

# ADER 构造过程

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# ADER 基本思想

对于黎曼问题:

$$\begin{aligned}\partial_t U + \partial_x F(U) &= 0 \\ U(x, 0) &= \begin{cases} U_L(x), x < x_{i+1/2} \\ U_R(x), x > x_{i+1/2} \end{cases}\end{aligned}\quad (1)$$

根据泰勒展开, 在拉格朗日体系下, 可以得到  $\tau$  时刻近似为

$$U(x_{i+1/2}, \tau) = U(x_{i+1/2}, 0+) + \sum_{k=1}^{r-1} \left[ \frac{d^k}{dt^k} U(x, t)(x_{i+1/2}, 0+) \right] \frac{\tau^k}{k!} \quad (2)$$

其中  $U(x_{i+1/2}, 0+)$  是通过  $U_L(x_{i+1/2})$  和  $U_L(x_{i-1/2})$  构造的黎曼解求得的 Godunov 状态。式中全导数可以通过如下形式得到,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \quad (3)$$

# 高阶全导数

因此可以给出一到二阶高阶全导数形式如下

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} \quad (4)$$

$$\begin{aligned} \frac{d^2 U}{dt^2} &= \frac{d}{dt} \left( \frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} \right) \\ &= \frac{\partial^2 U}{\partial t^2} + \frac{\partial u}{\partial t} \frac{\partial U}{\partial x} + 2u \frac{\partial^2 U}{\partial x \partial t} + u \frac{\partial u}{\partial x} \frac{\partial U}{\partial x} + u^2 \frac{\partial^2 U}{\partial x^2} \end{aligned} \quad (5)$$

# 高阶时间导数

高阶时间导数求法:

$$\begin{aligned}\frac{\partial U}{\partial t} &= -A \frac{\partial U}{\partial x} \\ \frac{\partial^2 U}{\partial x \partial t} &= -\frac{\partial A}{\partial x} \frac{\partial U}{\partial x} - A \frac{\partial^2 U}{\partial x^2} \\ \frac{\partial^2 U}{\partial t^2} &= -\frac{\partial A}{\partial t} \frac{\partial U}{\partial x} - A \frac{\partial^2 U}{\partial x \partial t} \\ \frac{\partial^3 U}{\partial x^2 \partial t} &= -\frac{\partial^2 A}{\partial x^2} \frac{\partial U}{\partial x} - 2 \frac{\partial A}{\partial x} \frac{\partial^2 U}{\partial x^2} - A \frac{\partial^3 U}{\partial x^3} \\ \frac{\partial^3 U}{\partial x \partial t^2} &= -\frac{\partial^2 A}{\partial x \partial t} \frac{\partial U}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial^2 U}{\partial x \partial t} - \frac{A}{\partial t} \frac{\partial U^2}{\partial x^2} - A \frac{\partial^3 U}{\partial x^2 \partial t} \\ \frac{\partial^3 U}{\partial t^3} &= -\frac{\partial^2 A}{\partial t^2} \frac{\partial U}{\partial x} - \frac{\partial A}{\partial t} \frac{\partial^2 U}{\partial x \partial t} - \frac{A}{\partial t} \frac{\partial U^2}{\partial x \partial t} - A \frac{\partial^3 U}{\partial x \partial t^2}\end{aligned}\tag{6}$$

## 高阶空间导数和广义黎曼解

通过广义黎曼解，界面上空间导数发展方程为

$$\begin{aligned} \frac{\partial U_x^{(k)}}{\partial \tau} + A(U(x_{i+1/2}(0+), 0+)) \frac{\partial U_x^{(k)}}{\partial x} &= 0 \\ U_x^{(k)}(x_{i+1/2}, 0) &= \begin{cases} P_L^{(k)}(x_{i+1/2}), & \text{if } x < x_{i+1/2}, \\ P_R^{(k)}(x_{i+1/2}), & \text{if } x > x_{i+1/2}, \end{cases} \end{aligned} \quad (7)$$

其中  $A$  为 Jacob 矩阵,  $A = \frac{\partial F}{\partial U}$ ,  $P_{L(R)}^{(k)} = \frac{\partial^k U_{L(R)}(x)}{\partial x^k}$  通过广义黎曼求解器可以得到更高阶空间导数。

