,

ADER 构造过程

November 5, 2018

ADER 基本思想

对于黎曼问题：

$$
\begin{aligned}
&\partial_t U + \partial_x F(U) = 0 \\
&U(x,0) = \begin{cases} U_L(x), x < x_{i+1/2} \\ U_R(x), x > x_{i+1/2} \end{cases}
\end{aligned}
\tag{1}
$$

根据泰勒展开，在拉格朗日体系下，可以得到 τ 时刻近似为

$$
U(x_{i+1/2}, \tau) = U(x_{i+1/2}, 0+) + \sum_{k=1}^{r-1} \left[\frac{d^k}{dt^k} U(x,t)(x_{i+1/2}, 0+) \right] \frac{\tau^k}{k!} \tag{2}
$$

其中 $U(x_{i+1/2}, 0+)$ 是通过 $U_L(x_{i+1/2})$ 和 $U_L(x_{i-1/2})$ 构造的黎曼解求得的 Godunov 状态。式中全导数可以通过如下形式得到，

$$
\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \tag{3}
$$

高阶全导数

因此可以给出一到二阶高阶全导数形式如下

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} \tag{4}$$

$$\begin{aligned}\frac{d^2U}{dt^2} &= \frac{d}{dt}(\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x}) \\ &= \frac{\partial^2 U}{\partial t^2} + \frac{\partial u}{\partial t}\frac{\partial U}{\partial x} + 2u\frac{\partial^2 U}{\partial x \partial t} + u\frac{\partial u}{\partial x}\frac{\partial U}{\partial x} + u^2\frac{\partial^2 U}{\partial x^2}\end{aligned} \tag{5}$$

高阶时间导数

高阶时间导数求法：

$$
\begin{aligned}
\frac{\partial_U}{\partial t} &= -A\frac{\partial U}{\partial x} \\
\frac{\partial^2 U}{\partial x \partial t} &= -\frac{\partial A}{\partial x}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x^2} \\
\frac{\partial^2 U}{\partial t^2} &= -\frac{\partial A}{\partial t}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x \partial t} \\
\frac{\partial^3 U}{\partial x^2 \partial t} &= -\frac{\partial^2 A}{\partial x^2}\frac{\partial U}{\partial x} - 2\frac{\partial A}{\partial x}\frac{\partial^2 U}{\partial x^2} - A\frac{\partial U^3}{\partial x^3} \\
\frac{\partial^3 U}{\partial x \partial t^2} &= -\frac{\partial^2 A}{\partial x \partial t}\frac{\partial U}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial^2 U}{\partial x \partial t} - \frac{A}{\partial t}\frac{\partial U^2}{\partial x^2} - A\frac{\partial^3 U}{\partial x^2 \partial t} \\
\frac{\partial^3 U}{\partial t^3} &= -\frac{\partial^2 A}{\partial t^2}\frac{\partial U}{\partial x} - \frac{\partial A}{\partial t}\frac{\partial^2 U}{\partial x \partial t} - \frac{A}{\partial t}\frac{\partial U^2}{\partial x \partial t} - A\frac{\partial^3 U}{\partial x \partial t^2}
\end{aligned}
\tag{6}
$$

高阶空间导数和广义黎曼解

通过广义黎曼解，界面上空间导数发展方程为

$$\frac{\partial U_x^{(k)}}{\partial \tau} + A(U(x_{i+1/2}(0+), 0+))\frac{\partial U_x^{(k)}}{\partial x} = 0$$
$$U_x^{(k)}(x_{i+1/2}, 0) = \begin{cases} P_L^{(k)}(x_{i+1/2}), & \text{if } x < x_{i+1/2}, \\ P_R^{(k)}(x_{i+1/2}), & \text{if } x > x_{i+1/2}, \end{cases} \tag{7}$$

其中 A 为 Jacob 矩阵，$A = \frac{\partial F}{\partial U}$，$P_{L(R)}^{(k)} = \frac{\partial^k U_{L(R)}(x)}{\partial x^k}$ 通过广义黎曼求解器可以得到更高阶空间导数。
这里 $A(U(x_{i+1/2}(0+), 0+))$ 为冻结矩阵，则可以通过如 HLL 构造

$$U^{(k)}(x_{i+1/2}, 0) = \frac{s_R U_R^{(k)} - s_L U_L^{(k)} + AU_L^{(k)} - AU_R^{(k)}}{s_R - s_L} \tag{8}$$

