

Parallel Lines

Proofs updated.

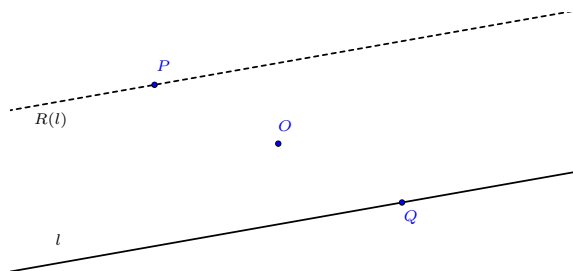
Axiom 1. *Parallel postulate (uniqueness of parallels): Given a line and a point not on the line, there is exactly one line through the given point parallel to the given line.*

Theorem 1. *A 180° rotation about a point on a line takes the line to itself.*

Proof Suppose point P is on line k . The point cuts the line into two opposite rays. A 180° rotation about P swaps the two opposite rays, thereby mapping the line onto itself. ■

Theorem 2. *A 180° rotation about a point not on a line takes the line to a parallel line.*

Proof Let O be a point not on line l . Let P be an arbitrary point on $R(l)$, the rotated image of l . To show that $R(l)$ is parallel to l , it is sufficient to show that P cannot lie also on l .

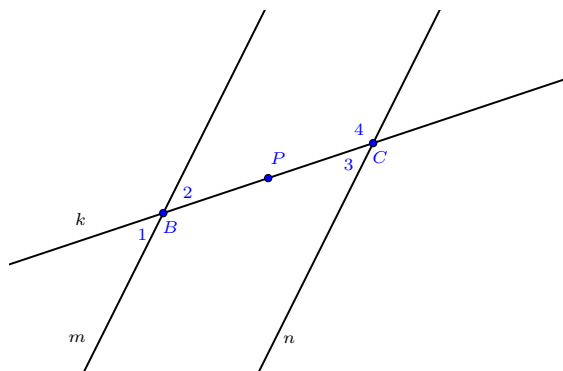


Because P is on $R(l)$, there is a point Q on l such that $P = R(Q)$. The rotated image of \overrightarrow{OQ} is \overrightarrow{OP} , and because $\angle QOP$ is 180° , it follows that Q , O , and P are collinear. Call that line k . We know line k is distinct from l because O is on k but not on l . Now, if P were on l , then points P and Q would be on two distinct lines, k and l , contradicting the assumption that on two points there is a unique line. The theorem is proved. ■

Theorem 3. *If two parallel lines are cut by a transversal alternate interior angles and corresponding angles are congruent.*

Learning outcomes:
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Proof Given that parallel lines m and n are cut by transversal k , prove that alternate interior angles are congruent.

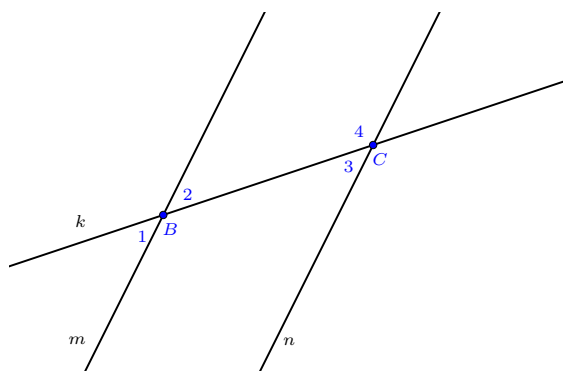


- (a) Let B and C be the intersections of transversal k with lines m and n , respectively. Let P be the midpoint of \overline{BC}
- (b) Rotate 180° about P , which takes k to (itself \checkmark / m / n).
- (c) The rotation maps B to C and C to B because distances are preserved.
- (d) The rotation maps m to a parallel line through C , which must be $(k / m / n \checkmark)$ by the uniqueness of parallels.
- (e) The rotation maps n to $(k / m \checkmark / n)$ by the same reasoning.
- (f) The rotation swaps $\angle 2$ and $(\angle 1 / \angle 2 / \angle 3 \checkmark / \angle 4)$. These alternate interior angles must be congruent because the rotation preserves angle measures.

■

Note: The congruence of corresponding angles now follows from the congruence of vertical angles. But here is a direct approach that uses a translation.

Proof Given that parallel lines m and n are cut by transversal k , prove that corresponding angles are congruent.



- (a) Let B and C be the intersections of transversal k with lines m and n , respectively.
- (b) Translate to the right along line k by distance BC , which takes k to (itself \checkmark / m / n).
- (c) The translation maps B to \boxed{C} , and it maps m to $(k / m / n \checkmark)$ because the translation maintains parallels, and there is a unique parallel to m through C .
- (d) The translation maps $\angle 1$ to $(\angle 1 / \angle 2 / \angle 3 \checkmark / \angle 4)$. These corresponding angles must be congruent because the translation preserves angle measures.

■

Theorem 4. *If two lines are cut by a transversal so that alternate interior (and corresponding) angles are congruent, then the lines are parallel.*