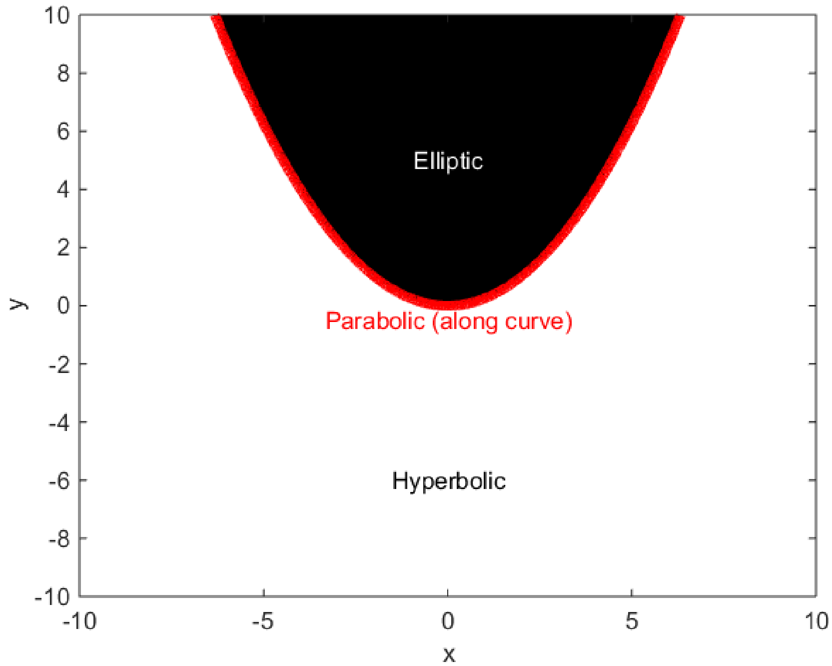
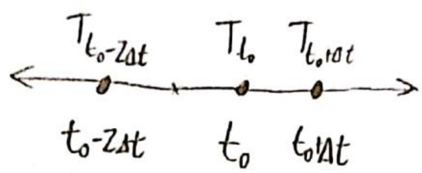
**Problem 1.**




```
%{
Brian Knisely
ME523, HW1, P1

The purpose of this code is to determine the regions in space in which a
PDE is characterized as elliptic, parabolic, or hyperbolic

Given PDE is
u_xx + x*u_xy + y*u_yy = 0
Typical format is
a*u_xx + b*u_xy + c*u_yy = f

Character determined by value of b^2 - 4*a*c
%}

clear; close all; format compact; home;

x = -10:0.01:10;  % Range of x
y = -10:0.01:10;  % Range of y
[b, c] = meshgrid(x, y);
% make mesh grid for x, y locations and define coefficients b and c

a = 1;  % Set first coefficient equal to 1

ch = b.^2 - 4.*a.*c;  % Compute value of character at every x-y location

v = [-100,0,200];  % Set contour levels (so the plot has distinct regions)
contourf(b, c, ch, v, 'linewidth',6,'linecolor','r'); colormap gray
% Plot filled contour in b, c (x, y) space with character as z-values
xlabel('x'); ylabel('y');  % Label axes

text(0, 5, 'Elliptic', 'horizontalAlignment', 'center', 'color', 'w');
text(0, -0.5, 'Parabolic (along curve)', ...
    'horizontalAlignment', 'center', 'color', 'r');
text(0, -6, 'Hyperbolic', 'horizontalAlignment', 'center', 'color', 'k');
% Add text to plot to show regions
```

Brian Knisely ME523 HW1

**Problem 2**




Begin with 1.2: Taylor series with undetermined coefficients: 1D uniform grid

$$A\,T_{t_0-2\Delta t} + \cancel{B}\,T_{t_0-\Delta t} + C\,T_{t_0+\Delta t} + \cancel{D}\,T_{t_0+2\Delta t} = (A + \cancel{B}_0 + C + \cancel{D}_0)\,T_{t_0}$$

