

# Parallel Lines

*Proofs updated.*

This page develops important results regarding parallel lines and transversals. **Read carefully, and complete the proofs.**

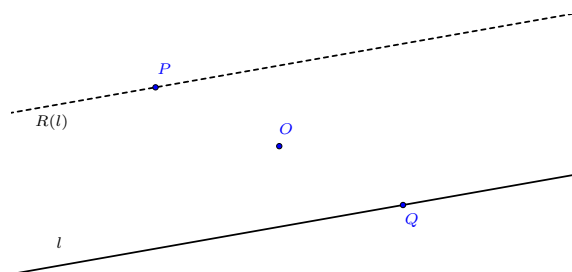
**Axiom 1.** *Parallel postulate (uniqueness of parallels): Given a line and a point not on the line, there is exactly one line through the given point parallel to the given line.*

**Theorem 1.** *A  $180^\circ$  rotation about a point on a line takes the line to itself.*

**Problem 1 Proof** Suppose point  $P$  is on line  $k$ . The point cuts the line into two opposite rays. A  $180^\circ$  rotation about  $P$  swaps the two opposite rays, thereby mapping the line onto itself. ■

**Theorem 2.** *A  $180^\circ$  rotation about a point not on a line takes the line to a parallel line.*

**Problem 2 Proof** Let  $O$  be a point not on line  $l$ . Let  $P$  be an arbitrary point on  $R(l)$ , the rotated image of  $l$ . To show that  $R(l)$  is parallel to  $l$ , it is sufficient to show that  $P$  cannot lie also on  $l$ .



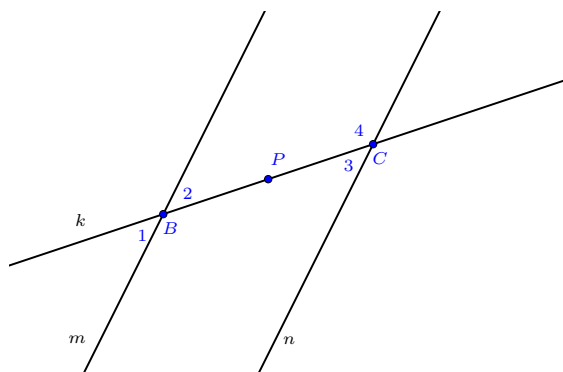
Because  $P$  is on  $R(l)$ , there is a point  $Q$  on  $l$  such that  $P = R(Q)$ . The rotated image of  $\overrightarrow{OQ}$  is  $(\overrightarrow{QO} / \overrightarrow{OP} \checkmark / \overrightarrow{QP})$ , and because  $\angle QOP$  is  $180^\circ$ , it follows that  $Q$ ,  $O$ , and  $P$  are collinear. Call that line  $k$ . We know line  $k$  is distinct from  $l$  because point O is on  $k$  but not on  $l$ . Now, if  $P$  were on  $l$ , then points  $P$  and  $Q$  would be on two distinct lines,  $k$  and  $l$ , contradicting the assumption that on two points there is a unique line. The theorem is proved. ■

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Learning outcomes:  
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**Theorem 3.** *If two parallel lines are cut by a transversal, alternate interior angles and corresponding angles are congruent.*

**Problem 3 Proof** Given that parallel lines  $m$  and  $n$  are cut by transversal  $k$ , prove that alternate interior angles are congruent.



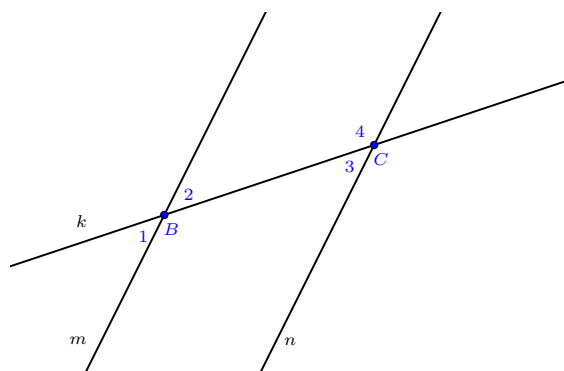
Let  $B$  and  $C$  be the intersections of transversal  $k$  with lines  $m$  and  $n$ , respectively. Let  $P$  be the midpoint of  $\overline{BC}$ .

