samba 公式推导

April 4, 2018

二维控制方程

运动方程：

$$\frac{d(r,z)}{dt} = (u,v) \tag{1}$$

质量守恒：

$$\frac{dm}{dt} = 0 \tag{2}$$

动量守恒方程：

$$\begin{aligned} \rho\frac{du}{dt} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \\ \rho\frac{dv}{dt} &= \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} \end{aligned} \tag{3}$$

能量守恒方程：

$$\begin{aligned} \rho\frac{de}{dt} &= \sigma_{rr}\frac{\partial u}{\partial r} + \sigma_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \\ &= -p\frac{dV}{dt} + s_{rr}\frac{\partial u}{\partial r} + s_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \end{aligned} \tag{4}$$

本构方程：

$$\begin{aligned} \frac{ds_{rr}}{dt} &= 2\mu\left(\frac{\partial u}{\partial r} - \frac{1}{3}\nabla\cdot\vec{u}\right) + s_{rz}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \\ \frac{ds_{zz}}{dt} &= 2\mu\left(\frac{\partial v}{\partial z} - \frac{1}{3}\nabla\cdot\vec{u}\right) - s_{rz}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \\ \frac{ds_{rz}}{dt} &= \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) + \frac{s_{rr}-s_{zz}}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \end{aligned} \tag{5}$$

Von Mises 屈服条件：

$$\frac{3}{2}(s_{rr}^2 + s_{zz}^2 + s_{\theta\theta}^2 + 2s_{xy}^2) \leq (Y^0)^2 \tag{6}$$

柱坐标下控制方程

在正交曲线坐标系 $(\xi_\alpha, \xi_\beta, \xi_\gamma)$ 中，存在坐标变换

$$x_i = x_i(\xi_\alpha, \xi_\beta, \xi_\gamma), \quad i = 1, 2, 3 \tag{7}$$

由于

$$\frac{\partial(x_1, x_2, x_3)}{\partial(\xi_\alpha, \xi_\beta, \xi_\gamma)} \neq 0 \tag{8}$$

因此可以解出

$$\xi_\nu = \xi_\nu(x_1, x_2, x_3), \quad \nu = 1, 2, 3 \tag{9}$$

定义 Lame 系数

$$h_\nu = \left[\sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial \xi_\nu}\right)^2\right]^{\frac{1}{2}}, \quad \nu = \alpha, \beta, \gamma \tag{10}$$

可以推出曲线坐标系下标量 ϕ 的梯度表达式

$$\nabla \phi = \frac{1}{h_\alpha} \frac{\partial \phi}{\partial \xi_\alpha} \boldsymbol{j}_\alpha + \frac{1}{h_\beta} \frac{\partial \phi}{\partial \xi_\beta} \boldsymbol{j}_\beta + \frac{1}{h_\gamma} \frac{\partial \phi}{\partial \xi_\gamma} \boldsymbol{j}_\gamma \tag{11}$$

对于向量函数 $\boldsymbol{f}$ 旋度表达式为

$$\begin{aligned}\nabla \times \boldsymbol{f} = &\frac{1}{h_\beta h_\gamma} \left[\frac{\partial}{\partial \beta}(h_\gamma f_\gamma) - \frac{\partial}{\partial \gamma}(h_\beta f_\beta)\right] \boldsymbol{j}_\alpha \\ &+ \frac{1}{h_\gamma h_\alpha} \left[\frac{\partial}{\partial \gamma}(h_\alpha f_\alpha) - \frac{\partial}{\partial \alpha}(h_\gamma f_\gamma)\right] \boldsymbol{j}_\beta \\ &+ \frac{1}{h_\alpha h_\beta} \left[\frac{\partial}{\partial \alpha}(h_\beta f_\beta) - \frac{\partial}{\partial \beta}(h_\alpha f_\alpha)\right] \boldsymbol{j}_\gamma\end{aligned} \tag{12}$$

散度表达式

$$\nabla \cdot \boldsymbol{f} = \frac{1}{h_\alpha h_\beta h_\gamma} \left[\frac{\partial}{\partial \xi_\alpha}(f_\alpha h_\beta h_\gamma) + \frac{\partial}{\partial \xi_\beta}(f_\beta h_\gamma h_\alpha) + \frac{\partial}{\partial \xi_\gamma}(f_\gamma h_\alpha h_\beta)\right] \tag{13}$$

对于曲线坐标系中的张量 $\boldsymbol{T}$

$$\boldsymbol{T} = \begin{pmatrix} T_{\alpha\alpha} & T_{\alpha\beta} & T_{\alpha\gamma} \\ T_{\beta\alpha} & T_{\beta\beta} & T_{\beta\gamma} \\ T_{\gamma\alpha} & T_{\gamma\beta} & T_{\gamma\gamma} \end{pmatrix} \tag{14}$$

散度三个分量为

$$\begin{aligned}(\nabla \cdot \boldsymbol{T})_\alpha = &\frac{1}{h_\alpha h_\beta h_\gamma} \left[\frac{\partial T_{\alpha\alpha} h_\beta h_\gamma}{\partial \xi_\alpha} + \frac{\partial T_{\beta\alpha} h_\gamma h_\alpha}{\partial \xi_\beta} + \frac{\partial T_{\gamma\alpha} h_\alpha h_\beta}{\partial \xi_\gamma}\right] \\ &+ \frac{T_{\alpha\beta}}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \xi_\beta} + \frac{T_{\alpha\gamma}}{h_\alpha h_\gamma} \frac{\partial h_\alpha}{\partial \xi_\gamma} - \frac{T_{\beta\beta}}{h_\alpha h_\beta} \frac{\partial h_\beta}{\partial \xi_\alpha} - \frac{T_{\gamma\gamma}}{h_\alpha h_\gamma} \frac{\partial h_\gamma}{\partial \xi_\alpha}\end{aligned} \tag{15}$$