这里  $A(U(x_{i+1/2}(0+), 0+))$  为冻结矩阵, 则可以通过如 HLL 构造

$$U^{(k)}(x_{i+1/2}, 0) = \frac{s_R U_R^{(k)} - s_L U_L^{(k)} + A U_L^{(k)} - A U_R^{(k)}}{s_R - s_L} \quad (8)$$

其中  $s_L(R)$  通过  $U_L$  和  $U_R$  构造。

# 有限体积离散

在拉氏框架下，网格点随时间运动方程为

$$\frac{dx(t)}{dt} = u(x, t) \quad (9)$$

对黎曼问题

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \\ \partial_t s_{xx} + u \partial_x s_{xx} - \frac{4}{3} \partial_x u = 0 \\ Q(x, t = 0) = \begin{cases} Q_L, & \text{if } x < 0 \\ Q_R, & \text{if } x > 0 \end{cases} \end{cases} \quad (10)$$

其中  $U = (\rho, \rho u, \rho E)$ ,  $F = (\rho u, \rho u^2 + p - s_{xx}, (\rho E + p - s_{xx})u)$ ,  
 $Q = (\rho, \rho u, \rho E)$ 。

# 有限体积离散

# 三阶 WENO 格式下二阶空间导数构造方法

We take the  $U_L^{(1)}$  as an example, the same to  $U_L$ , we use two stencils  $S(i-1, i) = c_{11}U_{i-1} + c_{12}U_i$  and  $S(i, i+1) = c_{21}U_i + c_{22}U_{i+1}$  to construct the  $U_{i+1/2}^{(1)L}$ .

In an uniform mesh, the coefficients  $c_{11}$  and  $c_{12}$  are constructed as

$$\frac{c_{11}U_{i-1} + c_{12}U_i - (c_{11}U_{i-2} + c_{12}U_{i-1})}{\Delta x^2} = \frac{\partial^2 U_i}{\partial x^2} + O(\Delta x) \quad (11)$$

We get  $c_{11} = -1$  and  $c_{12} = 1$ , in a similar process, we can get  $c_{21} = -1$  and  $c_{22} = 1$ , so

$$\begin{aligned} q_1 &= U_{i-1} - U_i \\ q_2 &= U_i - U_{i+1} \end{aligned} \quad (12)$$

Using  $S(i-1, i, i+1)$  we can construct a 2nd order approximation of  $U_L^{(1)}$

$$U_{i+1/2}^{(1)L} = \omega_1 q_1 + \omega_2 q_2 \quad (13)$$

where  $\omega_i = \frac{\alpha_k}{\alpha_1 + \alpha_2}$  and  $\alpha_k = \frac{d_k}{(\beta_k + \epsilon)^p}$ ,  $d_k$  is the linear weights of  $q_1$  and  $q_2$  can be solved as

$$\frac{d_1 q_1 + d_2 q_2}{\Delta x^2} = \frac{\partial^2 U_i}{\partial x^2} + O(\Delta x^2) \quad (14)$$

Then we get  $d_1 = \frac{1}{4}$  and  $d_2 = \frac{3}{4}$ . And the smoothness indicators are

$$\beta_1 = q_1^2, \beta_2 = q_2^2 \quad (15)$$



# 守恒方程有限体积离散

在区域  $(x_{i-1/2}^n, t^n) \rightarrow (x_{i+1/2}^n, t^n) \rightarrow (x_{i+1/2}^{n+1}, t^{n+1}) \rightarrow (x_{i-1/2}^{n+1}, t^{n+1})$  积分

$$\int \int_{\omega} \left( \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \right) dx dt = 0 \quad (16)$$

基于格林公式可以写作

$$\oint_{\partial \omega} (U dx - F dt) = 0 \quad (17)$$

展开有

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} U dx - \int_{x_{i-1/2}^n}^{x_{i+1/2}^n} U dx + \int_{x_{i-1/2}^{n+1}}^{x_{i-1/2}^n} U dx - \int_{x_{i+1/2}^{n+1}}^{x_{i+1/2}^n} U dx - \left( \int_{t^n}^{t^{n+1}} F(x_{i-1/2}, t) dt - \int_{t^n}^{t^{n+1}} F(x_{i+1/2}, t) dt \right) = 0 \quad (18)$$

将 Eq.(9) 带入有,

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} U dx - \int_{x_{i-1/2}^n}^{x_{i+1/2}^n} U dx - \left[ \begin{aligned} & \int_{t^n}^{t^{n+1}} F(x_{i-1/2}, t) - u(x_{i-1/2}, t) U dt \\ & - \int_{t^n}^{t^{n+1}} F(x_{i+1/2}, t) - u(x_{i+1/2}, t) U dt \end{aligned} \right] = 0 \quad (19)$$

