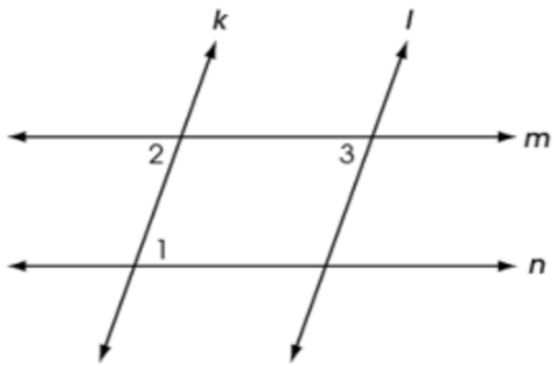
Quadrilaterals

Proof.

Problem 1 *Adapted from Ohio's 2017 Geometry released item 13.*

Two pairs of parallel lines intersect to form a parallelogram as shown.



Complete the following proof that opposite angles of a parallelogram are congruent:

(a) $\angle 1 \cong \angle 2$ *as (opposite angles/ alternate interior angles ✓/ corresponding angles) for parallel lines (*m *and* n ✓/ k *and* l*).*

(b) $\angle 3 \cong \angle 2$ *as (opposite angles/ alternate interior angles/ corresponding angles ✓)for parallel lines (*m *and* n/ k *and* l ✓*).*

(c) *Then* $\angle 1 \cong \angle 3$ *because they are both congruent to* $\angle 2$*.*

Problem 2 *Adapted from Ohio's 2018 Geometry released item 21.*

Given the parallelogram $WXYZ$*, prove that* $\overline{WX} \cong \overline{YZ}$*.*

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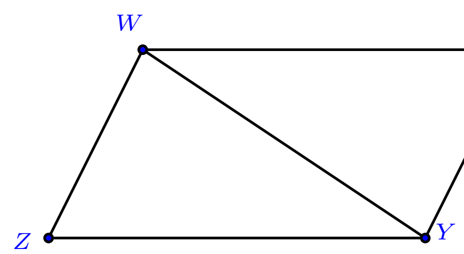


Fix note: It really would help to have an online environment that allows students to mark diagrams.

Complete the proof below:

(a) $\angle ZWY \cong \angle XYW$ as (alternate interior angles ✓/ corresponding angles/ opposite angles) for parallel segments ($\overline{WZ}$ and $\overline{XY}$ ✓/ $\overline{WX}$ and $\overline{YZ}$).

(b) $\angle ZYW \cong \angle XWY$ for the same reason, this time for parallel segments ($\overline{WZ}$ and $\overline{XY}$ / $\overline{WX}$ and $\overline{YZ}$ ✓).

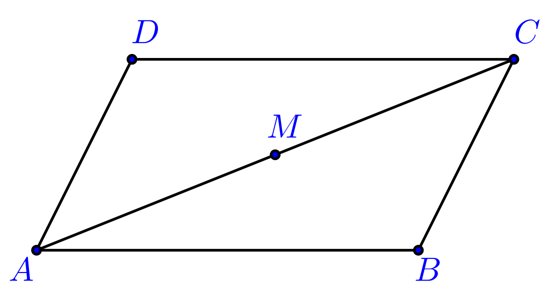
(c) $\overline{WY} \cong \overline{YW}$ because a segment is congruent to itself.

(d) $\triangle WYZ \cong \triangle YWX$ by (SAS/ ASA ✓/ SSS).

(e) Then $\overline{YZ} \cong \overline{WX}$ as corresponding parts of congruent triangles.

Fix note: Maybe number the angles.

Problem 3 *Use symmetry to prove properties of parallelograms.*



Consider a 180° *rotation about* M*, the midpoint of diagonal* $\overline{AC}$*. Show that this rotation maps the parallelogram onto itself.*

> *Fix note: The following details are subtle, especially distinguishing between mapping the sides (i.e., segments) and the lines containing the sides. Can any of this be omitted or abbreviated? Which parts might students supply?*

(a) *The rotation maps* A *to* C and C *to* A *because a* 180° *rotation about a point on a line takes the line to itself and preserves lengths.*