其中 $s_L(R)$ 通过 U_L 和 U_R 构造。

有限体积离散

在拉氏框架下，网格点随时间运动方程为

$$\frac{dx(t)}{dt} = u(x,t) \tag{9}$$

对黎曼问题

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial F}{\partial x} = 0 \\ \partial_t s_{xx} + u\partial_x s_{xx} - \dfrac{4}{3}\partial_x u = 0 \\ Q(x, t=0) = \begin{cases} Q_L, & \text{if } \ x < 0 \\ Q_R, & \text{if } \ x > 0 \end{cases} \end{cases} \tag{10}$$

其中 $U = (\rho, \rho u, \rho E)$， $F = (\rho u, \rho u^2 + p - s_{xx}, (\rho E + p - s_{xx})u)$, $Q = (\rho, \rho u, \rho E)$。

有限体积离散

三阶 WENO 格式下二阶空间导数构造方法

We take the $U_L^{(1)}$ as an example, the same to U_L, we use two stencils $S(i-1,i) = c_{11}U_{i-1} + c_{12}U_i$ and $S(i,i+1) = c_{21}U_i + c_{22}U_{i+1}$ to construct the $U_{i+1/2}^{(1)L}$,
In an uniform mesh, the coefficients c_{11} and c_{12} are constructed as

$$\frac{c_{11}U_{i-1} + c_{12}U_i - (c_{11}U_{i-2} + c_{12}U_{i-1})}{\Delta x^2} = \frac{\partial^2 U_i}{\partial x^2} + O(\Delta x) \tag{11}$$

We get $c_{11} = -1$ and $c_{12} = 1$, in a similar process, we can get $c_{21} = -1$ and $c_{22} = 1$, so

$$\begin{aligned} q_1 &= U_{i-1} - U_i \\ q_2 &= U_i - U_{i+1} \end{aligned} \tag{12}$$

Using $S(i-1,i,i+1)$ we can construct a 2nd order approximation of $U_L^{(1)}$

$$U_{i+1/2}^{(1)L} = \omega_1 q_1 + \omega_2 q_2 \tag{13}$$

where $\omega_i = \frac{\alpha_k}{\alpha_1+\alpha_2}$ and $\alpha_k = \frac{d_k}{(\beta_k+\varepsilon)^p}$, d_k is the linear weights of q_1 and q_2 can be solved as

$$\frac{d_1 q_1 + d_2 q_2}{\Delta x^2} = \frac{\partial^2 U_i}{\partial x^2} + O(\Delta x^2) \tag{14}$$

Then we get $d_1 = \frac{1}{4}$ and $d_2 = \frac{3}{4}$. And the smoothness indicatros are

$$\beta_1 = q_1^2, \beta_2 = q_2^2 \tag{15}$$

守恒方程有限体积离散

在区域 $(x^n_{i-1/2}, t^n) \rightarrow (x^n_{i+1/2}, t^n) \rightarrow (x^n_{i+1/2}, t^{n+1}) \rightarrow (x^{n+1}_{i+1/2}, t^{n+1})$ 积分

$$\int\int_\omega \left(\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x}\right) dxdt = 0 \tag{16}$$

基于格林公式可以写作

$$\oint_{\partial\omega} (Udx - Fdt) = 0 \tag{17}$$

展开有

$$\int_{x^{n+1}_{i-1/2}}^{x^{n+1}_{i+1/2}} Udx - \int_{x^n_{i-1/2}}^{x^n_{i+1/2}} Udx + \int_{x^{n+1}_{i-1/2}}^{x^n_{i-1/2}} Udx - \int_{x^{n+1}_{i+1/2}}^{x^n_{i+1/2}} Udx - \left(\int_{t^n}^{t^{n+1}} F(x_{i-1/2}, t)dt - \int_{t^n}^{t^{n+1}} F(x_{i+1/2}, t)dt\right) = 0 \tag{18}$$

将 Eq.(9) 带入有,

$$\int_{x^{n+1}_{i-1/2}}^{x^{n+1}_{i+1/2}} Udx - \int_{x^n_{i-1/2}}^{x^n_{i+1/2}} Udx - \begin{bmatrix} \int_{t^n}^{t^{n+1}} F(x_{i-1/2}, t) - u(x_{i-1/2}, t)Udt \\ -\int_{t^n}^{t^{n+1}} F(x_{i+1/2}, t) - u(x_{i+1/2}, t)Udt \end{bmatrix} = 0 \tag{19}$$

