samba 公式推导

March 28, 2018

二维控制方程

运动方程：

$$\frac{d(r,z)}{dt} = (u,v) \tag{1}$$

质量守恒：

$$\frac{dm}{dt} = 0 \tag{2}$$

动量守恒方程：

$$\begin{aligned}
\rho\frac{du}{dt} &= \frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \\
\rho\frac{dv}{dt} &= \frac{\partial\sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r}
\end{aligned} \tag{3}$$

能量守恒方程：

$$\begin{aligned}
\rho\frac{de}{dt} &= \sigma_{rr}\frac{\partial u}{\partial r} + \sigma_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \\
&= -p\frac{dV}{dt} + s_{rr}\frac{\partial u}{\partial r} + s_{zz}\frac{\partial v}{\partial z} + s_{rz}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)
\end{aligned} \tag{4}$$

本构方程：

$$\begin{aligned}
\frac{ds_{rr}}{dt} &= 2\mu\left(\frac{\partial u}{\partial r} - \frac{1}{3}\nabla\cdot\vec{u}\right) + s_{rz}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \\
\frac{ds_{zz}}{dt} &= 2\mu\left(\frac{\partial v}{\partial z} - \frac{1}{3}\nabla\cdot\vec{u}\right) - s_{rz}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) \\
\frac{ds_{rz}}{dt} &= \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) + \frac{s_{rr} - s_{zz}}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right)
\end{aligned} \tag{5}$$

Von Mises 屈服条件：

$$\frac{3}{2}(s_{rr}^2 + s_{zz}^2 + s_{\theta\theta}^2 + 2s_{xy}^2) \leq (Y^0)^2 \tag{6}$$

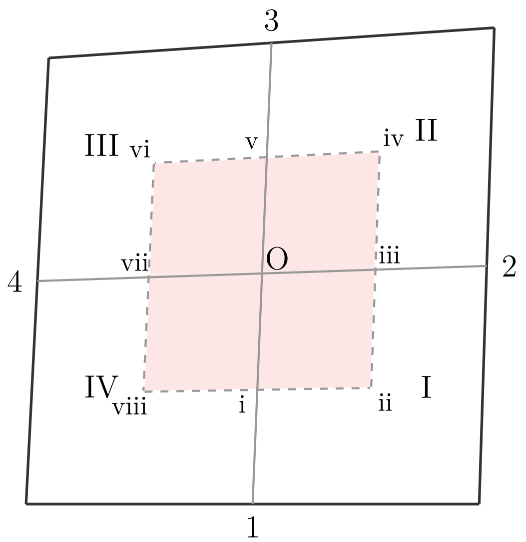


图 1: 有限体积方法求解动量方程控制单元示意图

离散方法

时间离散：预估校正方法

对于如下形式方程

$$\frac{\partial U}{\partial t} = F(U) \tag{7}$$

预估步：

$$\frac{U^{n+\frac{1}{2}} - U^n}{\frac{1}{2}\Delta t} = F(U^n) \tag{8}$$

校正步：

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+\frac{1}{2}}) \tag{9}$$

空间离散方法

动量方程离散

1. 有限体积离散

采用如图2所示非结构四边形单元，Wilkins 在 1963 年首先通过有限体积方法得到 Wilkins 格式 [?]。点 i，iii，v，vii 分别为线 O1，O2，O3，O4 的中点，ii，iv，vi，$viii$ 分别为区域 *I-IV* 的几何中心。

对于动量离散方程3，在八边形面回路 $\Omega = i \to ii \to \cdots \to vii \to i$ 上积分：

$$\int_\Omega \rho \frac{du}{dt} d\Omega = \int_\Omega \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega \tag{10}$$

$$\int_\Omega \rho \frac{dv}{dt} d\Omega = \int_\Omega \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} d\Omega \tag{11}$$

以方程10为例，左侧积分可以近似为

$$\int_\Omega \rho \frac{du}{dt} d\Omega = M \frac{du_o}{dt} \tag{12}$$

其中

$$M = \frac{1}{4}(A_I \rho_I + A_{II} \rho_{II} + A_{III} \rho_{III} + A_{IV} \rho_{IV}) \tag{13}$$

