Math 4480 Algebra for Teaching

Brad Findell

December 14, 2018

Contents

1 Ximera tutorial

After completing this section, students should be able to do the following.

- Type answers into Ximera.
- Update Ximera activities.

Author(s): Brad Findell

2 How to use Ximera

This course is built in Ximera.

Mathematics cannot be learned passively: it must be actively constructed by the person learning it. With this in mind, this course is built around solving problems!

Here are some examples. Play around with it, get it wrong, try the hints out. Don't be afraid to fail: **getting an answer wrong never hurts you.**

Example 1. *Some problems are multiple-choice:*

Multiple Choice:

(a) *Don't pick me.*

(b) *Not me either.*

(c) *Pick me!* ✓

(d) *Also an incorrect choice*

Feedback(*`attempt`*): *Click on the choice that says "Pick me!"*

Example 2. *Some problems are select-all that are correct:*

Select All Correct Answers:

(a) *Don't pick me.*

(b) *Pick me!* ✓

(c) *Pick me too!* ✓

(d) *I'm a correct choice too.* ✓

Feedback(*`attempt`*): *Click on the choices "Pick me!" "Pick me too!" and "I'm a correct choice too."*

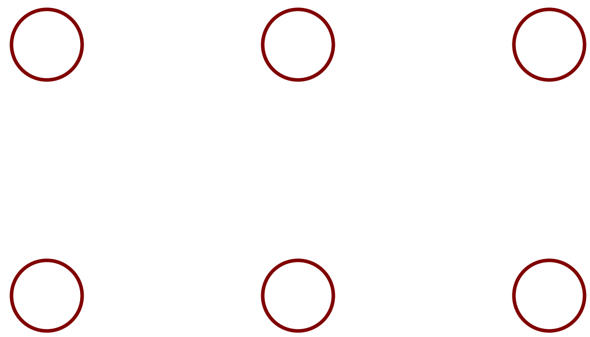
Example 3. *Some problems are fill in the blank:* $3 \times 2 = \boxed{6}$

Hint: *3×2 is the number of objects in 3 groups of 2 objects*

Author(s): Brad Findell

See constructed at http://en.wikipedia.org/wiki/Constructivism_(philosophy_of_education)

Hint:



Hint: $3 \times 2 = 6$

For this course, you should always have a paper and pencil near at hand to make notes, doodle pictures, or solve complicated equations. We **strongly** recommend that you really **grapple** with a problem before getting a hint, or moving on. The difference between what you learn by struggling with a problem on your own versus perusing someone else's solution is astonishing.

With that said, even if you get an answer right you should **always** try the hints out afterwards. They might explain the concept from a new point of view, or challenge you to think in a different way than you solved the problem.

Many fill-in-the-blank problems expect algebraic expressions for answers. In the examples below, try to retype the expression to the left of the equals sign:

Example 4. $\frac{x^2+y^2}{7} = \boxed{\frac{x^2+y^2}{7}}$

Feedback(*`attempt`*): *Type* `(x^2+y^2)/7`

Example 5. $\frac{\tan(x)}{2xa+b^2} = \boxed{\frac{\tan(x)}{2xa+b^2}}$

Feedback(*`attempt`*): *Type* `tan(x)/(2xa+b^2)`

Example 6. $\arcsin(x) = \boxed{\arcsin(x)}$

Feedback(*`attempt`*): *Type* `arcsin(x)`

Note that typing `sin^(-1)(x)` *does not work.*

Example 7. $|x| = \boxed{|x|}$

Feedback(*`attempt`*): *You can type* `|x|` *or* `abs(x)`*, but* `abs(x)` *may be preferable because it is easier to parse appropriately.*

Example 8. $\ln(x+1) = \boxed{\ln(x+1)}$

Feedback(*`attempt`*): *You could type* `ln(x+1)` *or* `log(x+1)`

Example 9. $\sin(\theta) = \boxed{\sin(\theta)}$

Feedback(*`attempt`*): *Type* `sin(theta)`

Example 10. $\varphi = \boxed{\varphi}$

Feedback(*`attempt`*): *Type* `phi`

Example 11. $\rho = \boxed{\rho}$

Feedback(*`attempt`*): *Type* `rho`

Example 12. $\sqrt{x} = \boxed{\sqrt{x}}$

Feedback(*`attempt`*): *Type* `sqrt(x)`

Feedback(*`attempt`*): *It would also work to type* `x^(1/2)`

Example 13. $\sqrt[3]{y} = \boxed{y^{\frac{1}{3}}}$

Feedback(*`attempt`*): *We do not have a "slick" way to enter this, so you should just type* `y^(1/3)`*, which is equivalent.*

Example 14. $DNE = \boxed{DNE}$

Feedback(*`attempt`*): *Type* `DNE`*, which means "Does Not Exist."*

Example 15. $\infty = \boxed{\infty}$

Feedback(*`attempt`*): *Type* `infty` *or* `infinity` *or* `oo`*.*

As you complete activities the green "completion bar" moves at the top of the page. This lets you know how close you are to being done with an activity.

