samba 公式推导

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二维控制方程

运动方程：

$$\frac{d(r,z)}{dt} = (u,v) \tag{1}$$

质量守恒：

$$\frac{dm}{dt} = 0 \tag{2}$$

动量守恒方程：

$$\begin{aligned} \rho\frac{du}{dt} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \\ \rho\frac{dv}{dt} &= \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} \end{aligned} \tag{3}$$

能量守恒方程：

$$\begin{aligned} \rho\frac{de}{dt} &= \sigma_{rr}\frac{\partial u}{\partial r} + \sigma_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \\ &= -p\frac{dV}{dt} + s_{rr}\frac{\partial u}{\partial r} + s_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \end{aligned} \tag{4}$$

本构方程：

$$\begin{aligned} \frac{ds_{rr}}{dt} &= 2\mu\left(\frac{\partial u}{\partial r} - \frac{1}{3}\nabla\cdot\vec{u}\right) + s_{rz}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \\ \frac{ds_{zz}}{dt} &= 2\mu\left(\frac{\partial v}{\partial z} - \frac{1}{3}\nabla\cdot\vec{u}\right) - s_{rz}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \\ \frac{ds_{rz}}{dt} &= \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) + \frac{s_{rr} - s_{zz}}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \end{aligned} \tag{5}$$

Von Mises 屈服条件：

$$\frac{3}{2}(s_{rr}^2 + s_{zz}^2 + s_{\theta\theta}^2 + 2s_{xy}^2) \le (Y^0)^2 \tag{6}$$

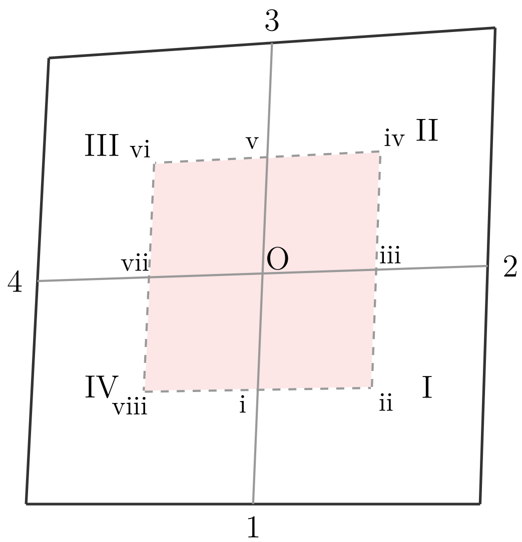

图 1: 有限体积方法求解动量方程控制单元示意图

离散方法

时间离散：预估校正方法

对于如下形式方程

$$\frac{\partial U}{\partial t} = F(U) \tag{7}$$

预估步：

$$\frac{U^{n+\frac{1}{2}} - U^n}{\frac{1}{2}\Delta t} = F(U^n) \tag{8}$$

校正步：

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+\frac{1}{2}}) \tag{9}$$

空间离散方法

运动方程离散

预估步:

$$\frac{(r,z)^{n+\frac{1}{2}} - (r,z)^n}{\frac{1}{2}\Delta t} = (u,v)^n \tag{10}$$

和 Eq.(7) 不同，$(u,v)^{n+1/2}$ 不能直接由 $(r,z)^{n+1/2}$ 求出。这里需要由动量方程求得 $(u,v)^{n+1}$，然后取平均值

$$(u,v)^{n+\frac{1}{2}} = \frac{(u,v)^n + (u,v)^{n+1}}{2} \tag{11}$$

校正步：

$$\frac{(r,z)^{n+1}-(r,z)^n}{\Delta t}=(u,v)^{n+\frac{1}{2}} \tag{12}$$

动量方程离散

1. 有限体积离散

采用如图 2所示非结构四边形单元，Wilkins 在 1963 年首先通过有限体积方法得到 Wilkins 格式 [2]。点 i，iii，v，vii 分别为线 O1，O2，O3，O4 的中点，ii，iv，vi，$viii$ 分别为区域 I-IV 的几何中心。

