samba 公式推导

March 21, 2018

1. 二维控制方程

运动方程:

$$\frac{d(r,z)}{dt} = (u,v) \tag{1}$$

质量守恒:

$$\frac{dm}{dt} = 0 \tag{2}$$

动量守恒方程:

$$\begin{aligned} \rho\frac{du}{dt} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \\ \rho\frac{dv}{dt} &= \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} \end{aligned} \tag{3}$$

能量守恒方程:

$$\begin{aligned} &\rho\frac{de}{dt} = \sigma_{rr}\frac{\partial u}{\partial r} + \sigma_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \\ &= -p\frac{dV}{dt} + s_{rr}\frac{\partial u}{\partial r} + s_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \end{aligned} \tag{4}$$

本构方程:

$$\begin{aligned} \frac{ds_{rr}}{dt} &= 2\mu\left(\frac{\partial u}{\partial r} - \frac{1}{3}\nabla\cdot\vec{u}\right) + s_{rz}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \\ \frac{ds_{zz}}{dt} &= 2\mu\left(\frac{\partial v}{\partial z} - \frac{1}{3}\nabla\cdot\vec{u}\right) - s_{rz}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \\ \frac{ds_{rz}}{dt} &= \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) + \frac{s_{rr} - s_{zz}}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \end{aligned} \tag{5}$$

2. 离散方法

2..1 时间离散：预估校正方法

对于如下形式方程

$$\frac{\partial U}{\partial t} = F(U) \tag{6}$$

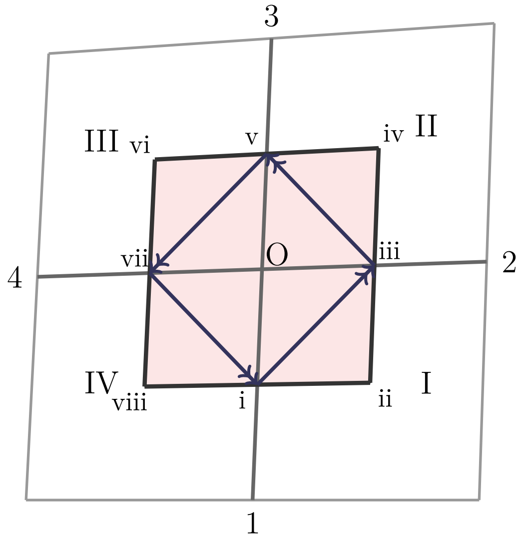

图 1: 控制单元示意图

预估步:

$$\frac{U^{n+\frac{1}{2}} - U^n}{\frac{1}{2}\Delta t} = F(U^n) \tag{7}$$

校正步:

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+\frac{1}{2}}) \tag{8}$$

2..2 空间离散方法

2..2.1 运动方程离散

预估步:

$$\frac{(r,z)^{n+\frac{1}{2}} - (r,z)^n}{\frac{1}{2}\Delta t} = (u,v)^n \tag{9}$$

和 Eq.(6) 不同，$(u,v)^{n+1/2}$ 不能直接由 $(r,z)^{n+1/2}$ 求出。这里需要由动量方程求得 $(u,v)^{n+1}$，然后取平均值

$$(u,v)^{n+\frac{1}{2}} = \frac{(u,v)^n + (u,v)^{n+1}}{2} \tag{10}$$

校正步:

