An exact Riemann solver for multi-material elastic-plastic flows with Mie-Grüneisen equation of state

1. Introduction

In this paper, an exact Riemann solver is built with the consideration of both elastic and plastic waves, for one-dimensional multi-material elastic-plastic materials with the Mie-Grüneisen EOS, isotropic elastic-plastic model [1] and the von Mises' yielding condition.

The elastic-plastic flow is used to describe the deformation process of solid materials under strong dynamics loading, such as explosive or high-speed impact. The simulation of elastic-plastic flows has important application backgrounds, espially in the Implosion Dynamics weapon and Innertial Confine Fusion (ICF). The first try of simulating the elastic-plastic flows was given by Wilkins [1] in 1960s.

Although the control equations of elastic-plastic flow is samilar to the equations in computational fluid dynamics (CFD). But there are mainly three difficulties of constructing Riemann solvers in elastic-plastic flows. Firstly, the constitutive models used in elastic-plastic flows are more complicated. Secondly, the EOS for solid materials are also with more characters. Finally, the yielding process of materials may lead to more wave structures.

Recently, a lot of works have been done in constructing approximate Riemann solvers for elastic-plastic flows with the considering of structures in the Riemann problem. For example, Gavrilyuk et al. [2] analyzed the structure of the Riemann solution and constructed a Riemann solver for the linear elastic system of hyperbolic non-conservative models for transverse waves. Cheng et al. [3] analyzed the wave structures of one-dimensional elastic-plastic flows and

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developed an effective two-rarefaction approximate Riemann solver with elastic waves (TRRSE). In [4], for one-dimensional elastic-plastic flows, Cheng introduced a HLLCE Riemann solver, which is fast and efficient in resolving elastic waves and plastic waves. Then in [5], Liu et.al remove the unreasonable assumution in HLLCE which may cause big error in solving multi-material problems and construct a multi-material HLLC elastic-plastic (MHLLCEP) solver.

As we know, in hydrodynamic numerical system, the exact Riemann solver is developed before approximate Riemann solvers [6], and it gives a guide and reference to the development of approximate solvers. However, in the elastic-plastic flows, for its complexity, there are only a few works have been done in constructing the exact Riemann solver. For example, Garaizar [7] designed an exact Riemann solution of isotropic hyperelastic material theoretically. Miller [8] presented a generial iterative method for the solution of the Riemann problem for hyperbolic elastic systems. Gao and Liu [9, 10] firstly considered the yielding condition in, and proposed an exact elastic-perfectly plastic solid Riemann solver with both elastic and plastic states. As many as thirty-six [9] and sixty-four [10] different solution types are considered in their works. However, in their papers, a linear "stiffened-gas" EOS was used in the plastic state and a more simple linear relation of pressure and strain rate is used in the elastic state. This symplification can truely deduce to simple relations across the shock wave and rarefaction wave, but we still need an exact Riemann solver with a uniform and more common used EOS and equations system for different elastic-plastic materials.

In this paper, we want to construct an exact Riemann solver for the elastic-plastic equations system with the isotropic elastic-plastic model and the von Mises' yielding condition. In this system, the Mie-Grüneisen EOS is used for both the elastic state and plastic state of the material, which is an adequate approximation to a wide variety of materials of interest, includes some gaseous or solid explosives and solid metals under high pressure **??**. According to both the theoretical analyzation and numerical tests [11], in this system, there may be three to five waves, including one contact wave and elastic wave (shock or rarefaction wave) or plastic wave or both the elastic and plastic waves in each side of the contact wave. We will prove this, the elastic wave always faster than the pastic wave in one side, so varing with the initial condition, there are 6×6 thirty-six possible cases of the structures in the solution. For a given case, we will obtain theoretical relations of all the states as functions of density. At last, through an iteration process of densities at both sides of the contact wave, the exact solution will be obtained. Also, the half Riemann problem and its exact solution are considered in this paper.

This paper is organized as follows. In section 2, we introduce the governing equations to be studied. In section 3, the Riemann problem and the relations for every wave type (contact wave, shock wave and rarefaction wave) is derived. Then, the exact Riemann solver is given in section 4. The half Riemann problem and its solver is introduced in section 5. Some numerical examples are presented to validate the method in section 6. Conclusions are shown in section 7.

2. Governing equations

In this paper, the elastic energy is not included in the total energy. The exclution of the elastic energy is usual for practical engineering problems [12] and is different from that in Ref.[2].

The governing equations system is given as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p - s_{xx}) = 0, \\ \partial_t(\rho E) + \partial_x\left[(\rho E + p - s_{xx})u\right] = 0, \\ \partial_t s_{xx} + u\partial_x s_{xx} - \dfrac{4}{3}\partial_x u = 0, \\ |s_{xx}| \leq \dfrac{2}{3}Y_0, \end{cases} \tag{1}$$

It contains the following parts.