右侧积分根据 Green 公式：

$$\int_\Omega \mathrm{grad} f d\Omega = \oint_{\partial\Omega} f \vec{n} dl \tag{14}$$

有

$$\int_\Omega \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega = -\oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr \tag{15}$$

积分

$$\int_{vii}^{i} \sigma_{rr} dz \approx \frac{1}{2} \sigma_{rrI}(z_2 - z_1) \tag{16}$$

同理可得

$$-\oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr = \frac{1}{2} \sum_{i=1}^{4} s_{rzi}(r_{i+1} - r_i) - \sigma_{rri}(z_{i+1} - z_i) \tag{17}$$

得到 wilkins 格式：

$$\frac{du_o}{dt} = \frac{1}{2M} \left[\sum_{i=1}^{4} s_{rzi}(r_{i+1} - r_i) - \sigma_{rri}(z_{i+1} - z_i) \right] \tag{18}$$

2: 有限元离散

该格式同样可以通过有限元方法得出 (Lascaux 1973 年) [?]。控制体同样取如图2，以 IV 区为例，可以化作有限元的单元积分：

$$\int_{\Omega_I V} \rho \frac{du}{dt} d\Omega = \int_{\Omega_I V} \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \right) d\Omega \tag{19}$$

其中

$$\phi_i = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta) \tag{20}$$

为基函数,$i=a,b,c,d$ 其中 $(\xi,\eta)=(-1,-1),(1,-1),(1,1),(-1,1)$

$$\begin{aligned} u &= \sum_{i=a}^{d} u_i\phi_i \\ r &= \sum_{i=a}^{d} r_i\phi_i \\ z &= \sum_{i=a}^{d} z_i\phi_i \end{aligned} \tag{21}$$

由于求解单元中心加速度，因此 $u=u_c\phi_c=u_O\phi_c$ 在 IV 区，$\phi_c=1/4(1+\xi)(1+\eta)$，因此公式19左侧可以化为:

$$\begin{aligned} \int_{\Omega_I V} \rho\frac{du}{dt}d\Omega &= \frac{du_O}{dt}\int_{\Omega_I V}\rho\phi_c d\Omega \\ &= \frac{du_O}{dt}M \end{aligned} \tag{22}$$

其中

$$M=\int_{-1}^{1}\int_{-1}^{1}\rho\phi_c J d\xi d\eta \tag{23}$$

J 为 Jacobi 矩阵

$$\begin{aligned} J =& \left\| \begin{matrix} \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{matrix} \right\| \\ =&\frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta}-\frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \\ =&\frac{1}{16}\sum_{i=a}^{d} r_i\xi_i(1+\eta_i\eta)\sum_{i=a}^{d} z_i\eta_i(1+\xi_i\xi) \\ -&\frac{1}{16}\sum_{i=a}^{d} r_i\eta_i(1+\xi_i\eta)\sum_{i=a}^{d} z_i\xi_i(1+\eta_i\eta) \\ =&\frac{1}{16}[(r_2+r_3-r_1-r_4)+\eta(r_1+r_3-r_2-r_4)] \\ &[(z_3+z_4-z_2-z_1)+\xi(z_1+z_3-z_2-z_4)] \\ -&\frac{1}{16}[(z_2+z_3-z_1-z_4)+\eta(z_1+z_3-z_2-z_4)] \\ &[(r_3+r_4-r_2-r_1)+\xi(r_1+r_3-r_2-r_4)] \\ =&\frac{1}{8}[(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3) \\ &+(\xi-\eta)(r_4-r_3)(z_2-z_1)+(\xi-\eta)(r_1-r_2)(z_4-z_3)] \end{aligned} \tag{24}$$

