

samba 公式推导

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1. 二维控制方程

运动方程:

$$\frac{d(r, z)}{dt} = (u, v) \quad (1)$$

质量守恒:

$$\frac{dm}{dt} = 0 \quad (2)$$

动量守恒方程:

$$\begin{aligned} \rho \frac{du}{dt} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \\ \rho \frac{dv}{dt} &= \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} \end{aligned} \quad (3)$$

能量守恒方程:

$$\begin{aligned} \rho \frac{de}{dt} &= \sigma_{rr} \frac{\partial u}{\partial r} + \sigma_{zz} \frac{\partial v}{\partial z} + s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \\ &= -p \frac{dV}{dt} + s_{rr} \frac{\partial u}{\partial r} + s_{zz} \frac{\partial v}{\partial z} + s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \end{aligned} \quad (4)$$

本构方程:

$$\begin{aligned} \frac{ds_{rr}}{dt} &= 2\mu \left(\frac{\partial u}{\partial r} - \frac{1}{3} \nabla \cdot \vec{u} \right) + s_{rz} \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} \right) \\ \frac{ds_{zz}}{dt} &= 2\mu \left(\frac{\partial v}{\partial z} - \frac{1}{3} \nabla \cdot \vec{u} \right) - s_{rz} \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} \right) \\ \frac{ds_{rz}}{dt} &= \mu \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} \right) + \frac{s_{rr} - s_{zz}}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} \right) \end{aligned} \quad (5)$$

2. 离散方法

2.1 时间离散: 预估校正方法

对于如下形式方程

$$\frac{\partial U}{\partial t} = F(U) \quad (6)$$

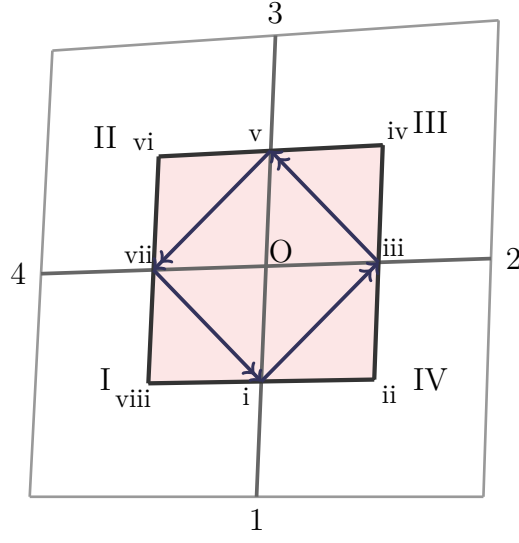


图 1: 控制单元示意图

预估步:

$$\frac{U^{n+\frac{1}{2}} - U^n}{\frac{1}{2}\Delta t} = F(U^n) \quad (7)$$

校正步:

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+\frac{1}{2}}) \quad (8)$$

2.2 空间离散方法

2.2.1 运动方程离散

预估步:

$$\frac{(r, z)^{n+\frac{1}{2}} - (r, z)^n}{\frac{1}{2}\Delta t} = (u, v)^n \quad (9)$$

和 Eq.(6) 不同, $(u, v)^{n+1/2}$ 不能直接由 $(r, z)^{n+1/2}$ 求出。这里需要由动量方程求得 $(u, v)^{n+1}$, 然后取平均值

$$(u, v)^{n+\frac{1}{2}} = \frac{(u, v)^n + (u, v)^{n+1}}{2} \quad (10)$$

校正步:

$$\frac{(r, z)^{n+1} - (r, z)^n}{\Delta t} = (u, v)^{n+\frac{1}{2}} \quad (11)$$

2.2.2 动量方程离散

采用如图 1所示非结构四边形单元, Wilkins 在 1963 年首先通过有限体积方法得到 Wilkins 格式 [1]。点 i, iii, v, vii 分别为

线 O1, O2, O3, O4 的中点, ii , iv , vi , $viii$ 分别为区域 $I-IV$ 的几何中心。

对于动量离散方程 3, 在八边形面回路 $\Omega = i \rightarrow ii \rightarrow \dots \rightarrow vii \rightarrow i$ 上积分:

$$\int_{\Omega} \rho \frac{du}{dt} d\Omega = \int_{\Omega} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega \quad (12)$$

$$\int_{\Omega} \rho \frac{dv}{dt} d\Omega = \int_{\Omega} \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} d\Omega \quad (13)$$

以方程 12 为例, 左侧积分可以近似为

$$\int_{\Omega} \rho \frac{du}{dt} d\Omega = M \frac{du_o}{dt} \quad (14)$$

其中

$$M = \frac{1}{4}(A_I \rho_I + A_{II} \rho_{II} + A_{III} \rho_{III} + A_{IV} \rho_{IV}) \quad (15)$$

右侧积分根据 Green 公式:

$$\int_{\Omega} \text{grad} f d\Omega = \oint_{\partial\Omega} f \vec{n} dl \quad (16)$$

有

$$\int_{\Omega} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega = - \oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr \quad (17)$$

