UNIVERSITY OF PENNSYLVANIA

ESE 303: Homework 8

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ESE 303: Stochastic Systems
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A CTMC states

Given the integer nature of the problem, $X(t)$ will be integer values between 0 and max.

B Transition times out of given state

The transition time will be the first occurrence of anything that changes the value of X. Given that the chance of a claim, premium or dividend is paid are all exponentially distributed, the transition time will also be exponential. Lack of transition is the case in which no event happens. Exponential distributions are additive, therefore,

$$T_{leave} = min\{T_p, T_c, T_d\} \qquad T_{stay} = e^{-(\lambda+\alpha+\beta)t}$$

Range	Event	v_x
A	premium	λ
B	premium & claim paid @ $X(t)$	$\lambda + \alpha$
C	premium & claim paid	$\lambda + \alpha$
E	claim, dividend	$\lambda + \beta$

C Possible states going out of X(t) = x

Given that there are only four options for each transition out of x,

Range	Event	Possible States
A	premium	x+1
B	premium & claim paid @ $X(t)$	x+1, net 0
C	premium & claim paid	x+1-c
E	claim, dividend	x-c-d

D Transition probabilities

Given Baye's theory,

$$\text{P\{premium payment| a transition happens\}} = \frac{\text{P\{premium payment\}} \cap \text{P\{transition happens\}}}{\text{P\{transition happens\}}}$$

$$= \frac{\lambda}{\lambda + \alpha + \beta}$$

Range	State j	Transition P
A	1	$\frac{\lambda}{\lambda}$
B	x+1	$\frac{\lambda}{\lambda+\alpha}$
B	0	$\frac{\alpha}{\lambda+\alpha}$
C	x+1	$\frac{\lambda}{\lambda+\alpha}$
C	x-c	$\frac{\alpha}{\lambda+\alpha}$
E	x-c	$\frac{\alpha}{\beta+\alpha}$
E	x-d	$\frac{\beta}{\beta+\alpha}$

E Simulation

```
function [X,T] = cfSim(lambda,beta, alpha, X0, c, d, Xr, Xmax, Tmax)

i = 1;
T(i) = 0;
X(i) = X0;
add = [1, -c, -d];

while T(i) < Tmax
        x=X(i);
        tp = exprnd(1/lambda);
        tc = exprnd(1/alpha);
        td = exprnd(1/beta);

        if x==0
                T(i+1) = T(i) + tp;
                X(i+1)=x+1;
        elseif  0<x && x<Xr
                [m, ind]=min([tp,tc]);
                T(i+1) = T(i) + m;
                X(i+1) = x + add(ind);
        elseif Xr <= x && x < Xmax
                [m, ind]=min([tp,tc,td]);
                T(i+1) = T(i) + m;
                X(i+1) = x + add(ind);
        elseif x==Xmax
                [m, ind] = min([realmax,tc,td]);
                T(i+1) = T(i) + m;
                X(i+1) = x + add(ind);
        else
                break;
        end
        i=i+1;
end
```

The code above was used to generate:

