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HLLC-EP 构造过程

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HLLC-EP

一维控制方程为

$$
\begin{cases}
\partial_t \rho + \partial_x(\rho u) = 0 \\
\partial_t(\rho u) + \partial_x(\rho u^2 + p - s_{xx}) = 0 \\
\partial_t(\rho E) + \partial_x([\rho E + p - s_{xx}]u) = 0 \\
\partial_t s_{xx} + u\partial_x s_{xx} - \dfrac{4}{3}\partial_x u = 0
\end{cases}
\tag{1}
$$

状态方程为 $p = p(\rho, e)$。根据 Runkine-Hugoniot 关系有

$$
\begin{aligned}
F_L^* &= F_L + s_L(U_L^* - U_L) \\
F_R^* &= F_L^* + s^*(U_R^* - U_L^*) \\
F_R^* &= F_R + s_R(U_R^* - U_R)
\end{aligned}
\tag{2}
$$

左右密度和接触间断速度估计

先通过三波系统估计接触间断左右状态，从上式 (2) 中第一式的第一项有

$$\rho_L^* s^* = \rho_L u_L + s_L(\rho_L^* - \rho_L) \tag{3}$$

可得

$$\rho_L^*(s^* - s_L) = \rho_L(u_L - s_L) \tag{4}$$

从第一式的第二项有

$$\rho_L^* s^{*2} - \sigma_L^* = \rho_L u_L^2 - \sigma_L + s_L(\rho_L^* s^* - \rho_L u_L) \tag{5}$$

可得

$$\sigma_L^* = \sigma_L + (s_L - u_L)\rho_L u_L + \rho_L^* s^*(s^* - s_L) \tag{6}$$

将 Eq.(4) 带入可得

$$\sigma_L^* = \sigma_L - \rho_L(s_L - u_L)(s^* - u_L) \tag{7}$$

其中 $\sigma = -p + s_{xx}$ 且接触间断两侧 $\sigma_L^* = \sigma_R^* = \sigma^*$。
根据 Eq.(2) 的第三式第一二项同理可得

$$\sigma_R^* = \sigma_R - \rho_R(s_R - u_R)(s^* - u_R) \tag{8}$$

因此可以得到

$$s^* = \frac{\sigma_L - \sigma_R + \rho_L u_L(s_L - u_L) - \rho_R u_R(s_R - u_R)}{\rho_L(s_L - u_L) - \rho_R(s_R - u_R)} \tag{9}$$

通过 Eq.(4) 可以得到

$$\rho_L^* = \frac{\rho_L(u_L - s_L)}{s^* - s_L}, \rho_R^* = \frac{\rho_R(u_R - s_R)}{s^* - s_R} \tag{10}$$

由 $s^* > s_L$ 和 $s^* < s_R$ 可知

$$\begin{matrix} u_L > s^* & \text{if} & \rho_L^* > \rho_L \\ u_L < s^* & \text{if} & \rho_L^* < \rho_L \end{matrix}, \begin{matrix} u_R < s^* & \text{if} & \rho_R^* > \rho_R \\ u_R < s^* & \text{if} & \rho_R^* < \rho_R \end{matrix} \tag{11}$$

根据估计密度关系确定塑性波

根据 Eq.(1) 中密度守恒可以导出

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{\partial u}{\partial x}=0 \tag{12}$$

同样，根据第四项可得

$$\frac{ds_{xx}}{dt}=\frac{4}{3}\mu\frac{\partial u}{\partial x} \tag{13}$$

因此在屈服前满足关系

$$\frac{ds_{xx}}{dt}=-\frac{4}{3}\mu\frac{1}{\rho}\frac{d\rho}{dt} \tag{14}$$

即

$$s_{xx}-s_{xx0}=-\frac{4}{3}\mu(\ln(\rho)-\ln(\rho_0)) \tag{15}$$

由于波前状态为 ρ_L 和 s_{xxL} 可得

$$s_{xxL}^{*}=-\frac{4}{3}\mu\ln(\frac{\rho_L^{*}}{\rho_L})+s_{xxL} \tag{16}$$

该关系成立条件为 $|s_{xxL}|\leq\frac{2}{3}Y_0$。而又由于 s_{xxL}^{*} 同样需要满足屈服条件，有

$$\left|-\frac{4}{3}\mu\ln(\frac{\rho_L^{*}}{\rho_L})+s_{xxL}\right|=|s_{xxL}^{*}|\leq\frac{2}{3}Y_0 \tag{17}$$