对于动量离散方程 3，在八边形面回路 $\Omega=i\to ii\to\cdots\to vii\to i$ 上积分：

$$\int_\Omega \rho\frac{du}{dt}d\Omega=\int_\Omega \frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial s_{rz}}{\partial z}d\Omega \tag{13}$$

$$\int_\Omega \rho\frac{dv}{dt}d\Omega=\int_\Omega \frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial s_{rz}}{\partial r}d\Omega \tag{14}$$

以方程 13为例，左侧积分可以近似为

$$\int_\Omega \rho\frac{du}{dt}d\Omega=M\frac{du_o}{dt} \tag{15}$$

其中

$$M=\frac{1}{4}(A_I\rho_I+A_{II}\rho_{II}+A_{III}\rho_{III}+A_{IV}\rho_{IV}) \tag{16}$$

右侧积分根据 Green 公式：

$$\int_\Omega \mathrm{grad} f d\Omega=\oint_{\partial\Omega} f\overrightarrow{n}dl \tag{17}$$

有

$$\int_\Omega \frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial s_{rz}}{\partial z}d\Omega=-\oint_{\partial\Omega}\sigma_{rr}dz+\oint_{\partial\Omega}s_{rz}dr \tag{18}$$

积分

$$\int_{vii}^{i}\sigma_{rr}dz\approx\frac{1}{2}\sigma_{rrI}(z_2-z_1) \tag{19}$$

同理可得

$$-\oint_{\partial\Omega}\sigma_{rr}dz+\oint_{\partial\Omega}s_{rz}dr=\frac{1}{2}\sum_{i=1}^{4}s_{rzi}(r_{i+1}-r_i)-\sigma_{rri}(z_{i+1}-z_i) \tag{20}$$

得到 wilkins 格式：

$$\frac{du_o}{dt}=\frac{1}{2M}\left[\sum_{i=1}^{4}s_{rzi}(r_{i+1}-r_i)-\sigma_{rri}(z_{i+1}-z_i)\right] \tag{21}$$

2: 有限元离散

该格式同样可以通过有限元方法得出 (Lascaux 1973 年) [1]。控制体同样取如图 2，以 IV 区为例，可以化作有限元的单元积分：

$$\int_{\Omega_{IV}} \rho \frac{du}{dt} d\Omega = \int_{\Omega_{IV}} \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \right) d\Omega \tag{22}$$

其中

$$\phi_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \tag{23}$$

为基函数, $i = a, b, c, d$ 其中 $(\xi, \eta) = (-1,-1), (1,-1), (1,1), (-1,1)$

$$\begin{aligned} u &= \sum_{i=a}^{d} u_i\phi_i \\ r &= \sum_{i=a}^{d} r_i\phi_i \\ z &= \sum_{i=a}^{d} z_i\phi_i \end{aligned} \tag{24}$$

由于求解单元中心加速度，因此 $u = u_c\phi_c = u_O\phi_c$ 在 IV 区，$\phi_c = 1/4(1+\xi)(1+\eta)$，因此公式 22左侧可以化为：

$$\begin{aligned} \int_{\Omega_{IV}} \rho \frac{du}{dt} d\Omega &= \frac{du_O}{dt} \int_{\Omega_{IV}} \rho\phi_c d\Omega \\ &= \frac{du_O}{dt} M \end{aligned} \tag{25}$$

