Identities and Inverses, Part 1

Identities and inverses arise in lots of mathematical settings. What is the same across these settings?

How is an inverse matrix like a multiplicative inverse or an inverse function?

When using the terms *identity* and *inverse*, it is important to be clear about the objects (e.g., numbers, matrices, functions) and operations involved. Also, because the meaning of inverse depends upon an identity element, it is good practice to discuss the identity first.

Definition 1. *We call* 0 *the* additive identity *because when the operation is addition* 0 *behaves as an identity element in most sets of numbers. In other words,* $x + 0 = x$ *and* $0 + x = x$ *for all* x *in the relevant number set (e.g., whole numbers, integers, rational numbers, or real numbers).*

Question 1 *Name some sets of numbers that don't have an additive identity.*

In your favorite set of numbers, when you have an additive identity, you can define the term additive inverse.

Definition 2. *Given a number* x*, a number* y *is said to be the* additive inverse *of* x *if* $x + y = 0$ *and* $y + x = 0$.

If y is the additive inverse of x, it follows immediately that x is the additive inverse of y. Thus, when both of these conditions hold, we say that x and y are additive inverses of each other.

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Note that the additive identity is a single element that works for all elements of the set, whereas "additive inverse" is a relationship between a pair of elements. We can put both ideas together as follows: When we add additive inverses, we get the additive identity.

Question 2 *Are there numbers that are their own additive inverses? If not, explain. If so, name all that you can, and explain how you know you have them all.*

Question 3 *Name some sets of numbers in which some of the elements don't have additive inverses.*

Question 4 *Define* multiplicative identity *for your favorite set of numbers.*

Question 5 *Define* multiplicative inverse *for your favorite set of numbers.*

Question 6 *Identity elements are sometimes called "do-nothing elements." Use that idea to talk about both the additive identity and the multiplicative identity for real numbers.*

Question 7 *Are there numbers that are their own multiplicative inverses? If not, explain. If so, name all that you can, and explain how you know you have them all.*

When discussing matrices, there are lots of size restrictions regarding whether two matrices can be added or multiplied. To keep the discussion simple, let's consider only 2×2 matrices. The ideas generalize fairly easily to $n \times n$ matrices, where n is a counting number. Non-square matrices are more problematic.

When we discuss an *identity matrix* and the *inverse of a matrix*, the operation is assumed to be matrix multiplication. In other words, the 2×2 identity matrix might be more appropriately called the *multiplicative identity matrix.*

Question 8 *What would be the 2×2 additive identity matrix? Show how you know.*

Question 9 *What would be the additive inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$? Show how you know.*

Question 10 *What is the 2×2 (multiplicative) identity matrix? Show how you know.*

Question 11 *What does it mean for matrices A and B to be (multiplicative) inverses of each other?*

A joke. After writing two matrices on the board, a professor asks a student, "Are these matrices inverses?" The student answers, "The first one is, and the second one isn't."

Question 12 *Why is this joke funny?*

Question 13 *Are there matrices that are their own (multiplicative) inverses? If not, explain. If so, list a few, and describe them all.*

Question 14 *Describe what "identity" and "inverse" mean across all of these settings.*