定义

$$f = F - uU \tag{20}$$

有

$$\int_{x^{n+1}_{i-1/2}}^{x^{n+1}_{i+1/2}} Udx - \int_{x^n_{i-1/2}}^{x^n_{i+1/2}} Udx - \left[\int_{t^n}^{t^{n+1}} f(x_{i-1/2}, t)Udt - \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t)Udt\right] = 0 \tag{21}$$

令单元平均值

$$\bar{U}_i^n = \frac{1}{\Delta x^n} \int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} U(x, t^n) dx \tag{22}$$

其中 $\Delta x^n = x_{i+1/2}^n - x_{i-1/2}^n$。因此 Eq.(21) 变为

$$\bar{U}_i^{n+1} \Delta x_i^{n+1} - \bar{U}_i^n \Delta x_i^n - (F_{i-1/2} - F_{i+1/2}) = 0 \tag{23}$$

其中

$$F_{i+1/2} = \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t) dt \tag{24}$$

Constitute equation

In the Lagrangian frame, the equation of the constitute model can be written as

$$\frac{\mathrm{d}s_{xx}}{\mathrm{d}t} = \frac{4\mu}{3}\frac{\partial u}{\partial x} \tag{25}$$

For the cell-centered Lagrangian scheme, the geometrical conservation law is very important, that is,

$$\frac{\dot{V}}{V} = \frac{\partial u}{\partial x} \tag{26}$$

should be satisfied. In the one-dimensional case, the volume of the cell I_i is evaluated by

$$V_i(t) = x_{i+1/2}(t) - x_{i-1/2}(t) \tag{27}$$

Taking the material derivative on the both sides of Eq.27, we have

$$\dot{V}_i(t) = u_{i+1/2}(t) - u_{i+1/2}(t) \tag{28}$$

We can get

$$\frac{\partial u}{\partial t} = \frac{\dot{V}_i(t)}{V_i} = \frac{u_{i+1/2}(t) - u_{i-1/2}(t)}{x_{i+1/2} - x_{i-1/2}} \tag{29}$$

Then, The semi-discrete formulation for Eq.25 has the following form:

$$\frac{ds_{xx}}{dt} = \frac{4\mu}{3}\frac{u_{i+1/2} - u_{i-1/2}}{\Delta x_i} \tag{30}$$

高斯积分

对于 $F_{i+1/2}$ 采用高斯积分

$$F_{i+1/2} = \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t)dt = \sum_{g=1}^{\alpha} \omega_g f(U(x_{i+1/2(t_g)}, t_g))\Delta t \tag{31}$$

其中 α 为高斯点个数，ω_g 为 g 点加权值。

Gaussian quadrature

In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration. An n-point Gaussian quadrature rule named after Carl Friedrich Gauss, is a quadrature rule constructed to yield an exact result for polynomials of degree $2n-1$ or less by a suitable choice of the nodes x_i and weightes w_i for $i = 1, , n$. The most common domain of integration for such a rule is taken as $[-1, 1]$, so the rule is stated as

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{n} \omega_i f(x_i) \tag{32}$$

which is exact for polynomials of degree $2n-1$ or less. This exact rule is known as the Gauss-Legendre quadrature rule.

Guass-Legendre quadrature

For the simplest integration problem stated above, i.e., $f(x)$ is well-approximated by polynomials on $[-1,1]$, the associated orthogonal polynomials are Legrendre-polynomials, denoted by $P_n(x)$. The i-th Gauss node, x_i is the i-th root of P_n and the weights are given by the formula(Abramowitz Stegun 1972, p.887)

$$\omega_i = \frac{2}{(1-x_i^2)[P_n'(x_i)]^2} \tag{33}$$

Some lower-order quadrature rules are tabulated below

Number of points, n	Points, x_i,	Weights, ω_i
1	0	2
2	$\pm\frac{1}{\sqrt{3}}$	1
3	0 $(\pm\sqrt{\frac{3}{5}})$	$\frac{8}{9}$ ($\frac{5}{9}$)
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$ ($\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$)	$\frac{18-\sqrt{30}}{36}$ ($\frac{18+\sqrt{30}}{36}$)

Change of interval
An integral over $[a,b]$ must be changed into an integral over $[-1,1]$ before applying the Gaussian quadrature rule. This change of interval can be done in the fllowing way

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f(\frac{b-a}{2}x+\frac{a+b}{2})dx \tag{34}$$