其中

$$M = \int_{-1}^{1}\int_{-1}^{1} \rho\phi_c J d\xi d\eta \tag{26}$$

J 为 Jacobi 矩阵

$$\begin{aligned}
J =& \begin{Vmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{Vmatrix} \\
=& \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \\
=& \frac{1}{16}\sum_{i=a}^{d} r_i\xi_i(1+\eta_i\eta)\sum_{i=a}^{d} z_i\eta_i(1+\xi_i\xi) \\
-& \frac{1}{16}\sum_{i=a}^{d} r_i\eta_i(1+\xi_i\eta)\sum_{i=a}^{d} z_i\xi_i(1+\eta_i\eta) \\
=& \frac{1}{16}[(r_2+r_3-r_1-r_4)+\eta(r_1+r_3-r_2-r_4)] \\
& [(z_3+z_4-z_2-z_1)+\xi(z_1+z_3-z_2-z_4)] \\
-& \frac{1}{16}[(z_2+z_3-z_1-z_4)+\eta(z_1+z_3-z_2-z_4)] \\
& [(r_3+r_4-r_2-r_1)+\xi(r_1+r_3-r_2-r_4)] \\
=& \frac{1}{8}[(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3) \\
& +(\xi-\eta)(r_4-r_3)(z_2-z_1)+(\xi-\eta)(r_1-r_2)(z_4-z_3)]
\end{aligned} \tag{27}$$

公式 (25) 可以化为:

$$\begin{aligned}
& \frac{du_O}{dt}\int_{-1}^{1}\int_{-1}^{1}\rho\phi_c J d\xi d\eta \\
=& \frac{1}{32}\frac{du_O}{dt}\int_{-1}^{1}\int_{-1}^{1}\rho(1+\xi)(1+\eta)[B+C(\xi-\eta)]d\xi d\eta
\end{aligned} \tag{28}$$

其中 $B=(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3)$, $C=(r_4-r_3)(z_2-z_1)+(r_1-r_2)(z_4-z_3)$ 公式 (28) 经过积分得到

$$\begin{aligned}
M =& \frac{1}{8}\rho_{IV}B \\
=& \frac{1}{8}\rho_{IV}[(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3)] = \frac{1}{4}\rho_{IV}A_{IV}
\end{aligned} \tag{29}$$

右侧以第一项为例：

$$
\begin{aligned}
&\int_{\Omega_{IV}} \frac{\partial \sigma_{rr}}{\partial r} d\Omega \\
=&\int_{-1}^{1}\int_{-1}^{1} \frac{1}{J}\frac{\partial(z,\sigma_{rr})}{\partial(\xi,\eta)} J d\xi d\eta \\
=&\int_{-1}^{1}\int_{-1}^{1} \frac{\partial(z,\sigma_{rr})}{\partial(\xi,\eta)} d\xi d\eta \\
=&\int_{-1}^{1}\int_{-1}^{1} \frac{\partial z}{\partial \xi}\frac{\partial \sigma_{rr}}{\partial \eta} - \frac{\partial z}{\partial \eta}\frac{\partial \sigma_{rr}}{\partial \xi} d\xi d\eta \\
=&\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \sum_{i=a}^{d} z_i\xi_i(1+\eta_i\eta)\frac{\partial \sigma_{rr}}{\partial \eta} - \sum_{i=a}^{d} z_i\eta_i(1+\xi_i\xi)\frac{\partial \sigma_{rr}}{\partial \xi} d\xi d\eta
\end{aligned}
\tag{30}
$$

由于 σ_{rr} 求的是中心点，在 IV 区为 $\sigma_{rr}=\phi_c\sigma_{rrc}=\frac{1}{4}(1+\xi)(1+\eta)\sigma_{rrO}$ 上式变为

$$
\begin{aligned}
&\frac{1}{16}\sigma_{rrO}\int_{-1}^{1}\int_{-1}^{1} \sum_{i=a}^{d} z_i\xi_i(1+\eta_i\eta)(1+\xi) - \sum_{i=a}^{d} z_i\eta_i(1+\xi_i\xi)(1+\eta) d\xi d\eta \\
=&\frac{1}{4}\sigma_{rrO}(\sum_{i=a}^{d} z_i\xi_i - \sum_{i=a}^{d} z_i\eta_i) \\
=&\frac{1}{2}\sigma_{rrO}(z_b - z_d) \\
=&\frac{1}{2}\sigma_{rrO}(z_1 - z_4)
\end{aligned}
\tag{31}
$$

同理 IV 区右侧右侧第二项可以化为：

$$
\int_{\Omega_{IV}} \frac{\partial s_{rz}}{\partial z} d\Omega = \frac{1}{2} s_{rzO}(r_4 - r_1) \tag{32}
$$

