

samba 公式推导

April 2, 2018

二维控制方程

运动方程:

$$\frac{d(r, z)}{dt} = (u, v) \quad (1)$$

质量守恒:

$$\frac{dm}{dt} = 0 \quad (2)$$

动量守恒方程:

$$\begin{aligned} \rho \frac{du}{dt} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \\ \rho \frac{dv}{dt} &= \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} \end{aligned} \quad (3)$$

能量守恒方程:

$$\begin{aligned} \rho \frac{de}{dt} &= \sigma_{rr} \frac{\partial u}{\partial r} + \sigma_{zz} \frac{\partial v}{\partial z} + s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \\ &= -p \frac{dV}{dt} + s_{rr} \frac{\partial u}{\partial r} + s_{zz} \frac{\partial v}{\partial z} + s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \end{aligned} \quad (4)$$

本构方程:

$$\begin{aligned} \frac{ds_{rr}}{dt} &= 2\mu \left(\frac{\partial u}{\partial r} - \frac{1}{3} \nabla \cdot \vec{u} \right) + s_{rz} \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} \right) \\ \frac{ds_{zz}}{dt} &= 2\mu \left(\frac{\partial v}{\partial z} - \frac{1}{3} \nabla \cdot \vec{u} \right) - s_{rz} \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} \right) \\ \frac{ds_{rz}}{dt} &= \mu \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} \right) + \frac{s_{rr} - s_{zz}}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} \right) \end{aligned} \quad (5)$$

Von Mises 屈服条件:

$$\frac{3}{2} (s_{rr}^2 + s_{zz}^2 + s_{\theta\theta}^2 + 2s_{xy}^2) \leq (Y^0)^2 \quad (6)$$

柱坐标下控制方程

在正交曲线坐标系 $(\xi_\alpha, \xi_\beta, \xi_\gamma)$ 中, 存在坐标变换

$$x_i = x_i(\xi_\alpha, \xi_\beta, \xi_\gamma), \quad i = 1, 2, 3 \quad (7)$$

由于

$$\frac{\partial(x_1, x_2, x_3)}{\partial(\xi_\alpha, \xi_\beta, \xi_\gamma)} \neq 0 \quad (8)$$

因此可以解出

$$\xi_\nu = \xi_\nu(x_1, x_2, x_3), \quad \nu = 1, 2, 3 \quad (9)$$

定义 Lamé 系数

$$h_\nu = \left[\sum_{i=1}^3 \left(\frac{\partial x_i}{\partial \xi_\nu} \right)^2 \right]^{\frac{1}{2}}, \quad \nu = \alpha, \beta, \gamma \quad (10)$$

可以推出曲线坐标系下标量 ϕ 的梯度表达式

$$\nabla \phi = \frac{1}{h_\alpha} \frac{\partial \phi}{\partial \xi_\alpha} \mathbf{j}_\alpha + \frac{1}{h_\beta} \frac{\partial \phi}{\partial \xi_\beta} \mathbf{j}_\beta + \frac{1}{h_\gamma} \frac{\partial \phi}{\partial \xi_\gamma} \mathbf{j}_\gamma \quad (11)$$

对于向量函数 \mathbf{f} 旋度表达式为

$$\begin{aligned} \nabla \times \mathbf{f} = & \frac{1}{h_\beta h_\gamma} \left[\frac{\partial}{\partial \beta} (h_\gamma f_\gamma) - \frac{\partial}{\partial \gamma} (h_\beta f_\beta) \right] \mathbf{j}_\alpha \\ & + \frac{1}{h_\gamma h_\alpha} \left[\frac{\partial}{\partial \gamma} (h_\alpha f_\alpha) - \frac{\partial}{\partial \alpha} (h_\gamma f_\gamma) \right] \mathbf{j}_\beta \\ & + \frac{1}{h_\alpha h_\beta} \left[\frac{\partial}{\partial \alpha} (h_\beta f_\beta) - \frac{\partial}{\partial \beta} (h_\alpha f_\alpha) \right] \mathbf{j}_\gamma \end{aligned} \quad (12)$$

散度表达式

$$\nabla \cdot \mathbf{f} = \frac{1}{h_\alpha h_\beta h_\gamma} \left[\frac{\partial}{\partial \xi_\alpha} (f_\alpha h_\beta h_\gamma) + \frac{\partial}{\partial \xi_\beta} (f_\beta h_\gamma h_\alpha) + \frac{\partial}{\partial \xi_\gamma} (f_\gamma h_\alpha h_\beta) \right] \quad (13)$$