$$\frac{(r,z)^{n+1} - (r,z)^n}{\Delta t} = (u,v)^{n+\frac{1}{2}} \tag{11}$$

2..2.2 动量方程离散

采用如图 2所示非结构四边形单元，Wilkins 在 1963 年首先通过有限体积方法得到 Wilkins 格式 [2]。点 i，iii，v，vii 分别为

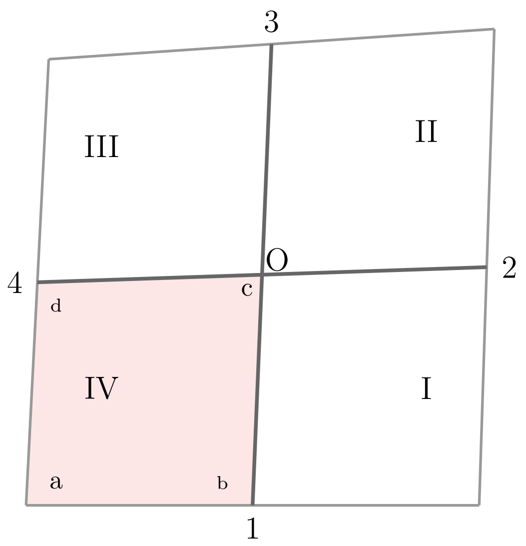

图 2: 控制单元示意图

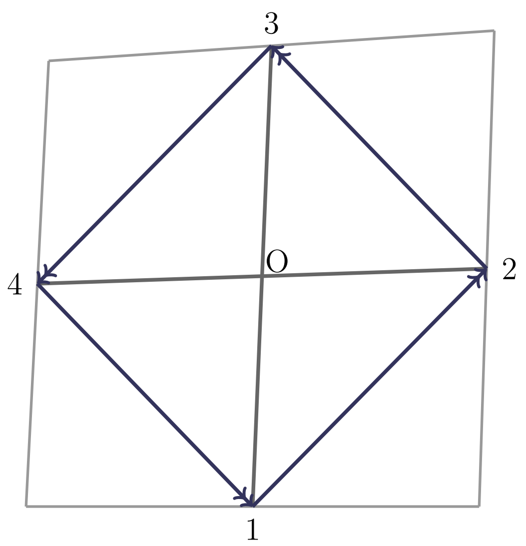

图 3: 控制单元示意图

线 O1，O2，O3，O4 的中点，ii，iv，vi，$viii$ 分别为区域 I-IV 的几何中心。

对于动量离散方程 3，在八边形面回路 $\Omega = i \rightarrow ii \rightarrow \cdots \rightarrow vii \rightarrow i$ 上积分：

$$\int_\Omega \rho \frac{du}{dt} d\Omega = \int_\Omega \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega \tag{12}$$

$$\int_\Omega \rho \frac{dv}{dt} d\Omega = \int_\Omega \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} d\Omega \tag{13}$$

以方程 12为例，左侧积分可以近似为

$$\int_\Omega \rho \frac{du}{dt} d\Omega = M \frac{du_o}{dt} \tag{14}$$

其中

$$M = \frac{1}{4}(A_I \rho_I + A_{II} \rho_{II} + A_{III} \rho_{III} + A_{IV} \rho_{IV}) \tag{15}$$

右侧积分根据 Green 公式：

$$\int_\Omega \mathrm{grad} f d\Omega = \oint_{\partial\Omega} f \overrightarrow{n} dl \tag{16}$$

有

$$\int_\Omega \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega = -\oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr \tag{17}$$

由于线回路积分和所选回路无关，选择回路 $i \rightarrow iii \rightarrow v \rightarrow vii \rightarrow i$ 积分，以 $i \rightarrow iii$ 为例由于其平行且为 $\overrightarrow{12}$ 的一半，积分

$$\int_{vii}^{i} \sigma_{rr} dz \approx \frac{1}{2} \sigma_{zzI} (z_2 - z_1) \tag{18}$$

同理可得

$$-\oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr = \frac{1}{2} \sum_{i=1}^{4} s_{rzi}(r_{i+1} - r_i) - \sigma_{rri}(z_{i+1} - z_i) \tag{19}$$

得到 wilkins 格式：

$$\frac{du_o}{dt} = \frac{1}{2M} \left[\sum_{i=1}^{4} s_{rzi}(r_{i+1} - r_i) - \sigma_{rri}(z_{i+1} - z_i) \right] \tag{20}$$

该格式同样可以通过有限元方法得出 (Lascaux 1973 年) [1]。控制体同样取如图 2，以 I 区为例，在该区通过虚功原理，可以化作有限元的单元积分：

$$\int_{\Omega_I} \rho \frac{du}{dt} \phi_k d\Omega = \int_{\Omega_I} \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \right) \phi_k d\Omega \tag{21}$$