You advance through pages either by completing them and clicking the "next activity" button, or by navigating on the little scroll bar at the top of the page.

3 How is my work scored?

We explain how your work is scored.

We want you to learn from this text. Hence, we ask questions to “push” your thinking, and leave blanks in examples to ensure you are following along. We encourage you to **keep a notebook** where you write each question and your answers, along with each major theorem and example. In essence we want you to imagine that **we are writing mathematics together**, and thus are exploring a new world of mathematics together.

With this in mind, your work is graded on the basis of its correct **completion**. The green bar above

tells you how close you are to completion. We hope that you can complete each activity and see a full green bar:

Author(s): Brad Findell

However, sometimes there is a bug that prohibits a "full green bar." In that case, do not despair, as we take these bugs into account when grading. Moreover, please **let us know** any issues you are having. If possible **we will fix the issue**.

If a correction is made then we may make an update. In this case an orange button will appear at the top of the screen:

If you click the "update" button, a dialog will appear:

updateDialog.png

If you update your work, the current activity will be replaced by a new activity. Since your previous work was for the previous incarnation of the activity, your previous work will be deleted. However, if you've completed the activity, **your record of completion remains**. You can witness this by selecting another activity, and observing your green-bar for the updated activity. Unfortunately, if your activity was not complete before you updated and you want a full green-bar on the updated activity, you will have to complete the activity again.

We simply want you to learn and to provide you with the best possible learning experience.

Problem 1 *Are you ready to start doing some math!?!*

Multiple Choice:

(a) *Yes I am!* ✓

Hint: *We promise the real course will not be this corny.*

4 Quick Questions

Many teachers have quick answers to the following questions.

Please provide quick answers and one-sentence explanations, when requested. Answer off the top of your head, **without a calculator**, and spend **no more than 40 minutes** on these.

Question 2 *Evaluate* $-x^2$ *when* $x = 9$.

Free Response:

Question 3 *Evaluate* x^{-2} *when* $x = 9$.

Free Response:

Question 4 *Evaluate* $x^{1/2}$ *when* $x = 9$.

Free Response:

Question 5 *Evaluate* $\frac{2}{0}$ *and explain your answer.*

Free Response:

Question 6 *Evaluate* $\frac{0}{0}$ *and explain your answer.*

Free Response:

Question 7 *Evaluate* $\frac{0}{2}$ *and explain your answer.*

Author(s): Brad Findell

Free Response:

Question 8 *Is 0 even, odd, neither, or both? Explain.*

Free Response:

Question 9 *Give another explanation for the previous question.*

Free Response:

Question 10 *Is $\sqrt{4} = \pm 2$? Explain.*

Free Response:

Question 11 *To divide fractions, is it okay to convert to a common denominator and then ignore the denominators and divide the numerators? Explain.*

Free Response:

Question 12 *To divide fractions, it is okay to divide the numerators and divide the denominators? Explain.*

Free Response:

Question 13 *Write a "story problem" for $1\frac{3}{4} \div \frac{1}{2}$.*

Free Response:

Question 14 *Is $15 \equiv 7 \pmod 4$? Explain.*

Free Response:

Question 15 *Is* $2 \equiv 17 \pmod 5$*? Explain.*

Free Response:

For the following three questions, suppose f is a function with a domain and range that are both subsets of the real numbers and that $f(3) = 2$. Based on this information:

Question 16 *Where is the* 3*?*

Free Response:

Question 17 *Where is the* 2*?*

Free Response:

Question 18 *Where is the* $f(3)$*?*

Free Response:

Question 19 *Is* $0.99999\cdots = 1$*? Explain.*

Free Response:

Question 20 *Why is* $a^0 = 1$*. Does it matter what* a *is?*

Free Response:

Question 21 *Why is* $a^{-n} = \dfrac{1}{a^n}$. *Does it matter what* a *is? Does it matter what* n *is?*

Free Response:

Question 22 *What does it mean for a number to be irrational?*

Free Response:

Question 23 *How long did you spend on these questions?*

Free Response:

5 Identities and Inverses, Part 1

Identities and inverses arise in lots of mathematical settings. What is the same across these settings?