对于曲线坐标系中的张量 \mathbf{T}

$$\mathbf{T} = \begin{pmatrix} T_{\alpha\alpha} & T_{\alpha\beta} & T_{\alpha\gamma} \\ T_{\beta\alpha} & T_{\beta\beta} & T_{\beta\gamma} \\ T_{\gamma\alpha} & T_{\gamma\beta} & T_{\gamma\gamma} \end{pmatrix} \quad (14)$$

散度三个分量为

$$\begin{aligned} (\nabla \cdot \mathbf{T})_\alpha = & \frac{1}{h_\alpha h_\beta h_\gamma} \left[\frac{\partial T_{\alpha\alpha} h_\beta h_\gamma}{\partial \xi_\alpha} + \frac{\partial T_{\beta\alpha} h_\gamma h_\alpha}{\partial \xi_\beta} + \frac{\partial T_{\gamma\alpha} h_\alpha h_\beta}{\partial \xi_\gamma} \right] \\ & + \frac{T_{\alpha\beta}}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \xi_\beta} + \frac{T_{\alpha\gamma}}{h_\alpha h_\gamma} \frac{\partial h_\alpha}{\partial \xi_\gamma} - \frac{T_{\beta\beta}}{h_\alpha h_\beta} \frac{\partial h_\beta}{\partial \xi_\alpha} - \frac{T_{\gamma\gamma}}{h_\alpha h_\gamma} \frac{\partial h_\gamma}{\partial \xi_\alpha} \end{aligned} \quad (15)$$

$$\begin{aligned}
(\nabla \cdot \mathbf{T})_\beta &= \frac{1}{h_\alpha h_\beta h_\gamma} \left[\frac{\partial T_{\alpha\beta} h_\beta h_\gamma}{\partial \xi_\alpha} + \frac{\partial T_{\beta\beta} h_\gamma h_\alpha}{\partial \xi_\beta} + \frac{\partial T_{\gamma\beta} h_\alpha h_\beta}{\partial \xi_\gamma} \right] \\
&+ \frac{T_{\beta\alpha}}{h_\alpha h_\beta} \frac{\partial h_\beta}{\partial \xi_\alpha} + \frac{T_{\beta\gamma}}{h_\beta h_\gamma} \frac{\partial h_\beta}{\partial \xi_\gamma} - \frac{T_{\alpha\alpha}}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \xi_\beta} - \frac{T_{\gamma\gamma}}{h_\beta h_\gamma} \frac{\partial h_\gamma}{\partial \xi_\beta}
\end{aligned} \tag{16}$$

$$\begin{aligned}
(\nabla \cdot \mathbf{T})_\gamma &= \frac{1}{h_\alpha h_\beta h_\gamma} \left[\frac{\partial T_{\alpha\gamma} h_\beta h_\gamma}{\partial \xi_\alpha} + \frac{\partial T_{\beta\gamma} h_\gamma h_\alpha}{\partial \xi_\beta} + \frac{\partial T_{\gamma\gamma} h_\alpha h_\beta}{\partial \xi_\gamma} \right] \\
&+ \frac{T_{\gamma\alpha}}{h_\gamma h_\alpha} \frac{\partial h_\gamma}{\partial \xi_\alpha} + \frac{T_{\gamma\beta}}{h_\beta h_\gamma} \frac{\partial h_\gamma}{\partial \xi_\beta} - \frac{T_{\alpha\alpha}}{h_\alpha h_\gamma} \frac{\partial h_\alpha}{\partial \xi_\gamma} - \frac{T_{\beta\beta}}{h_\beta h_\gamma} \frac{\partial h_\beta}{\partial \xi_\gamma}
\end{aligned} \tag{17}$$

具体求解过程见附录4。

离散方法

时间离散：预估校正方法

对于如下形式方程

$$\frac{\partial U}{\partial t} = F(U) \tag{18}$$

预估步：

$$\frac{U^{n+\frac{1}{2}} - U^n}{\frac{1}{2}\Delta t} = F(U^n) \tag{19}$$

校正步：

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+\frac{1}{2}}) \tag{20}$$

空间离散方法

动量方程离散

1. 有限体积离散

采用如图2所示非结构四边形单元，Wilkins 在 1963 年首先通过有限体积方法得到 Wilkins 格式 [2]。点 i , iii , v , vii 分别为线 O1, O2, O3, O4 的中点， ii , iv , vi , $viii$ 分别为区域 I-IV 的几何中心。

对于动量离散方程3，在八边形面回路 $\Omega = i \rightarrow ii \rightarrow \dots \rightarrow vii \rightarrow i$ 上积分：

$$\int_{\Omega} \rho \frac{du}{dt} d\Omega = \int_{\Omega} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega \tag{21}$$

$$\int_{\Omega} \rho \frac{dv}{dt} d\Omega = \int_{\Omega} \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial s_{rz}}{\partial r} d\Omega \tag{22}$$

