmetry.*

Question 3 *We often say a figure is "symmetric" when we notice that it has symmetry, but now we want to be more precise:*

A symmetry *of a figure is a (reflection/ rotation/ transformation ✓/ translation) that maps a figure (to its opposite/ onto itself ✓/ to another figure).*

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Question 4 *Explain why a sequence of two symmetries of a figure must also be a symmetry of that figure.*

Free Response: ***Hint:*** *The first transformation leaves the figure unchanged and the second transformation leaves the figure unchanged, so the sequence of two transformations leaves the figure unchanged.*

Question 5 *Explain why the identity transformation should be considered a symmetry of* any *figure.*

Free Response: ***Hint:*** *Some possible explanations:*

- *The identity transformation satisfies the definition of a symmetry: It maps the figure onto itself.*
- *If a figure has reflection symmetry R_k about a line k, then R_k followed by R_k is the identity transformation. And by the previous result, this sequence of symmetries must also be a symmetry.*
- *If a figure has rotational symmetry R_α by some angle α about some center, then it must also have a rotational symmetry $R_{-\alpha}$ by the angle $-\alpha$ about the same center. R_α followed by $R_{-\alpha}$ is the identity transformation. And by the previous result, this sequence of symmetries must also be a symmetry.*

Note: If the identity transformation is the only *symmetry of a figure, we usually say the figure is asymmetric or has no symmetry.*

Question 6 *It is reasonable to call the identity transformation a translation because it is a translation of magnitude $\boxed{0}$ in any direction.*

It is reasonable to call the identity transformation a rotation because it is a rotation of $\boxed{0}$ degrees about any center.

Question 7 *Indicate the number of rotation and reflection symmetries of the following figures (including the identity rotation):*

(a) *An equilateral triangle: $\boxed{3}$ rotation(s) and $\boxed{3}$ reflection(s).*

(b) *An isosceles triangle that is not equilateral: $\boxed{1}$ rotation(s) and $\boxed{1}$ reflection(s).*

(c) *A square: $\boxed{4}$ rotation(s) and $\boxed{4}$ reflection(s).*

(d) *A rectangle that is not a square: $\boxed{2}$ rotation(s) and $\boxed{2}$ reflection(s).*

(e) *A rhombus that is not a square:* $\boxed{2}$ *rotation(s) and* $\boxed{2}$ *reflection(s).*

(f) *A (non-special) parallelogram:* $\boxed{2}$ *rotation(s) and* $\boxed{0}$ *reflection(s).*

(g) *A regular* $n-$*gon:* $\boxed{n}$ *rotation(s) and* $\boxed{n}$ *reflection(s).*

Symmetries of polygons.

Question 8 *Suppose that quadrilateral $ABCD$ has exactly one rotation symmetry (other than the identity transformation) and no reflection symmetry. What kind(s) of quadrilateral could it be? Explain your reasoning.*

Free Response: ***Hint:*** *If a rotation of α is a symmetry of a figure, then a rotation of 2α must also be a symmetry. Thus, for the quadrilateral to have only one (non-identity) rotation symmetry, it must be that $2\alpha = 360^\circ$, so that $\alpha = 180^\circ$. A 180° rotation will swap opposite vertices, which implies that the center of rotation is the midpoint of each diagonal, so that the diagonals must bisect each other. Thus, quadrilateral $ABCD$ is a parallelogram. It has to be a non-special parallelogram or it would have additional symmetry.*

Question 9 *Suppose that quadrilateral $ABCD$ has exactly one reflection symmetry and no rotation symmetry (other than the identity transformation). What kind(s) of quadrilateral could it be? Explain your reasoning.*

Free Response: ***Hint:*** *A parallelogram has 180° rotational symmetry, so quadrilateral $ABCD$ cannot be a parallelogram, which also excludes the special cases: rhombus, rectangle, and square. If the line of symmetry goes through two vertices, it must be a kite that is not a rhombus. If the line of symmetry goes through two sides, it must be an isosceles trapezoid.*

Question 10 *What are the symmetries of a circle?*

Free Response: ***Hint:*** *A circle has rotational symmetry by any angle about its center. A circle has reflection symmetry about any line through its center. A circle does not have translation symmetry.*

Question 11 *How can you use the symmetries of a circle to determine whether a figure is indeed a circle?*

Free Response: ***Hint:*** *Perform any of the symmetry transformations to be sure that the circle is actually mapped onto itself.*

Question 12 *What are the symmetries of a line?*

(a) *Describe all translation symmetries.*

Free Response: ***Hint:*** *A line has translation symmetry by a vector of any length parallel to the line.*

(b) *Describe all rotation symmetries.*

Free Response: ***Hint:*** *A line has 180° rotational symmetry about any point on the line.*

(c) *Describe two types of reflection symmetries.*

Free Response: ***Hint:*** *A line has reflection symmetry about any perpendicular to the line. A line also has reflection symmetry about itself.*

(d) *Given a line, describe a rotation symmetry and a reflection symmetry that have the same effect on a line. How do the corresponding transformations differ in what they do to the surrounding space?*

Free Response: ***Hint:*** *Given a point P on the line k, let j be the unique line through P perpendicular to k. A 180° rotation about P and a reflection about j have the same effect on the line k, even though those transformations have different effects on the rest of the plane.*

Free Response: ***Hint:*** *A line has translation symmetry by a vector of any length parallel to the line. A line has 180° rotational symmetry about any point on the line. A line has reflection symmetry about any perpendicular to the line.*