2.1. Conservation terms

For the continuous one-dimensional solid, the conservation terms in differential form can be given as

$$\partial_t \mathbf{U} + \partial_x \boldsymbol{F}(\mathbf{U}) = 0,\ x \in\ \Omega \subset \mathbf{R},\ t > 0,$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 - \sigma \\ (\rho E - \sigma) u \end{bmatrix}, \tag{2}$$

ρ, u, σ and E are the density, velocity in $x-$direction, Cauchy stress and total energy per unit volume, respectively, E has the relation with specific internal energy e as

$$E = e + \frac{1}{2}u^2, \tag{3}$$

$$\sigma = -p + s_{xx}, \tag{4}$$

where p and s_{xx} denote hydrostatic pressure and deviatoric stress in the $x-$direction, respectively.

2.2. The equation of state (EOS)

The relation of the pressure with the density and the specific internal energy is gotten from the equation of state (EOS). In this paper, we consider the Mie-Grüneisen EOS,

$$p(\rho, e) = \rho_0 a_0^2 f(\eta) + \rho_0 \Gamma_0 e, \tag{5}$$

where $f(\eta) = \frac{(\eta-1)(\eta-\Gamma_0(\eta-1)/2)}{(\eta-s(\eta-1))^2}$, $\eta = \frac{\rho}{\rho_0}$, ρ_0, a_0, s and Γ_0 are constant parameters of the Mie-Grüneisen EOS.

2.3. The constitutive relation

Hooke's law is used here to describe the relationship between the deviatoric stress and the strain,

$$\dot{s}_{xx} = 2\mu \left(\dot{\varepsilon}_x - \frac{1}{3}\frac{\dot{V}}{V} \right), \tag{6}$$

where μ is the shear modulus, V is the volume, and the dot means the material time derivative,

$$\dot{()} = \frac{\partial ()}{\partial t} + u\frac{\partial ()}{\partial t}, \tag{7}$$

and

$$\dot{\varepsilon}_x = \frac{\partial u}{\partial x}, \quad \frac{\dot{V}}{V} = \frac{\partial u}{\partial x}. \tag{8}$$

By using Eq.(8), Eq.(6) can be rewritten as

$$\frac{\partial s_{xx}}{\partial t} + u\frac{\partial s_{xx}}{\partial t} = \frac{4}{3}\mu\frac{\partial u}{\partial x}. \tag{9}$$

2.4. The yielding condition

The Von Mises' yielding condition is used here to describe the elastic limit. In one spatial dimension, the von Mises' yielding criterion is given by

$$|s_{xx}| \leq \frac{2}{3}Y_0, \tag{10}$$

where Y_0 is the yield strength of the material in simple tension.

3. The Riemann problem

The Riemann problem for the 1D time dependent elastic-plastic equations is given as follows:

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p - s_{xx}) = 0, \\ \partial_t(\rho E) + \partial_x\left[(\rho E + p - s_{xx})u\right] = 0, \\ \partial_t s_{xx} + u\partial_x s_{xx} - \frac{4}{3}\partial_x u = 0, \\ |s_{xx}| \leq \frac{2}{3}Y_0, \\ Q(x, t=0) = \begin{cases} Q_L, \text{ if } x < 0, \\ Q_R, \text{ if } x \geq 0, \end{cases} \end{cases} \tag{11}$$

where $Q = (\rho, \rho u, \rho E, s_{xx})^T$.

According to the yielding of the material, the euqations may have different Jacobian matrix and different sonic velocities. We will discuss them seprately.

3.1. Elastic state

3.1.1. Jacobian matrix in elastic regions

For the Mie-Grüneisen EOS, if the material is not yielding and

$$|s_{xx}| < \frac{2}{3} Y_0, \tag{12}$$

the system (11) can be written as

$$\partial_t \mathbf{Q} + \mathbf{J}(\mathbf{Q}) \partial_x \mathbf{Q} = 0, \tag{13}$$

where $Q = (\rho, \rho u, \rho E, s_{xx})$, and the Jacobian matrix is

$$J(Q) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 + \frac{\partial p}{\partial \rho} + \Gamma(\frac{u^2}{2} - e) & u(2-\Gamma) & \Gamma & -1 \\ (\Gamma(\frac{u^2}{2} - e) - e - \frac{u^2}{2} + \frac{\sigma}{\rho} + \frac{\partial p}{\partial \rho})u & -\Gamma u^2 - \frac{\sigma}{\rho} + \frac{u^2}{2} + e & (1+\Gamma)u & -u \\ \frac{4}{3}\mu\frac{u}{\rho} & -\frac{4}{3}\mu\frac{1}{\rho} & 0 & u \end{bmatrix}, \tag{14}$$

where $\Gamma = \frac{\Gamma_0 \rho_0}{\rho}$.