$$(\nabla\cdot\boldsymbol{T})_\beta=\frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial T_{\alpha\beta}h_\beta h_\gamma}{\partial\xi_\alpha}+\frac{\partial T_{\beta\beta}h_\gamma h_\alpha}{\partial\xi_\beta}+\frac{\partial T_{\gamma\beta}h_\alpha h_\beta}{\partial\xi_\gamma}\right]$$
$$+\frac{T_{\beta\alpha}}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial\xi_\alpha}+\frac{T_{\beta\gamma}}{h_\beta h_\gamma}\frac{\partial h_\beta}{\partial\xi_\gamma}-\frac{T_{\alpha\alpha}}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial\xi_\beta}-\frac{T_{\gamma\gamma}}{h_\beta h_\gamma}\frac{\partial h_\gamma}{\partial\xi_\beta} \tag{16}$$

$$(\nabla\cdot\boldsymbol{T})_\gamma=\frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial T_{\alpha\gamma}h_\beta h_\gamma}{\partial\xi_\alpha}+\frac{\partial T_{\beta\gamma}h_\gamma h_\alpha}{\partial\xi_\beta}+\frac{\partial T_{\gamma\gamma}h_\alpha h_\beta}{\partial\xi_\gamma}\right]$$
$$+\frac{T_{\gamma\alpha}}{h_\gamma h_\alpha}\frac{\partial h_\gamma}{\partial\xi_\alpha}+\frac{T_{\gamma\beta}}{h_\beta h_\gamma}\frac{\partial h_\gamma}{\partial\xi_\beta}-\frac{T_{\alpha\alpha}}{h_\alpha h_\gamma}\frac{\partial h_\alpha}{\partial\xi_\gamma}-\frac{T_{\beta\beta}}{h_\beta h_\gamma}\frac{\partial h_\beta}{\partial\xi_\gamma} \tag{17}$$

具体求解过程见附录4.。

离散方法

时间离散：预估校正方法

对于如下形式方程

$$\frac{\partial U}{\partial t}=F(U) \tag{18}$$

预估步：

$$\frac{U^{n+\frac{1}{2}}-U^n}{\frac{1}{2}\Delta t}=F(U^n) \tag{19}$$

校正步：

$$\frac{U^{n+1}-U^n}{\Delta t}=F(U^{n+\frac{1}{2}}) \tag{20}$$

空间离散方法

动量方程离散

1. 有限体积离散

采用如图2所示非结构四边形单元，Wilkins 在 1963 年首先通过有限体积方法得到 Wilkins 格式 [2]。点 i，iii，v，vii 分别为线 O1，O2，O3，O4 的中点，ii，iv，vi，$viii$ 分别为区域 I-IV 的几何中心。

对于动量离散方程3，在八边形面回路 $\Omega=i\to ii\to\cdots\to vii\to i$ 上积分：

$$\int_\Omega\rho\frac{du}{dt}d\Omega=\int_\Omega\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial s_{rz}}{\partial z}d\Omega \tag{21}$$

$$\int_\Omega\rho\frac{dv}{dt}d\Omega=\int_\Omega\frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial s_{rz}}{\partial r}d\Omega \tag{22}$$

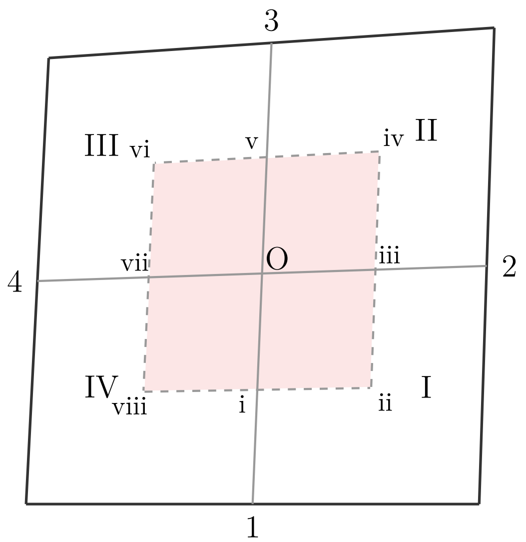


图 1: 有限体积方法求解动量方程控制单元示意图

以方程21为例，左侧积分可以近似为

$$\int_{\Omega} \rho \frac{du}{dt} d\Omega = M \frac{du_o}{dt} \tag{23}$$

其中

$$M = \frac{1}{4}(A_I \rho_I + A_{II} \rho_{II} + A_{III} \rho_{III} + A_{IV} \rho_{IV}) \tag{24}$$

右侧积分根据 Green 公式：

$$\int_{\Omega} \mathrm{grad} f d\Omega = \oint_{\partial\Omega} f \overrightarrow{n} dl \tag{25}$$

有

$$\int_{\Omega} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega = -\oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr \tag{26}$$

积分

$$\int_{vii}^{i} \sigma_{rr} dz \approx \frac{1}{2} \sigma_{rrI}(z_2 - z_1) \tag{27}$$

同理可得

$$-\oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr = \frac{1}{2} \sum_{i=1}^{4} s_{rzi}(r_{i+1} - r_i) - \sigma_{rri}(z_{i+1} - z_i) \tag{28}$$

得到 wilkins 格式：

$$\frac{du_o}{dt} = \frac{1}{2M} \left[\sum_{i=1}^{4} s_{rzi}(r_{i+1} - r_i) - \sigma_{rri}(z_{i+1} - z_i) \right] \tag{29}$$

2: 有限元离散

该格式同样可以通过有限元方法得出 (Lascaux 1973 年）[1]。控制体同样取如图2，以 IV 区为例，可以化作有限元的单元积分：

$$\int_{\Omega_{IV}} \rho \frac{du}{dt}\phi_O d\Omega = \int_{\Omega_{IV}} \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z}\right)\phi_O d\Omega \tag{30}$$

其中 ϕ_o 在 IV 区为

$$\phi_O = \phi_c = \frac{1}{4}(1+\xi)(1+\eta) \tag{31}$$

为基函数,$i = a, b, c, d$ 其中 $(\xi, \eta) = (-1, -1), (1, -1), (1, 1), (-1, 1)$

$$\begin{aligned} u &= \sum_{i=a}^{d} u_i \phi_i \\ r &= \sum_{i=a}^{d} r_i \phi_i \\ z &= \sum_{i=a}^{d} z_i \phi_i \end{aligned} \tag{32}$$