Question 13 *How can you use the symmetries of a line to determine whether a figure is indeed a line?*

Free Response: ***Hint:*** *Perform any of the symmetry transformations to be sure that the line is actually mapped onto itself.*

Question 14 *Find some tessellations. For each tessellation, describe all of its symmetries.*

Free Response: ***Hint:***

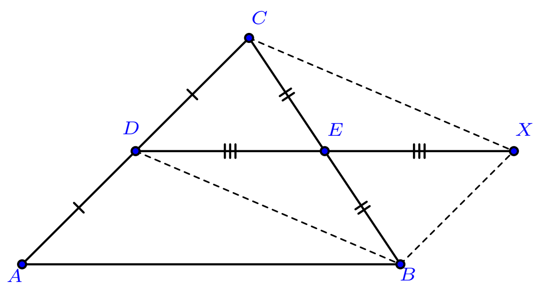
Midsegment Theorem

Proofs updated.

Theorem 1. *Midsegment Theorem: The segment joining the midpoints of two sides of a triangle is parallel to and half the length of the third side.*

In preparation for the midsegment theorem, the class proved several useful theorems about parallelograms.

Problem 15 *To prove the midsegment theorem for $\triangle ABC$ with midpoints D and E of sides AC and BC, respectively, Mitch extended $\overline{DE}$ to a point X such that $EX = DE$, as shown in the marked figure. Then he added dotted lines to the figure to show parallelograms.*



Mitch organized his reasoning in the following flow chart:

Fix note: The flowchart omits reasons to reduce clutter. The most significant steps are green whereas the details are blue.

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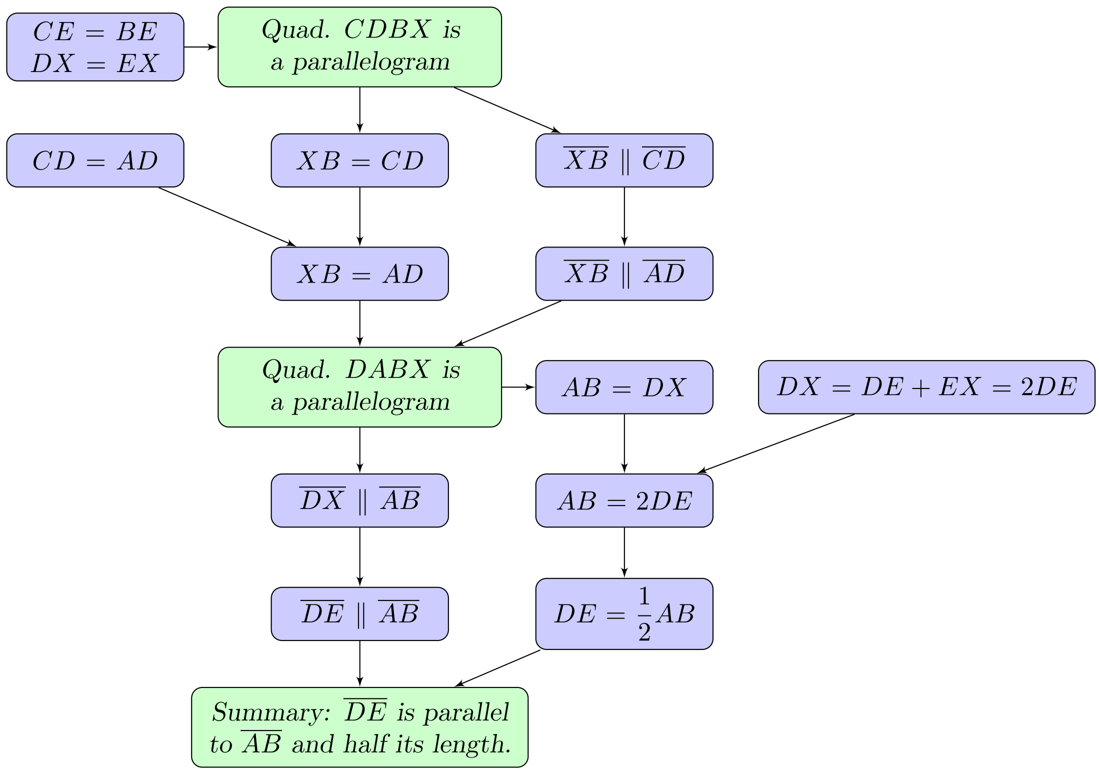

In the proof above, which theorem may Mitch use to conclude that quadrilateral $CDBX$ a parallelogram?

Multiple Choice:

(a) *If a pair of sides of a quadrilateral are congruent and parallel, then it is a parallelogram.*

(b) *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.* ✓

(c) *If opposite sides of a quadrilateral are congruent, then it is a parallelogram.*

(d) *If opposite angles of a quadrilateral are congruent, then it is a parallelogram.*

(e) *The Pythagorean Theorem.*

(f) *None of these.*

In the proof above, which theorem may Mitch use to conclude that quadrilateral $DABX$ a parallelogram?

Multiple Choice:

(a) *If one pair of sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.* ✓

(b) *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

(c) *If opposite sides of a quadrilateral are congruent, then it is a parallelogram.*

(d) *If opposite angles of a quadrilateral are congruent, then it is a parallelogram.*

(e) *The Pythagorean Theorem.*

(f) *None of these.*