0, no $t-\Delta t$ term

0, no $t+2\Delta t$ term

$$+\left(-2A - \cancel{B}_0 + C + 2\cancel{D}_0\right)\Delta t \left.\frac{dT}{dt}\right|_{t_0}$$

$$+\left(2A + \tfrac{1}{2}\cancel{B}_0 + \tfrac{1}{2}C + 2\cancel{D}_0\right)(\Delta t)^2 \left.\frac{d^2T}{dt^2}\right|_{t_0}$$

$$+\left(-\tfrac{4}{3}A - \tfrac{1}{6}\cancel{B}_0 + \tfrac{1}{6}C + \tfrac{4}{3}\cancel{D}_0\right)(\Delta t)^3 \left.\frac{d^3T}{dt^3}\right|_{t_0}$$

$$+\left(\tfrac{2}{3}A + \tfrac{1}{24}\cancel{B}_0 + \tfrac{1}{24}C + \tfrac{2}{3}\cancel{D}_0\right)(\Delta t)^4 \left.\frac{d^4T}{dt^4}\right|_{t_0} + \text{HOTs}$$

Keep A, we have the $t-2\Delta t$ term

Keep C, we have the $t+\Delta t$ term

$B=0$, we have no $t-\Delta t$ term

$D=0$, we have no $t+2\Delta t$ term

Solve for first derivative term and try to cancel higher-order error terms

$$\left.\frac{\partial T}{\partial t}\right|_{t_0} = \frac{1}{(-2A+C)}\left\{\frac{A\,T_{t_0-2\Delta t} - (A+C)\,T_{t_0} + C\,T_{t_0+\Delta t}}{\Delta t} - \underbrace{\left(2A + \tfrac{1}{2}C\right)}_{(1)}\Delta t \left.\frac{d^2T}{dt^2}\right|_{t_0}\right.$$

$$\left. - \left(-\tfrac{4}{3}A + \tfrac{1}{6}C\right)(\Delta t)^2 \left.\frac{d^3T}{dt^3}\right|_{t_0} - \left(\tfrac{2}{3}A + \tfrac{1}{24}C\right)(\Delta t)^3 \left.\frac{d^4T}{dt^4}\right|_{t_0} + \text{HOTs}\right\}$$

$2A + \frac{1}{2}C = 0 \rightarrow C = -4A \rightarrow$ cancel (1)

$$\frac{\partial T}{\partial t} = \frac{1}{-6\cancel{A}}\left\{\frac{\cancel{A}\,T_{t_0-2\Delta t} - (\cancel{A}-4\cancel{A})\,T_{t_0} - 4\cancel{A}\,T_{t_0+\Delta t}}{\Delta t} - \underbrace{\left(-\tfrac{4}{3}A - \tfrac{4}{6}A\right)(\Delta t)^2 \left.\frac{d^3T}{dt^3}\right|_{t_0} + \text{HOTs}}\right\}$$

$$\boxed{\frac{\partial T}{\partial t} \approx \frac{1}{6}\left\{\frac{T_{t_0-2\Delta t} + 3T_{t_0} - 4\,T_{t_0+\Delta t}}{\Delta t}\right\}}$$

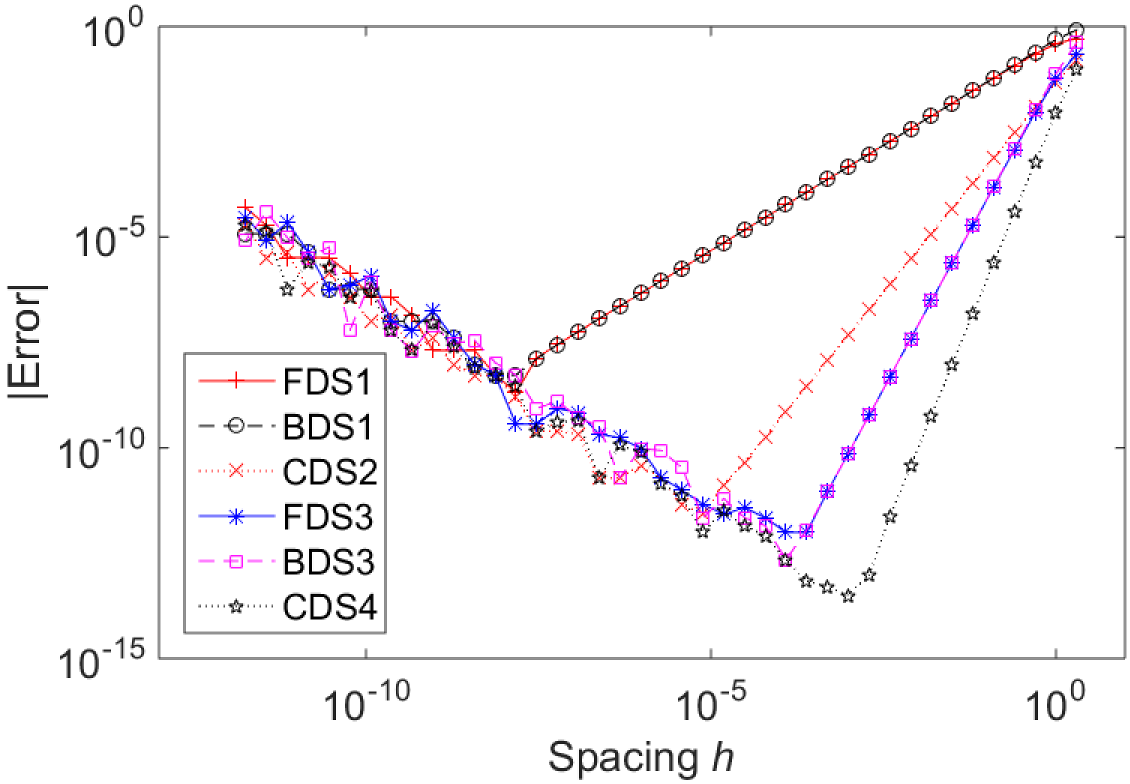
★ Truncation error is $\mathcal{O}((\Delta t)^2)$

