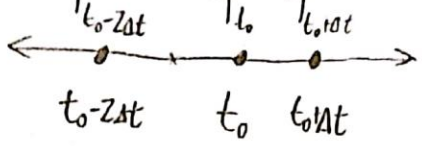
Brian Knisely ME523 HW1

Problem 2



Begin with 1.2: Taylor series with undetermined coefficients: 1D uniform grid

$$A\,T_{t_0-2\Delta t} + \cancel{B}\,T_{t_0-\Delta t} + C\,T_{t_0+\Delta t} + \cancel{D}\,T_{t_0+2\Delta t} = (A + \cancel{B}_0 + C + \cancel{D}_0)\,T_{t_0}$$

$B \to 0$, no $t-\Delta t$ term; $D \to 0$, no $t+2\Delta t$ term

$$+ (-2A - \cancel{B}_0 + C + 2\cancel{D}_0)\,\Delta t \left.\frac{dT}{dt}\right|_{t_0}$$

Keep A, we have the $t-2\Delta t$ term

Keep C, we have the $t+\Delta t$ term

$B=0$, we have no $t-\Delta t$ term

$D=0$, we have no $t+2\Delta t$ term

$$+ \left(2A + \tfrac{1}{2}\cancel{B}_0 + \tfrac{1}{2}C + 2\cancel{D}_0\right)(\Delta t)^2 \left.\frac{d^2T}{dt^2}\right|_{t_0}$$

$$+ \left(-\tfrac{4}{3}A - \tfrac{1}{6}\cancel{B}_0 + \tfrac{1}{6}C + \tfrac{4}{3}\cancel{D}_0\right)(\Delta t)^3 \left.\frac{d^3T}{dt^3}\right|_{t_0}$$

$$+ \left(\tfrac{2}{3}A + \tfrac{1}{24}\cancel{B}_0 + \tfrac{1}{24}C + \tfrac{2}{3}\cancel{D}_0\right)(\Delta t)^4 \left.\frac{d^4T}{dt^4}\right|_{t_0} + HOTs$$

Solve for first derivative term and try to cancel higher-order error terms

$$\left.\frac{\partial T}{\partial t}\right|_{t_0} = \frac{1}{(-2A+C)}\left\{\frac{A\,T_{t_0-2\Delta t} - (A+C)\,T_{t_0} + C\,T_{t_0+\Delta t}}{\Delta t} - \underbrace{\left(2A + \tfrac{1}{2}C\right)\Delta t \left.\frac{d^2T}{dt^2}\right|_{t_0}}_{(1)}\right.$$

$$\left. - \left(-\tfrac{4}{3}A + \tfrac{1}{6}C\right)(\Delta t)^2 \left.\frac{d^3T}{dt^3}\right|_{t_0} - \left(\tfrac{2}{3}A + \tfrac{1}{24}C\right)(\Delta t)^3 \left.\frac{d^4T}{dt^4}\right|_{t_0} + HOTs\right\}$$

$2A + \frac{1}{2}C = 0 \to C = -4A \to$ cancel (1)

$$\frac{\partial T}{\partial t} = \frac{1}{-6A}\left\{\frac{A\,T_{t_0-2\Delta t} - (A-4A)\,T_{t_0} - 4A\,T_{t_0+\Delta t}}{\Delta t} - \underbrace{\left(-\tfrac{4}{3}A - \tfrac{4}{6}A\right)(\Delta t)^2 \left.\frac{d^3T}{dt^3}\right|_{t_0}}_{\text{Truncation error is } \mathcal{O}((\Delta t)^2)} + HOTs\right\}$$

$$\frac{\partial T}{\partial t} \approx \frac{1}{6}\left\{\frac{T_{t_0-2\Delta t} + 3T_{t_0} - 4T_{t_0+\Delta t}}{\Delta t}\right\}$$

★ Truncation error is $\mathcal{O}((\Delta t)^2)$

★ 2nd-Order Accurate