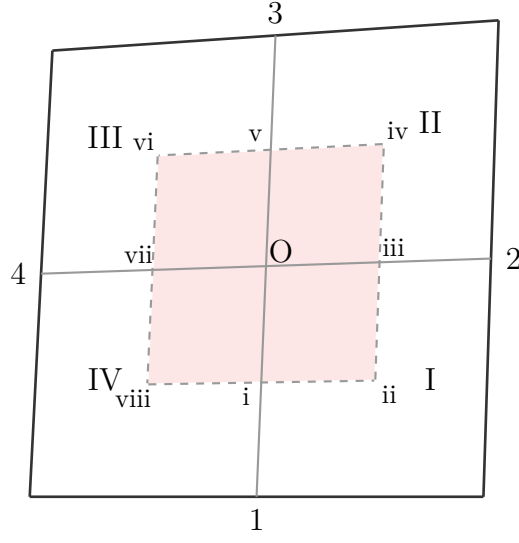


图 1: 有限体积方法求解动量方程控制单元示意图

以方程21为例，左侧积分可以近似为

$$\int_{\Omega} \rho \frac{du}{dt} d\Omega = M \frac{du_o}{dt} \quad (23)$$

其中

$$M = \frac{1}{4}(A_I \rho_I + A_{II} \rho_{II} + A_{III} \rho_{III} + A_{IV} \rho_{IV}) \quad (24)$$

右侧积分根据 Green 公式：

$$\int_{\Omega} \text{grad} f d\Omega = \oint_{\partial\Omega} f \vec{n} dl \quad (25)$$

有

$$\int_{\Omega} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} d\Omega = - \oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr \quad (26)$$

积分

$$\int_{vii}^i \sigma_{rr} dz \approx \frac{1}{2} \sigma_{rrI} (z_2 - z_1) \quad (27)$$

同理可得

$$- \oint_{\partial\Omega} \sigma_{rr} dz + \oint_{\partial\Omega} s_{rz} dr = \frac{1}{2} \sum_{i=1}^4 s_{rzi} (r_{i+1} - r_i) - \sigma_{rri} (z_{i+1} - z_i) \quad (28)$$

得到 wilkins 格式：

$$\frac{du_o}{dt} = \frac{1}{2M} \left[\sum_{i=1}^4 s_{rzi} (r_{i+1} - r_i) - \sigma_{rri} (z_{i+1} - z_i) \right] \quad (29)$$

2: 有限元离散

该格式同样可以通过有限元方法得出 (Lascaux 1973 年) [1]。控制体同样取如图2, 以 IV 区为例, 可以化作有限元的单元积分:

$$\int_{\Omega_{IV}} \rho \frac{du}{dt} \phi_O d\Omega = \int_{\Omega_{IV}} \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \right) \phi_O d\Omega \quad (30)$$

其中 ϕ_o 在 IV 区为

$$\phi_O = \phi_c = \frac{1}{4}(1 + \xi)(1 + \eta) \quad (31)$$

为基函数, $i = a, b, c, d$ 其中 $(\xi, \eta) = (-1, -1), (1, -1), (1, 1), (-1, 1)$

$$\begin{aligned} u &= \sum_{i=a}^d u_i \phi_i \\ r &= \sum_{i=a}^d r_i \phi_i \\ z &= \sum_{i=a}^d z_i \phi_i \end{aligned} \quad (32)$$

公式30左侧可以化为:

$$\begin{aligned} \int_{\Omega_{IV}} \rho \phi_c \frac{du}{dt} d\Omega &= \frac{du}{dt} \int_{\Omega_{IV}} \rho \phi_c d\Omega \\ &= \frac{du}{dt} M \end{aligned} \quad (33)$$

其中

$$M = \int_{-1}^1 \int_{-1}^1 \rho \phi_c J d\xi d\eta \quad (34)$$

J 为 Jacobi 矩阵

$$\begin{aligned}
J &= \left\| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{array} \right\| \\
&= \frac{\partial r}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta} \frac{\partial z}{\partial \xi} \\
&= \frac{1}{16} \sum_{i=a}^d r_i \xi_i (1 + \eta_i \eta) \sum_{i=a}^d z_i \eta_i (1 + \xi_i \xi) \\
&\quad - \frac{1}{16} \sum_{i=a}^d r_i \eta_i (1 + \xi_i \eta) \sum_{i=a}^d z_i \xi_i (1 + \eta_i \eta) \\
&= \frac{1}{16} [(r_2 + r_3 - r_1 - r_4) + \eta(r_1 + r_3 - r_2 - r_4)] \\
&\quad [(z_3 + z_4 - z_2 - z_1) + \xi(z_1 + z_3 - z_2 - z_4)] \\
&\quad - \frac{1}{16} [(z_2 + z_3 - z_1 - z_4) + \eta(z_1 + z_3 - z_2 - z_4)] \\
&\quad [(r_3 + r_4 - r_2 - r_1) + \xi(r_1 + r_3 - r_2 - r_4)] \\
&= \frac{1}{8} [(z_1 - z_3)(r_4 - r_2) + (z_2 - z_4)(r_1 - r_3) \\
&\quad + (\xi - \eta)(r_4 - r_3)(z_2 - z_1) + (\xi - \eta)(r_1 - r_2)(z_4 - z_3)] \\
&\hspace{15em} (35)
\end{aligned}$$