★ 2nd-Order Accurate

**Problem 3.**

Output from code:

```
i |      h      | Error:FDS1 | Error:BDS1 | Error:CDS2 | Error:FDS3 | Error:BDS3 | Error:CDS4
 1, 2.00000e+00,-5.15286e-01, 8.37611e-01, 1.61162e-01,-2.21578e-01, 4.17056e-01, 9.77388e-02
 2, 1.00000e+00,-3.92317e-01, 4.86015e-01, 4.68485e-02,-5.85506e-02, 7.60385e-02, 8.74393e-03
 3, 5.00000e-01,-2.21739e-01, 2.46060e-01, 1.21603e-02,-8.94686e-03, 1.01421e-02, 5.97608e-04
 4, 2.50000e-01,-1.15728e-01, 1.21865e-01, 3.06873e-03,-1.19284e-03, 1.26922e-03, 3.81939e-05
 5, 1.25000e-01,-5.88618e-02, 6.03998e-02, 7.68983e-04,-1.52686e-04, 1.57487e-04, 2.40048e-06
 6, 6.25000e-02,-2.96522e-02, 3.00369e-02, 1.92358e-04,-1.92735e-05, 1.95740e-05, 1.50239e-07
 7, 3.12500e-02,-1.48778e-02, 1.49740e-02, 4.80966e-05,-2.41976e-06, 2.43854e-06, 9.39324e-09
 8, 1.56250e-02,-7.45139e-03, 7.47544e-03, 1.20246e-05,-3.03094e-07, 3.04268e-07, 5.87129e-10
 9, 7.81250e-03,-3.72876e-03, 3.73477e-03, 3.00618e-06,-3.79246e-08, 3.79980e-08, 3.66953e-11
10, 3.90625e-03,-1.86514e-03, 1.86664e-03, 7.51546e-07,-4.74289e-09, 4.74751e-09, 2.30260e-12
11, 1.95313e-03,-9.32758e-04, 9.33133e-04, 1.87887e-07,-5.93073e-10, 5.93268e-10, 9.75886e-14
12, 9.76563e-04,-4.66426e-04, 4.66520e-04, 4.69717e-08,-7.41683e-11, 7.42688e-11, 3.12528e-14
13, 4.88281e-04,-2.33225e-04, 2.33248e-04, 1.17430e-08,-9.17738e-12, 9.27780e-12, 5.01821e-14
14, 2.44141e-04,-1.16615e-04, 1.16621e-04, 2.93577e-09,-9.91929e-13, 1.13021e-12, 6.91669e-14
15, 1.22070e-04,-5.83084e-05, 5.83098e-05, 7.33880e-10,-9.91929e-13, 2.20712e-13, 2.20712e-13
16, 6.10352e-05,-2.91544e-05, 2.91547e-05, 1.83635e-10,-2.20457e-12, 1.43341e-12, 8.27061e-13
17, 3.05176e-05,-1.45772e-05, 1.45773e-05, 4.63018e-11, 3.85869e-12,-2.20457e-12, 1.43341e-12
18, 1.52588e-05,-7.28862e-06, 7.28865e-06, 1.35600e-11, 2.64605e-12, 6.28403e-12, 3.25240e-12
19, 7.62939e-06,-3.64431e-06, 3.64432e-06, 2.64605e-12,-4.62991e-12,-2.20457e-12,-9.91929e-13
20, 3.81470e-06,-1.82215e-06, 1.82214e-06,-4.62991e-12, 9.92201e-12,-3.37337e-11,-7.05525e-12
21, 1.90735e-06,-9.11085e-07, 9.11047e-07,-1.91818e-11, 1.96233e-11,-8.70908e-11,-1.43312e-11
22, 9.53674e-07,-4.55436e-07, 4.55514e-07, 3.90258e-11, 9.72335e-11, 9.72335e-11, 7.78310e-11
23, 4.76837e-07,-2.27728e-07, 2.27689e-07,-1.91818e-11,-1.74402e-10, 1.96233e-11,-1.16195e-10
24, 2.38419e-07,-1.13873e-07, 1.13835e-07,-1.91818e-11, 2.13649e-10,-3.29623e-10, 1.96233e-11
25, 1.19209e-07,-5.61314e-08, 5.65587e-08, 2.13649e-10, 6.79310e-10, 5.24090e-10, 4.46479e-10
