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ADER 构造过程

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ADER 基本思想

对于黎曼问题：

$$\partial_t U + \partial_x F(U) = 0$$

$$U(x,0) = \left\{ \begin{array}{l} U_L(x), x < x_{i+1/2} \\ U_R(x), x > x_{i+1/2} \end{array} \right. \tag{1}$$

根据泰勒展开，在拉格朗日体系下，可以得到 τ 时刻近似为

$$U(x_{i+1/2}, \tau) = U(x_{i+1/2}, 0+) + \sum_{k=1}^{r-1} \left[\frac{d^k}{dt^k} U(x,t)(x_{i+1/2}, 0+) \right] \frac{\tau^k}{k!} \tag{2}$$

其中 $U(x_{i+1/2}, 0+)$ 是通过 $U_L(x_{i+1/2})$ 和 $U_L(x_{i-1/2})$ 构造的黎曼解求得的 Godunov 状态。式中全导数可以通过如下形式得到，

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \tag{3}$$

高阶全导数

因此可以给出一到二阶高阶全导数形式如下

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} \tag{4}$$

$$\begin{aligned}
\frac{d^2 U}{dt^2} &= \frac{d}{dt}\left(\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x}\right) \\
&= \frac{\partial^2 U}{\partial t^2} + \frac{\partial u}{\partial t}\frac{\partial U}{\partial x} + 2u\frac{\partial^2 U}{\partial x\partial t} + u\frac{\partial u}{\partial x}\frac{\partial U}{\partial x} + u^2\frac{\partial^2 U}{\partial x^2}
\end{aligned} \tag{5}$$

高阶时间导数

高阶时间导数求法：

$$\frac{\partial U}{\partial t} = -A\frac{\partial U}{\partial x}$$

$$\frac{\partial^2 U}{\partial x \partial t} = -\frac{\partial A}{\partial x}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x^2}$$

$$\frac{\partial^2 U}{\partial t^2} = -\frac{\partial A}{\partial t}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x \partial t}$$

$$\frac{\partial^3 U}{\partial x^2 \partial t} = -\frac{\partial^2 A}{\partial x^2}\frac{\partial U}{\partial x} - 2\frac{\partial A}{\partial x}\frac{\partial^2 U}{\partial x^2} - A\frac{\partial U^3}{\partial x^3}$$

$$\frac{\partial^3 U}{\partial x \partial t^2} = -\frac{\partial^2 A}{\partial x \partial t}\frac{\partial U}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial^2 U}{\partial x \partial t} - \frac{A}{\partial t}\frac{\partial U^2}{\partial x^2} - A\frac{\partial^3 U}{\partial x^2 \partial t}$$

$$\frac{\partial^3 U}{\partial t^3} = -\frac{\partial^2 A}{\partial t^2}\frac{\partial U}{\partial x} - \frac{\partial A}{\partial t}\frac{\partial^2 U}{\partial x \partial t} - \frac{A}{\partial t}\frac{\partial U^2}{\partial x \partial t} - A\frac{\partial^3 U}{\partial x \partial t^2}$$

(6)

高阶空间导数的求法

通过广义黎曼解，界面上空间导数发展方程为

$$\frac{\partial U_x^{(k)}}{\partial \tau} + A(U(x_{i+1/2}(0+), 0+))\frac{\partial U_x^{(k)}}{\partial x} = 0$$

$$U_x^{(k)}(x_{i+1/2}, 0) = \begin{cases} P_L^{(k)}(x_{i+1/2}), & \text{if } x < x_{i+1/2}, \\ P_R^{(k)}(x_{i+1/2}), & \text{if } x > x_{i+1/2}, \end{cases}$$

$$(7)$$

其中 A 为 Jacob 矩阵，$A = \frac{\partial F}{\partial U}$，$P_{L(R)}^{(k)} = \frac{\partial^k U_{L(R)}(x)}{\partial x^k}$ 通过广义黎曼求解器可以得到更高阶空间导数。

有限体积离散

在拉氏框架下，网格点随时间运动方程为

$$\frac{dx(t)}{dt} = u(x, t) \tag{8}$$

对黎曼问题

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial F}{\partial x} = 0 \\[2mm] \partial_t s_{xx} + u \partial_x s_{xx} - \dfrac{4}{3} \partial_x u = 0 \\[2mm] Q(x, t = 0) = \begin{cases} Q_L, & \text{if } x < 0 \\ Q_R, & \text{if } x > 0 \end{cases} \end{cases} \tag{9}$$

其中 $U = (\rho, \rho u, \rho E)$, $F = (\rho u, \rho u^2 + p - s_{xx}, (\rho E + p - s_{xx})u)$, $Q = (\rho, \rho u, \rho E)$。

三阶 WENO 格式下二阶空间导数构造方法

We take the $U_L^{(1)}$ as an example, the same to U_L, we use two stencils $S(i-1, i) = c_{11}U_{i-1} + c_{12}U_i$ and $S(i, i+1) = c_{21}U_i + c_{22}U_{i+1}$ to construct the $U_{i+1/2}^{(1)L}$,

In an uniform mesh, the coefficients c_{11} and c_{12} are constructed as

$$\frac{c_{11}U_{i-1} + c_{12}U_i - (c_{11}U_{i-2} + c_{12}U_{i-1})}{\Delta x^2} = \frac{\partial^2 U_i}{\partial x^2} + O(\Delta x) \tag{10}$$

We get $c_{11} = -1$ and $c_{12} = 1$, in a similar process, we can get $c_{21} = -1$ and $c_{22} = 1$, so

$$
\begin{aligned}
q_1 &= U_{i-1} - U_i \\
q_2 &= U_i - U_{i+1}
\end{aligned}
\tag{11}
$$