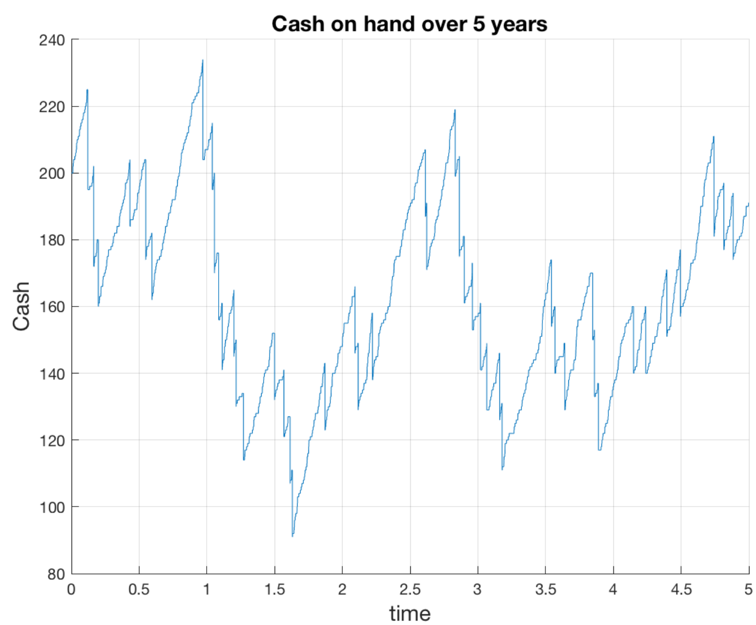

Figure 1: Cash On Hand @ time t

F Kolmogorov's Forward Equation

Given the lecture notes,

Range	State j @ t+1	$v_x \cdot p_{xy}$	q_{xy}
A	1	$\frac{\lambda}{\lambda} \cdot \lambda$	λ
B	x+1	$\frac{\lambda}{\lambda+\alpha} \cdot (\lambda+\alpha)$	λ
B	0	$\frac{\alpha}{\lambda+\alpha} \cdot (\lambda+\alpha)$	α
C	x+1	$\frac{\lambda}{\lambda+\alpha} \cdot (\lambda+\alpha)$	λ
C	x-c	$\frac{\alpha}{\lambda+\alpha} \cdot (\lambda+\alpha)$	α
D	x-c	$\frac{\alpha}{\lambda+\beta+\alpha} \cdot (\lambda+\alpha+\beta)$	α
D	x+1	$\frac{\lambda}{\lambda+\beta+\alpha} \cdot (\lambda+\alpha+\beta)$	λ
D	x-d	$\frac{\beta}{\lambda+\beta+\alpha} \cdot (\lambda+\alpha+\beta)$	β
E	x-d	$\frac{\beta}{\beta+\alpha} \cdot (\beta+\alpha)$	β
E	x-c	$\frac{\alpha}{\beta+\alpha} \cdot (\beta+\alpha)$	α

Therefore, by plugging the various probabilities into the Kolmogorov's equations, I can generate the following:

F.1 Range A, Y = 0

$$\alpha \sum_{k=1}^{c} P_{xk} - \lambda P_{xy}$$

F.2 Range B, 0 < Y < c

$$\lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda+\alpha) P_{xY}$$

F.3 Range C, c ≤ Y < X$_r$

Given the dividend cannot surpass X$_r$,

$$\lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda+\alpha) P_{xY}$$

Given the dividend can surpass X$_r$,

$$\lambda P_{x,Y-1} + \alpha P_{x,Y+c} + \beta P_{x,Y+d} - (\lambda+\alpha) P_{xY}$$

F.4 Range D, X$_r$ < Y ≤ Xmax

The transition is close enough to Y that a claim or dividend can cause the transition:

$$\lambda P_{x,Y-1} + \alpha P_{x,Y+c} + \beta P_{x,Y+d} - (\lambda+\alpha+\beta) P_{xY}$$

When the transition cannot be caused by a claim or dividend:

$$\lambda P_{x,Y-1} - (\lambda+\alpha+\beta) P_{xY}$$

When a transition can only be caused a claim or premium:

$$\lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda+\alpha+\beta) P_{xY}$$

When the transition can only be caused by a dividend or premium:

$$\lambda P_{x,Y-1} + \beta P_{x,Y+d} - (\lambda+\alpha+\beta) P_{xY}$$

F.5 Range E, Y = Xmax

$$\lambda P_{x,Y-1} - (\alpha+\beta) P_{xY}$$

G Kolmogorov's Backwards Equations

G.1 Range A, X = 0

$$\lambda P_{X+1,y} - P_{X,y}$$

G.2 Range B, 0 ¡ X ¡ c

$$\lambda P_{X+1,y} + \alpha P_{0,y} - (\lambda + \alpha) P_{X,y}$$

G.3 Range C, c $\leq$ X ¡ X_r

$$\lambda P_{X+1,y} + \alpha P_{x-c,y} - (\lambda + \alpha) P_{X,y}$$

G.4 Range D, $X_r \leq$ X $<$ Xmax

$$\lambda P_{X+1,y} + \alpha P_{x-c,y} + \beta P_{x-d,y} - (\lambda + \alpha + \beta) P_{X,y}$$

G.5 Range E, X = Xmax

$$\alpha P_{x-c,y} + \beta P_{x-d,y} - (\alpha + \beta) P_{X,y}$$

H Solution of Kolmogorovs equations

```
function [R]=Kolm(lambda,alpha,beta,c,d,Xr,Xmax)
        %UNTITLED2 Summary of this function goes here
        %   Detailed explanation goes here
        R=zeros(Xmax+1);
        R(1, 1) = -lambda;
        R(Xmax+1, Xmax)=lambda;
        R(Xmax+1, Xmax+1)=-(alpha+beta);

        for i=1:Xmax+1
                if i>=2 && i <=Xr-d
                        R(i,i)=-(lambda+alpha);
                        R(i, i-1) = lambda;
                        R(i, i+c) = alpha;
                elseif i>=Xr-d+1 && i<=Xmax-d+1
                        R(i, i-1) = lambda;
                        R(i,i+c)=alpha;
                        R(i,i+d)=beta;
                        if i>= Xr+1
                                R(i,i)=-(lambda+alpha+beta);
                        else
                                R(i,i)=-(lambda+alpha);
                end
                        elseif i >= Xmax-d+2 && i<= Xmax-c+1
                        R(i,i-1)=lambda;
                        R(i,i+c)=alpha;
                        R(i,i)=-(lambda+alpha+beta);
                elseif i>= Xmax-c+2 && i<=Xmax
                        R(i,i-1)=lambda;
                        R(i,i)=-(lambda+alpha+beta);
                end
        end
end
```