26, 5.96046e-08,-2.81917e-08, 2.76877e-08,-2.52012e-10,-8.72894e-10, 1.30019e-09,-4.07233e-10
27, 2.98023e-08,-1.32905e-08, 1.27865e-08,-2.52012e-10, 3.68869e-10,-8.72894e-10,-2.52012e-10
28, 1.49012e-08,-2.11466e-09, 5.33592e-09, 1.61063e-09, 3.68869e-10, 5.33592e-09, 2.85240e-09
29, 7.45058e-09, 5.33592e-09, 5.33592e-09, 5.33592e-09, 5.33592e-09, 1.03030e-08, 5.33592e-09
30, 3.72529e-09, 2.02371e-08,-9.56524e-09, 5.33592e-09,-9.56524e-09, 3.51382e-08, 7.81945e-09
31, 1.86265e-09, 2.02371e-08,-3.93676e-08,-9.56524e-09,-3.93676e-08,-2.94335e-08,-2.44664e-08
32, 9.31323e-10, 2.02371e-08,-9.89722e-08,-3.93676e-08,-1.78445e-07, 7.98417e-08,-8.90381e-08
33, 4.65661e-10, 1.39446e-07,-9.89722e-08, 2.02371e-08, 5.99735e-08,-1.94993e-08, 2.02371e-08
34, 2.32831e-10, 3.77865e-07,-9.89722e-08, 1.39446e-07,-9.89722e-08, 5.99735e-08, 5.99735e-08
35, 1.16415e-10, 3.77865e-07,-5.75809e-07,-9.89722e-08,-1.21159e-06, 8.54702e-07,-4.96337e-07
36, 5.82077e-11, 1.33154e-06,-5.75809e-07, 3.77865e-07, 6.95756e-07, 5.99735e-08, 3.77865e-07
37, 2.91038e-11, 3.23889e-06,-5.75809e-07, 1.33154e-06,-5.75809e-07, 5.78202e-06, 1.96732e-06
38, 1.45519e-11, 3.23889e-06,-4.39051e-06,-5.75809e-07,-4.39051e-06,-3.11894e-06,-2.48316e-06
39, 7.27596e-12, 3.23889e-06,-1.20199e-05,-4.39051e-06,-2.21924e-05,-9.47677e-06,-5.75809e-07
40, 3.63798e-12, 1.84977e-05,-1.20199e-05, 3.23889e-06, 8.32515e-06, 3.88427e-05, 1.34114e-05
41, 1.81899e-12, 4.90153e-05,-1.20199e-05, 1.84977e-05, 2.86702e-05, 8.32515e-06, 1.84977e-05
```




Discussion:

The results confirm the expected order of accuracy for each scheme. It is clear that the higher-order schemes have lower error than the lower-order schemes for the same spacing (at low values of spacing). At large values of h, the error in estimating the derivative is high. Decreasing the size of spacing causes the error to decrease, to an extent. At a certain minimum spacing, the solution begins losing accuracy. This may be due to either the finite-precision number systems used in calculations or due to a lack of stability in the numerical model. At extremely small increments of *h*, the finite-precision number system may not be able to resolve the exact numbers accurately, causing errors to be amplified. Comparing the absolute minimum error, the highest-order scheme is able to reach a lower minimum error than any lower-order scheme. We also see that for schemes of the same order, e.g. FDS3 and BDS3, the reduction in error with spacing is the same.