公式 (22) 可以化为:

$$\begin{aligned}&\frac{du_O}{dt}\int_{-1}^{1}\int_{-1}^{1}\rho\phi_c J d\xi d\eta\\&=\frac{1}{32}\frac{du_O}{dt}\int_{-1}^{1}\int_{-1}^{1}\rho(1+\xi)(1+\eta)[B+C(\xi-\eta)]d\xi d\eta\end{aligned}\tag{25}$$

其中 $B=(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3)$, $C=(r_4-r_3)(z_2-z_1)+(r_1-r_2)(z_4-z_3)$ 公式 (25) 经过积分得到

$$\begin{aligned}M=&\frac{1}{8}\rho_{IV}B\\=&\frac{1}{8}\rho_{IV}[(z_1-z_3)(r_4-r_2)+(z_2-z_4)(r_1-r_3)]=\quad\frac{1}{4}\rho_{IV}A_{IV}\end{aligned}\tag{26}$$

右侧以第一项为例:

$$\begin{aligned}&\int_{\Omega_IV}\frac{\partial\sigma_{rr}}{\partial r}d\Omega\\=&\int_{-1}^{1}\int_{-1}^{1}\frac{1}{J}\frac{\partial(z,\sigma_{rr})}{\partial(\xi,\eta)}Jd\xi d\eta\\=&\int_{-1}^{1}\int_{-1}^{1}\frac{\partial(z,\sigma_{rr})}{\partial(\xi,\eta)}d\xi d\eta\\=&\int_{-1}^{1}\int_{-1}^{1}\frac{\partial z}{\partial\xi}\frac{\partial\sigma_{rr}}{\partial\eta}-\frac{\partial z}{\partial\eta}\frac{\partial\sigma_{rr}}{\partial\xi}d\xi d\eta\\=&\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\sum_{i=a}^{d}z_i\xi_i(1+\eta_i\eta)\frac{\partial\sigma_{rr}}{\partial\eta}-\sum_{i=a}^{d}z_i\eta_i(1+\xi_i\xi)\frac{\partial\sigma_{rr}}{\partial\xi}d\xi d\eta\end{aligned}\tag{27}$$

由于 σ_{rr} 求的是中心点, 在 IV 区为 $\sigma_{rr}=\phi_c\sigma_{rrc}=\frac{1}{4}(1+\xi)(1+\eta)\sigma_{rrO}$ 上式变为

$$\begin{aligned}&\frac{1}{16}\sigma_{rrO}\int_{-1}^{1}\int_{-1}^{1}\sum_{i=a}^{d}z_i\xi_i(1+\eta_i\eta)(1+\xi)-\sum_{i=a}^{d}z_i\eta_i(1+\xi_i\xi)(1+\eta)d\xi d\eta\\=&\frac{1}{4}\sigma_{rrO}(\sum_{i=a}^{d}z_i\xi_i-\sum_{i=a}^{d}z_i\eta_i)\\=&\frac{1}{2}\sigma_{rrO}(z_b-z_d)\\=&\frac{1}{2}\sigma_{rrO}(z_1-z_4)\end{aligned}\tag{28}$$

同理 IV 区右侧右侧第二项可以化为:

$$\int_{\Omega_{IV}}\frac{\partial s_{rz}}{\partial z}d\Omega=\frac{1}{2}s_{rzO}(r_4-r_1)\tag{29}$$