公式30左侧可以化为：

$$\begin{aligned} \int_{\Omega_{IV}} \rho \phi_c \frac{du}{dt} d\Omega &= \frac{du}{dt}\int_{\Omega_{IV}} \rho \phi_c d\Omega \\ &= \frac{du}{dt} M \end{aligned} \tag{33}$$

其中

$$M = \int_{-1}^{1}\int_{-1}^{1} \rho \phi_c J d\xi d\eta \tag{34}$$

J 为 Jacobi 矩阵

$$
\begin{aligned}
J = & \left\| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{array} \right\| \\
= & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \\
= & \frac{1}{16}\sum_{i=a}^{d} r_i \xi_i (1+\eta_i \eta) \sum_{i=a}^{d} z_i \eta_i (1+\xi_i \xi) \\
- & \frac{1}{16}\sum_{i=a}^{d} r_i \eta_i (1+\xi_i \eta) \sum_{i=a}^{d} z_i \xi_i (1+\eta_i \eta) \\
= & \frac{1}{16}[(r_2+r_3-r_1-r_4)+\eta(r_1+r_3-r_2-r_4)] \\
& [(z_3+z_4-z_2-z_1)+\xi(z_1+z_3-z_2-z_4)] \\
- & \frac{1}{16}[(z_2+z_3-z_1-z_4)+\eta(z_1+z_3-z_2-z_4)] \\
& [(r_3+r_4-r_2-r_1)+\xi(r_1+r_3-r_2-r_4)] \\
= & \frac{1}{8}[(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3) \\
& +(\xi-\eta)(r_4-r_3)(z_2-z_1)+(\xi-\eta)(r_1-r_2)(z_4-z_3)]
\end{aligned} \tag{35}
$$

公式 (33) 可以化为:

$$
\begin{aligned}
& \frac{du_O}{dt}\int_{-1}^{1}\int_{-1}^{1}\rho\phi_c J d\xi d\eta \\
= & \frac{1}{32}\frac{du_O}{dt}\int_{-1}^{1}\int_{-1}^{1}\rho(1+\xi)(1+\eta)[B+C(\xi-\eta)]d\xi d\eta
\end{aligned} \tag{36}
$$

其中 $B=(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3)$, $C=(r_4-r_3)(z_2-z_1)+(r_1-r_2)(z_4-z_3)$ 公式 (36) 经过积分得到

$$
\begin{aligned}
M = & \frac{1}{8}\rho_{IV} B \\
= & \frac{1}{8}\rho_{IV}[(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3)] = \quad \frac{1}{4}\rho_{IV} A_{IV}
\end{aligned} \tag{37}
$$

右侧以第一项为例：

$$
\begin{aligned}
&\int_{\Omega_{IV}} \frac{\partial\sigma_{rr}}{\partial r}\phi_c d\Omega \\
&= \int_{\Omega_{IV}} \frac{\partial\sigma_{rr}\phi_c}{\partial r} d\Omega - \int_{\Omega_{IV}} \frac{\partial\phi_c}{\partial r}\sigma_{rr} d\Omega \\
=& -\sigma_{rr}\int_{-1}^{1}\int_{-1}^{1}\frac{1}{J}\frac{\partial(z,\phi_c)}{\partial(\xi,\eta)} J d\xi d\eta \\
=& -\sigma_{rr}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial(z,\phi_c)}{\partial(\xi,\eta)} d\xi d\eta \\
=& -\sigma_{rr}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial z}{\partial\xi}\frac{\partial\phi_c}{\partial\eta} - \frac{\partial z}{\partial\eta}\frac{\partial\phi_c}{\partial\xi} d\xi d\eta \\
=& -\frac{1}{4}\sigma_{rr}\int_{-1}^{1}\int_{-1}^{1}\sum_{i=a}^{d} z_i\xi_i(1+\eta_i\eta)\frac{\partial\phi_c}{\partial\eta} - \sum_{i=a}^{d} z_i\eta_i(1+\xi_i\xi)\frac{\partial\phi_c}{\partial\xi} d\xi d\eta
\end{aligned}
\tag{38}
$$

将 ϕ_c 带入

$$
\begin{aligned}
&-\frac{1}{16}\sigma_{rr}\int_{-1}^{1}\int_{-1}^{1}\sum_{i=a}^{d} z_i\xi_i(1+\eta_i\eta)(1+\xi) - \sum_{i=a}^{d} z_i\eta_i(1+\xi_i\xi)(1+\eta) d\xi d\eta \\
=& -\frac{1}{4}\sigma_{rr}\left(\sum_{i=a}^{d} z_i\xi_i - \sum_{i=a}^{d} z_i\eta_i\right) \\
=& \frac{1}{2}\sigma_{rr}(z_d - z_b) \\
=& \frac{1}{2}\sigma_{rr}(z_4 - z_1)
\end{aligned}
\tag{39}
$$

同理 IV 区右侧右侧第二项可以化为：

$$
\int_{\Omega_{IV}} \frac{\partial s_{rz}}{\partial z}\phi_c d\Omega = \frac{1}{2}s_{rzO}(r_1 - r_4) \tag{40}
$$

在第 I 区，O 点为第 d 节点，因此 $\phi_d = 1/4(1-\xi)(1+\eta)$ 其他求解过程和 IV 区相同，最后右侧积分变为：

$$
\begin{aligned}
&\int_{\Omega_I}\left(\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z}\right)\phi_d d\Omega = \frac{1}{2}s_{rzO}(r_4 - r_1) \\
=& \frac{1}{2}\sigma_{rrO}(z_a - z_c) + \frac{1}{2}s_{rzO}(z_c - z_a) \\
=& \frac{1}{2}\sigma_{rrO}(z_1 - z_2) + \frac{1}{2}s_{rzO}(z_2 - z_1)
\end{aligned}
\tag{41}
$$