对于区域 $[0,\tau]$

求解流程

$$
\begin{aligned}
x_{i+1/2}^{n+1} &= x_{i+1/2}^{n} + \sum_{g=1}^{2} u(x_{i+1/2}, t_g)\omega_g \Delta t \\
\Delta x_i^{n+1} &= x_{i+1/2}^{n+1} - x_{i-1/2}^{n+1} \\
\Delta x_i^{n+1} \overline{U}_i^{n+1} &= \Delta x_i^{n} \overline{U}_i^{n} + F_{i+1/2} - F_{i-1/2} \\
\overline{\hat{s}_{xx}}_i^{n+1} &= \overline{s_{xx}}_i^{n} + \frac{\Theta_{i+1/2} - \Theta_{i-1/2}}{\Delta x^n} \\
\overline{s_{xx}}_i^{n+1} &= \Gamma(\overline{\hat{s}_{xx}}_i^{n+1})
\end{aligned}
\tag{35}
$$

其中

$$
\begin{aligned}
F_{i+1/2} &= \sum_{g=1}^{2} f(U(x_{i+1/2}), t_g)\omega_g \Delta t \\
\Theta_{i+1/2} &= \sum_{g=1}^{2} \frac{4}{3}\mu u(x_{i+1/2}, t_g)\omega_g \Delta t
\end{aligned}
\tag{36}
$$

通过泰勒展开

$$
U(x_{i+1/2}, \tau) = U(x_{i+1/2}, 0) + \sum_{k=1}^{n-1} \frac{d^k U}{dt^k}\frac{\tau^n}{n!} \tag{37}
$$

求解流程

其中全导数

$$
\begin{aligned}
\frac{dU}{dt} &= \frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} \\
\frac{d^2U}{dt^2} &= \frac{d}{dt}(\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x}) \\
&= \frac{\partial^2 U}{\partial t^2} + \frac{\partial u}{\partial t}\frac{\partial U}{\partial x} + 2u\frac{\partial^2 U}{\partial x\partial t} + u\frac{\partial u}{\partial x}\frac{\partial U}{\partial x} + u^2\frac{\partial^2 U}{\partial x^2}
\end{aligned}
\tag{38}
$$

时间导数又可以转化为空间导数

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= -A\frac{\partial U}{\partial x} \\
\frac{\partial^2 U}{\partial x\partial t} &= -\frac{\partial A}{\partial x}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x^2} \\
\frac{\partial^2 U}{\partial t^2} &= -\frac{\partial A}{\partial t}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x\partial t}
\end{aligned}
\tag{39}
$$

最终需要构造高阶空间导数。根据重构如 WENO3 可以构造二阶精度的 $U_L(x_{i+1/2}, 0)$ 和 $U_R(x_{i+1/2}, 0)$ 同样可以构造一阶精度的 $U_L^{(1}(x_{i+1/2}, 0)$ 和 $U_R^{(1)}(x_{i+1/2}, 0)$。WENO5 可以构造四阶的 $U_{L(R)}(x_{i+1/2}, 0)$ ... 一阶精度的 $U_{L(R)}^{(3)}(x_{i+1/2}, 0)$。通过黎曼求解器可以根据 $U_{L(R)}(x_{i+1/2}, 0)$ 求得 $U(x_{i+1/2}, 0)$，同样的通过广义黎曼求解器可以根据 $U_{L(R)}^{(k)}(x_{i+1/2}, 0)$ 得到 $U^{(k)}(x_{1+1/2})$。

A 和 dA

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -u^2+\frac{\partial p}{\partial \rho}+\Gamma(\frac{u^2}{2}-e) & u(2-\Gamma) & \Gamma & -1 \\ (\Gamma(\frac{u^2}{2}-e)-e+\frac{\sigma_x}{\rho}+\frac{\partial p}{\partial \rho})u & -\Gamma u^2-\frac{\sigma_x}{\rho}+e & (1+\Gamma)u & -u \\ \frac{4}{3}\mu\frac{u}{\rho} & -\frac{4}{3}\mu\frac{1}{\rho} & 0 & u \end{pmatrix} \tag{40}$$

where, $e=E-\frac{u^2}{2}$, $\Gamma=\frac{\Gamma_0\rho_0}{\rho}$, $\sigma_x=-p+s_{xx}$, $\frac{\partial p}{\partial \rho}=a_0^2 f'(\eta)$