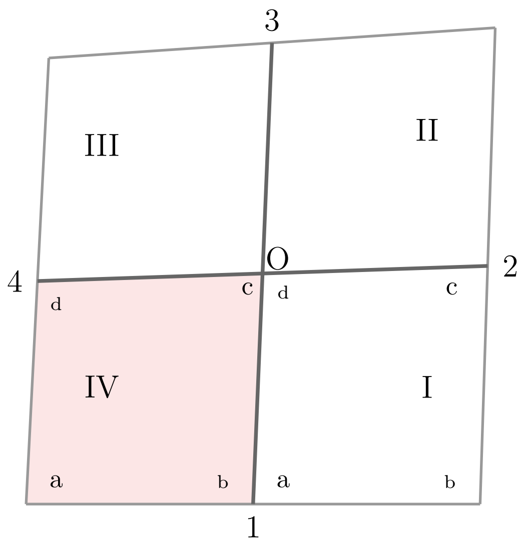


图 2: 有限元求解动量方程控制单元示意图

在第 I 区，O 点为第 d 节点，因此 $\phi_d = 1/4(1-\xi)(1+\eta)$ 其他求解过程和 IV 区相同，最后右侧积分变为：

$$\begin{aligned}\int_{\Omega_I} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega &= \frac{1}{2} s_{rzO}(r_4 - r_1) \\ =&\frac{1}{2}\sigma_{rrO}(z_a - z_c) + \frac{1}{2}s_{rzO}(z_c - z_a) \\ =&\frac{1}{2}\sigma_{rrO}(z_1 - z_2) + \frac{1}{2}s_{rzO}(z_2 - z_1)\end{aligned} \tag{30}$$

将 I-IV 区左右项加和可得：

$$\frac{du_o}{dt} = \frac{1}{2M}\left[\sum_{i=1}^{4} s_{rzO}(r_{i+1} - r_i) - \sigma_{rrO}(z_{i+1} - z_i)\right] \tag{31}$$

其中

$$M = \frac{1}{4}(A_I\rho_I + A_{II}\rho_{II} + A_{III}\rho_{III} + A_{IV}\rho_{IV}) \tag{32}$$

和有限体积方法求得的公式 (18) 相同。

在计算中需要在动量方程中增加沙漏粘性 h 和人工粘性 q，公式 (18) 变为：

$$M\frac{du}{dt} = \sum_{i=1}^{4}\left(h_i + \frac{1}{2}[s_{rz}(r_{i+1} - r_i) - (\sigma_{rr} - q_i)(z_{i+1} - z_i)]\right) \tag{33}$$

能量方程离散

能量守恒方程：

$$
\begin{aligned}
\rho \frac{de}{dt} &= \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} \\
&= \sigma_{rr} \frac{\partial u}{\partial r} + \sigma_{zz} \frac{\partial v}{\partial z} + \sigma_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \\
&= -p \nabla \cdot \overrightarrow{u} + s_{rr} \frac{\partial u}{\partial r} + s_{zz} \frac{\partial v}{\partial z} + s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right)
\end{aligned}
\tag{34}
$$

通过质量守恒方程：

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overrightarrow{u}) = 0 \tag{35}
$$

有

$$
\frac{\partial \rho}{\partial t} + \overrightarrow{u} \cdot \nabla \rho + \rho \nabla \cdot \overrightarrow{u} = 0 \tag{36}
$$

根据全微分关系

$$
\frac{d\rho}{dt} + \rho \nabla \cdot \overrightarrow{u} = 0 \tag{37}
$$

所以

$$
\begin{aligned}
\nabla \cdot \overrightarrow{u} &= -\frac{1}{\rho} \frac{d\rho}{dt} \\
&= -V \frac{d\frac{1}{V}}{dt} \\
&= \frac{1}{V} \frac{dV}{dt}
\end{aligned}
\tag{38}
$$

所以方程 (34) 可以化为

$$
\rho \frac{de}{dt} = -\frac{p}{V} \frac{dV}{dt} + s_{rr} \frac{\partial u}{\partial r} + s_{zz} \frac{\partial v}{\partial z} + s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \tag{39}
$$

既

$$
\frac{dE}{dt} = -p \frac{dV}{dt} + V s_{rr} \frac{\partial u}{\partial r} + V s_{zz} \frac{\partial v}{\partial z} + V s_{rz} \left(\frac{\partial u}{\partial z} + V \frac{\partial v}{\partial r} \right) \tag{40}
$$

其中 $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{s}$。

采用有限体积方法，在图3 所示整个单元上进行积分有，其中除速度外其它量均集中在单元中心 O，因此有：

$$
\int_\Omega \frac{dE}{dt} d\Omega = \int_\Omega -p \frac{dV}{dt} + V s_{rr} \frac{\partial u}{\partial r} + V s_{zz} \frac{\partial v}{\partial z} + V s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) d\Omega \tag{41}
$$