公式 (33) 可以化为:

$$\begin{aligned}
&\frac{du_O}{dt} \int_{-1}^1 \int_{-1}^1 \rho \phi_c J d\xi d\eta \\
&= \frac{1}{32} \frac{du_O}{dt} \int_{-1}^1 \int_{-1}^1 \rho (1 + \xi)(1 + \eta) [B + C(\xi - \eta)] d\xi d\eta
\end{aligned} \tag{36}$$

其中 $B = (z_1 - z_3)(r_4 - r_2) + (z_2 - z_4)(r_1 - r_3)$, $C = (r_4 - r_3)(z_2 - z_1) + (r_1 - r_2)(z_4 - z_3)$ 公式 (36) 经过积分得到

$$\begin{aligned}
M &= \frac{1}{8} \rho_{IV} B \\
&= \frac{1}{8} \rho_{IV} [(z_1 - z_3)(r_4 - r_2) + (z_2 - z_4)(r_1 - r_3)] = \frac{1}{4} \rho_{IV} A_{IV}
\end{aligned} \tag{37}$$

右侧以第一项为例：

$$\begin{aligned}
& \int_{\Omega_{IV}} \frac{\partial \sigma_{rr}}{\partial r} \phi_c d\Omega \\
&= \int_{\Omega_{IV}} \frac{\partial \sigma_{rr} \phi_c}{\partial r} d\Omega - \int_{\Omega_{IV}} \frac{\partial \phi_c}{\partial r} \sigma_{rr} d\Omega \\
&= -\sigma_{rr} \int_{-1}^1 \int_{-1}^1 \frac{1}{J} \frac{\partial(z, \phi_c)}{\partial(\xi, \eta)} J d\xi d\eta \\
&= -\sigma_{rr} \int_{-1}^1 \int_{-1}^1 \frac{\partial(z, \phi_c)}{\partial(\xi, \eta)} d\xi d\eta \\
&= -\sigma_{rr} \int_{-1}^1 \int_{-1}^1 \frac{\partial z}{\partial \xi} \frac{\partial \phi_c}{\partial \eta} - \frac{\partial z}{\partial \eta} \frac{\partial \phi_c}{\partial \xi} d\xi d\eta \\
&= -\frac{1}{4} \sigma_{rr} \int_{-1}^1 \int_{-1}^1 \sum_{i=a}^d z_i \xi_i (1 + \eta_i \eta) \frac{\partial \phi_c}{\partial \eta} - \sum_{i=a}^d z_i \eta_i (1 + \xi_i \xi) \frac{\partial \phi_c}{\partial \xi} d\xi d\eta
\end{aligned} \tag{38}$$

将 ϕ_c 带入

$$\begin{aligned}
& -\frac{1}{16} \sigma_{rr} \int_{-1}^1 \int_{-1}^1 \sum_{i=a}^d z_i \xi_i (1 + \eta_i \eta) (1 + \xi) - \sum_{i=a}^d z_i \eta_i (1 + \xi_i \xi) (1 + \eta) d\xi d\eta \\
&= -\frac{1}{4} \sigma_{rr} \left(\sum_{i=a}^d z_i \xi_i - \sum_{i=a}^d z_i \eta_i \right) \\
&= \frac{1}{2} \sigma_{rr} (z_d - z_b) \\
&= \frac{1}{2} \sigma_{rr} (z_4 - z_1)
\end{aligned} \tag{39}$$

同理 IV 区右侧第二项可以化为：

$$\int_{\Omega_{IV}} \frac{\partial s_{rz}}{\partial z} \phi_c d\Omega = \frac{1}{2} s_{rzO} (r_1 - r_4) \tag{40}$$

在第 I 区， O 点为第 d 节点，因此 $\phi_d = 1/4(1 - \xi)(1 + \eta)$ 其他求解过程和 IV 区相同，最后右侧积分变为：

$$\begin{aligned}
& \int_{\Omega_I} \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} \right) \phi_d d\Omega = \frac{1}{2} s_{rzO} (r_4 - r_1) \\
&= \frac{1}{2} \sigma_{rrO} (z_a - z_c) + \frac{1}{2} s_{rzO} (z_c - z_a) \\
&= \frac{1}{2} \sigma_{rrO} (z_1 - z_2) + \frac{1}{2} s_{rzO} (z_2 - z_1)
\end{aligned} \tag{41}$$