(b) *The rotation maps* $\overleftrightarrow{AB}$ *to a parallel line through* C *(the image of* A*), which by the uniqueness of parallels must be* $\overleftrightarrow{CD}$*. Similarly, the rotation maps* $\overleftrightarrow{CD}$ *to* $\overleftrightarrow{AB}$*,* $\overleftrightarrow{AD}$ *to* $\overleftrightarrow{CB}$*, and* $\overleftrightarrow{CB}$ *to* $\overleftrightarrow{AD}$*.*

(c) *Furthermore, the intersection of* $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$*, which is* B*, must map to the intersection of their images,* $\overleftrightarrow{CD}$ and $\overleftrightarrow{AD}$*, which is* D*. And likewise,* D *must map to* B*.*

(d) *Because vertices are mapped to vertices, sides are mapped to opposite sides, angles are mapped to opposite angles, and thus the parallelogram is mapped onto itself.*

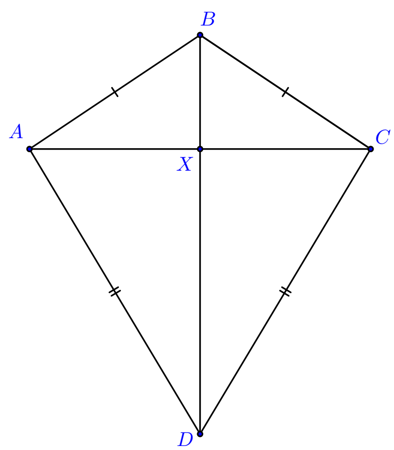
Because the 180° *rotation about* M *maps the parallelogram onto itself, we can conclude that*

Now we see that the symmetry provides the properties for free:

- *opposite sides are congruent (sides are mapped to opposite sides),*
- *opposite angles are congruent (angles are mapped to opposite angles), and*
- *the diagonals bisect each other.*

  Detail: The 180° *rotation about* M *swaps* $\overrightarrow{MB}$ and $\overrightarrow{MD}$*, so they must be opposite rays, and thus* B*,* M*, and* D *are collinear.*
  Because the rotation preserves lengths, $MB = MD$*, so that* M *is also the midpoint of* $\overline{BD}$*, which means that the diagonals bisect each other.*

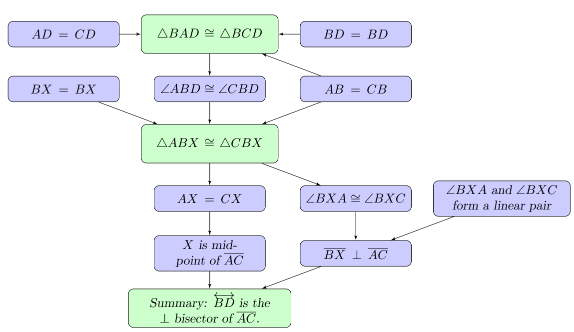
Problem 4 *Quadrilateral* $ABCD$ *is a kite as marked. Prove that* $\overleftrightarrow{BD}$ *is the perpendicular bisector of* $\overline{AC}$*.*



Key theorem: The points on a perpendicular bisector are exactly those that are equidistant from the endpoints of a segment.

Because B and D are each equidistant from A and C, they each must lie on the perpendicular bisector of $\overline{AC}$, which implies that $\overleftrightarrow{BD}$ is its perpendicular bisector.

Problem 5 *Another proof:*



Fix note: Do we need a step about $\overleftrightarrow{BX}$ and $\overleftrightarrow{BD}$ being the same line?

Fix note: Here, for the record, are the steps for a paragraph proof.

(a) $\overline{BD} \cong \overline{BD}$, *so that* $\triangle BAD \cong \triangle BCD$ *by SSS.*

(b) $\angle ABD \cong \angle CBD$ *by CPCTC.*

(c) $\overline{BX} \cong \overline{BX}$, *so that* $\triangle ABX \cong \triangle CBX$ *by SAS.*

(d) $\angle BXA \cong \angle BXC$ *by CPCTC, and they are a linear pair, so* $\overline{BX} \perp \overline{AC}$.

(e) $\overline{AX} \cong \overline{CX}$ *by CPCTC, so* X *is the midpoint of* $\overline{AC}$.

(f) *Thus,* $\overleftrightarrow{BD}$ *is the perpendicular bisector of* $\overline{AC}$.