既

$$
\frac{dE}{dt} = -p \frac{dV}{dt} + \frac{1}{A} V s_{rr} \int_\Omega \frac{\partial u}{\partial r} d\Omega + V s_{zz} \frac{1}{A} \int_\Omega \frac{\partial v}{\partial z} d\Omega + V s_{rz} \frac{1}{A} \int_\Omega \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) d\Omega \tag{42}
$$

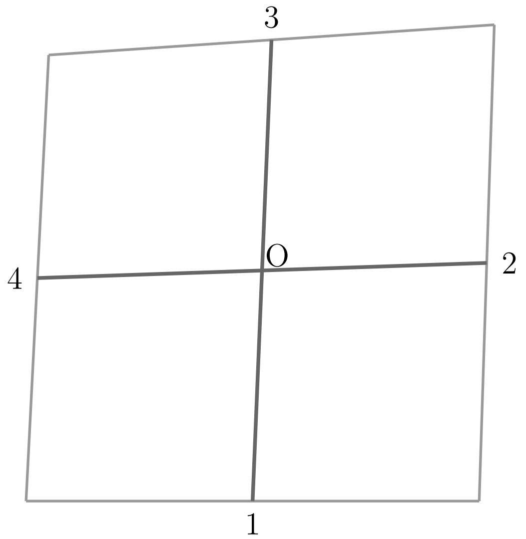


图 3: 能量方程离散控制单元示意图

以其中

$$\frac{1}{A}\int_{\Omega}\frac{\partial u}{\partial r}d\Omega \tag{43}$$

的离散为例：

$$\begin{aligned}\frac{1}{A}\int_{\Omega}\frac{\partial u}{\partial r}d\Omega &= -\frac{1}{A}\oint_{\partial\omega}udz \\ &\approx \frac{1}{A}(\frac{1}{2}(u_1+u_2)(z_2-z_1)+\frac{1}{2}(u_2+u_3)(z_3-z_2) \\ &+(u_3+u_4)(z_4-z_3)+\frac{1}{2}(u_4+u_1)(u_1-u_4)) \\ &= -\frac{1}{A}\left((u_1-u_3)(z_2-z_4)+(u_2-u_4)(z_3-z_1)\right)\end{aligned} \tag{44}$$

并将其记作 $\bar{u}_r$。

能量方程空间离散为

$$\frac{dE}{dt} = -p\frac{dV}{dt}+V\boldsymbol{s}\cdot\boldsymbol{\varepsilon} \tag{45}$$