$$
\begin{aligned}
dA(1,1) &= 0 \quad dA(1,2) = 0 \quad dA(1,3) = 0 \quad dA(1,4) = 0 \\
dA(2,1) &= -2udu + a_0^2 f''(\eta)\frac{d\rho}{\rho_0} + d\Gamma(\frac{u^2}{2} - e) + \Gamma(udu - de) \\
dA(2,2) &= du(2-\Gamma) - ud\Gamma \\
dA(2,3) &= d\Gamma \\
dA(2,4) &= 0 \\
dA(3,1) &= (\Gamma(\frac{u^2}{2} - e) - e + \frac{\sigma_x}{\rho} + \frac{\partial p}{\partial \rho})du + (d\Gamma(\frac{u^2}{2} - e) + \Gamma(udu - de) - de + \frac{d\sigma_x}{\rho} - \frac{\sigma_x d\rho}{\rho^2} + a_0^2 f''(\eta)\frac{d\rho}{\rho_0})u \\
dA(3,2) &= -d\Gamma u^2 - 2u\Gamma - \frac{d\sigma_x}{\rho} + \frac{\sigma_x d\rho}{\rho^2} + de \\
dA(3,3) &= (1+\Gamma)du + d\Gamma u \\
dA(3,4) &= -du \\
dA(4,1) &= \frac{4}{3}\mu\frac{du}{\rho} - \frac{4}{3}\mu\frac{ud\rho}{\rho^2} \\
dA(4,2) &= \frac{4}{3}\mu\frac{d\rho}{\rho^2} \\
dA(4,3) &= 0 \\
dA(4,4) &= du
\end{aligned} \tag{41}
$$

where $de = dE - udu, d\Gamma = -\frac{\Gamma_0 \rho_0 d\rho}{\rho^2}$, $d\sigma_x = -dp + ds_{xx}$.

sub-cell WENO-3 construct

First, define sub-cell points The sub-cell points $x_i^{(k)}$, $k = 1, \ldots, 2r-1$ in I_i are given by

$$x_i^{(k)} = x_{i-1/2} + \frac{k-1}{2r-3}\Delta x \tag{42}$$

For third-order scheme there are is no need of sub-cell points, $x_i^{(1)} = x_{(i-1/2)}$,$x_i^2 = x_{(i+1/2)}$.
Step 2 Reconstruct (2r-2) point-wise values of u Using p_{WENO}^{2r-1} to get $u_i^{(k)}$

$$p_{WENO}^{2r-1} = \sum_{k=0}^{r-1} \omega_k q_k^r(x, \bar{u}_{i-r+1+k}, \cdots, \bar{u}_{i+k}) \tag{43}$$

$$u_i^{(k)} = u(x_i^{(k)}) + e_i^{(k)}(\Delta x)^{2r-1} \tag{44}$$

Step 3, Construct $p_i(x)$ Ansatz $p_i(x)$ of u in I_i as follows

$$p_i(x) = \sum_{l=0}^{2r-2} a_l\left(\frac{x - x_{i-1/2}}{\Delta x}\right)^l \tag{45}$$

By the equation

$$\begin{cases} p_i(x_i^{(l)}) = u_i^{(l)}, & l = 0, \cdots, 2r-2 \\ \dfrac{1}{\Delta x}\displaystyle\int_{I_i} p_i(x)dx = \bar{u}_i \end{cases} \tag{46}$$

For the third-order WENO scheme

$$\begin{aligned} U_i^{(1)} &= U_{(i-1/2)} \\ U_i^{(2)} &= U_{(i+1/2)} \end{aligned} \tag{47}$$

$$\begin{aligned} a_0 &= U_i^{(1)} \\ a_1 &= 6\bar{U} - 2U_i^{(2)} - 4U_i^{(1)} \\ a_2 &= 3(U_i^{(1)} + U_i^{(2)} - 2\bar{U}) \end{aligned} \tag{48}$$

