,

ADER 构造过程

October 18, 2018

ADER 基本思想

对于黎曼问题：

$$\partial_t U + \partial_x F(U) = 0$$
$$U(x,0) = \begin{cases} U_L(x), x < x_{i+1/2} \\ U_R(x), x > x_{i+1/2} \end{cases} \tag{1}$$

根据泰勒展开，在拉格朗日体系下，可以得到 τ 时刻近似为

$$U(x_{i+1/2},\tau) = U(x_{i+1/2},0+) + \sum_{k=1}^{r-1} \left[\frac{d^k}{dt^k} U(x,t)(x_{i+1/2},0+) \right] \frac{\tau^k}{k!} \tag{2}$$

其中 $U(x_{i+1/2},0+)$ 是通过 $U_L(x_{i+1/2})$ 和 $U_L(x_{i-1/2})$ 构造的黎曼解求得的 Godunov 状态。式中全导数可以通过如下形式得到，

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{3}$$

高阶全导数

因此可以给出一到二阶高阶全导数形式如下

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} \tag{4}$$

$$\begin{aligned}\frac{d^2U}{dt^2} &= \frac{d}{dt}(\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x}) \\ &= \frac{\partial^2 U}{\partial t^2} + \frac{\partial u}{\partial t}\frac{\partial U}{\partial x} + 2u\frac{\partial^2 U}{\partial x \partial t} + u\frac{\partial u}{\partial x}\frac{\partial U}{\partial x} + u^2\frac{\partial^2 U}{\partial x^2}\end{aligned} \tag{5}$$

高阶时间导数

高阶时间导数求法：

$$
\begin{aligned}
\frac{\partial_U}{\partial t} &= -A\frac{\partial U}{\partial x} \\
\frac{\partial^2 U}{\partial x \partial t} &= -\frac{\partial A}{\partial x}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x^2} \\
\frac{\partial^2 U}{\partial t^2} &= -\frac{\partial A}{\partial t}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x \partial t} \\
\frac{\partial^3 U}{\partial x^2 \partial t} &= -\frac{\partial^2 A}{\partial x^2}\frac{\partial U}{\partial x} - 2\frac{\partial A}{\partial x}\frac{\partial^2 U}{\partial x^2} - A\frac{\partial U^3}{\partial x^3} \\
\frac{\partial^3 U}{\partial x \partial t^2} &= -\frac{\partial^2 A}{\partial x \partial t}\frac{\partial U}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial^2 U}{\partial x \partial t} - \frac{A}{\partial t}\frac{\partial U^2}{\partial x^2} - A\frac{\partial^3 U}{\partial x^2 \partial t} \\
\frac{\partial^3 U}{\partial t^3} &= -\frac{\partial^2 A}{\partial t^2}\frac{\partial U}{\partial x} - \frac{\partial A}{\partial t}\frac{\partial^2 U}{\partial x \partial t} - \frac{A}{\partial t}\frac{\partial U^2}{\partial x \partial t} - A\frac{\partial^3 U}{\partial x \partial t^2}
\end{aligned}
\tag{6}
$$

高阶空间导数和广义黎曼解

通过广义黎曼解，界面上空间导数发展方程为

$$
\begin{aligned}
&\frac{\partial U_x^{(k)}}{\partial \tau} + A(U(x_{i+1/2}(0+),0+))\frac{\partial U_x^{(k)}}{\partial x} = 0 \\
&U_x^{(k)}(x_{i+1/2},0) = \begin{cases} P_L^{(k)}(x_{i+1/2}), & \text{if } \ x < x_{i+1/2}, \\ P_R^{(k)}(x_{i+1/2}), & \text{if } \ x > x_{i+1/2}, \end{cases}
\end{aligned} \tag{7}
$$

其中 A 为 Jacob 矩阵，$A = \frac{\partial F}{\partial U}$, $P_{L(R)}^{(k)} = \frac{\partial^k U_{L(R)}(x)}{\partial x^k}$ 通过广义黎曼求解器可以得到更高阶空间导数。
这里 $A(U(x_{i+1/2}(0+),0+))$ 为冻结矩阵，则可以通过如 HLL 构造

$$
U^{(k)}(x_{i+1/2},0) = \frac{s_R U_R^{(k)} - s_L U_L^{(k)} + AU_L^{(k)} - AU_R^{(k)}}{s_R - s_L} \tag{8}
$$