将 $I-IV$ 区左右项加和可得：

$$\frac{du_o}{dt} = \frac{1}{2M} \left[\sum_{i=1}^4 s_{rzO} (r_{i+1} - r_i) - \sigma_{rrO} (z_{i+1} - z_i) \right] \tag{42}$$

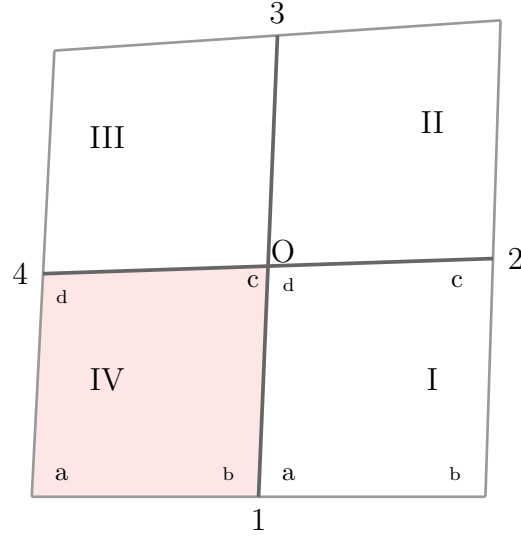


图 2: 有限元求解动量方程控制单元示意图

其中

$$M = \frac{1}{4}(A_I \rho_I + A_{II} \rho_{II} + A_{III} \rho_{III} + A_{IV} \rho_{IV}) \quad (43)$$

和有限体积方法求得的公式 (29) 相同。

在计算中需要在动量方程中增加沙漏粘性 h 和人工粘性 q , 公式 (29) 变为:

$$M \frac{du}{dt} = \sum_{i=1}^4 \left(h_i + \frac{1}{2} [s_{rz}(r_{i+1} - r_i) - (\sigma_{rr} - q_i)(z_{i+1} - z_i)] \right) \quad (44)$$

能量方程离散

能量守恒方程:

$$\begin{aligned} \rho \frac{de}{dt} &= \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \\ &= \sigma_{rr} \frac{\partial u}{\partial r} + \sigma_{zz} \frac{\partial v}{\partial z} + \sigma_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \\ &= -p \nabla \cdot \vec{u} + s_{rr} \frac{\partial u}{\partial r} + s_{zz} \frac{\partial v}{\partial z} + s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \end{aligned} \quad (45)$$

通过质量守恒方程:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad (46)$$

有

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0 \quad (47)$$

根据全微分关系

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{u} = 0 \quad (48)$$

所以

$$\begin{aligned} \nabla \cdot \vec{u} &= -\frac{1}{\rho} \frac{d\rho}{dt} \\ &= -\frac{V}{m} \frac{d^m}{dt} \\ &= \frac{1}{V} \frac{dV}{dt} \end{aligned} \quad (49)$$

所以方程 (45) 可以化为

$$\rho \frac{de}{dt} = -\frac{p}{V} \frac{dV}{dt} + s_{rr} \frac{\partial u}{\partial r} + s_{zz} \frac{\partial v}{\partial z} + s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \quad (50)$$

既

$$\frac{dE}{dt} = -p \frac{dV}{dt} + V s_{rr} \frac{\partial u}{\partial r} + V s_{zz} \frac{\partial v}{\partial z} + V s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) \quad (51)$$

其中 $\sigma = -p\mathbf{I} + \mathbf{s}$ 。

采用有限体积方法，在图3所示整个单元上进行积分有，其中除速度外其它量均集中在单元中心 O ，因此有：

$$\int_{\Omega} \frac{dE}{dt} d\Omega = \int_{\Omega} -p \frac{dV}{dt} + V s_{rr} \frac{\partial u}{\partial r} + V s_{zz} \frac{\partial v}{\partial z} + V s_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) d\Omega \quad (52)$$

既

$$\frac{dE}{dt} = -p \frac{dV}{dt} + \frac{1}{A} V s_{rr} \int_{\Omega} \frac{\partial u}{\partial r} d\Omega + V s_{zz} \frac{1}{A} \int_{\Omega} \frac{\partial v}{\partial z} d\Omega + V s_{rz} \frac{1}{A} \int_{\Omega} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) d\Omega \quad (53)$$