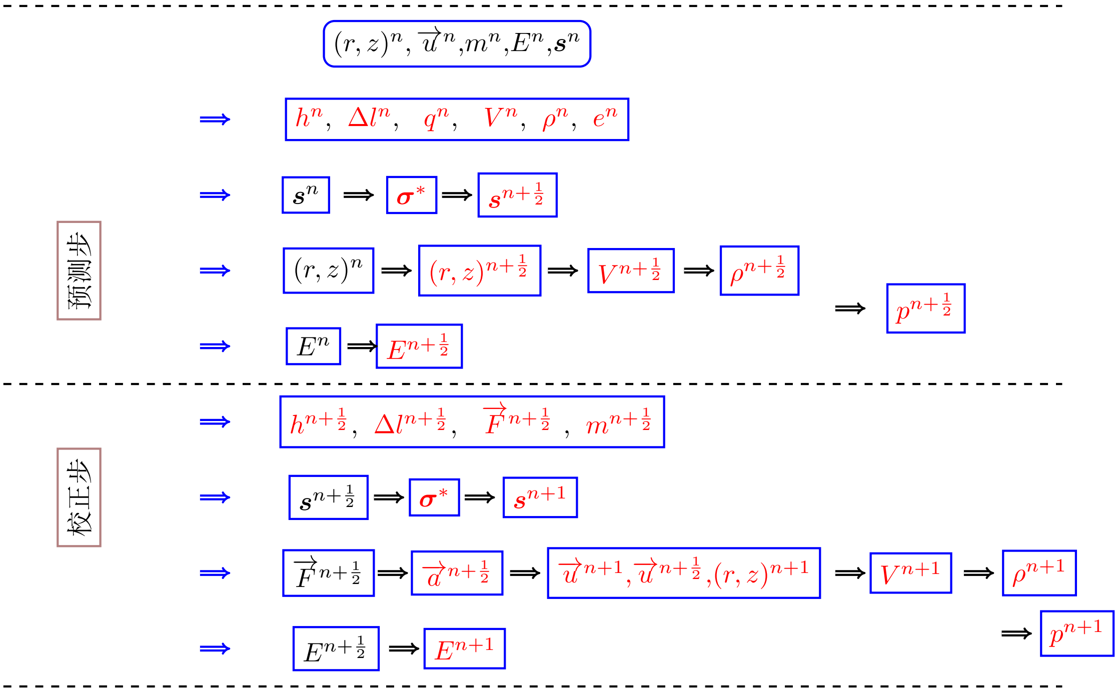
其中

$$\boldsymbol{s}\cdot\boldsymbol{\varepsilon} = s_{rr}\bar{u}_r+s_{zz}\bar{v}_z+s_{rz}(\bar{u}_z+\bar{v}_r) \tag{46}$$

本构方程离散

和能量方程离散类似，以 $\frac{ds_{rr}}{dt}$ 为例

$$\frac{ds_{rr}}{dt} = 2\mu\left(\bar{u}_r^n-\frac{1}{3}(\bar{u}_r^n+\bar{v}_z^n)\right)+s_{rz}^n(\bar{u}_z^n-\bar{v}_r^n) \tag{47}$$



求解流程

Feahgq：计算沙漏粘性 h

fe2al：计算特征长度 Δl

Bukq：计算人工粘性 q

体积 V: 通过 (r,z) 坐标位置，计算单元体积

密度 ρ: 通过 $\rho=\frac{m}{V}$ 计算单元密度

比内能 e: 通过 $e=\frac{E}{m}$ 计算

rstres: 更新应力旋转项：

$$
\begin{aligned}
\sigma^*_{rr} &= \sigma^n_{rr} + s^n_{rz}(u^n_z - v^n_r)\frac{\Delta t}{2} \\
\sigma^*_{zz} &= \sigma^n_{zz} + s^n_{rz}(u^n_z - v^n_r)\frac{\Delta t}{2} \\
s^*_{rz} &= s^n_{rz} + \frac{s^n_{zz} - s^n_{rr}}{2}(u^n_z - v^n_r)\frac{\Delta t}{2}
\end{aligned} \tag{48}
$$

fe2d10：更新偏应力：

$$
\begin{aligned}
s_{rr}^{n+\frac{1}{2}} &= \sigma_{rr}^{*} + p^{n} + 2\mu\left(u_r^n - \frac{1}{3}\nabla(u^n, v^n)\right)\frac{\Delta t}{2} \\
s_{zz}^{n+\frac{1}{2}} &= \sigma_{zz}^{*} + p^{n} + 2\mu\left(v_z^n - \frac{1}{3}\nabla(u^n, v^n)\right)\frac{\Delta t}{2} \\
s_{rr}^{n+\frac{1}{2}} &= s_{rz}^{*} + \mu(u_z^n + v_r^n)\frac{\Delta t}{2} \\
(s_{rr}^{n+\frac{1}{2}}, s_{zz}^{n+\frac{1}{2}}, s_{rz}^{n+\frac{1}{2}}) &= \Upsilon(s_{rr}^{n+\frac{1}{2}}, s_{zz}^{n+\frac{1}{2}} s_{rz}^{n+\frac{1}{2}}, Y^0)
\end{aligned}
\tag{49}
$$

更新坐标

$$
\begin{aligned}
r^{n+\frac{1}{2}} &= r^n + \frac{\Delta t}{2}u^n \\
z^{n+\frac{1}{2}} &= z^n + \frac{\Delta t}{2}v^n
\end{aligned}
\tag{50}
$$