确定塑性波

最后得到关系

$$s^*_{xxL} = \begin{cases} \frac{2}{3}Y_0 & \text{else if } -\frac{4}{3}\mu\ln(\frac{\rho^*_L}{\rho_L}) + s_{xxL} \geq \frac{2}{3}Y_0 \\ -\frac{2}{3}Y_0 & \text{else if } -\frac{4}{3}\mu\ln(\frac{\rho^*_L}{\rho_L}) + s_{xxL} \leq -\frac{2}{3}Y_0 \\ -\frac{4}{3}\mu\ln(\frac{\rho^*_L}{\rho_L}) + s_{xxL} & \text{else} \end{cases} \tag{18}$$

如果 $|s_{xxL}| \leq \frac{2}{3}Y_0$ 且 $\left|-\frac{4}{3}\mu\ln(\frac{\rho^*_L}{\rho_L}) + s_{xxL}\right| \geq \frac{2}{3}Y_0$，即存在弹性波又存在塑性波。

塑性波前后条件

以左侧为例，塑性波波前条件为

$$\rho_L, u_L, p_L, \sigma_{xxL} \tag{19}$$

根据 R-H 条件，有

$$\begin{aligned}
\hat{\rho}_L(\hat{u}_L - \hat{s}_L) &= \rho_L(u_L - \hat{s}_L) \\
\hat{\rho}_L\hat{u}_L(\hat{u}_L - \hat{s}_L) &= \rho_L u_L(u_L - \hat{s}_L) + \hat{\sigma}_L - \sigma_L \\
\hat{\rho}_L\hat{E}(\hat{u}_L - \hat{s}_L) &= \rho_L E_L(u_L - \hat{s}_L) + \hat{\sigma}_L\hat{u}_L - \sigma_L u_L
\end{aligned} \tag{20}$$

其中

$$\begin{aligned}
\hat{s}_{xxL} &= -\frac{2}{3}Y_0 \quad \text{if} \quad \rho_L^* > \rho_L \\
\hat{s}_{xxL} &= \frac{2}{3}Y_0 \quad \text{if} \quad \rho_L^* < \rho_L
\end{aligned} \tag{21}$$

$$\begin{aligned}
\hat{\rho}_L &= \rho_L \exp\left(\frac{Y_0}{2\mu} + \frac{3S_{xxL}}{4\mu}\right) \quad \text{if} \quad \rho_L^* > \rho_L \\
\hat{\rho}_L &= \rho_L \exp\left(-\frac{Y_0}{2\mu} + \frac{3S_{xxL}}{4\mu}\right) \quad \text{if} \quad \rho_L^* < \rho_L
\end{aligned} \tag{22}$$

未知量仅为波速 $\hat{s}_L$、波后压力 $\hat{p}_L$ 和波后速度 $\hat{u}_L$。

塑性波波后状态推导

下面为了推导方便用 $_2$ 表示波后条件用 $_1$ 表示波前条件，s 表示波速根据 Eq.(20).1 有

$$s = \frac{\rho_2 u_2 - \rho_1 u_1}{\rho_2 - \rho_1} \tag{23}$$

和

$$u_1 - s = \frac{(u_1 - u_2)\rho_2}{\rho_2 - \rho_1} \tag{24}$$

将公式 Eq.(20).1 带入公式 Eq.(20).2 可得

$$\rho_1(u_2 - u_1)(u_1 - s) = \sigma_2 - \sigma_1 \tag{25}$$

继续将 Eq.(24) 带入上式，有

$$-t(u_2 - u_1)^2 = \sigma_2 - \sigma_1 \tag{26}$$

其中 $t = \frac{\rho_1 \rho_2}{\rho_2 - \rho_1}$。同理 Eq.(20).2 可以化为

$$t(u_1 - u_2)(E_2 - E_1) = \sigma_2 u_2 - \sigma_1 u_1 \tag{27}$$

又知道 $E = e + \frac{1}{2}u^2$，有

$$e_2 - e_1 = -\frac{\sigma_1 + \sigma_2}{2t} \tag{28}$$

根据状态方程

$$e = c_0 p - c_1 f(\rho) \tag{29}$$

其中 $c_0 = \frac{1}{\rho_0 \Gamma_0}$，$c_1 = \frac{a_0^2}{\Gamma_0}$ 而 $\sigma = -p + s_{xx}$，所以可解得

$$p_2 = \frac{2t(c_1 f(\rho_2) + e_1) - (\sigma_1 + s_{xx2})}{2tc_0 - 1} \tag{30}$$