How is an inverse matrix like a multiplicative inverse or an inverse function?

When using the terms *identity* and *inverse*, it is important to be clear about the objects (e.g., numbers, matrices, functions) and operations involved. Also, because the meaning of inverse depends upon an identity element, it is good practice to discuss the identity first.

Definition 1. *We call* 0 *the* additive identity *because when the operation is addition* 0 *behaves as an identity element in most sets of numbers. In other words,* $x + 0 = x$ *and* $0 + x = x$ *for all* x *in the relevant number set (e.g., whole numbers, integers, rational numbers, or real numbers).*

Question 24 *Name some sets of numbers that don't have an additive identity.*

Free Response: ***Hint:*** *Possible answers: counting numbers, odd numbers, non-zero rational numbers.*

In your favorite set of numbers, when you have an additive identity, you can define the term additive inverse.

Definition 2. *Given a number* x*, a number* y *is said to be the* additive inverse *of* x *if* $x + y = 0$ *and* $y + x = 0$*.*

If y is the additive inverse of x, it follows immediately that x is the additive inverse of y. Thus, when both of these conditions hold, we say that x and y are additive inverses of each other.

Note that the additive identity is a single element that works for all elements of the set, whereas "additive inverse" is a relationship between a pair of elements. We can put both ideas together as follows: When we add additive inverses, we get the additive identity.

Question 25 *Are there numbers that are their own additive inverses? If not, explain. If so, name all that you can, and explain how you know you have them all.*

Author(s): Brad Findell

Free Response: ***Hint:*** *A number x that is its own additive inverse must be a solution to the equation $x + x = 0$. This is equivalent to $2x = 0$, and the only solution is $x = 0$. Thus, 0 is the only number that is its own additive inverse.*

Question 26 *Name some sets of numbers in which some of the elements don't have additive inverses.*

Free Response: ***Hint:*** *Possible answers: counting numbers, whole numbers, the set $1, 3, 4$.*

Question 27 *Define* multiplicative identity *for your favorite set of numbers.*

Free Response: ***Hint:*** *In the real numbers, we call 1 the* multiplicative identity *because when the operation is multiplication 1 behaves as an identity element. In other words, $x \cdot 1 = x$ and $1 \cdot x = x$ for all real numbers x.*

Question 28 *Define* multiplicative inverse *for your favorite set of numbers.*

Free Response: ***Hint:*** *Given a real number x, a real number y is the multiplicative inverse of x if $xy = 1$ and $yx = 1$.*

Question 29 *Identity elements are sometimes called "do-nothing elements." Use that idea to talk about both the additive identity and the multiplicative identity for real numbers.*

Free Response: ***Hint:*** *Given any number x, adding the additive identity yields x, so the identity "does nothing." Similarly, multiplying x by the multiplicative identity yields x. So 1 is a do-nothing element for multiplication.*

Question 30 *Are there numbers that are their own multiplicative inverses? If not, explain. If so, name all that you can, and explain how you know you have them all.*

Free Response: ***Hint:*** *A number x that is its own multiplicative inverse would be a solution to the equation $x \cdot x = 1$ or $x^2 = 1$. In the real numbers, this equation has exactly two solutions: $x = \pm 1$.*

When discussing matrices, there are lots of size restrictions regarding whether two matrices can be added or multiplied. To keep the discussion simple, let's consider only 2×2 matrices. The ideas generalize fairly easily to $n \times n$ matrices, where n is a counting number. Non-square matrices are more problematic.

When we discuss an *identity matrix* and the *inverse of a matrix*, the operation is assumed to be matrix multiplication. In other words, the 2×2 identity matrix might be more appropriately called the *multiplicative identity matrix*.

Question 31 *What would be the 2×2 additive identity matrix? Show how you know.*

Free Response: ***Hint:*** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *is the additive identity matrix because*

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *for all a, b, c, d.*

Note that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, *this can be written simply:*

$A + Z = A$ *and* $Z + A = A$.

Question 32 *What would be the additive inverse of the matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$*? Show how you know.*

Free Response: ***Hint:*** $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ *is the additive inverse of* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *because*

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *for all a, b, c, d.*

Note that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$, *this can be written simply:*

$A + -A = Z$ *and* $-A + A = Z$.