以其中

$$\frac{1}{A} \int_{\Omega} \frac{\partial u}{\partial r} d\Omega \quad (54)$$

的离散为例：

$$\begin{aligned} \frac{1}{A} \int_{\Omega} \frac{\partial u}{\partial r} d\Omega &= -\frac{1}{A} \oint_{\partial\omega} u dz \\ &\approx \frac{1}{A} \left(\frac{1}{2} (u_1 + u_2) (z_2 - z_1) + \frac{1}{2} (u_2 + u_3) (z_3 - z_2) \right. \\ &\quad \left. + (u_3 + u_4) (z_4 - z_3) + \frac{1}{2} (u_4 + u_1) (u_1 - u_4) \right) \\ &= -\frac{1}{A} ((u_1 - u_3) (z_2 - z_4) + (u_2 - u_4) (z_3 - z_1)) \end{aligned} \quad (55)$$

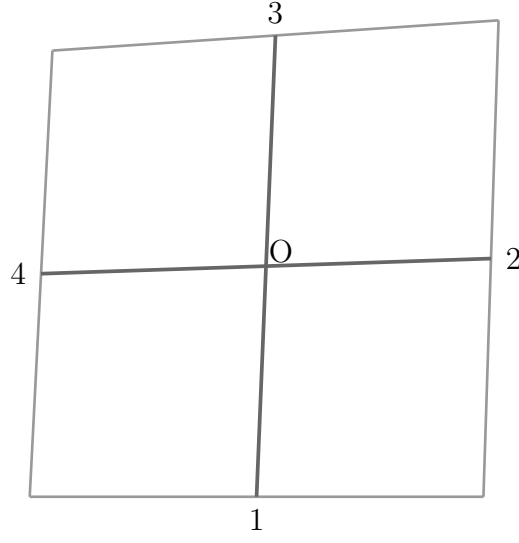


图 3: 能量方程离散控制单元示意图

并将其记作 \bar{u}_r 。

能量方程空间离散为

$$\frac{dE}{dt} = -p \frac{dV}{dt} + V \mathbf{s} \cdot \boldsymbol{\varepsilon} \quad (56)$$

其中

$$\mathbf{s} \cdot \boldsymbol{\varepsilon} = s_{rr} \bar{u}_r + s_{zz} \bar{v}_z + s_{rz} (\bar{u}_z + \bar{v}_r) \quad (57)$$

本构方程离散

和能量方程离散类似，以 $\frac{ds_{rr}}{dt}$ 为例

$$\frac{ds_{rr}}{dt} = 2\mu \left(\bar{u}_r^n - \frac{1}{3}(\bar{u}_r^n + \bar{v}_z^n) \right) + s_{rz}^n (\bar{u}_z^n - \bar{v}_r^n) \quad (58)$$