将 I-IV 区左右项加和可得：

$$
\frac{du_o}{dt} = \frac{1}{2M}\left[\sum_{i=1}^{4} s_{rzO}(r_{i+1} - r_i) - \sigma_{rrO}(z_{i+1} - z_i)\right] \tag{42}
$$

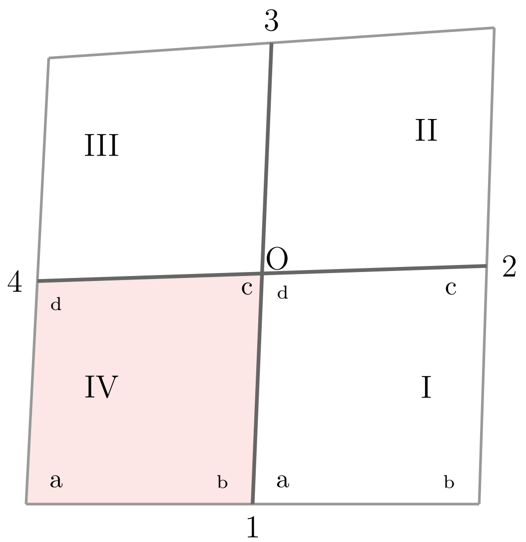


图 2: 有限元求解动量方程控制单元示意图

其中

$$M = \frac{1}{4}(A_I\rho_I + A_{II}\rho_{II} + A_{III}\rho_{III} + A_{IV}\rho_{IV}) \tag{43}$$

和有限体积方法求得的公式 (29) 相同。

在计算中需要在动量方程中增加沙漏粘性 h 和人工粘性 q，公式 (29) 变为：

$$M\frac{du}{dt} = \sum_{i=1}^{4}\left(h_i + \frac{1}{2}[s_{rz}(r_{i+1} - r_i) - (\sigma_{rr} - q_i)(z_{i+1} - z_i)]\right) \tag{44}$$

能量方程离散

能量守恒方程：

$$\begin{aligned} \rho\frac{de}{dt} &= \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} \\ &= \sigma_{rr}\frac{\partial u}{\partial r} + \sigma_{zz}\frac{\partial v}{\partial z} + \sigma_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \\ &= -p\nabla\cdot\overrightarrow{u} + s_{rr}\frac{\partial u}{\partial r} + s_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \end{aligned} \tag{45}$$

通过质量守恒方程：

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\overrightarrow{u}) = 0 \tag{46}$$

有

$$\frac{\partial \rho}{\partial t} + \overrightarrow{u} \cdot \nabla \rho + \rho \nabla \cdot \overrightarrow{u} = 0 \tag{47}$$

根据全微分关系

$$\frac{d\rho}{dt} + \rho \nabla \cdot \overrightarrow{u} = 0 \tag{48}$$

所以

$$\begin{aligned}\nabla \cdot \overrightarrow{u} &= -\frac{1}{\rho}\frac{d\rho}{dt} \\ &= -\frac{V}{m}\frac{d\frac{m}{V}}{dt} \\ &= \frac{1}{V}\frac{dV}{dt}\end{aligned} \tag{49}$$

所以方程 (45) 可以化为

$$\rho\frac{de}{dt} = -\frac{p}{V}\frac{dV}{dt} + s_{rr}\frac{\partial u}{\partial r} + s_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \tag{50}$$

既

$$\frac{dE}{dt} = -p\frac{dV}{dt} + Vs_{rr}\frac{\partial u}{\partial r} + Vs_{zz}\frac{\partial v}{\partial z} + Vs_{rz}\left(\frac{\partial u}{\partial z} + V\frac{\partial v}{\partial r}\right) \tag{51}$$

其中 $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{s}$。

采用有限体积方法，在图3 所示整个单元上进行积分有，其中除速度外其它量均集中在单元中心 O，因此有：

$$\int_\Omega \frac{dE}{dt}d\Omega = \int_\Omega -p\frac{dV}{dt}+Vs_{rr}\frac{\partial u}{\partial r}+Vs_{zz}\frac{\partial v}{\partial z}+Vs_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)d\Omega \tag{52}$$

既

$$\frac{dE}{dt} = -p\frac{dV}{dt}+\frac{1}{A}Vs_{rr}\int_\Omega \frac{\partial u}{\partial r}d\Omega+Vs_{zz}\frac{1}{A}\int_\Omega \frac{\partial v}{\partial z}d\Omega+Vs_{rz}\frac{1}{A}\int_\Omega \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)d\Omega \tag{53}$$

以其中

$$\frac{1}{A}\int_\Omega \frac{\partial u}{\partial r}d\Omega \tag{54}$$

的离散为例：

$$\begin{aligned}\frac{1}{A}\int_\Omega \frac{\partial u}{\partial r}d\Omega &= -\frac{1}{A}\oint_{\partial\omega} udz \\ &\approx \frac{1}{A}(\frac{1}{2}(u_1+u_2)(z_2-z_1) + \frac{1}{2}(u_2+u_3)(z_3-z_2) \\ &+ (u_3+u_4)(z_4-z_3) + \frac{1}{2}(u_4+u_1)(u_1-u_4)) \\ &= -\frac{1}{A}\left((u_1-u_3)(z_2-z_4) + (u_2-u_4)(z_3-z_1)\right)\end{aligned} \tag{55}$$

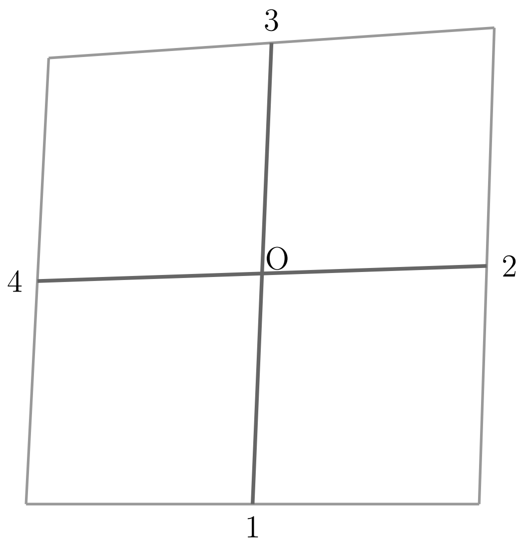


图 3: 能量方程离散控制单元示意图

并将其记作 $\bar{u}_r$。

能量方程空间离散为

$$\frac{dE}{dt} = -p\frac{dV}{dt} + V\boldsymbol{s}\cdot\boldsymbol{\varepsilon} \tag{56}$$