$U_L^{(i)}$

For each cell I_j, we have two stencils $S_j = (x_{j-\frac{3}{2}}, x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and $S_{j+1} = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}, x_{j+\frac{3}{2}})$ corresponding to $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. On these two stencils, we obtain two interpolations

$$p_j'(x) = u_j + \frac{\bar{u}_j - \bar{u}_{j-1}}{h}(x - x_j), p_{j+1}'(x) = \bar{u}_j + \frac{\bar{u}_{j+1} - \bar{u}_j}{h}(x - x_j) \tag{49}$$

with the smoothness indicators $IS_j = (\bar{u}_j - \bar{u}_{j-1})^2$ and $IS_{j+1} = (\bar{u}_{j+1} - \bar{u}_j)^2$ Then we can get the left-hand reconstruction

$$R_j(x) = \frac{\alpha_0}{\alpha_0 + \alpha_1} p_j'(x) + \frac{\alpha_1}{\alpha_0 + \alpha_1} p_{j+1}'(x) \tag{50}$$

where $\alpha_0 = C_0/(\varepsilon + IS_j)^2$, $\alpha_1 = C_1/(\varepsilon + IS_{j+1})$, C_0 and C_1 are the optimal weight of u at position x, given by solving

$$q^3(x, \bar{u}_{i-1}, \bar{u}_i, \bar{u}_{i+1}) = \sum_{k=0}^{1} C_0 q_0(x, \bar{u}_{i-1+k}, \bar{u}_{i+k}) \tag{51}$$

For convenient to understand, here we set $\beta = (x - x_i)/\Delta x$, C_0 and C_1 are

$$C_0 = 1 - \frac{\beta_a}{\beta} C_1 = \frac{\beta_a}{\beta} \tag{52}$$

where $\beta_a = 3\beta^2 + 3\beta - \frac{1}{4}$.
The we can get the subcell approximation of u in I_j as

$$p_j(x) = \sum_{l=0}^{2} a_l (\frac{x - x_{j+1/2}}{\Delta x})^l \tag{53}$$

where

$$
\begin{aligned}
a_0 &= U_i^{(1)} \\
a_1 &= 6\bar{U} - 2U_i^{(2)} - 4U_i^{(1)} \\
a_2 &= 3(U_i^{(1)} + U_i^{(2)} - 2\bar{U})
\end{aligned}
\tag{54}
$$

and $u_i^{(1)} = R_j(x_{i-1/2})$,$u_i^{(2)} = R_j(x_{i+1/2})$.
So

$$
\begin{aligned}
\frac{\partial p_j}{\partial x}(x) &= \frac{a_1}{\Delta x} + \frac{2a_2(x - x_{i-1/2})}{\Delta x^2} \\
\frac{\partial^2 p_j}{\partial x^2}(x) &= \frac{2a_2}{\Delta x^2}
\end{aligned}
\tag{55}
$$

$U_R^{(i)}$

For each cell I_j, we have two stencils $S_j = (x_{j-\frac{3}{2}}, x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and $S_{j+1} = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}, x_{j+\frac{3}{2}})$ corresponding to $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. On these two stencils, we obtain two interpolations

$$p_j'(x) = u_j + \frac{\bar{u}_j - \bar{u}_{j-1}}{h}(x - x_j) \tag{56}$$

and

$$p_{j+1}'(x) = \bar{u}_j + \frac{\bar{u}_{j+1} - \bar{u}_j}{h}(x - x_j) \tag{57}$$

with the smoothness indicators $IS_j = (\bar{u}_j - \bar{u}_{j-1})^2$ and $IS_{j+1} = (\bar{u}_{j+1} - \bar{u}_j)^2$ Then we can get the left-hand reconstruction

$$u_L(x) = \frac{\alpha_0}{\alpha_0 + \alpha_1} p_j'(x) + \frac{\alpha_1}{\alpha_0 + \alpha_1} p_{j+1}'(x) \tag{58}$$

where $\alpha_0 = C_0/(\varepsilon + IS_j)^2$, $\alpha_1 = C_1/(\varepsilon + IS_{j+1})$, C_0 and C_1 are the optimal weight of u at position x, given by solving

$$q^3(x, \bar{u}_{i-1}, \bar{u}_i, \bar{u}_{i+1}) = \sum_{k=0}^{1} C_0 q_0(x, \bar{u}_{i-1+k}, \bar{u}_{i+k}) \tag{59}$$

For convenient to understand, here we set $\beta = (x - x_i)/\Delta x$, C_0 and C_1 are

$$C_0 = 1 - \frac{\beta_a}{\beta} C_1 = \frac{\beta_a}{\beta} \tag{60}$$

where $\beta_a = 3\beta^2 + 3\beta - \frac{1}{4}$.
The we can get the subcell approximation of u in I_j as

$$p_j(x) = \sum_{l=0}^{2} a_l \left(\frac{x - x_{j+1/2}}{\Delta x}\right)^l \tag{61}$$