其中 $s_L(R)$ 通过 U_L 和 U_R 构造。

有限体积离散

在拉氏框架下，网格点随时间运动方程为

$$\frac{dx(t)}{dt} = u(x,t) \tag{9}$$

对黎曼问题

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial F}{\partial x} = 0 \\ \partial_t s_{xx} + u\partial_x s_{xx} - \dfrac{4}{3}\partial_x u = 0 \\ Q(x, t=0) = \begin{cases} Q_L, & \text{if } \ x < 0 \\ Q_R, & \text{if } \ x > 0 \end{cases} \end{cases} \tag{10}$$

其中 $U = (\rho, \rho u, \rho E)$， $F = (\rho u, \rho u^2 + p - s_{xx}, (\rho E + p - s_{xx})u)$，
$Q = (\rho, \rho u, \rho E)$。

有限体积离散

三阶 WENO 格式下二阶空间导数构造方法

We take the $U_L^{(1)}$ as an example, the same to U_L, we use two stencils $S(i-1,i) = c_{11}U_{i-1} + c_{12}U_i$ and $S(i,i+1) = c_{21}U_i + c_{22}U_{i+1}$ to construct the $U_{i+1/2}^{(1)L}$,
In an uniform mesh, the coefficients c_{11} and c_{12} are constructed as

$$\frac{c_{11}U_{i-1} + c_{12}U_i - (c_{11}U_{i-2} + c_{12}U_{i-1})}{\Delta x^2} = \frac{\partial^2 U_i}{\partial x^2} + O(\Delta x) \tag{11}$$

We get $c_{11} = -1$ and $c_{12} = 1$, in a similar process, we can get $c_{21} = -1$ and $c_{22} = 1$, so

$$\begin{aligned} q_1 &= U_{i-1} - U_i \\ q_2 &= U_i - U_{i+1} \end{aligned} \tag{12}$$

Using $S(i-1,i,i+1)$ we can construct a 2nd order approximation of $U_L^{(1)}$

$$U_{i+1/2}^{(1)L} = \omega_1 q_1 + \omega_2 q_2 \tag{13}$$

where $\omega_i = \frac{\alpha_k}{\alpha_1+\alpha_2}$ and $\alpha_k = \frac{d_k}{(\beta_k+\varepsilon)^p}$, d_k is the linear weights of q_1 and q_2 can be solved as

$$\frac{d_1 q_1 + d_2 q_2}{\Delta x^2} = \frac{\partial^2 U_i}{\partial x^2} + O(\Delta x^2) \tag{14}$$

Then we get $d_1 = \frac{1}{4}$ and $d_2 = \frac{3}{4}$. And the smoothness indicatros are

$$\beta_1 = q_1^2, \beta_2 = q_2^2 \tag{15}$$

守恒方程有限体积离散

在区域 $(x_{i-1/2}^n, t^n) \rightarrow (x_{i+1/2}^n, t^n) \rightarrow (x_{i+1/2}^n, t^{n+1}) \rightarrow (x_{i+1/2}^{n+1}, t^{n+1})$ 积分

$$\int\int_\omega (\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x}) dxdt = 0 \tag{16}$$

基于格林公式可以写作

$$\oint_{\partial\omega} (Udx - Fdt) = 0 \tag{17}$$

展开有

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} Udx - \int_{x_{i-1/2}^{n}}^{x_{i+1/2}^{n}} Udx + \int_{x_{i-1/2}^{n+1}}^{x_{i-1/2}^{n}} Udx - \int_{x_{i+1/2}^{n+1}}^{x_{i+1/2}^{n}} Udx - \left(\int_{t^n}^{t^{n+1}} F(x_{i-1/2}, t) dt - \int_{t^n}^{t^{n+1}} F(x_{i+1/2}, t) dt \right) = 0 \tag{18}$$