其中

$$\boldsymbol{s}\cdot\boldsymbol{\varepsilon} = s_{rr}\bar{u}_r + s_{zz}\bar{v}_z + s_{rz}(\bar{u}_z + \bar{v}_r) \tag{57}$$

本构方程离散

和能量方程离散类似，以 $\frac{ds_{rr}}{dt}$ 为例

$$\frac{ds_{rr}}{dt} = 2\mu\left(\bar{u}_r^n - \frac{1}{3}(\bar{u}_r^n + \bar{v}_z^n)\right) + s_{rz}^n(\bar{u}_z^n - \bar{v}_r^n) \tag{58}$$

求解流程

Feahgq：计算沙漏粘性 h

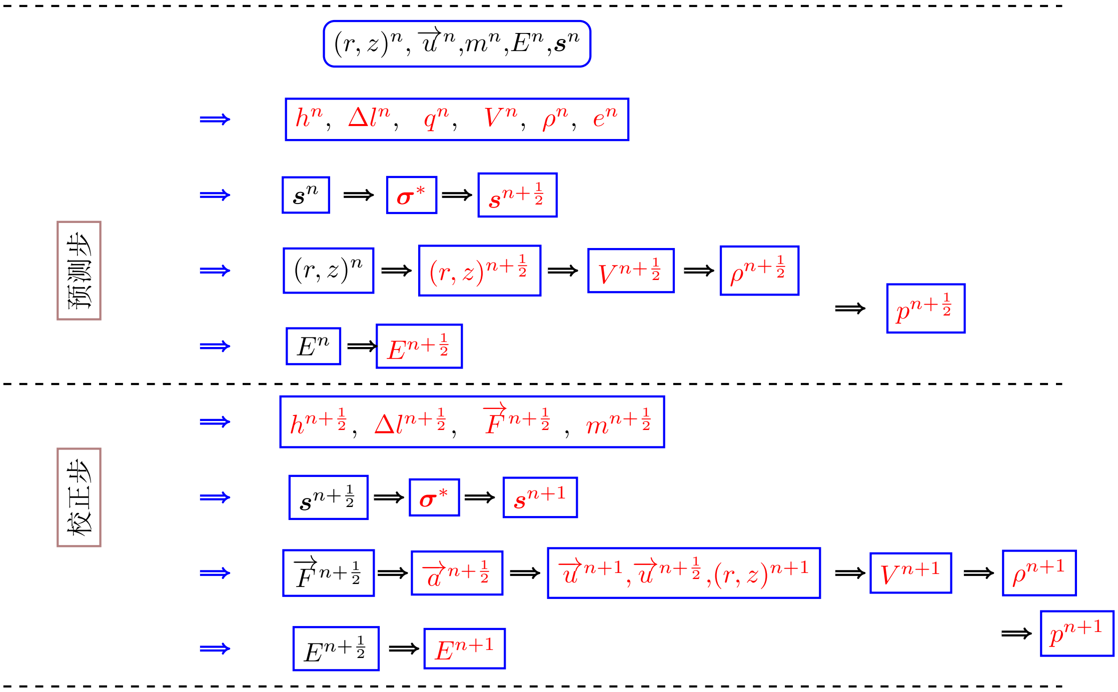
fe2al：计算特征长度 Δl

Bukq：计算人工粘性 q

体积 V: 通过 (r,z) 坐标位置，计算单元体积

密度 ρ: 通过 $\rho = \frac{m}{V}$ 计算单元密度

比内能 e: 通过 $e = \frac{E}{m}$ 计算



rstres: 更新应力旋转项：

$$
\begin{aligned}
\sigma_{rr}^* &= \sigma_{rr}^n + s_{rz}^n(u_z^n - v_r^n)\frac{\Delta t}{2} \\
\sigma_{zz}^* &= \sigma_{zz}^n + s_{rz}^n(u_z^n - v_r^n)\frac{\Delta t}{2} \\
s_{rz}^* &= s_{rz}^n + \frac{s_{zz}^n - s_{rr}^n}{2}(u_z^n - v_r^n)\frac{\Delta t}{2}
\end{aligned} \tag{59}
$$

fe2d10：更新偏应力：

$$
\begin{aligned}
s_{rr}^{n+\frac{1}{2}} &= \sigma_{rr}^* + p^n + 2\mu\left(u_r^n - \frac{1}{3}\nabla(u^n, v^n)\right)\frac{\Delta t}{2} \\
s_{zz}^{n+\frac{1}{2}} &= \sigma_{zz}^* + p^n + 2\mu\left(v_z^n - \frac{1}{3}\nabla(u^n, v^n)\right)\frac{\Delta t}{2} \\
s_{rr}^{n+\frac{1}{2}} &= s_{rz}^* + \mu(u_z^n + v_r^n)\frac{\Delta t}{2} \\
(s_{rr}^{n+\frac{1}{2}}, s_{zz}^{n+\frac{1}{2}}, s_{rz}^{n+\frac{1}{2}}) &= \Upsilon(s_{rr}^{n+\frac{1}{2}}, s_{zz}^{n+\frac{1}{2}} s_{rz}^{n+\frac{1}{2}}, Y^0)
\end{aligned} \tag{60}
$$

更新坐标

$$
\begin{aligned}
r^{n+\frac{1}{2}} &= r^n + \frac{\Delta t}{2}u^n \\
z^{n+\frac{1}{2}} &= z^n + \frac{\Delta t}{2}v^n
\end{aligned} \tag{61}
$$