通过坐标 (r, z) 求解体积 V。

通过 V 求解 ρ:

$$
\rho = \frac{m}{V} \tag{51}
$$

ieupd_ correct：求解半点时刻能量 能量方程空间离散过程见公式 (45)，

$$
E^{n+\frac{1}{2}} = E^n + \boldsymbol{\sigma}^n : \boldsymbol{\epsilon}^n \frac{\Delta t}{2} \tag{52}
$$

Eqos: 通过 $p^{n+\frac{1}{2}} = p(\rho^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})$ 更新压力

Arfrc 计算节点力 $F^{n+\frac{1}{2}}$ 体力计算的具体公式见 (33):

$$
F^{n+\frac{1}{2}} = \sum_{i=1}^{4}\left(h_i^{n+\frac{1}{2}} + .5\frac{1}{2}[s_{rz}^{n+\frac{1}{2}}(r_{i+1}^{n+\frac{1}{2}} - r_i^{n+\frac{1}{2}}) - (\sigma_{rr}^{n+\frac{1}{2}} - q_i^{n+\frac{1}{2}})(z_{i+1}^{n+\frac{1}{2}} - z_i^{n+\frac{1}{2}})]\right) \tag{53}
$$

质量可以通过质量守恒定律 (2) 求得:

$$
\frac{dm}{dt} = 0 \tag{54}
$$

rstres: 更新应力旋转项：

$$
\begin{aligned}
\sigma_{rr}^{*} &= \sigma_{rr}^{n} + s_{rz}^{n+\frac{1}{2}}(u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}})\Delta t \\
\sigma_{zz}^{*} &= \sigma_{zz}^{n} + s_{rz}^{n+\frac{1}{2}}(u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}})\Delta t \\
s_{rz}^{*} &= s_{rz}^{n} + \frac{s_{zz}^{n+\frac{1}{2}} - s_{rr}^{n+\frac{1}{2}}}{2}(u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}})\Delta t
\end{aligned}
\tag{55}
$$

fe2d10：更新偏应力：

$$
\begin{aligned}
& s_{rr}^{n+1}=\sigma_{rr}^{*}+p^{n}+2\mu\left(u_{r}^{n+\frac{1}{2}}-\frac{1}{3}\nabla(u^{n+\frac{1}{2}},v^{n+\frac{1}{2}})\right)\Delta t \\
& s_{zz}^{n+1}=\sigma_{zz}^{*}+p^{n}+2\mu\left(v_{z}^{n+\frac{1}{2}}-\frac{1}{3}\nabla(u^{n+\frac{1}{2}},v^{n+\frac{1}{2}})\right)\Delta t \\
& s_{rr}^{n+1}=s_{rz}^{*}+\mu(u_{z}^{n+\frac{1}{2}}+v_{r}^{n+\frac{1}{2}})\Delta t \\
& (s_{rr}^{n1},s_{zz}^{n+1},s_{rz}^{n+1})=\Upsilon(s_{rr}^{n+1},s_{zz}^{n+1}s_{rz}^{n+1},Y^{0})
\end{aligned}
\tag{56}
$$

更新加速度 $\overrightarrow{a}=\overrightarrow{F}/m$

update_cc：

$$
\begin{aligned}
(u,v)^{n+1} &= (u,v)^{n}+\Delta t(a_r,a_z)^{n+\frac{1}{2}} \\
(u,v)^{n+\frac{1}{2}} &= \frac{(u,v)^{n}+(u,v)^{n+1}}{2} \\
(r,z)^{n+1} &= (r,z)^{n}+\Delta t(u,v)^{n+\frac{1}{2}}
\end{aligned}
\tag{57}
$$

ieupd_ correct：求解半点时刻能量 能量方程空间离散过程见公式（45），

$$
E^{n+1}=E^{n}+\boldsymbol{\sigma}^{n+\frac{1}{2}}:\boldsymbol{\epsilon}^{n+\frac{1}{2}}\Delta t \tag{58}
$$

Eqos: 通过 $p^{n+1}=p(\rho^{n+1},e^{n+1})$ 更新压力