定义

$$f = F - uU \quad (20)$$

有

$$\int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} U dx - \int_{x_{i-1/2}^n}^{x_{i+1/2}^n} U dx - \left[ \int_{t^n}^{t^{n+1}} f(x_{i-1/2}, t) U dt - \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t) U dt \right] = 0 \quad (21)$$

令单元平均值

$$\bar{U}_i^n = \frac{1}{\Delta x^n} \int_{x_{i-1/2}^{n+1}}^{x_{i+1/2}^{n+1}} U(x, t^n) dx \quad (22)$$

其中  $\Delta x^n = x_{i+1/2}^n - x_{i-1/2}^n$ 。因此 Eq.(21) 变为

$$\bar{U}_i^{n+1} \Delta x_i^{n+1} - \bar{U}_i^n \Delta x_i^n - (F_{i-1/2} - F_{i+1/2}) = 0 \quad (23)$$

其中

$$F_{i+1/2} = \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t) dt \quad (24)$$

# Constitute equation

In the Lagrangian frame, the equation of the constitute model can be written as

$$\frac{ds_{xx}}{dt} = \frac{4\mu}{3} \frac{\partial u}{\partial x} \quad (25)$$

For the cell-centered Lagrangian scheme, the geometrical conservation law is very important, that is,

$$\frac{\dot{V}}{V} = \frac{\partial u}{\partial x} \quad (26)$$

should be satisfied. In the one-dimensional case, the volume of the cell  $I_i$  is evaluated by

$$V_i(t) = x_{i+1/2}(t) - x_{i-1/2}(t) \quad (27)$$

Taking the material derivative on the both sides of Eq.27, we have

$$\dot{V}_i(t) = u_{i+1/2}(t) - u_{i-1/2}(t) \quad (28)$$

We can get

$$\frac{\partial u}{\partial t} = \frac{\dot{V}_i(t)}{V_i} = \frac{u_{i+1/2}(t) - u_{i-1/2}(t)}{x_{i+1/2} - x_{i-1/2}} \quad (29)$$

Then, The semi-discrete formulation for Eq.25 has the following form:

$$\frac{ds_{xx}}{dt} = \frac{4\mu}{3} \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x_i} \quad (30)$$

# 高斯积分

对于  $F_{i+1/2}$  采用高斯积分

$$F_{i+1/2} = \int_{t^n}^{t^{n+1}} f(x_{i+1/2}, t) dt = \sum_{g=1}^{\alpha} \omega_g f(U(x_{i+1/2}(t_g), t_g)) \Delta t \quad (31)$$

其中  $\alpha$  为高斯点个数,  $\omega_g$  为  $g$  点加权值。

# Gaussian quadrature

In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration. An  $n$ -point Gaussian quadrature rule named after Carl Friedrich Gauss, is a quadrature rule constructed to yield an exact result for polynomials of degree  $2n - 1$  or less by a suitable choice of the nodes  $x_i$  and weights  $w_i$  for  $i = 1, \dots, n$ . The most common domain of integration for such a rule is taken as  $[-1, 1]$ , so the rule is stated as

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i) \quad (32)$$

which is exact for polynomials of degree  $2n - 1$  or less. This exact rule is known as the Gauss-Legendre quadrature rule.

# Guass-Legendre quadrature

For the simplest integration problem stated above, i.e.,  $f(x)$  is well-approximated by polynomials on  $[-1, 1]$ , the associated orthogonal polynomials are Legendre-polynomials, denoted by  $P_n(x)$ . The  $i$ -th Gauss node,  $x_i$  is the  $i$ -th root of  $P_n$  and the weights are given by the formula (Abramowitz Stegun 1972, p.887)

$$\omega_i = \frac{2}{(1 - x_i^2)[P'_n(x_i)]^2} \quad (33)$$

Some lower-order quadrature rules are tabulated below

Number of points, $n$	Points, $x_i$ ,	Weights, $\omega_i$
1	0	2
2	$\pm \frac{1}{\sqrt{3}}$	1
3	0 ( $\pm \sqrt{\frac{3}{5}}$ )	$\frac{8}{9}$ ( $\frac{5}{9}$ )
4	$\pm \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$ ( $\pm \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$ )	$\frac{18-\sqrt{30}}{36}$ ( $\frac{18+\sqrt{30}}{36}$ )