将 Eq.(9) 带入有，

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} Udx - \int_{x_{i-1/2}^{n}}^{x_{i+1/2}^{n}} Udx - \begin{bmatrix} \int_{t^n}^{t^{n+1}} F(x_{i-1/2}, t) - u(x_{i-1/2}, t) Udt \\ - \int_{t^n}^{t^{n+1}} F(x_{i+1/2}, t) - u(x_{i+1/2}, t) Udt \end{bmatrix} = 0 \tag{19}$$

定义

$$f = F - uU \tag{20}$$

有

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} Udx - \int_{x_{i-1/2}^{n}}^{x_{i+1/2}^{n}} Udx - \left[\int_{t^n}^{t^{n+1}} f(x_{i-1/2}, t) Udt - \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t) Udt \right] = 0 \tag{21}$$

令单元平均值

$$\bar{U}_i^n = \frac{1}{\Delta x^n} \int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} U(x, t^n) dx \tag{22}$$

其中 $\Delta x^n = x_{i+1/2}^n - x_{i-1/2}^n$。因此 Eq.(21) 变为

$$\bar{U}_i^{n+1} \Delta x_i^{n+1} - \bar{U}_i^n \Delta x_i^n - (F_{i-1/2} - F_{i+1/2}) = 0 \tag{23}$$

其中

$$F_{i+1/2} = \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t) dt \tag{24}$$

Constitute equation

In the Lagrangian frame, the equation of the constitute model can be written as

$$\frac{\mathrm{d}s_{xx}}{\mathrm{d}t} = \frac{4\mu}{3}\frac{\partial u}{\partial x} \tag{25}$$

For the cell-centered Lagrangian scheme, the geometrical conservation law is very important, that is,

$$\frac{\dot{V}}{V} = \frac{\partial u}{\partial x} \tag{26}$$

should be satisfied. In the one-dimensional case, the volume of the cell I_i is evaluated by

$$V_i(t) = x_{i+1/2}(t) - x_{i-1/2}(t) \tag{27}$$

Taking the material derivative on the both sides of Eq.27, we have

$$\dot{V}_i(t) = u_{i+1/2}(t) - u_{i+1/2}(t) \tag{28}$$

We can get

$$\frac{\partial u}{\partial t} = \frac{\dot{V}_i(t)}{V_i} = \frac{u_{i+1/2}(t) - u_{i-1/2}(t)}{x_{i+1/2} - x_{i-1/2}} \tag{29}$$

Then, The semi-discrete formulation for Eq.25 has the following form:

$$\frac{ds_{xx}}{dt} = \frac{4\mu}{3}\frac{u_{i+1/2} - u_{i-1/2}}{\Delta x_i} \tag{30}$$

高斯积分

对于 $F_{i+1/2}$ 采用高斯积分

$$F_{i+1/2} = \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t)dt = \sum_{g=1}^{\alpha} \omega_g f(U(x_{i+1/2(t_g)}, t_g))\Delta t \tag{31}$$

其中 α 为高斯点个数，ω_g 为 g 点加权值。

Gaussian quadrature

In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration. An n-point Gaussian quadrature rule named after Carl Friedrich Gauss, is a quadrature rule constructed to yield an exact result for polynomials of degree $2n-1$ or less by a suitable choice of the nodes x_i and weightes w_i for $i = 1, , n$. The most common domain of integration for such a rule is taken as $[-1, 1]$, so the rule is stated as

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{n} \omega_i f(x_i) \tag{32}$$

which is exact for polynomials of degree $2n-1$ or less. This exact rule is known as the Gauss-Legendre quadrature rule.

Guass-Legendre quadrature

For the simplest integration problem stated above, i.e., $f(x)$ is well-approximated by polynomials on $[-1, 1]$, the associated orthogonal polynomials are Legrendre-polynomials, denoted by $P_n(x)$. The i-th Gauss node, x_i is the i-th root of P_n and the weights are given by the formula(Abramowitz Stegun 1972, p.887)

$$\omega_i = \frac{2}{(1 - x_i^2)[P_n'(x_i)]^2} \tag{33}$$

Some lower-order quadrature rules are tabulated below

Number of points, n	Points, x_i,	Weights, ω_i
1	0	2
2	$\pm\frac{1}{\sqrt{3}}$	1
3	0 $(\pm\sqrt{\frac{3}{5}})$	$\frac{8}{9}$ $(\frac{5}{9})$
4	$\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$ $(\pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}})$	$\frac{18-\sqrt{30}}{36}$ $(\frac{18+\sqrt{30}}{36})$