Question 33 *What is the 2×2 (multiplicative) identity matrix? Show how you know.*

Free Response: ***Hint:*** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is the multiplicative identity matrix because for any 2×2 matrix A, $AI = A$ and $IA = A$. Here is more detail:*

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *for all a, b, c, d.*

Question 34 *What does it mean for matrices A and B to be (multiplicative) inverses of each other?*

Free Response: ***Hint:*** *Matrices A and B are inverses of each other if $AB = I$ and $BA = I$.*

A joke. After writing two matrices on the board, a professor asks a student, "Are these matrices inverses?" The student answers, "The first one is, and the second one isn't."

Question 35 *Why is this joke funny?*

Free Response: ***Hint:*** *The student thinks that "inverse" can apply to a single object, but "inverse" is a relationship between a pair of objects.*

Question 36 *Are there matrices that are their own (multiplicative) inverses? If not, explain. If so, list a few, and describe them all.*

Free Response: ***Hint:*** *A matrix A that is its own multiplicative inverse would be a solution to the equation $A \cdot A = I$ or $A^2 = I$. There are an infinite number of matrices that satisfy this equation. Here are four convenient ones:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

Question 37 *Describe what "identity" and "inverse" mean across all of these settings.*

Free Response:

6 Identities and Inverses, Part 2

The identity function and inverses of functions as examples of the concepts of identity and inverse.

In a previous activity, we explored identities and inverses through a careful process that involved four steps:

(a) Specifying the objects,

(b) Specifying the operation,

(c) Defining the identity with respect to that operation on those objects, and

(d) Defining the meaning of inverse with respect to that operation on those objects.

In this activity, we explore identities and inverses for functions.

Remark 1. *In the Common Core State Standards (CCSS), the expectations regarding inverses of functions are quite modest for the audience of all students, requiring only that they "Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse (F-BF.4.a)." Students in Precalculus and Calculus need more, of course.*

These notes and questions are intended to help teachers understand these subtle ideas well enough to make wise instructional decisions for both populations of students.

Functions as Objects

To begin our exploration, we first decide that *functions are the objects* under consideration. Thinking of functions as objects can be something of a challenge, however, because much classroom experience emphasizes formulas and computing particular function values.

The CCSS includes the following standards about functions:

- 8.F.1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

Author(s): Brad Findell

- F-IF.1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

Question 38 *Write a definition of* function *that you used in an upper-level undergraduate mathematics class. Explain how your definition is or is not consistent with the above definitions.*

Free Response: ***Hint:*** *Possible answer: For any sets A and B,* a function f from A to B *is a subset of the Cartesian product $A \times B$ such that every $a \in A$ appears exactly once as the first element of an ordered pair (a, b) in f. When $(a, b) \in f$, the notation $f(a)$ means the corresponding b.*

In the less formal definitions from the CCSS, A is the set of 'inputs' or the elements of the domain. The 'outputs' are the elements of the range, which is a subset of B.

Question 39 *Many students think of function as synonymous with formula. Describe some advantages to thinking of functions as broader than formulas.*

Free Response: ***Hint:*** *Possible answer: In many modeling situations, formulas are not suitable. For example, suppose $T(n)$ denotes the high temperature at an Arps Hall weather station on the n^{th} day of the year.*

Question 40 *Suppose g is a function, $g(3) = 5$ and $1 = g(3) + g(a)$. What can you say about $g(a)$? What can you say about a? About g?*

Free Response: ***Hint:*** *From the two equations, we can conclude that $g(a) = -4$, but we know nothing else about a and nothing else about g.*

To think of a function as an object, it helps to imagine the function as a whole. Saying "f is the squaring function" is a statement about the whole function, whereas $f(3) = 9$ is a statement about a specific function value. In general, a stand-alone letter, such as f, is used to indicate the function as a whole, whereas $f(a)$ indicates the function value for the particular input value a.

To think about the whole function, imagine varying through all possible input values. The whole function is not just the output values but rather all the correspondences between the input values and output values. Thus, if the domain of f is D, we may write $f = \{(x, y) | x \in D, y = f(x)\}$, which is to say the function f is this set of ordered pairs.

A Note on Notation

Mathematicians and teachers are sometimes sloppy regarding the notational distinction between a function value and the function as a whole, allowing $f(x) = x^2$, for example, to be taken as a statement about the whole function.