Change of interval

An integral over  $[a, b]$  must be changed into an integral over  $[-1, 1]$  before applying the Gaussian quadrature rule. This change of interval can be done in the following way

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) dx \quad (34)$$

对于区域  $[0, \tau]$

# 求解流程

$$\begin{aligned}
 x_{i+1/2}^{n+1} &= x_{i+1/2}^n + \sum_{g=1}^2 u(x_{i+1/2}, t_g) \omega_g \Delta t \\
 \Delta x_i^{n+1} &= x_{i+1/2}^{n+1} - x_{i-1/2}^{n+1} \\
 \Delta x_i^{n+1} \overline{U}_i^{n+1} &= \Delta x_i^n \overline{U}_i^n + F_{i+1/2} - F_{i-1/2} \\
 \overline{s}_{xx_i}^{n+1} &= \overline{s}_{xx_i}^n + \frac{\Theta_{i+1/2} - \Theta_{i-1/2}}{\Delta x^n} \\
 \overline{s}_{xx_i}^{n+1} &= \Gamma(\overline{s}_{xx_i}^{n+1})
 \end{aligned} \tag{35}$$

其中

$$\begin{aligned}
 F_{i+1/2} &= \sum_{g=1}^2 f(U(x_{i+1/2}), t_g) \omega_g \Delta t \\
 \Theta_{i+1/2} &= \sum_{g=1}^2 \frac{4}{3} \mu u(x_{i+1/2}, t_g) \omega_g \Delta t
 \end{aligned} \tag{36}$$

通过泰勒展开

$$U(x_{i+1/2}, \tau) = U(x_{i+1/2}, 0) + \sum_{k=1}^{n-1} \frac{d^k U}{dt^k} \frac{\tau^k}{k!} \tag{37}$$

# 求解流程

其中全导数

$$\begin{aligned}\frac{dU}{dt} &= \frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} \\ \frac{d^2 U}{dt^2} &= \frac{d}{dt} \left( \frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} \right) \\ &= \frac{\partial^2 U}{\partial t^2} + \frac{\partial u}{\partial t} \frac{\partial U}{\partial x} + 2u \frac{\partial^2 U}{\partial x \partial t} + u \frac{\partial u}{\partial x} \frac{\partial U}{\partial x} + u^2 \frac{\partial^2 U}{\partial x^2}\end{aligned}\tag{38}$$

时间导数又可以转化为空间导数

$$\begin{aligned}\frac{\partial U}{\partial t} &= -A \frac{\partial U}{\partial x} \\ \frac{\partial^2 U}{\partial x \partial t} &= -\frac{\partial A}{\partial x} \frac{\partial U}{\partial x} - A \frac{\partial^2 U}{\partial x^2} \\ \frac{\partial^2 U}{\partial t^2} &= -\frac{\partial A}{\partial t} \frac{\partial U}{\partial x} - A \frac{\partial^2 U}{\partial x \partial t}\end{aligned}\tag{39}$$

最终需要构造高阶空间导数。根据重构如 WENO3 可以构造二阶精度的  $U_L(x_{i+1/2}, 0)$  和  $U_R(x_{i+1/2}, 0)$  同样可以构造一阶精度的  $U_L^{(1)}(x_{i+1/2}, 0)$  和  $U_R^{(1)}(x_{i+1/2}, 0)$ 。WENO5 可以构造四阶的  $U_{L(R)}(x_{i+1/2}, 0) \dots$  一阶精度的  $U_{L(R)}^{(3)}(x_{i+1/2}, 0)$ 。通过黎曼求解器可以根据  $U_{L(R)}(x_{i+1/2}, 0)$  求得  $U(x_{i+1/2}, 0)$ ，同样的通过广义黎曼求解器可以根据  $U_{L(R)}^{(k)}(x_{i+1/2}, 0)$  得到  $U^{(k)}(x_{i+1/2}, 0)$ 。



