Online HW 9: Measurement and Scaling

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Measurement

Short-answer problems about measurement.

Numbers, Units, and Quantities

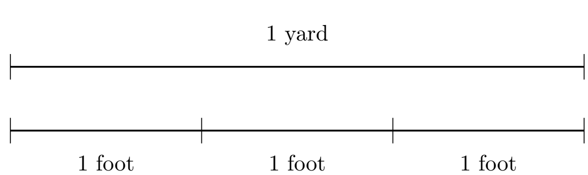
In this section, we develop and explain the algebra of measurement calculations and unit conversions.

Question 1 *Brad measured his driveway to be 21 feet long. How long is it in yards?* $\boxed{?}$ *yards.*

We all know that 3 feet and 1 yard express the same length. Let's write an equation to express this equality of lengths:

$$3 \text{ ft} = 1 \text{ yd}$$

Here is a picture illustrating this idea:



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The equation 3 ft = 1 yd seems to suggest that you (multiply/divide) feet by 3 to get yards. But when converting feet to yards in the question above you (multiplied/divided) the number of feet by 3.

Why do these answers appear to be opposites of each other?

The resolution of this conundrum requires that we distinguish numbers (e.g., 21) from units (e.g., feet). Quantities measuring, say, length, weight, or speed involve both numbers and units, but the numbers behave differently from the units, as we see in the example above.

With a little algebraic manipulation, the equation 3 ft = 1 yd is equivalent to the following equations:

$$\frac{3 \text{ feet}}{1 \text{ yard}} = 1 \qquad \text{and} \qquad \frac{1 \text{ yard}}{3 \text{ feet}} = 1.$$

In both equations, the 1 on the right is "dimensionless" in the sense that it is without units. Think of it as a scale factor of 1, which leaves lengths unchanged. These equations are useful for demonstrating the conversion from feet to yards and vice versa:

$$21 \text{ feet} = 21 \cancel{\text{feet}} \cdot \frac{1 \text{ yard}}{3 \cancel{\text{feet}}} = 7 \text{ yards}$$

$$7 \text{ yards} = 7 \cancel{\text{yards}} \cdot \frac{3 \text{ feet}}{1 \cancel{\text{yard}}} = 21 \text{ feet}$$

These examples illustrate an *algebra of units*, in which units behave much like algebraic variables. In both conversions, we multiply a quantity by a dimensionless 1, so that the calculation doesn't change the amount that the quantity represents. The cancellation of units helps to confirm that we are doing the calculation correctly.

At a conceptual level, this algebra of units helps illuminate how the numbers and units behave in apparently opposite ways:

- Yards are three times as big as feet, so there are one-third as many in a given length.
- Feet are one-third the size of yards, so there are three times as many in a given length.

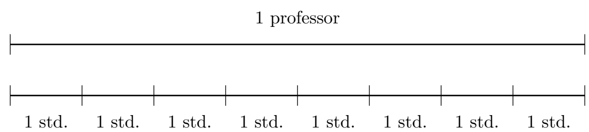
Thus, we must be careful to consider whether the letters represent numbers or units.

Let's try another problem.

Question 2 *At Big State University, the student-professor ratio is 8 to 1. Write an equation relating the number of students, s, to the number of professors, p.*

In response to this question, it is tempting to write $8s = p$, and this turns out to be a very common incorrect answer among students. But let's try some specific numbers: If there are 160 students, then there should be 20 professors, but the equation says that $p = 8s = 8 \cdot 160 = 1280$, which would mean 1280 professors. Clearly the equation is backwards: It caused us to multiply the number of students by 8 when we should have divided.

But here is another way of thinking about this incorrect answer. If "students" and "professors" were units (like yards and feet), then the equation would be correct. Here is a picture to help:



If professors and students are units of measurement with this consistent relationship between them, then to "convert" 20 professors to students, we proceed just as we converted yards to feet:

$$20 \text{ professors} = 20 \cancel{\text{professors}} \cdot \frac{8 \text{ students}}{1 \cancel{\text{professor}}} = 160 \text{ students}$$

The above question states, however, that the letters are to be the *numbers* of students and professors, not units of measurement. So let's repeat the previous calculation generally, where p represents the number of professors and s represents the number of students:

$$p \text{ professors} = p \cancel{\text{professors}} \cdot \frac{8 \text{ students}}{1 \cancel{\text{professor}}} = 8p \text{ students} = s \text{ students}$$

So the correct equation is indeed $s = 8p$. But again, notice how the numbers and the units work in apparently opposite ways.

Because of the common confusion between numbers and units, some mathematics educators recommend against using the first letter of a unit to indicate the number of that unit in a measurement. Indeed, the expressions "p professors" and "s students" are hard to read and interpret—and "$8p$ students" might be worse.

Whether you accept this recommendation or not, here is some good advice:

- When making calculations with quantities that include units, be very clear and careful about whether the letters represent units or numbers.

- When writing equations, it is very easy to get the relationships backwards, so check the equations with easy numbers.