由于线回路积分和所选回路无关, 选择回路 $i \rightarrow iii \rightarrow v \rightarrow vii \rightarrow i$ 积分, 以 $i \rightarrow iii$ 为例由于其平行且为 $\bar{12}$ 的一半, 积分

$$\int_{vii}^i \sigma_{rr} dz \approx \frac{1}{2} \sigma_{zzI} (z_2 - z_1) \quad (18)$$

同理可得

$$- \oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr = \frac{1}{2} \sum_{i=1}^4 s_{rzi} (r_{i+1} - r_i) - \sigma_{rri} (z_{i+1} - z_i) \quad (19)$$

得到 wilkins 格式:

$$\frac{du_o}{dt} = \frac{1}{2M} \left[\sum_{i=1}^4 s_{rzi} (r_{i+1} - r_i) - \sigma_{rri} (z_{i+1} - z_i) \right] \quad (20)$$

lascaux application, 该格式同样可以通过有限元方法得出 (Lascaux 1973 年) ??。控制体同样取如图 1, 以 I 区为例, 在该区通过虚功原理, 可以化作有限元的单元积分:

$$\int_{\Omega_I} \rho \frac{du}{dt} \phi_k d\Omega = \int_{\Omega_I} \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \right) \phi_k d\Omega \quad (21)$$

其中

$$\phi_i = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta) \quad (22)$$

为基函数, $i = a, b, c, d$ 如图?? 其中 $(\xi, \eta) = (-1, -1), (1, -1), (1, 1), (-1, 1)$

$$\begin{aligned} u &= \sum_{i=a}^d u_i \phi_i \\ r &= \sum_{i=a}^d r_i \phi_i \\ z &= \sum_{i=a}^d z_i \phi_i \end{aligned} \quad (23)$$

公式 21左侧可以化为：

$$\begin{aligned} \int_{\Omega_I} \rho \frac{du}{dt} \phi_k d\Omega &= \int_{\Omega_I} \rho \sum_{i=a}^d \frac{du_i}{dt} \phi_i \phi_k dr dz \\ &= \sum_{i=a}^d \frac{du_i}{dt} \int_{-1}^1 \int_{-1}^1 \rho \phi_i \phi_k J d\xi d\eta \end{aligned} \quad (24)$$

其中

$$J = \left\| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{array} \right\| \quad (25)$$

右侧以第一项为例：

$$\begin{aligned} &\int_{\Omega_I} \frac{\partial \sigma_{rr}}{\partial r} + \phi_k d\Omega = \\ &\int_{\Omega_I} \frac{\partial(\sigma_{rr} \phi_k)}{\partial r} d\Omega - \int_{\Omega_I} \frac{\partial \phi_k}{\partial r} \sigma_{rr} d\Omega \\ &= - \int_{\partial \Omega_I} \sigma_{rr} \phi_k dz - \int_{\Omega_I} \frac{\partial \phi_k}{\partial r} \sigma_{rr} d\Omega \quad (26) \\ &= - \int_{\partial \Omega_I} \sigma_{rr} \phi_k dz - \int_{\Omega_I} \frac{1}{J} \frac{\partial(z, \phi_k)}{\partial(\xi, \eta)} \sigma_{rr} J d\xi d\eta \\ &= - \int_{\partial \Omega_I} \sigma_{rr} \phi_k dz - \int_{-1}^1 \int_{-1}^1 \frac{\partial(z, \phi_k)}{\partial(\xi, \eta)} \sigma_{rr} d\xi d\eta \end{aligned}$$

2.2.3 能量方程离散

采用有限体积方法，和动量方程离散相似，以偏导数 $\frac{\partial u}{\partial r}$ 为例，通过 Green 积分可以化为：

$$\begin{aligned}\frac{\partial u}{\partial r} &= -\frac{1}{A} \oint_{1 \rightarrow 2 \dots \rightarrow 1} u dz \\ &\approx -\frac{1}{A} \left(\frac{1}{2}(u_1 + u_2)(z_2 - z_1) + \frac{1}{2}(u_2 + u_3)(z_3 - z_2) + \frac{1}{2}(u_3 + u_4)(z_4 - z_3) + \frac{1}{2}(u_4 + u_1)(u_1 - u_4) \right) \\ &= -\frac{1}{A} ((u_1 - u_3)(z_2 - z_4) + (u_2 - u_4)(z_3 - z_1))\end{aligned}\quad (27)$$

并将其记作 \bar{u}_r 。

能量方程空间离散为

$$\frac{dE}{dt} = -p^n \frac{dV}{dt} + V^n + s_r r^n \bar{u}_r^n + s_z z^n \bar{v}_z^n + s_{rz}^n (\bar{u}_z^n + \bar{v}_r^n) \quad (28)$$

2.2.4 本构方程离散

和能量方程离散类似，以 $\frac{ds_{rr}}{dt}$ 为例

$$\frac{ds_{rr}}{dt} = 2\mu \left(\bar{u}_r^n - \frac{1}{3}(\bar{u}_r^n + \bar{v}_z^n) \right) + s_{rz}^n (\bar{u}_z^n - \bar{v}_r^n) \quad (29)$$

参考文献

- [1] Mark L Wilkins. Calculation of elastic-plastic flow. Technical report, California Univ Livermore Radiation Lab, 1963.