塑性波波后状态推导

可以求得 $\sigma_2 = -p_2 + s_{xx2}$ 根据公式 Eq.(26) 有

$$(u_2 - u_1)^2 = \frac{\sigma_1 - \sigma_2}{t} \tag{31}$$

由 Eq.(11) 关系可以得到，对于左行波

$$\begin{aligned} u_2 &= u_1 + \sqrt{\frac{\sigma_1 - \sigma_2}{t}} \quad \text{if } \rho_R^* > \rho_R \\ u_2 &= u_1 - \sqrt{\frac{\sigma_1 - \sigma_2}{t}} \quad \text{if } \rho_R^* < \rho_R \end{aligned} \tag{32}$$

由此可以求得所有塑性波波后条件。

左侧塑性波波后条件

如果 $|sxxL| \geq \frac{2}{3}Y_0$，左侧已经屈服所以不存在塑性波，同理，如果 $\left|-\frac{4}{3}\mu\ln(\frac{\rho_L^*}{\rho_L})+s_{xxL}\right| \leq \frac{2}{3}Y_0$ 左侧均为未屈服状态，因此

$$\hat{\rho}_L=\rho_L,\hat{u}_L=u_L,\hat{p}_L=p_L,\hat{s}_{xxL}=s_{xxL} \tag{33}$$

当 $|sxxL|<\frac{2}{3}Y_0$ 同时 $\left|-\frac{4}{3}\mu\ln(\frac{\rho_L^*}{\rho_L})+s_{xxL}\right|>\frac{2}{3}Y_0$ 时，波前为未屈服状态，波后为屈服状态，因此存在塑性波。此时

$$\hat{s}_{xxL}=\begin{cases}-\dfrac{2}{3}Y_0 & \text{if} \quad \rho_L^*>\rho_L\\ \dfrac{2}{3}Y_0 & \text{if} \quad \rho_L^*<\rho_L\end{cases},\hat{\rho}_L=\begin{cases}\rho_L\exp(\dfrac{Y_0}{2\mu}+\dfrac{3S_{xxL}}{4\mu}) & \text{if} \quad \rho_L^*>\rho_L\\ \rho_L\exp(-\dfrac{Y_0}{2\mu}+\dfrac{3S_{xxL}}{4\mu}) & \text{if} \quad \rho_L^*<\rho_L\end{cases} \tag{34}$$

$$\hat{p}_L=\frac{2t(c_1f(\hat{\rho}_L)+e_L)-(\sigma_L+\hat{s}_{xxL})}{2tc_0-1} \tag{35}$$

其中 $t=\frac{\rho_L\hat{\rho}_L}{\hat{\rho}_L-\rho_L}$，$c_0=\frac{1}{\rho_0\Gamma_0}$，$c_1=\frac{a_0^2}{\Gamma_0}$。柯西应力为 $\hat{\sigma}_L=-\hat{p}_L+\hat{s}_{xxL}$, 速度

$$\hat{u}_L=\begin{cases}u_L+\sqrt{\dfrac{\sigma_L-\hat{\sigma}_L}{t}} & \text{if} \quad \rho_L^*>\rho_L\\ u_L-\sqrt{\dfrac{\sigma_L-\hat{\sigma}_L}{t}} & \text{if} \quad \rho_L^*<\rho_L\end{cases} \tag{36}$$

右侧塑性波波后条件

如果 $|sxxR| \geq \frac{2}{3}Y_0$，右侧已经屈服所以不存在塑性波，同理，如果 $\left|-\frac{4}{3}\mu\ln(\frac{\rho_R^*}{\rho_R}) + s_{xxL}\right| \leq \frac{2}{3}Y_0$ 左侧均为未屈服状态，因此

$$\hat{\rho}_R = \rho_R, \hat{u}_R = u_R, \hat{p}_R = p_R, \hat{s}_{xxR} = s_{xxR} \tag{37}$$