Using $S(i-1, i, i+1)$ we can construct a 2nd order approximation of $U_L^{(1)}$

$$U_{i+1/2}^{(1)L} = \omega_1 q_1 + \omega_2 q_2 \tag{12}$$

where $\omega_i = \frac{\alpha_k}{\alpha_1 + \alpha_2}$ and $\alpha_k = \frac{d_k}{(\beta_k + \varepsilon)^p}$, d_k is the linear weights of q_1 and q_2 can be solved as

$$\frac{d_1 q_1 + d_2 q_2}{\Delta x^2} = \frac{\partial^2 U_i}{\partial x^2} + O(\Delta x^2) \tag{13}$$

Then we get $d_1 = \frac{1}{4}$ and $d_2 = \frac{3}{4}$. And the smoothness indicatros are

$$\beta_1 = q_1^2, \beta_2 = q_2^2 \tag{14}$$

有限体积离散

在区域 $(x_{i-1/2}^n, t^n) \rightarrow (x_{i+1/2}^n, t^n) \rightarrow (x_{i+1/2}^n, t^{n+1}) \rightarrow (x_{i+1/2}^{n+1}, t^{n+1})$ 积分

$$\int\int_\omega \left(\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x}\right)dxdt = 0 \tag{15}$$

基于格林公式可以写作

$$\oint_{\partial\omega} (Udx - Fdt) = 0 \tag{16}$$

展开有

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} Udx - \int_{x_{i-1/2}^n}^{x_{i+1/2}^n} Udx + \int_{x_{i-1/2}^{n+1}}^{x_{i-1/2}^n} Udx - \int_{x_{i+1/2}^{n+1}}^{x_{i+1/2}^n} Udx - \left(\int_{t^n}^{t^{n+1}} F(x_{i-1/2}, t)dt - \int_{t^n}^{t^{n+1}} F(x_{i+1/2}, t)dt\right) = 0 \tag{17}$$

将 Eq.(9) 带入有,

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} Udx - \int_{x_{i-1/2}^n}^{x_{i+1/2}^n} Udx - \left[\begin{array}{l} \int_{t^n}^{t^{n+1}} F(x_{i-1/2}, t) - u(x_{i-1/2}, t)Udt \\ - \int_{t^n}^{t^{n+1}} F(x_{i+1/2}, t) - u(x_{i+1/2}, t)Udt \end{array}\right] = 0 \tag{18}$$

定义

$$f = F - uU \tag{19}$$

有

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} Udx - \int_{x_{i-1/2}^n}^{x_{i+1/2}^n} Udx - \left[\int_{t^n}^{t^{n+1}} f(x_{i-1/2}, t)Udt - \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t)Udt\right] = 0 \tag{20}$$

令单元平均值

$$\bar{U}_i^n = \frac{1}{\Delta x^n} \int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} U(x, t^n) dx \tag{21}$$

其中 $\Delta x^n = x_{i+1/2}^n - x_{i-1/2}^n$。因此 Eq.(21) 变为

$$\bar{U}_i^{n+1} \Delta x_i^{n+1} - \bar{U}_i^n \Delta x_i^n - (F_{i-1/2} - F_{i+1/2}) = 0 \tag{22}$$

其中

$$F_{i+1/2} = \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t) dt \tag{23}$$

高斯积分

对于 $F_{i+1/2}$ 采用高斯积分

$$F_{i+1/2} = \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t) dt = \sum_{g=1}^{\alpha} \omega_g f(U(x_{i+1/2(t_g)}, t_g)) \Delta t \qquad (24)$$

其中 α 为高斯点个数，ω_g 为 g 点加权值。

Gaussian quadrature

In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration. An n-point Gaussian quadrature rule named after Carl Friedrich Gauss, is a quadrature rule constructed to yield an exact result for polynomials of degree $2n - 1$ or less by a suitable choice of the nodes x_i and weightes w_i for $i = 1, , n$. The most common domain of integration for such a rule is taken as $[-1, 1]$, so the rule is stated as

$$\int_{-1}^{1} f(x) dx \approx \sum_{i=0}^{n} \omega_i f(x_i) \tag{25}$$

which is exact for polynomials of degree $2n - 1$ or less. This exact rule is known as the Gauss-Legendre quadrature rule.

Guass-Legendre quadrature

For the simplest integration problem stated above, i.e., $f(x)$ is well-approximated by polynomials on $[-1, 1]$, the associated orthogonal polynomials are Legrendre-polynomials, denoted by $P_n(x)$. The i-th Gauss node, x_i is the i-th root of P_n and the weights are given by the formula(Abramowitz Stegun 1972, p.887)

$$\omega_i = \frac{2}{(1 - x_i^2)[P_n'(x_i)]^2} \tag{26}$$

Some lower-order quadrature rules are tabulated below

Number of points, n	Points, x_i,	Weights, ω_i
1	0	2
2	$\pm\frac{1}{\sqrt{3}}$	1
3	$0\ (\pm\sqrt{\frac{3}{5}})$	$\frac{8}{9}\ (\ \frac{5}{9})$
4	$\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\ (\ \pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}})$	$\frac{18-\sqrt{30}}{36}\ (\ \frac{18+\sqrt{30}}{36}\ )$

Change of interval
An integral over $[a, b]$ must be changed into an integral over $[-1, 1]$ before applying the Gaussian quadrature rule. This change of interval can be done in the fllowing way

$$\int_a^b f(x)\,dx = \frac{b-a}{2} \int_{-1}^1 f(\frac{b-a}{2}x + \frac{a+b}{2})\,dx \tag{27}$$