Consider the following expressions: $f(a)$, $f(x_0)$, and $f(x)$. Without any additional context, many mathematicians and teachers interpret the first two as particular output values, because it is customary to use the letter a and the subscripted x_0 to denote particular values, considered one at a time and conceived as "fixed" while reasoning through a problem. The expression $f(x)$, on the other hand, is ambiguous, for it sometimes denotes a particular output value, yet other times denotes the function as a whole.

Some mathematicians occasionally rail at the use of $f(x)$ for the function as a whole, while others are content that the meaning is usually clear from the context. When specifying a function, some authors and computer algebra systems avoid the ambiguous "specification formula" $f(x) = x^2$ and instead use the notation $f : x \mapsto x^2$, which can be read, "f maps x to x^2."

This distinction and the "maps to" notation are likely too subtle when high school students are first learning function notation, because students already have plenty of difficulty with simple uses of the notation. The distinction can be useful in calculus, however, and it becomes necessary in upper-level undergraduate mathematics courses. And it is important that teachers understand the distinction because some of their students' difficulties will involve this issue.

Specifying the Operation

Once we consider function to be objects, we can specify an operation for combining such objects. To reach the goal of discussing inverses of functions, we must agree that *the operation is function composition.* The open circle symbol, $\circ$, is often used to indicate function composition, so that $f \circ g$ indicates the composition of functions f and g, taken as whole objects. The expression $f(g(x_0))$, in contrast, is about particular function values. The statement $(f \circ g)(x) = f(g(x))$ indicates how the two ideas are related.

Question 41 *Suppose $f : x \mapsto x^2$ and $g : x \mapsto x + 3$. Compute $f \circ g$ and $g \circ f$. What do you notice?*

Free Response: ***Hint:*** *First, $f(g(x)) = f(x + 3) = (x + 3)^2$. So $f \circ g : x \mapsto (x + 3)^2$.*

Second, $g(f(x)) = g(x^2) = x^2 + 3$. So $g \circ f : x \mapsto x^2 + 3$.

Note that $f \circ g \neq g \circ f$, which is to say that function composition is not commutative.

Defining Identity and Inverse

At last, we can define identity function. To keep the discussion within the realm of school mathematics, let's consider only real-valued functions of a real variable. In other words, both the input and output values are assumed to be real numbers, so that both the domain and the range are subsets of the real numbers.

Definition 3. *A function I is said to be an* identity function *on a domain D if $f \circ I = f$ and $I \circ f = f$ for any function f with domain D. Note that these are statements about whole functions.*

Question 42 *Describe the similarities and differences between this definition of identity function and your definition of multiplicative identity.*

Free Response: ***Hint:*** *It is essentially the same idea, with appropriate substitutions. Replace function composition with multiplication, I with 1, and f as any function with x as any number.*

Question 43 *Restate the definition of identity function so that it involves statements about function values.*

Free Response: ***Hint:*** *If function I is called the identity function on domain D if $f(I(x)) = f(x)$ and $I(f(x)) = f(x)$ for all $x \in D$ and for any function f with domain D.*

Question 44 *What is the identity function on the real numbers? Call it I.*

Free Response: ***Hint:*** $I(x) = x$.

Before we define the inverse of a function, we must first acknowledge that not all functions have inverses. We will address this issue in more detail later in the course. For now, let's restrict our attention to a particular collection of invertible functions: those that are both one-to-one and onto a domain D that is a subset of $\mathbb{R}$.

Remark 2. *If the function $f : A \to B$ is not one-to-one, it can be made one-to-one by restricting its domain to a subset X of A. To ensure the function is onto, let $R = f(X)$, the actual range of the restricted function. Then the function $g : X \to R$ given by $x \mapsto f(x)$ for all $x \in X$ is both one-to-one and onto, and hence it is invertible.*

Definition 4. *Suppose $f : D \to D$ is one-to-one and onto. A function $g : D \to D$ is the* inverse *of f if $g \circ f = I$ and $f \circ g = I$. Note that f is then also the inverse of g, and hence f and g are* inverse functions *in the sense that they are inverses of each other.*

Question 45 *Describe the similarities and differences between this definition of the inverse of a function and your definition of multiplicative inverse.*

Free Response: ***Hint:*** *It is essentially the same idea, with appropriate substitutions. Replace function composition with multiplication, I with 1, f with x, and g (the inverse of f) with y (the multiplicative inverse of x).*

Remark 3. *This analogy between multiplicative inverse and inverse of a function is what is behind the confusing notation f^{-1} for the inverse of a function. If a is a nonzero real number, then a^{-1} denotes the inverse of a with respect to multiplication. Similarly, if f is an invertible function, then f^{-1} means the inverse of f with respect to function composition.*

We should apologize to students for the fact that in $\sin^2 x$, the exponent is about multiplication, whereas in $\sin^{-1} x$, the exponent is about function composition. Because these notations are incompatible, some mathematicians and teachers use the notation $\arcsin x$ instead.