通过坐标 (r,z) 求解体积 V。

通过 V 求解 ρ:

$$\rho = \frac{m}{V} \tag{62}$$

ieupd_ correct：求解半点时刻能量 能量方程空间离散过程见公式 (56)，

$$E^{n+\frac{1}{2}} = E^n + \boldsymbol{\sigma}^n : \boldsymbol{\epsilon}^n \frac{\Delta t}{2} \tag{63}$$

Eqos: 通过 $p^{n+\frac{1}{2}} = p(\rho^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})$ 更新压力

Arfrc 计算节点力 $F^{n+\frac{1}{2}}$ 体力计算的具体公式见 (44):

$$F^{n+\frac{1}{2}} = \sum_{i=1}^{4} \left(h_i^{n+\frac{1}{2}} + .5\frac{1}{2}[s_{rz}^{n+\frac{1}{2}}(r_{i+1}^{n+\frac{1}{2}} - r_i^{n+\frac{1}{2}}) - (\sigma_{rr}^{n+\frac{1}{2}} - q_i^{n+\frac{1}{2}})(z_{i+1}^{n+\frac{1}{2}} - z_i^{n+\frac{1}{2}})] \right) \tag{64}$$

质量可以通过质量守恒定律 (2) 求得:

$$\frac{dm}{dt} = 0 \tag{65}$$

rstres: 更新应力旋转项:

$$\begin{aligned}
\sigma_{rr}^* &= \sigma_{rr}^n + s_{rz}^{n+\frac{1}{2}}(u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}})\Delta t \\
\sigma_{zz}^* &= \sigma_{zz}^n + s_{rz}^{n+\frac{1}{2}}(u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}})\Delta t \\
s_{rz}^* &= s_{rz}^n + \frac{s_{zz}^{n+\frac{1}{2}} - s_{rr}^{n+\frac{1}{2}}}{2}(u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}})\Delta t
\end{aligned} \tag{66}$$

fe2d10：更新偏应力:

$$\begin{aligned}
s_{rr}^{n+1} &= \sigma_{rr}^* + p^n + 2\mu \left(u_r^{n+\frac{1}{2}} - \frac{1}{3}\nabla(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) \right) \Delta t \\
s_{zz}^{n+1} &= \sigma_{zz}^* + p^n + 2\mu \left(v_z^{n+\frac{1}{2}} - \frac{1}{3}\nabla(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) \right) \Delta t \\
s_{rr}^{n+1} &= s_{rz}^* + \mu(u_z^{n+\frac{1}{2}} + v_r^{n+\frac{1}{2}})\Delta t \\
(s_{rr}^{n1}, s_{zz}^{n+1}, s_{rz}^{n+1}) &= \Upsilon(s_{rr}^{n+1}, s_{zz}^{n+1} s_{rz}^{n+1}, Y^0)
\end{aligned} \tag{67}$$

更新加速度 $\overrightarrow{a} = \overrightarrow{F}/m$

update_cc：

$$\begin{aligned}
(u,v)^{n+1} &= (u,v)^n + \Delta t(a_r, a_z)^{n+\frac{1}{2}} \\
(u,v)^{n+\frac{1}{2}} &= \frac{(u,v)^n + (u,v)^{n+1}}{2} \\
(r,z)^{n+1} &= (r,z)^n + \Delta t(u,v)^{n+\frac{1}{2}}
\end{aligned} \tag{68}$$

ieupd_ correct：求解半点时刻能量 能量方程空间离散过程见公式 (56)，

$$E^{n+1} = E^n + \boldsymbol{\sigma}^{n+\frac{1}{2}} : \boldsymbol{\epsilon}^{n+\frac{1}{2}} \Delta t \tag{69}$$

Eqos: 通过 $p^{n+1} = p(\rho^{n+1}, e^{n+1})$ 更新压力

附录 I

Lame 系数

在沿 ξ_α 坐标曲线上任取两点 $M(x,y,z)$ 和 $N(x+dx, y+dy, z+dz)$，那么 MN 沿 ξ_α 弧微分为 $ds_\alpha = \sqrt{dx^2+dy^2+dz^2}$ ，由于曲线坐标系三个方向正交，在 ξ_α 上有 $d\xi_\beta = 0, d\xi_\gamma = 0$，因此有 $dx = \frac{\partial x}{\partial \xi_\alpha} d\xi_\alpha$，同理 $dy = \frac{\partial y}{\partial \xi_\alpha} d\xi_\alpha$ 和 $dz = \frac{\partial z}{\partial \xi_\alpha} d\xi_\alpha$，即

$$ds_\alpha = \sqrt{\left(\frac{\partial x}{\partial \xi_\alpha}\right)^2 + \left(\frac{\partial y}{\partial \xi_\alpha}\right)^2 + \left(\frac{\partial z}{\partial \xi_\alpha}\right)^2} d\xi_\alpha \tag{70}$$

令

$$h_\alpha = \sqrt{\left(\frac{\partial x}{\partial \xi_\alpha}\right)^2 + \left(\frac{\partial y}{\partial \xi_\alpha}\right)^2 + \left(\frac{\partial z}{\partial \xi_\alpha}\right)^2} \tag{71}$$

和

$$h_\beta = \sqrt{\left(\frac{\partial x}{\partial \xi_\beta}\right)^2 + \left(\frac{\partial y}{\partial \xi_\beta}\right)^2 + \left(\frac{\partial z}{\partial \xi_\beta}\right)^2} \tag{72}$$

$$h_\gamma = \sqrt{\left(\frac{\partial x}{\partial \xi_\gamma}\right)^2 + \left(\frac{\partial y}{\partial \xi_\gamma}\right)^2 + \left(\frac{\partial z}{\partial \xi_\gamma}\right)^2} \tag{73}$$

为 Lame 系数。

$$\begin{aligned} ds_\alpha &= h_\alpha d\xi_\alpha \\ ds_\beta &= h_\beta d\xi_\beta \\ ds_\gamma &= h_\gamma d\xi_\gamma \end{aligned} \tag{74}$$