在第 I 区，O 点为第 d 节点，因此 $\phi_d = 1/4(1-\xi)(1+\eta)$ 其他求解过程和 IV 区相同，最后右侧积分变为：

$$
\begin{aligned}
&\int_{\Omega_I} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega = \frac{1}{2} s_{rzO}(r_4 - r_1) \\
=&\frac{1}{2}\sigma_{rrO}(z_a - z_c) + \frac{1}{2} s_{rzO}(z_c - z_a) \\
=&\frac{1}{2}\sigma_{rrO}(z_1 - z_2) + \frac{1}{2} s_{rzO}(z_2 - z_1)
\end{aligned}
\tag{33}
$$

将 I-IV 区左右项加和可得：

$$
\frac{du_o}{dt} = \frac{1}{2M}\left[\sum_{i=1}^{4} s_{rzO}(r_{i+1} - r_i) - \sigma_{rrO}(z_{i+1} - z_i)\right] \tag{34}
$$

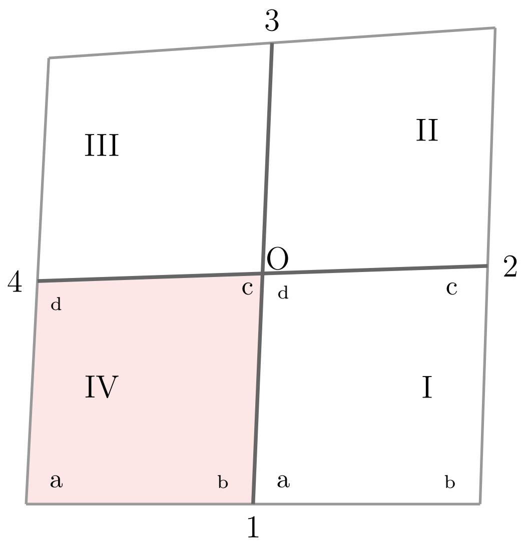

图 2: 有限元求解动量方程控制单元示意图

其中

$$M=\frac{1}{4}(A_I\rho_I+A_{II}\rho_{II}+A_{III}\rho_{III}+A_{IV}\rho_{IV}) \tag{35}$$

和有限体积方法求得的公式 (21) 相同。

能量方程离散

能量守恒方程:

$$\begin{aligned}
&\rho\frac{de}{dt}=\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\\
&=\sigma_{rr}\frac{\partial u}{\partial r}+\sigma_{zz}\frac{\partial v}{\partial z}+\sigma_{rz}\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)\\
&=-p\nabla\cdot\overrightarrow{u}+s_{rr}\frac{\partial u}{\partial r}+s_{zz}\frac{\partial v}{\partial z}+s_{rz}\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)
\end{aligned} \tag{36}$$

通过质量守恒方程:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\overrightarrow{u})=0 \tag{37}$$

有

$$\frac{\partial\rho}{\partial t}+\overrightarrow{u}\cdot\nabla\rho+\rho\nabla\cdot\overrightarrow{u}=0 \tag{38}$$

根据全微分关系

$$\frac{d\rho}{dt}+\rho\nabla\cdot\overrightarrow{u}=0 \tag{39}$$

所以

$$
\begin{aligned}
\nabla \cdot \overrightarrow{u} &= -\frac{1}{\rho}\frac{d\rho}{dt} \\
&= -V\frac{d\frac{1}{V}}{dt} \\
&= \frac{1}{V}\frac{dV}{dt}
\end{aligned}
\tag{40}
$$

所以方程 (36) 可以化为

$$
\rho\frac{de}{dt} = -\frac{p}{V}\frac{dV}{dt} + s_{rr}\frac{\partial u}{\partial r} + s_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \tag{41}
$$

既

$$
\frac{de}{dt} = -p\frac{dV}{dt} + Vs_{rr}\frac{\partial u}{\partial r} + Vs_{zz}\frac{\partial v}{\partial z} + Vs_{rz}\left(\frac{\partial u}{\partial z} + V\frac{\partial v}{\partial r}\right) \tag{42}
$$