Question 46 *Give a definition of* inverse function *that involves statements about function values.*

Free Response: ***Hint:*** *Given a function $f : D \to D$, a function $g : D \to D$ is the inverse of f if $f(g(x)) = x$ and $g(f(x)) = x$ for all $x \in D$.*

A joke. After writing two matrices on the board, a professor asks a student, “Are these matrices inverses?” The student answers, “The first one is, and the second one isn’t.”

Question 47 *Rewrite the joke as a joke about functions.*

Free Response: ***Hint:*** *After writing two functions on the board, a professor asks a student, “Are these functions inverses?” The student answers, “The first one is, and the second one isn’t.”*

Question 48 *Suppose a function composed with itself is the identity function. What can you say about the inverse of the function? Can you think of such a function?*

Hint: *Call the function f. Compare the following: (1) that f composed with itself is the identity function, and (2) the definition of the inverse of f.*

Free Response: ***Hint:*** *This is subtle and worth some thought.*

Examples of Inverses of Functions

Addition is a "binary operation," which is to say it takes two numbers (as inputs) and returns their sum (as output). In other words, addition of real numbers is a function with the domain $\mathbb{R} \times \mathbb{R}$, pairs of real numbers, and the range $\mathbb{R}$. How, then, should we think about the oft-heard claim that "Subtraction is the inverse of addition"?

When considering the expression $5+3$, we can "undo" the addition by subtracting 3, and we get back to where we started, which happens to be at 5 but which could have been any number. In other words, we are thinking of the "add 3" function $f : x \mapsto x + 3$.

Question 49 *Use these ideas to determine the inverse of the function $f : x \mapsto x+3$? Use the definition of inverse function to demonstrate that you are correct.*

Free Response: ***Hint:*** *The inverse of f is the the "subtract 3" function $g : x \mapsto x - 3$.*

Check that $g \circ f = I$ as follows: $g(f(x)) = g(x + 3) = (x + 3) - 3 = x$.

Check that $f \circ g = I$ as follows: $f(g(x)) = f(x - 3) = (x - 3) + 3 = x$.

Note the way in which g "undoes" or "reverses" the effect of f. If we know the output of f was 12, we can compute $g(12) = 9$ to know that the input must have been 9.

Question 50 *Suppose $f : x \mapsto 2x - 5$. Use verbal descriptions of f and the idea of "undoing" to determine the inverse of f. Explain your reasoning.*

Free Response: ***Hint:*** *Think of f as "multiply by 2 then subtract 5." To undo that, we would "add 5 then divide by 2." So $g : x \mapsto \dfrac{x+5}{2}$ is the inverse of f.*

Remark 4. *Notice that figuring out the inverse of f does not require that misunderstood procedure that begins by swapping x and y.*

When someone says that "Subtraction is the inverse of addition," the point is more general than what we have just seen. We are really talking about a family of addition functions, $f_a : x \mapsto x + a$, where a is a parameter that indicates a particular addition function, each of which has its own inverse, $g_a : x \mapsto x - a$. We need the following definition:

Definition 5. *A* family of functions *is a parametrized set of related functions.*

Much of high school mathematics is organized around particular families of functions, such as linear functions, quadratic functions, exponential functions, or trigonometric functions. The quadratic functions, for example, are the family $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The parameters a, b, and c, when conceived as "fixed," indicate which particular function we are talking about. Note that "trigonometric functions" are not a family in the sense of the above definition.

Question 51 *In the family of quadratic functions described above, why is it important to state that $a \neq 0$?*

Free Response: ***Hint:*** *If $a = 0$ the function is not quadratic but linear.*

The concept of a family of functions is much more varied than the standard families that populate high school mathematics. For example, in problem-solving situations, it might be useful to consider the family of cubic functions with zeros at 0 and 1, which could be parametrized as $f : x \mapsto ax(x-1)(x-r)$.

Question 52 *Use families of functions to explain the statement "division is the inverse of multiplication."*

Free Response: ***Hint:*** *Consider the family of "multiply by a" functions, $f_a : x \mapsto ax$, where $a \neq 0$. The inverse of f_a is the "divide by a" function $g : x \mapsto \dfrac{x}{a}$.*