其中
$$\phi_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \tag{22}$$
为基函数,$i=a,b,c,d$ 其中 $(\xi,\eta)=(-1,-1),(1,-1),(1,1),(-1,1)$

$$\begin{aligned} u &= \sum_{i=a}^{d} u_i\phi_i \\ r &= \sum_{i=a}^{d} r_i\phi_i \\ z &= \sum_{i=a}^{d} z_i\phi_i \end{aligned} \tag{23}$$

公式 21左侧可以化为:

$$\begin{aligned} \int_{\Omega_I} \rho\frac{du}{dt}\phi_k d\Omega &= \int_{\Omega_I} \rho\sum_{i=a}^{d}\frac{du_i}{dt}\phi_i\phi_k drdz \\ &= \sum_{i=a}^{d}\frac{du_i}{dt}\int_{-1}^{1}\int_{-1}^{1}\rho\phi_i\phi_k Jd\xi d\eta \end{aligned} \tag{24}$$

其中
$$J = \left\| \begin{matrix} \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{matrix} \right\| \tag{25}$$

右侧以第一项为例:

$$\begin{aligned} \int_{\Omega_I} \frac{\partial \sigma_{rr}}{\partial r} + \phi_k d\Omega = \\ \int_{\Omega_I} \frac{\partial(\sigma_{rr}\phi_k)}{\partial r}d\Omega - \int_{\Omega_I}\frac{\partial \phi_k}{\partial r}\sigma_{rr}d\Omega \\ = -\int_{\partial\Omega_I}\sigma_{rr}\phi_k dz - \int_{\Omega_I}\frac{\partial \phi_k}{\partial r}\sigma_{rr}d\Omega \\ = -\int_{\partial\Omega_I}\sigma_{rr}\phi_k dz - \int_{\Omega_I}\frac{1}{J}\frac{\partial(z,\phi_k)}{\partial(\xi,\eta)}\sigma_{rr}Jd\xi d\eta \\ = -\int_{\partial\Omega_I}\sigma_{rr}\phi_k dz - \int_{-1}^{1}\int_{-1}^{1}\frac{\partial(z,\phi_k)}{\partial(\xi,\eta)}\sigma_{rr}d\xi d\eta \end{aligned} \tag{26}$$

2..2.3 能量方程离散

采用有限体积方法，和动量方程离散相似，以偏导数 $\frac{\partial u}{\partial r}$ 为例，通过 Green 积分可以化为：

$$
\begin{aligned}
\frac{\partial u}{\partial r} &= -\frac{1}{A}\oint_{1\to 2\cdots\to 1} u dz \\
&\approx -\frac{1}{A}(\frac{1}{2}(u_1+u_2)(z_2-z_1)+\frac{1}{2}(u_2+u_3)(z_3-z_2) \\
&+\frac{1}{2}(u_3+u_4)(z_4-z_3)+\frac{1}{2}(u_4+u_1)(u_1-u_4)) \\
&= -\frac{1}{A}\left((u_1-u_3)(z_2-z_4)+(u_2-u_4)(z_3-z_1)\right)
\end{aligned}
\tag{27}
$$

并将其记作 $\bar{u}_r$。

能量方程空间离散为

$$
\frac{dE}{dt} = -p^n\frac{dV}{dt}+V^n+s_r r^n\bar{u}_r^n+s_z z^n\bar{v}_z^n+s_{rz}^n(\bar{u}_z^n+\bar{v}_r^n) \tag{28}
$$

2..2.4 本构方程离散

和能量方程离散类似，以 $\frac{ds_{rr}}{dt}$ 为例

$$
\frac{ds_{rr}}{dt} = 2\mu\left(\bar{u}_r^n-\frac{1}{3}(\bar{u}_r^n+\bar{v}_z^n)\right)+s_{rz}^n(\bar{u}_z^n-\bar{v}_r^n) \tag{29}
$$

参考文献

[1] P Lascaux. Application of the finite elements method in two-dimensional hydrodynamics using the lagrange variables. Technical report, CEA Centre d'Etudes de Limeil.

[2] Mark L Wilkins. Calculation of elastic-plastic flow. Technical report, California Univ Livermore Radiation Lab, 1963.