当 $|sxxR| < \frac{2}{3}Y_0$ 同时 $\left|-\frac{4}{3}\mu\ln(\frac{\rho_R^*}{\rho_R}) + s_{xxR}\right| > \frac{2}{3}Y_0$ 时，波前为未屈服状态，波后为屈服状态，因此存在塑性波。此时

$$\hat{s}_{xxR} = \begin{cases} \frac{2}{3}Y_0 & \text{if} \quad \rho_R^* > \rho_R \\ -\frac{2}{3}Y_0 & \text{if} \quad \rho_R^* < \rho_R \end{cases}, \hat{\rho}_R = \begin{cases} \rho_R\exp(\frac{Y_0}{2\mu} + \frac{3S_{xxR}}{4\mu}) & \text{if} \quad \rho_R^* > \rho_R \\ \rho_R\exp(-\frac{Y_0}{2\mu} + \frac{3S_{xxR}}{4\mu}) & \text{if} \quad \rho_R^* < \rho_R \end{cases} \tag{38}$$

$$\hat{p}_R = \frac{2t(c_1 f(\hat{\rho}_R) + e_R) - (\sigma_R + \hat{s}_{xxR})}{2tc_0 - 1} \tag{39}$$

其中 $t = \frac{\rho_R\hat{\rho}_R}{\hat{\rho}_R - \rho_R}$，$c_0 = \frac{1}{\rho_0\Gamma_0}$，$c_1 = \frac{a_0^2}{\Gamma_0}$。柯西应力为 $\hat{\sigma}_R = -\hat{p}_R + \hat{s}_{xxR}$, 速度

$$\hat{u}_R = \begin{cases} u_R - \sqrt{\frac{\sigma_R - \hat{\sigma}_R}{t}} & \text{if} \quad \rho_R^* > \rho_R \\ u_R + \sqrt{\frac{\sigma_R - \hat{\sigma}_R}{t}} & \text{if} \quad \rho_R^* < \rho_R \end{cases} \tag{40}$$

求解弹性波状态

估计左行波和右行波波速，

$$s_L = \min(\hat{u}_L - \hat{c}_L, \hat{u}_R - \hat{c}_R), s_R = \max(\hat{u}_L + \hat{c}_L, \hat{u}_R + \hat{c}_R), \tag{41}$$

根据 Eq.(9) 有

$$s^* = \frac{\hat{\sigma}_L - \hat{\sigma}_R + \hat{\rho}_L\hat{u}_L(s_L - u_L) - \hat{\rho}_R\hat{u}_R(s_R - u_R)}{\hat{\rho}_L(s_L - u_L) - \hat{\rho}_R(s_R - u_R)} \tag{42}$$

因此可以得到

$$\rho_L^* = \frac{\hat{\rho}_L(\hat{u}_L - s_L)}{s^* - s_L}, \rho_R^* = \frac{\hat{\rho}_R(u_R - s_R)}{s^* - s_R} \tag{43}$$

求波后偏应力

$$s_{xxL}^* = \begin{cases} \frac{2}{3}Y0 & \text{if} \quad -\frac{4}{3}\mu\ln(\frac{\rho_L^*}{\hat{\rho}_L}) + \hat{s}_{xxL} \geq \frac{2}{3}Y_0 \\ -\frac{2}{3}Y0 & \text{if} \quad -\frac{4}{3}\mu\ln(\frac{\rho_L^*}{\hat{\rho}_L}) + \hat{s}_{xxL} \leq -\frac{2}{3}Y_0 \\ -\frac{4}{3}\mu\ln(\frac{\rho_L^*}{\hat{\rho}_L}) + \hat{s}_{xxL} & \text{else} \end{cases}, s_{xxR}^* = \begin{cases} \frac{2}{3}Y0 & \text{if} \quad -\frac{4}{3}\mu\ln(\frac{\rho_R^*}{\hat{\rho}_R}) + \hat{s}_{xxR} \geq \\ -\frac{2}{3}Y0 & \text{if} \quad -\frac{4}{3}\mu\ln(\frac{\rho_R^*}{\hat{\rho}_R}) + \hat{s}_{xxR} \leq \\ -\frac{4}{3}\mu\ln(\frac{\rho_R^*}{\hat{\rho}_R}) + \hat{s}_{xxR} & \text{else} \end{cases} \tag{44}$$

求波后柯西应力

$$\sigma_L^* = \sigma_R^* = \hat{\sigma}_L - \hat{\rho}_L(s_L - u_L)(s^* - u_L) \tag{45}$$

波后压力

$$p_L^* = s_{xxL}^* - \sigma_L^*, p_R^* = s_{xxR}^* - \sigma_R^* \tag{46}$$