Problem 3 Source code:

```
%{
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ME523, HW1, P3
January 21, 2018

The purpose of this code is to compare five differencing schemes for
taking the first derivative with respect to x of cos(x) at x = 0.3
and calculating the corresponding errors. The schemes to be compared are
BDS1, CDS2, FDS3, BDS3, and CDS4.
%}

clear; close all; format compact; home;

n = 41;  % Number of iterations to compute
```

```
x = 0.3;  % Location to evaluate derivative
h = zeros(n, 1);  % Initialize array of spacings
h(1) = 2;  % Initialize first value of spacing to be 2
for i = 1:n-1
    h(i+1) = h(i)/2;  % Reduce size of spacing by 50% each iteration
end


% Initialize arrays for each scheme
% The first column in each array holds its value, and the second column
% stores its error
bds1 = zeros(n, 2);  % 1st order backward difference
cds2 = zeros(n, 2);  % 2nd order central difference
fds3 = zeros(n, 2);  % 3rd order forward difference
bds3 = zeros(n, 2);  % 3rd order backward difference
cds4 = zeros(n, 2);  % 4th order central difference


for i = 1:n
    exact = -sin(0.3);  % Compute analytical derivative
    % Approximate derivative with each scheme
    fds1(i, 1) = (cos(x+h(i))-cos(x)) / h(i);
    bds1(i, 1) = (cos(x)-cos(x-h(i))) / h(i);
    cds2(i, 1) = (cos(x+h(i))-cos(x-h(i))) / (2*h(i));
    fds3(i, 1) = (-cos(x+2*h(i))+6*cos(x+h(i))-3*cos(x)-2*cos(x-h(i))) /...
        (6*h(i));
    bds3(i, 1) = (2*cos(x+h(i))+3*cos(x)-6*cos(x-h(i))+cos(x-2*h(i))) / ...
        (6*h(i));
    cds4(i, 1) = (-cos(x+2*h(i))+8*cos(x+h(i))-8*cos(x-h(i))+...
        cos(x-2*h(i))) / (12*h(i));
    % Compute errors for each scheme
    fds1(i, 2) = fds1(i, 1) - exact;
    bds1(i, 2) = bds1(i, 1) - exact;
    cds2(i, 2) = cds2(i, 1) - exact;
    fds3(i, 2) = fds3(i, 1) - exact;
    bds3(i, 2) = bds3(i, 1) - exact;
    cds4(i, 2) = cds4(i, 1) - exact;
end

figure(1);  % Create figure to show error
loglog(h, abs(fds1(:, 2)), 'r+-', h, abs(bds1(:, 2)), 'ko--', ...
    h, abs(cds2(:, 2)), 'rx:', h, abs(fds3(:, 2)), 'b*-', ...
    h, abs(bds3(:, 2)), 'ms--', h, abs(cds4(:, 2)), 'kp:');
xlabel('Spacing \it{h}'); ylabel('|Error|');
legend('FDS1', 'BDS1', 'CDS2', 'FDS3', 'BDS3', 'CDS4', ...
    'location', 'southwest');
set(gca, 'fontsize', 16);
xlim([1e-13, 10]);
set(gcf, 'outerposition', [50 50 850 650])

fid = fopen('results.txt', 'w');  % Open a file to store results

% Print error results
fprintf(fid,'%-2s|%-12s|%-12s|%-12s|%-12s|%-12s|%-12s|%-12s\n', ...
    'i', '      h', ' Error:FDS1', ' Error:BDS1', ' Error:CDS2', ...
    ' Error:FDS3', ' Error:BDS3', ' Error:CDS4');
for i = 1:n
    fprintf(fid,'%2.0f,%12.5e,%12.5e,%12.5e,%12.5e,%12.5e,%12.5e,%12.5e', ...
    i, h(i), fds1(i, 2), bds1(i, 2), cds2(i, 2), fds3(i, 2), bds3(i, 2), cds4(i, 2));
    fprintf(fid,'\n');
```

```
end

fclose(fid);  % Close the file
```