- Rotate  $180^\circ$  about  $P$ , which takes  $k$  to (itself  $\checkmark$   $m/n$ ).
- The rotation maps  $B$  to  $C$  because  $PB = PC$  and the rotation preserves distances.
- Because  $P$  is not on  $m$ , the rotation maps  $m$  to a parallel line through  $C$ , which must be  $(k/m/n \checkmark)$  by the uniqueness of parallels.
- Thus, the rotation maps  $\angle 2$  to  $(\angle 1 / \angle 2 / \angle 3 \checkmark / \angle 4)$ . These alternate interior angles must be congruent because the rotation preserves angle measures.

■

**Note:** The congruence of corresponding angles now follows from the congruence of vertical angles. But the next problem is another approach that uses a translation.

**Problem 4 Proof** Given that parallel lines  $m$  and  $n$  are cut by transversal  $k$ , prove that corresponding angles are congruent.



Let  $B$  and  $C$  be the intersections of transversal  $k$  with lines  $m$  and  $n$ , respectively.

- Translate to the right along line  $k$  by distance  $BC$ , which takes  $k$  to (itself  $\checkmark$  /  $m$  /  $n$ ).
- The translation maps  $B$  to  $\boxed{C}$ , and it maps  $m$  to  $(k$  /  $m$  /  $n$   $\checkmark$ ) because the translation maintains parallels, and there is a unique parallel to  $m$  through  $C$ .
- The translation maps  $\angle 1$  to  $(\angle 1$  /  $\angle 2$  /  $\angle 3$   $\checkmark$  /  $\angle 4$ ). These corresponding angles must be congruent because the translation preserves angle measures.

■

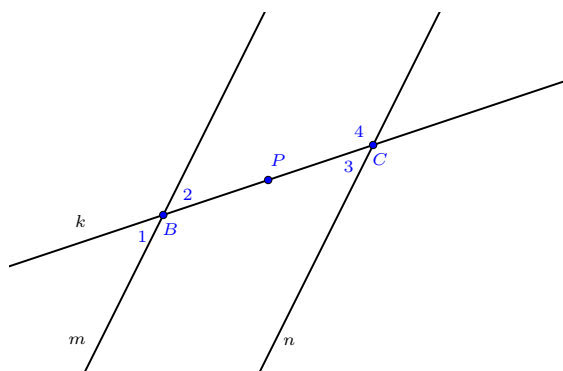
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**Theorem 4.** If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.

**Question 5** This theorem is the converse of the previous theorem about alternate interior angles.

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**Problem 6 Proof** Given that  $m$  and  $n$  are cut by transversal  $k$  with alternate interior angles congruent, prove that lines  $m$  and  $n$  are parallel.



Let  $B$  and  $C$  be the intersections of transversal  $k$  with lines  $m$  and  $n$ , respectively. Let  $P$  be the midpoint of  $\overline{BC}$ .

- (a) Rotate  $180^\circ$  about  $P$ , which takes  $k$  to (itself  $\checkmark$   $m/n$ ), and which swaps  $B$  and  $C$  because distances are preserved.
- (b) Because  $\angle 2 \cong \angle 3$  and because a side of  $\angle 2$  (i.e.,  $\overrightarrow{BP}$ ) is mapped to a side of  $\angle 3$  (i.e.,  $(\overrightarrow{CP} \checkmark \overrightarrow{PC} / \overrightarrow{BP})$ ), it must be that the other side of  $\angle 2$  (which lies on  $m$ ) is mapped to the other side of  $\angle 3$  (which lies on line  $n$ ). Thus,  $n$  is the image of  $m$ .
- (c) Because  $P$  is not on  $m$ , the  $180^\circ$  rotation maps  $m$  to a parallel line through  $C$ . Thus,  $n$  must be parallel to  $m$ .

■