Change of interval

An integral over $[a, b]$ must be changed into an integral over $[-1, 1]$ before applying the Gaussian quadrature rule. This change of interval can be done in the fllowing way

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f(\frac{b-a}{2}x + \frac{a+b}{2})dx \tag{34}$$

对于区域 $[0, \tau]$

求解流程

$$
\begin{aligned}
x_{i+1/2}^{n+1} &= x_{i+1/2}^{n} + \sum_{g=1}^{2} u(x_{i+1/2}, t_g)\omega_g \Delta t \\
\Delta x_i^{n+1} &= x_{i+1/2}^{n+1} - x_{i-1/2}^{n+1} \\
\Delta x_i^{n+1} \overline{U}_i^{n+1} &= \Delta x_i^{n} \overline{U}_i^{n} + F_{i+1/2} - F_{i-1/2} \\
\overline{\hat{s}_{xx}}_i^{n+1} &= \overline{s_{xx}}_i^{n} + \frac{\Theta_{i+1/2} - \Theta_{i-1/2}}{\Delta x^n} \\
\overline{s_{xx}}_i^{n+1} &= \Gamma(\overline{\hat{s}_{xx}}_i^{n+1})
\end{aligned} \tag{35}
$$

其中

$$
\begin{aligned}
F_{i+1/2} &= \sum_{g=1}^{2} f(U(x_{i+1/2}), t_g)\omega_g \Delta t \\
\Theta_{i+1/2} &= \sum_{g=1}^{2} \frac{4}{3}\mu u(x_{i+1/2}, t_g)\omega_g \Delta t
\end{aligned} \tag{36}
$$

通过泰勒展开

$$
U(x_{i+1/2}, \tau) = U(x_{i+1/2}, 0) + \sum_{k=1}^{n-1} \frac{d^k U}{dt^k} \frac{\tau^n}{n!} \tag{37}
$$

求解流程

其中全导数

$$
\begin{aligned}
\frac{dU}{dt} &= \frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} \\
\frac{d^2U}{dt^2} &= \frac{d}{dt}(\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x}) \\
&= \frac{\partial^2 U}{\partial t^2} + \frac{\partial u}{\partial t}\frac{\partial U}{\partial x} + 2u\frac{\partial^2 U}{\partial x \partial t} + u\frac{\partial u}{\partial x}\frac{\partial U}{\partial x} + u^2\frac{\partial^2 U}{\partial x^2}
\end{aligned} \tag{38}
$$

时间导数又可以转化为空间导数

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= -A\frac{\partial U}{\partial x} \\
\frac{\partial^2 U}{\partial x \partial t} &= -\frac{\partial A}{\partial x}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x^2} \\
\frac{\partial^2 U}{\partial t^2} &= -\frac{\partial A}{\partial t}\frac{\partial U}{\partial x} - A\frac{\partial^2 U}{\partial x \partial t}
\end{aligned} \tag{39}
$$

最终需要构造高阶空间导数。根据重构如 WENO3 可以构造二阶精度的 $U_L(x_{i+1/2}, 0)$ 和 $U_R(x_{i+1/2}, 0)$ 同样可以构造一阶精度的 $U_L^{(1}(x_{i+1/2}, 0)$ 和 $U_R^{(1)}(x_{i+1/2}, 0)$。WENO5 可以构造四阶的 $U_{L(R)}(x_{i+1/2}, 0)$ ... 一阶精度的 $U_{L(R)}^{(3)}(x_{i+1/2}, 0)$。通过黎曼求解器可以根据 $U_{L(R)}(x_{i+1/2}, 0)$ 求得 $U(x_{i+1/2}, 0)$，同样的通过广义黎曼求解器可以根据 $U_{L(R)}^{(k)}(x_{i+1/2}, 0)$ 得到 $U^{(k)}(x_{1+1/2})$。