其中 $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{s}$。

采用有限体积方法，在图 3 所示整个单元上进行积分有，其中除速度外其它量均集中在单元中心 O，因此有：

$$
\int_\Omega \frac{de}{dt}d\Omega = \int_\Omega -p\frac{dV}{dt} + Vs_{rr}\frac{\partial u}{\partial r} + Vs_{zz}\frac{\partial v}{\partial z} + Vs_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)d\Omega \tag{43}
$$

既

$$
\frac{de}{dt} = -p\frac{dV}{dt} + \frac{1}{A}Vs_{rr}\int_\Omega \frac{\partial u}{\partial r}d\Omega + Vs_{zz}\frac{1}{A}\int_\Omega \frac{\partial v}{\partial z}d\Omega + Vs_{rz}\frac{1}{A}\int_\Omega \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)d\Omega \tag{44}
$$

以其中

$$
\frac{1}{A}\int_\Omega \frac{\partial u}{\partial r}d\Omega \tag{45}
$$

的离散为例：

$$
\begin{aligned}
\frac{1}{A}\int_\Omega \frac{\partial u}{\partial r}d\Omega = -\frac{1}{A}\oint_{\partial\omega} udz \\
&\approx \frac{1}{A}(\frac{1}{2}(u_1+u_2)(z_2-z_1) + \frac{1}{2}(u_2+u_3)(z_3-z_2) \\
&+ (u_3+u_4)(z_4-z_3) + \frac{1}{2}(u_4+u_1)(u_1-u_4)) \\
&= -\frac{1}{A}\left((u_1-u_3)(z_2-z_4) + (u_2-u_4)(z_3-z_1)\right)
\end{aligned}
\tag{46}
$$

并将其记作 $\bar{u}_r$。

能量方程空间离散为

$$
\frac{de}{dt} = -p\frac{dV}{dt} + V\boldsymbol{s}\cdot\boldsymbol{\varepsilon} \tag{47}
$$

其中

$$
\boldsymbol{s}\cdot\boldsymbol{\varepsilon} = s_{rr}\bar{u}_r + s_{zz}\bar{v}_z + s_{rz}(\bar{u}_z + \bar{v}_r) \tag{48}
$$

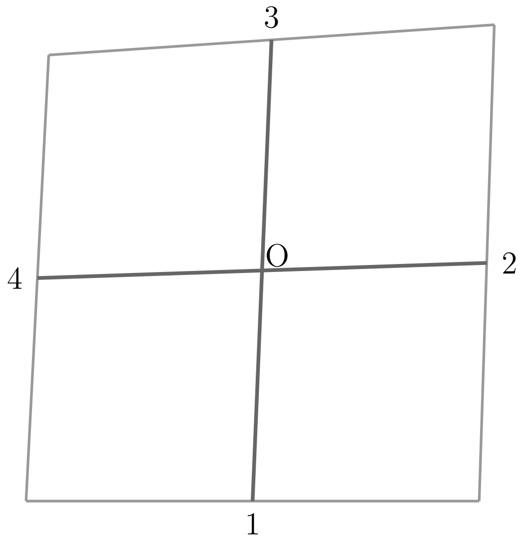

图 3: 能量方程离散控制单元示意图

本构方程离散

和能量方程离散类似，以 $\frac{ds_{rr}}{dt}$ 为例

$$\frac{ds_{rr}}{dt} = 2\mu\left(\bar{u}_r^n - \frac{1}{3}(\bar{u}_r^n + \bar{v}_z^n)\right) + s_{rz}^n(\bar{u}_z^n - \bar{v}_r^n) \tag{49}$$

参考文献

[1] P Lascaux. Application of the finite elements method in two-dimensional hydrodynamics using the lagrange variables. Technical report, CEA Centre d'Etudes de Limeil.

[2] Mark L Wilkins. Calculation of elastic-plastic flow. Technical report, California Univ Livermore Radiation Lab, 1963.