求解流程

Feahgq: 计算沙漏粘性 h

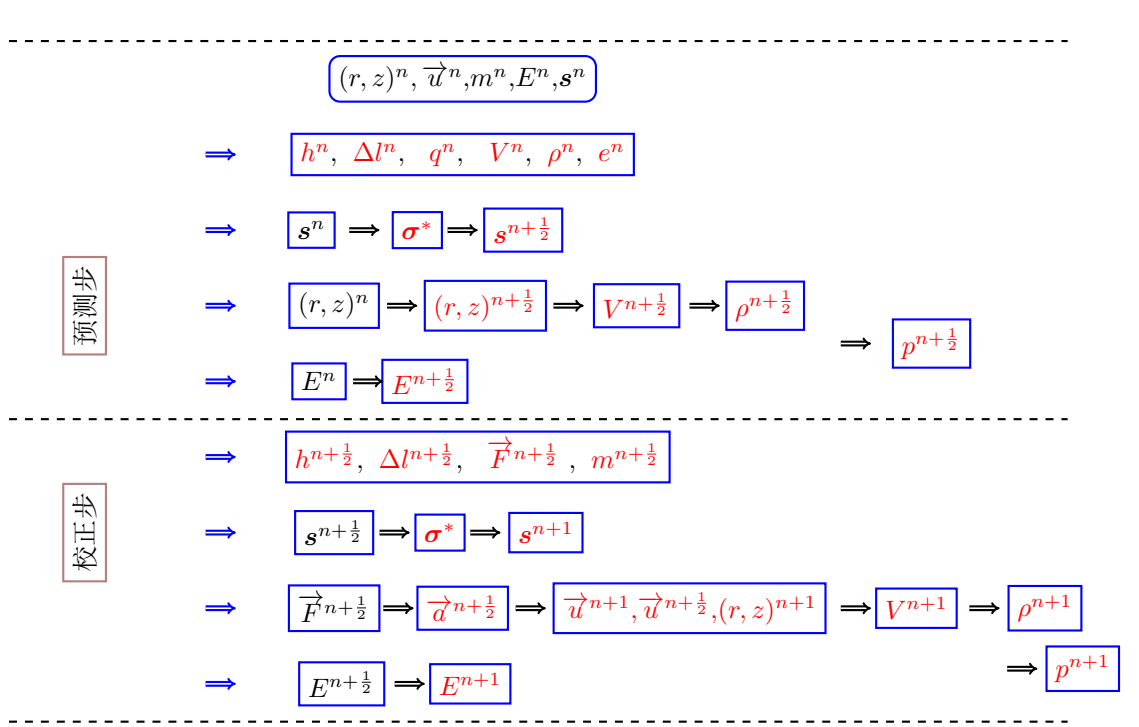
fe2al: 计算特征长度 Δl

Bukq: 计算人工粘性 q

体积 V : 通过 (r, z) 坐标位置，计算单元体积

密度 ρ : 通过 $\rho = \frac{m}{V}$ 计算单元密度

比内能 e : 通过 $e = \frac{E}{m}$ 计算



rstres: 更新应力旋转项:

$$\begin{aligned}
 \sigma_{rr}^* &= \sigma_{rr}^n + s_{rz}^n (u_z^n - v_r^n) \frac{\Delta t}{2} \\
 \sigma_{zz}^* &= \sigma_{zz}^n + s_{rz}^n (u_z^n - v_r^n) \frac{\Delta t}{2} \\
 s_{rz}^* &= s_{rz}^n + \frac{s_{zz}^n - s_{rr}^n}{2} (u_z^n - v_r^n) \frac{\Delta t}{2}
 \end{aligned} \tag{59}$$

fe2d10: 更新偏应力:

$$\begin{aligned}
 s_{rr}^{n+\frac{1}{2}} &= \sigma_{rr}^* + p^n + 2\mu \left(u_r^n - \frac{1}{3} \nabla(u^n, v^n) \right) \frac{\Delta t}{2} \\
 s_{zz}^{n+\frac{1}{2}} &= \sigma_{zz}^* + p^n + 2\mu \left(v_z^n - \frac{1}{3} \nabla(u^n, v^n) \right) \frac{\Delta t}{2} \\
 s_{rr}^{n+\frac{1}{2}} &= s_{rz}^* + \mu (u_z^n + v_r^n) \frac{\Delta t}{2} \\
 (s_{rr}^{n+\frac{1}{2}}, s_{zz}^{n+\frac{1}{2}}, s_{rz}^{n+\frac{1}{2}}) &= \Upsilon(s_{rr}^{n+\frac{1}{2}}, s_{zz}^{n+\frac{1}{2}}, s_{rz}^{n+\frac{1}{2}}, Y^0)
 \end{aligned} \tag{60}$$

更新坐标

$$\begin{aligned}
 r^{n+\frac{1}{2}} &= r^n + \frac{\Delta t}{2} u^n \\
 z^{n+\frac{1}{2}} &= z^n + \frac{\Delta t}{2} v^n
 \end{aligned} \tag{61}$$

通过坐标 (r, z) 求解体积 V 。

通过 V 求解 ρ :

$$\rho = \frac{m}{V} \quad (62)$$

ieupd_correct : 求解半点时刻能量 能量方程空间离散过程
见公式 (56) ,

$$E^{n+\frac{1}{2}} = E^n + \sigma^n : \epsilon^n \frac{\Delta t}{2} \quad (63)$$

Eqos: 通过 $p^{n+\frac{1}{2}} = p(\rho^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})$ 更新压力

Arfr 计算节点力 $F^{n+\frac{1}{2}}$ 体力计算的具体公式见 (44):

$$F^{n+\frac{1}{2}} = \sum_{i=1}^4 \left(h_i^{n+\frac{1}{2}} + .5 \frac{1}{2} [s_{rz}^{n+\frac{1}{2}} (r_{i+1}^{n+\frac{1}{2}} - r_i^{n+\frac{1}{2}}) - (\sigma_{rr}^{n+\frac{1}{2}} - q_i^{n+\frac{1}{2}})(z_{i+1}^{n+\frac{1}{2}} - z_i^{n+\frac{1}{2}})] \right) \quad (64)$$