The code above was used to generate the following:

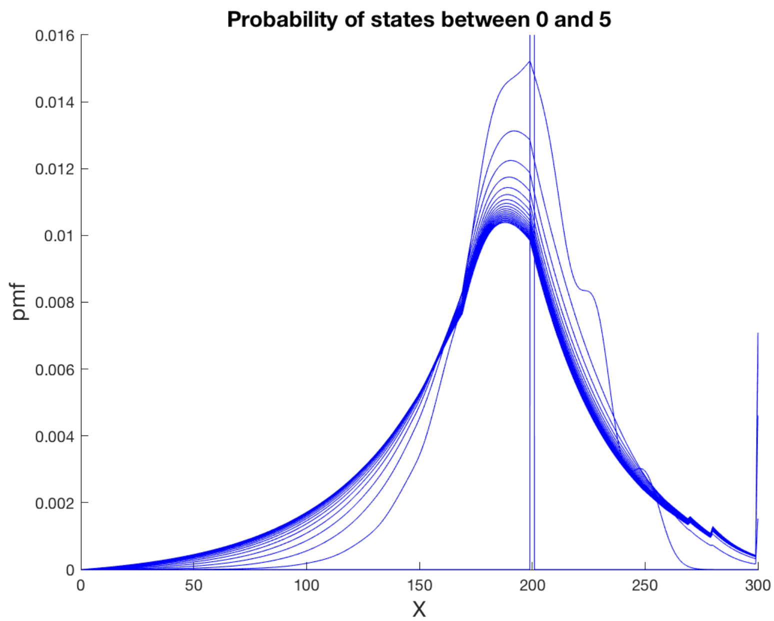

Figure 2: Probability Mass Function of Realizations

I PMF of Dividends

```
s = 0;
for i=1:1000
[x,~] = cfSim(lambda,beta, alpha, X0, c, d, Xr, Xmax, Tmax);
a = x(2:end);
diff = a-x(1:end-1);
if ~isempty(find(diff == -d, 1))
s = s+1;
end
end
disp(s/100);

%%
pmf = zeros(100, 21);
for i=1:100
        [x,t] = cfSim(lambda,beta, alpha, X0, c, d, Xr, Xmax, Tmax);
        a = x(2:end);
        diff = a-x(1:end-1);

        T = .25:.25:5;
        edges = zeros(1, length(T));
        for tm = 1:length(T)
                [~,ind] = find(t<=T(tm), 1, 'last');
                edges(tm) = ind;
                end

                edges(edges > length(diff)) = length(diff);

                edges = [1,edges];
                p = zeros(1, 21);
                for e=2:21
                p(e-1) = length(find(diff(edges(e-1):edges(e)) == -d));
        end
        pmf(i, :) = p/sum(p);
end
pmf(any(isnan(pmf), 2),:)=[];
avg = mean(pmf);
stairs(0:20, avg);
grid on;
title('average pmf of paying a dividend at in quarter q (100 trials)');
xlabel('q');
ylabel('pmf');
saveas(gcf, 'pmfDividend.png');
```

The code above was used to generate the following:

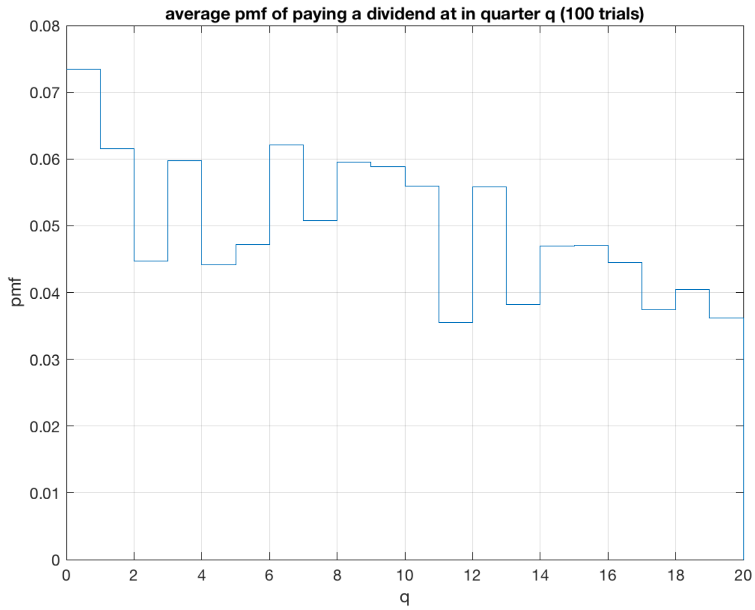

Figure 3: Average pmf, 100 trials

This indicates that over 20 quarters, the insurance company will almost always pay a dividend. However, the probability that the company will pay a dividend in quarter q varies from aprox. 7.5% to 3.75%