

Parallel Lines

Proofs.

This page develops important results regarding parallel lines and transversals. **Read carefully, and complete the proofs.**

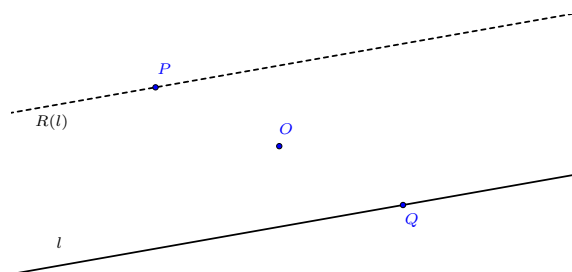
Axiom 1. *Parallel postulate (uniqueness of parallels): Given a line and a point not on the line, there is exactly one line through the given point parallel to the given line.*

Theorem 1. *A 180° rotation about a point on a line takes the line to itself.*

Problem 1 Proof Suppose point P is on line k . The point cuts the line into two opposite rays. A 180° rotation about P swaps the two opposite rays, thereby mapping the line onto itself. ■

Theorem 2. *A 180° rotation about a point not on a line takes the line to a parallel line.*

Problem 2 Proof Let O be a point not on line l . Let P be an arbitrary point on $R(l)$, the rotated image of l . To show that $R(l)$ is parallel to l , it is sufficient to show that P cannot lie also on l .

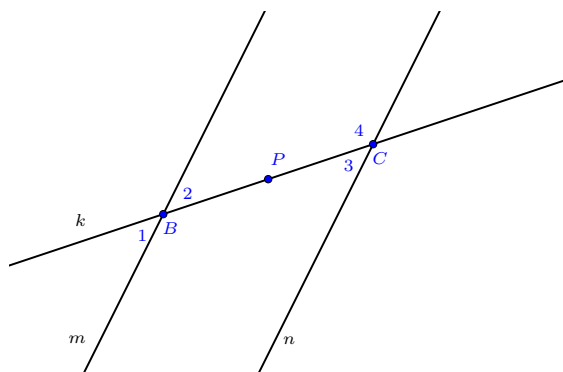


Because P is on $R(l)$, there is a point Q on l such that $P = R(Q)$. The rotated image of \overrightarrow{OQ} is $(\overrightarrow{OQ} / \overrightarrow{OP} \checkmark / \overrightarrow{QP})$, and because $\angle QOP$ is 180° , it follows that Q , O , and P are collinear. Call that line k . We know line k is distinct from l because point O is on k but not on l . Now, if P were on l , then points P and Q would be on two distinct lines, k and l , contradicting the assumption that on two points there is a unique line. The theorem is proved. ■

Learning outcomes:
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Theorem 3. *If two parallel lines are cut by a transversal, alternate interior angles and corresponding angles are congruent.*

Problem 3 Proof Given that parallel lines m and n are cut by transversal k , prove that alternate interior angles are congruent.



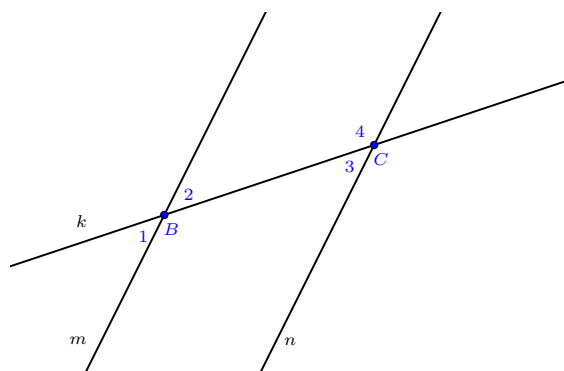
Let B and C be the intersections of transversal k with lines m and n , respectively. Let P be the midpoint of \overline{BC} .

- (a) Rotate 180° about P , which takes k to (itself \checkmark / m / n).
- (b) The rotation maps B to \boxed{C} because $PB = PC$ and the rotation preserves distances.
- (c) Because P is not on m , the rotation maps m to a parallel line through C , which must be $(k / m / n \checkmark)$ by the uniqueness of parallels.
- (d) Thus, the rotation maps $\angle 2$ to $(\angle 1 / \angle 2 / \angle 3 \checkmark / \angle 4)$. These alternate interior angles must be congruent because the rotation preserves angle measures.

■

Note: The congruence of corresponding angles now follows from the congruence of vertical angles. But the next problem is another approach that uses a translation.

Problem 4 Proof Given that parallel lines m and n are cut by transversal k , prove that corresponding angles are congruent.



Let B and C be the intersections of transversal k with lines m and n , respectively.

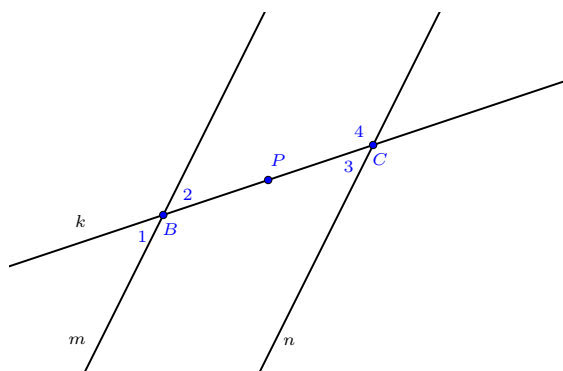
- (a) Translate to the right along line k by distance BC , which takes k to (itself \checkmark / m / n).
- (b) The translation maps B to \boxed{C} , and it maps m to $(k / m / n \checkmark)$ because the translation maintains parallels, and there is a unique parallel to m through C .
- (c) The translation maps $\angle 1$ to $(\angle 1 / \angle 2 / \angle 3 \checkmark / \angle 4)$. These corresponding angles must be congruent because the translation preserves angle measures.

■

Theorem 4. If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.

Question 5 This theorem is the converse of the previous theorem about alternate interior angles.

Problem 6 Proof Given that m and n are cut by transversal k with alternate interior angles congruent, prove that lines m and n are parallel.



Let B and C be the intersections of transversal k with lines m and n , respectively. Let P be the midpoint of \overline{BC} .

- (a) Rotate 180° about P , which takes k to (itself \checkmark m/n), and which swaps B and C because distances are preserved.
- (b) Because $\angle 2 \cong \angle 3$ and because a side of $\angle 2$ (i.e., \overrightarrow{BP}) is mapped to a side of $\angle 3$ (i.e., $(\overrightarrow{CP} \checkmark \overrightarrow{PC} / \overrightarrow{BP})$), it must be that the other side of $\angle 2$ (which lies on m) is mapped to the other side of $\angle 3$ (which lies on line n). Thus, n is the image of m .
- (c) Because P is not on m , the 180° rotation maps m to a parallel line through C . Thus, n must be parallel to m .

■