质量可以通过质量守恒定律 (2) 求得:

$$\frac{dm}{dt} = 0 \quad (65)$$

rstres: 更新应力旋转项:

$$\begin{aligned} \sigma_{rr}^* &= \sigma_{rr}^n + s_{rz}^{n+\frac{1}{2}} (u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}}) \Delta t \\ \sigma_{zz}^* &= \sigma_{zz}^n + s_{rz}^{n+\frac{1}{2}} (u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}}) \Delta t \\ s_{rz}^* &= s_{rz}^n + \frac{s_{zz}^{n+\frac{1}{2}} - s_{rr}^{n+\frac{1}{2}}}{2} (u_z^{n+\frac{1}{2}} - v_r^{n+\frac{1}{2}}) \Delta t \end{aligned} \quad (66)$$

fe2d10 : 更新偏应力:

$$\begin{aligned} s_{rr}^{n+1} &= \sigma_{rr}^* + p^n + 2\mu \left(u_r^{n+\frac{1}{2}} - \frac{1}{3} \nabla(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) \right) \Delta t \\ s_{zz}^{n+1} &= \sigma_{zz}^* + p^n + 2\mu \left(v_z^{n+\frac{1}{2}} - \frac{1}{3} \nabla(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) \right) \Delta t \\ s_{rr}^{n+1} &= s_{rz}^* + \mu (u_z^{n+\frac{1}{2}} + v_r^{n+\frac{1}{2}}) \Delta t \end{aligned} \quad (67)$$

$$(s_{rr}^{n+1}, s_{zz}^{n+1}, s_{rz}^{n+1}) = \Upsilon(s_{rr}^{n+1}, s_{zz}^{n+1}, s_{rz}^{n+1}, Y^0)$$

更新加速度 $\vec{a} = \vec{F}/m$

update_cc :

$$\begin{aligned} (u, v)^{n+1} &= (u, v)^n + \Delta t (a_r, a_z)^{n+\frac{1}{2}} \\ (u, v)^{n+\frac{1}{2}} &= \frac{(u, v)^n + (u, v)^{n+1}}{2} \\ (r, z)^{n+1} &= (r, z)^n + \Delta t (u, v)^{n+\frac{1}{2}} \end{aligned} \quad (68)$$

ieupd__correct : 求解半点时刻能量 能量方程空间离散过程
见公式 (56) ,

$$E^{n+1} = E^n + \sigma^{n+\frac{1}{2}} : \epsilon^{n+\frac{1}{2}} \Delta t \quad (69)$$

Eqs: 通过 $p^{n+1} = p(\rho^{n+1}, e^{n+1})$ 更新压力

附录 I

Lame 系数

在沿 ξ_α 坐标曲线上任取两点 $M(x, y, z)$ 和 $N(x + dx, y + dy, z + dz)$, 那么 MN 沿 ξ_α 弧微分为 $ds_\alpha = \sqrt{dx^2 + dy^2 + dz^2}$, 由于曲线坐标系三个方向正交, 在 ξ_α 上有 $d\xi_\beta = 0, d\xi_\gamma = 0$, 因此有 $dx = \frac{\partial x}{\partial \xi_\alpha} d\xi_\alpha$, 同理 $dy = \frac{\partial y}{\partial \xi_\alpha} d\xi_\alpha$ 和 $dz = \frac{\partial z}{\partial \xi_\alpha} d\xi_\alpha$, 即

$$ds_\alpha = \sqrt{\left(\frac{\partial x}{\partial \xi_\alpha}\right)^2 + \left(\frac{\partial y}{\partial \xi_\alpha}\right)^2 + \left(\frac{\partial z}{\partial \xi_\alpha}\right)^2} d\xi_\alpha \quad (70)$$

令

$$h_\alpha = \sqrt{\left(\frac{\partial x}{\partial \xi_\alpha}\right)^2 + \left(\frac{\partial y}{\partial \xi_\alpha}\right)^2 + \left(\frac{\partial z}{\partial \xi_\alpha}\right)^2} \quad (71)$$

和

$$h_\beta = \sqrt{\left(\frac{\partial x}{\partial \xi_\beta}\right)^2 + \left(\frac{\partial y}{\partial \xi_\beta}\right)^2 + \left(\frac{\partial z}{\partial \xi_\beta}\right)^2} \quad (72)$$

$$h_\gamma = \sqrt{\left(\frac{\partial x}{\partial \xi_\gamma}\right)^2 + \left(\frac{\partial y}{\partial \xi_\gamma}\right)^2 + \left(\frac{\partial z}{\partial \xi_\gamma}\right)^2} \quad (73)$$

为 Lamé 系数。

$$\begin{aligned} ds_\alpha &= h_\alpha d\xi_\alpha \\ ds_\beta &= h_\beta d\xi_\beta \\ ds_\gamma &= h_\gamma d\xi_\gamma \end{aligned} \quad (74)$$

因此有

$$\frac{\partial \mathbf{r}}{\partial \xi_\nu} = \frac{\partial \mathbf{r}}{\partial \xi_\nu} \frac{ds_\nu}{d\xi_\nu} = h_\nu \mathbf{e}_\nu \quad (75)$$

梯度公式

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \quad (76)$$

又有

$$\begin{aligned} d\mathbf{r} &= dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \\ d\phi &= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \end{aligned} \quad (77)$$

可以得到

参考文献

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