# A 和 dA

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -u^2 + \frac{\partial p}{\partial \rho} + \Gamma(\frac{u^2}{2} - e) & u(2 - \Gamma) & \Gamma & -1 \\ (\Gamma(\frac{u^2}{2} - e) - e + \frac{\sigma_x}{\rho} + \frac{\partial p}{\partial \rho})u & -\Gamma u^2 - \frac{\sigma_x}{\rho} + e & (1 + \Gamma)u & -u \\ \frac{4}{3}\mu\frac{u}{\rho} & -\frac{4}{3}\mu\frac{1}{\rho} & 0 & u \end{pmatrix} \quad (40)$$

where,  $e = E - \frac{u^2}{2}$ ,  $\Gamma = \frac{\Gamma_0 \rho_0}{\rho}$ ,  $\sigma_x = -p + s_{xx}$ ,  $\frac{\partial p}{\partial \rho} = a_0^2 f'(\eta)$

$$dA(1, 1) = 0 \quad dA(1, 2) = 0 \quad dA(1, 3) = 0 \quad dA(1, 4) = 0$$

$$dA(2, 1) = -2udu + a_0^2 f''(\eta) \frac{d\rho}{\rho_0} + d\Gamma(\frac{u^2}{2} - e) + \Gamma(udu - de)$$

$$dA(2, 2) = du(2 - \Gamma) - ud\Gamma$$

$$dA(2, 3) = d\Gamma$$

$$dA(2, 4) = 0$$

$$dA(3, 1) = (\Gamma(\frac{u^2}{2} - e) - e + \frac{\sigma_x}{\rho} + \frac{\partial p}{\partial \rho})du + (d\Gamma(\frac{u^2}{2} - e) + \Gamma(udu - de) - de + \frac{d\sigma_x}{\rho} - \frac{\sigma_x d\rho}{\rho^2} + a_0^2 f''(\eta) \frac{d\rho}{\rho_0})u$$

$$dA(3, 2) = -d\Gamma u^2 - 2u\Gamma - \frac{d\sigma_x}{\rho} + \frac{\sigma_x d\rho}{\rho^2} + de$$

$$dA(3, 3) = (1 + \Gamma)du + d\Gamma u$$

$$dA(3, 4) = -du$$

$$dA(4, 1) = \frac{4}{3}\mu \frac{du}{\rho} - \frac{4}{3}\mu \frac{ud\rho}{\rho^2}$$

$$dA(4, 2) = \frac{4}{3}\mu \frac{d\rho}{\rho^2}$$

$$dA(4, 3) = 0$$

$$dA(4, 4) = du$$

(41)

where  $de = dE - udu, d\Gamma = -\frac{\Gamma_0 \rho_0 d\rho}{\rho^2}, d\sigma_x = -dp + ds_{xx}$ .

# sub-cell WENO-3 construct

First, define sub-cell points The sub-cell points  $x_i^{(k)}$ ,  $k = 1, \dots, 2r - 1$  in  $I_i$  are given by

$$x_i^{(k)} = x_{i-1/2} + \frac{k-1}{2r-3} \Delta x \quad (42)$$

For third-order scheme there are no need of sub-cell points,  $x_i^{(1)} = x_{(i-1/2)}$ ,  $x_i^{(2)} = x_{(i+1/2)}$ .

Step 2 Reconstruct  $(2r-2)$  point-wise values of  $u$  Using  $p_{WENO}^{2r-1}$  to get  $u_i^{(k)}$

$$p_{WENO}^{2r-1} = \sum_{k=0}^{r-1} \omega_k q_k^r(x, \bar{u}_{i-r+1+k}, \dots, \bar{u}_{i+k}) \quad (43)$$

$$u_i^{(k)} = u(x_i^{(k)}) + e_i^{(k)} (\Delta x)^{2r-1} \quad (44)$$

Step 3, Construct  $p_i(x)$  Ansatz  $p_i(x)$  of  $u$  in  $I_i$  as follows

$$p_i(x) = \sum_{l=0}^{2r-2} a_l \left( \frac{x - x_{i-1/2}}{\Delta x} \right)^l \quad (45)$$

By the equation

$$\begin{cases} p_i(x_i^{(l)}) = u_i^{(l)}, & l = 0, \dots, 2r-2 \\ \frac{1}{\Delta x} \int_{I_i} p_i(x) dx = \bar{u}_i \end{cases} \quad (46)$$

For the third-order WENO scheme

$$\begin{aligned}U_i^{(1)} &= U_{(i-1/2)}^R \\U_i^{(2)} &= U_{(i+1/2)}^L\end{aligned}\tag{47}$$

$$\begin{aligned}a_0 &= U_i^{(1)} \\a_1 &= 6\bar{U} - 2U_i^{(2)} - 4U_i^{(1)} \\a_2 &= 3(U_i^{(1)} + U_i^{(2)} - 2\bar{U})\end{aligned}\tag{48}$$