因此有

$$\frac{\partial \boldsymbol{r}}{\partial \xi_\nu} = \frac{\partial \boldsymbol{r}}{\partial \xi_\nu} \frac{ds_\nu}{d\xi_\nu} = h_\nu \boldsymbol{e}_\nu \tag{75}$$

梯度公式

在笛卡尔坐标系中有

$$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k} \tag{76}$$

又有

$$\begin{aligned} d\boldsymbol{r} &= dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k} \\ d\phi &= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}dz \end{aligned} \tag{77}$$

可以得到

$$\nabla\phi \cdot d\boldsymbol{r} = d\phi \tag{78}$$

该公式和所用坐标系无关。同样，在曲线坐标系下，

$$\begin{aligned} d\phi =& \frac{\partial\phi}{\partial\xi_\alpha}d\xi_\alpha + \frac{\partial\phi}{\partial\xi_\beta}d\xi_\beta + \frac{\partial\phi}{\partial\xi_\gamma}d\xi_\gamma \\ d\boldsymbol{r} =& \frac{\partial\boldsymbol{r}}{\partial\xi_\alpha}d\xi_\alpha + \frac{\partial\boldsymbol{r}}{\partial\xi_\beta}d\xi_\beta + \frac{\partial\boldsymbol{r}}{\partial\xi_\gamma}d\xi_\gamma \\ =& h_\alpha d\xi_\alpha\boldsymbol{e}_\alpha + h_\beta d\xi_\beta\boldsymbol{e}_\beta + h_\gamma d\xi_\gamma\boldsymbol{e}_\gamma \end{aligned} \tag{79}$$

其中 $\boldsymbol{e}_\nu$ 是沿着坐标曲线的单位矢量。采用待定系数法，令 $\nabla f = \lambda_\alpha\boldsymbol{e}_\alpha + \lambda_\beta\boldsymbol{e}_\beta + \lambda_\gamma\boldsymbol{e}_\gamma$ 代入公式 (78) 有

$$\begin{aligned} d\phi =& \frac{\partial\phi}{\partial\xi_\alpha}d\xi_\alpha + \frac{\partial\phi}{\partial\xi_\beta}d\xi_\beta + \frac{\partial\phi}{\partial\xi_\gamma}d\xi_\gamma \\ =& (\lambda_\alpha\boldsymbol{e}_\alpha + \lambda_\beta\boldsymbol{e}_\beta + \lambda_\gamma\boldsymbol{e}_\gamma)\cdot(h_\alpha d\xi_\alpha\boldsymbol{e}_\alpha + h_\beta d\xi_\beta\boldsymbol{e}_\beta + h_\gamma d\xi_\gamma\boldsymbol{e}_\gamma) \\ =& \lambda_\alpha h_\alpha d\xi_\alpha + \lambda_\beta h_\beta d\xi_\beta + \lambda_\gamma h_\gamma d\xi_\gamma \end{aligned} \tag{80}$$

根据公式 (79) 有

$$\lambda_\alpha = \frac{1}{h_\alpha}\frac{\partial\phi}{\partial\xi_\alpha}, \quad \lambda_\beta = \frac{1}{h_\beta}\frac{\partial\phi}{\partial\xi_\beta}, \quad \lambda_\gamma = \frac{1}{h_\gamma}\frac{\partial\phi}{\partial\xi_\gamma} \tag{81}$$

一般曲线坐标系下散度公式

单位矢量的旋度

单位矢量的旋度

$$\nabla\xi_\nu = (\frac{1}{h_\alpha}\frac{\partial}{\partial\xi_\alpha}\boldsymbol{e}_\alpha + \frac{1}{h_\beta}\frac{\partial}{\partial\xi_\beta}\boldsymbol{e}_\beta + \frac{1}{h_\gamma}\frac{\partial}{\partial\xi_\gamma}\boldsymbol{e}_\gamma)\xi_\nu = \frac{1}{h_\nu}\boldsymbol{e}_\nu \tag{82}$$

因此有

$$\nabla\times\left(\frac{1}{h_\nu}\boldsymbol{e}_\nu\right)=\nabla\times\nabla\xi_\nu=0 \tag{83}$$

即矢量 $\frac{1}{h_\nu}\boldsymbol{e}_\nu$ 有零旋度。

单位矢量的散度

由于三个单位矢量 $\boldsymbol{e}_\nu$ 两两相交，有 $\boldsymbol{e}_\alpha=\boldsymbol{e}_\beta\times\boldsymbol{e}_\gamma$，因此

$$\frac{1}{h_\beta h_\gamma}\boldsymbol{e}_\alpha=\left(\frac{1}{h_\beta}\boldsymbol{e}_\beta\right)\times\left(\frac{1}{h_\gamma}\boldsymbol{e}_\gamma\right)=\nabla\xi_\beta\times\nabla\xi_\gamma \tag{84}$$

有

$$\nabla\cdot\frac{1}{h_\beta h_\gamma}\boldsymbol{e}_\alpha=\nabla\cdot(\nabla\xi_\beta\times\nabla\xi_\gamma)=\nabla\xi_\beta\cdot(\nabla\times\nabla\xi_\gamma)-\nabla\xi_\gamma\cdot(\nabla\times\nabla\xi_\beta)=0 \tag{85}$$

因此矢量 $\frac{1}{h_\beta h_\gamma}\boldsymbol{e}_\alpha$ 的散度为零。

曲线坐标系下的散度公式

令 $\boldsymbol{F}=F_\alpha\boldsymbol{e}_\alpha+F_\beta\boldsymbol{e}_\beta+F_\gamma\boldsymbol{e}_\gamma$ 则 $\nabla\cdot\boldsymbol{F}=\nabla\cdot(F_\alpha\boldsymbol{e}_\alpha)+\nabla\cdot(F_\beta\boldsymbol{e}_\beta)+\nabla\cdot(F_\gamma\boldsymbol{e}_\gamma)$