Math teachers benefit from knowing conversions from memory. Here are some that you might not know:

$$\begin{aligned} 1 \text{ mile} &= 5280 \text{ feet} \\ 1 \text{ inch} &= 2.54 \text{ cm (exactly)} \\ 1 \text{ kg} &\approx 2.205 \text{ lbs} \end{aligned}$$

And here is a useful uncommon conversion that makes use of common ones:

$$60 \text{ mph} = \frac{60 \text{ miles}}{1 \text{ hour}} = \frac{60 \cancel{\text{miles}}}{1 \cancel{\text{hour}}} \cdot \frac{1 \cancel{\text{hour}}}{3600 \text{ sec}} \cdot \frac{5280 \text{ feet}}{1 \cancel{\text{mile}}} = \frac{88 \text{ feet}}{1 \text{ sec}} = 88 \text{ fps}$$

Question 3 *Use the conversions above to convert 1 meter to inches and to yards.*

$$1 \textit{ meter} = \boxed{?} \textit{ inches} = \boxed{?} \textit{ yards}$$

Question 4 *College and school tracks used to be 1/4 mile around, or 440 yards, so that a half-mile race was two laps, and a one-mile race was four laps. Today, most college and school tracks have been converted to metric, with one lap measuring 400 meters, which is close to 440 yards. In high-school track, the "mile" is usually run as 1600 meters.*

The 1600 meters race is $\boxed{?}$ *yards* (shorter/longer) *than 1 mile.*

A four-minute miler should run 1600 meters about $\boxed{?}$ *seconds (to the closest 0.1 seconds)* (faster/slower) *than four minutes.*

Scaling

Short-answer problems about scaling.

Length and Area Under Scaling

In this section, we explore what happens to length, area, and other measures under scaling.

In a previous section, we defined similarity in terms of basic rigid motions and dilations, and we showed that this definition leads to well-known results about similarity, such as the AA criterion for triangle similarity and consistent ratios of lengths between and within similar figures. A key feature of this discussion was the notion of "scale factor," which describes what happens to lengths under a dilation. From the definition of a dilation, it is clear that segments on lines through the center of dilation scale by the scale factor. We used the side-splitter theorems to show that other segments are scaled by the same scale factor.

Question 5 *Two students claims that a 3×5 rectangle and a 4×6 rectangle are similar.*

Fred says that that they are similar because the angles are the same. How do you respond?

Ned says that they are similar because you can do the same thing (i.e., add 1) to "both sides" of the 3×5 rectangle to get the 4×6 rectangle. How do you respond?

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Question 6 *Complete the following sentences using words such as filling, falling, covering, wrapping, hiding, surrounding, or traveling:*

Area vs. perimeter can be thought of as $\boxed{?}$ *vs.* $\boxed{?}$*, respectively.*

Volume vs. surface area can be thought of as $\boxed{?}$ *vs.* $\boxed{?}$*, respectively.*

To explore how measures of figures change under scaling and non-scaling transformations, here are some useful strategies:

- Cutting the figures and rearranging the pieces.
- Using "rep-tiles."
- Using known formulas for perimeters, areas, volumes, or surface areas.
- Approximating with segments, squares, or cubes.

Question 7 *To estimate the length of a curve, imagine approximating it with many small segments. Now apply a similarity transformation with a scale factor of* k*. Each segment will scale by* $\boxed{?}$*, so the length of the curve will be* $\boxed{?}$ *times the original length.*

Question 8 *To estimate the area of a non-polygonal region, imagine covering it approximately with a grid of squares of side length s. Piecing together partial squares, suppose you count n squares. Your area estimate is then* $\boxed{?}$.

Now apply a similarity transformation of scale factor k to both the region and the grid. Each square in the scaled grid will have area $\boxed{?}$*, and piecing together partial squares there will be* $\boxed{?}$ *squares. Thus, we estimate the area of the scaled region to be* $\boxed{?}$*, which is* $\boxed{?}$ *times the area of the original region.*

Question 9 *When n copies of a plane figure can form a figure similar to the original, the figure is called a rep-n-tile. Explain briefly why any parallelogram is a rep-4-tile and also a rep-9-tile. Generalize.*

In general, if a parallelogram is scaled by a factor of k, then $\boxed{?}$ *copies of the original parallelogram can make the scaled version.*

Question 10 *Use formulas to determine what happens to the perimeter and* *area of a rectangle when it is scaled by k.*

Begin with a rectangle of base b and height h. After scaling the rectangle by k, the base will be $\boxed{?}$ *and the height will be* $\boxed{?}$.

The original perimeter is $\boxed{?}$. *After scaling, the perimeter will be* $\boxed{?}$, *which is precisely* $\boxed{?}$ *times the original perimeter.*

The original area is $\boxed{?}$. *After scaling, the area will be* $\boxed{?}$, *which is precisely* $\boxed{?}$ *times the original area.*

Question 11 *Use formulas to determine what happens to the circumference and area of a circle when it is scaled by k.*

Begin with a circle of radius r. After scaling the circle by k, its radius will be $\boxed{?}$.

The original circumference is $\boxed{?}$. *After scaling, the circumference will be* $\boxed{?}$, *which is precisely* $\boxed{?}$ *times the original circumference.*

The original area is $\boxed{?}$. *After scaling, the area will be* $\boxed{?}$, *which is precisely* $\boxed{?}$ *times the original area.*