对于上式第一项

$$\nabla\cdot(F_\alpha\boldsymbol{e}_\alpha)=\nabla\cdot\left(h_\beta h_\gamma F_\alpha\frac{\boldsymbol{e}_\alpha}{h_\beta h_\gamma}\right)=\frac{\boldsymbol{e}_\alpha}{h_\beta h_\gamma}\cdot\nabla(h_\beta h_\gamma F_\alpha)+h_\beta h_\gamma F_\alpha\nabla\cdot\frac{\boldsymbol{e}_\alpha}{h_\beta h_\gamma} \tag{86}$$

由于 $\frac{\boldsymbol{e}_\alpha}{h_\beta h_\gamma}$ 有零散度

$$\begin{aligned}\nabla\cdot(F_\alpha\boldsymbol{e}_\alpha)=&\frac{\boldsymbol{e}_\alpha}{h_\beta h_\gamma}\cdot\nabla(h_\beta h_\gamma F_\alpha)\\=&\frac{\boldsymbol{e}_\alpha}{h_\beta h_\gamma}\cdot\left(\frac{1}{h_\alpha}\frac{\partial}{\partial\xi_\alpha}\boldsymbol{e}_\alpha+\frac{1}{h_\beta}\frac{\partial}{\partial\xi_\beta}\boldsymbol{e}_\beta+\frac{1}{h_\gamma}\frac{\partial}{\partial\xi_\gamma}\boldsymbol{e}_\gamma\right)h_\beta h_\gamma F_\alpha\\=&\frac{1}{h_\alpha h_\beta h_\gamma}\frac{\partial h_\beta h_\gamma F_\alpha}{\partial\xi_\alpha}\end{aligned} \tag{87}$$

同理可得

$$\nabla\cdot(F_\beta\boldsymbol{e}_\beta)=\frac{1}{h_\alpha h_\beta h_\gamma}\frac{\partial h_\alpha h_\gamma F_\beta}{\partial\xi_\beta} \tag{88}$$

$$\nabla\cdot(F_\gamma\boldsymbol{e}_\gamma)=\frac{1}{h_\alpha h_\beta h_\gamma}\frac{\partial h_\alpha h_\beta F_\gamma}{\partial\xi_\gamma} \tag{89}$$

因此可得散度表达式

$$\nabla\cdot\boldsymbol{F}=\frac{1}{h_\alpha h_\beta h_\gamma}\left(\frac{\partial h_\beta h_\gamma F_\alpha}{\partial\xi_\alpha}+\frac{\partial h_\alpha h_\gamma F_\beta}{\partial\xi_\beta}+\frac{\partial h_\alpha h_\beta F_\gamma}{\partial\xi_\gamma}\right) \tag{90}$$

旋度公式

令 $\boldsymbol{F}=F_\alpha\boldsymbol{e}_\alpha+F_\beta\boldsymbol{e}_\beta+F_\gamma\boldsymbol{e}_\gamma$ 则 $\nabla\times\boldsymbol{F}=\nabla\times(F_\alpha\boldsymbol{e}_\alpha)+\nabla\times(F_\beta\boldsymbol{e}_\beta)+\nabla\times(F_\gamma\boldsymbol{e}_\gamma)$ 右端第一项

$$\begin{aligned}\nabla\times(F_\alpha\boldsymbol{e}_\alpha)=&\nabla(h_\alpha F_\alpha)\times\frac{\boldsymbol{e}_\alpha}{h_\alpha}+F_\alpha h_\alpha\nabla\times\frac{\boldsymbol{e}_\alpha}{h_\alpha}\\ =&\left(\frac{1}{h_\alpha}\frac{\partial}{\partial\xi_\alpha}\boldsymbol{e}_\alpha+\frac{1}{h_\beta}\frac{\partial}{\partial\xi_\beta}\boldsymbol{e}_\beta+\frac{1}{h_\gamma}\frac{\partial}{\partial\xi_\gamma}\boldsymbol{e}_\gamma\right)h_\alpha F_\alpha\times\frac{1}{h_\alpha}\boldsymbol{e}_\alpha+0\\ =&\frac{1}{h_\alpha h_\beta}\frac{\partial h_\alpha F_\alpha}{\partial\xi_\beta}(-\boldsymbol{e}_\gamma)+\frac{1}{h_\alpha h_\gamma}\frac{\partial h_\alpha F_\alpha}{\partial\xi_\gamma}(\boldsymbol{e}_\beta)\\ =&\frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial h_\alpha F_\alpha}{\partial\xi_\gamma}(h_\beta\boldsymbol{e}_\beta)-\frac{\partial h_\alpha F_\alpha}{\partial\xi_\beta}(h_\gamma\boldsymbol{e}_\gamma)\right]\end{aligned} \tag{91}$$

同理可得

$$\begin{aligned}\nabla\times(F_\beta\boldsymbol{e}_\beta)&=\frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial h_\beta F_\beta}{\partial\xi_\alpha}(h_\gamma\boldsymbol{e}_\gamma)-\frac{\partial h_\beta F_\beta}{\partial\xi_\gamma}(h_\alpha\boldsymbol{e}_\alpha)\right]\\ \nabla\times(F_\gamma\boldsymbol{e}_\gamma)&=\frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial h_\gamma F_\gamma}{\partial\xi_\beta}(h_\alpha\boldsymbol{e}_\alpha)-\frac{\partial h_\gamma F_\gamma}{\partial\xi_\alpha}(h_\beta\boldsymbol{e}_\beta)\right]\end{aligned} \tag{92}$$

可以写成简化形式

$$\nabla\times\boldsymbol{F}=\frac{1}{h_\alpha h_\beta h_\gamma}\begin{vmatrix}h_\alpha\boldsymbol{e}_\alpha & h_\beta\boldsymbol{e}_\beta & h_\gamma\boldsymbol{e}_\gamma\\ \frac{\partial}{\partial\xi_\alpha} & \frac{\partial}{\partial\xi_\beta} & \frac{\partial}{\partial\xi_\gamma}\\ h_\alpha F_\alpha & h_\beta F_\beta & h_\gamma F_\gamma\end{vmatrix} \tag{93}$$

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