The eigenvalues of the coefficient matrix $\mathbf{J}(\mathbf{Q})$ are given as

$$\lambda_1 = \lambda_2 = u, \quad \lambda_3 = u - c, \quad \lambda_4 = u + c, \tag{15}$$

where

$$\begin{cases} c_e = \sqrt{a^2 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} + \frac{4}{3}\frac{\mu}{\rho}}, \\ a^2 = \frac{\partial p}{\partial \rho} + \frac{p}{\rho^2}\frac{\partial p}{\partial e} = a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0\Gamma_0. \end{cases} \tag{16}$$

The corresponding right eigenvectors are

$$r_1 = \begin{bmatrix} \frac{1}{b_1} \\ \frac{u}{b_1} \\ 0 \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} -\frac{\Gamma}{b_1} \\ -\frac{\Gamma u}{b_1} \\ 1 \\ 0 \end{bmatrix}, \quad r_3 = \frac{1}{\phi^2} \begin{bmatrix} 1 \\ u - c_e \\ h - uc_e \\ \phi^2 \end{bmatrix}, \quad r_4 = \frac{1}{\phi^2} \begin{bmatrix} 1 \\ u + c_e \\ h + uc_e \\ \phi^2 \end{bmatrix}, \tag{17}$$

where

$$b_1 = \frac{\partial p}{\partial \rho} - \Gamma E, \quad h = E + \frac{p - s_{xx}}{\rho}, \tag{18}$$

and

$$\phi^2 = a^2 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} - c_e^2 = -\frac{4\mu}{3}\frac{1}{\rho}. \tag{19}$$

3.1.2. A relation between ρ and s_{xx}

Thanks to (7), the equations of the density and the deviatoric stress in Eq.(11) can be written as

$$\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{d\rho}{dt}, \tag{20}$$

and

$$\frac{ds_{xx}}{dt} = \frac{4}{3}\mu\frac{\partial u}{\partial x}. \tag{21}$$

Substituting (20) into (21) yields

$$\frac{ds_{xx}}{dt} = -\frac{4}{3}\mu\frac{1}{\rho}\frac{d\rho}{dt}. \tag{22}$$

Integrate the above equation from the data in front of a wave to the data behind the wave and perform some simple algebraic manipulations, one can get

$$s_{xx} + \frac{4}{3}\mu \ln(\rho) = \text{constant} \tag{23}$$

This relation always hold in the elastic state.

3.1.3. Relations across the contact wave

For a system without molecular diffusion, there is no materials convecting across the contact wave or interface, so the velocities on two sides of the discontinuity are always equal. This can also be verified by the eigenvectors in Eq.(17) and Eq.(70).

Using $\mathbf{W}_L^*$ and $\mathbf{W}_R^*$ to denote the two data states connected the contact wave, where $\mathbf{W} = (\rho, u, p, s_{xx})$.

Thanks to Eq.(17), for the λ_1-wave we have

$$\frac{d\rho}{\frac{1}{b_1}} = \frac{d\rho u}{\frac{u}{b_1}} = \frac{d\rho E}{0} = \frac{ds_{xx}}{1}. \tag{24}$$

From the above equations, we can easily deduce that

$$du = 0, \quad d(s_{xx} - p) = 0, \tag{25}$$

which means

$$u_L^* = u_R^*, \tag{26}$$

and

$$\sigma_{x,L}^* = \sigma_{x,R}^*, \tag{27}$$

where $()_L^*$ and $()_R^*$ denote $()$ in the region of $\mathbf{W}_L^*$ and $\mathbf{W}_R^*$, respectively. Here we do not show the details of the derivation for a simple presentation.

Similarly, for the λ_2-wave one has

$$\frac{d\rho}{\frac{-\Gamma}{b_1}} = \frac{d\rho u}{\frac{-u\Gamma}{b_1}} = \frac{d\rho E}{1} = \frac{ds_{xx}}{0}. \tag{28}$$

From the above equations, we can easily deduce that

$$du = 0, \quad dp = 0, \quad ds_{xx} = 0, \tag{29}$$

which means

$$u_L^* = u_R^*, \tag{30}$$

$$p_L^* = p_R^*, \quad s_{xx,L}^* = s_{xx,R}^*. \tag{31}$$

From Eq.(31), we get that

$$\sigma_{x,L}^* = \sigma_{x,R}^*. \tag{32}$$

At last, for the λ_1 and λ_2 waves, one can find that the following two relations always hold:

$$u_L^* = u_R^*, \quad \sigma_{x,L}^* = \sigma_{x,R}^*. \tag{33}$$

For convenience, we define

$$s^* = u_L^* = u_R^*. \tag{34}$$

where s^* denotes the velocity of the contact wave.

3.1.4. Relations across rarefaction waves

Left-going rarefaction wave Across the left wave associated with λ_3-wave, ($\lambda_3 = u - c_e$), we have

$$\frac{d\rho}{1} = \frac{d(\rho u)}{u - c_e} = \frac{d(\rho E)}{h - uc_e} = \frac{ds_{xx}}{-\frac{4\mu}{3}\frac{1}{\rho}}. \tag{35}$$

which lead to

$$\mathrm{d}u = -\frac{c_e}{\rho}\mathrm{d}\rho, \tag{36}$$

$$\mathrm{d}E = -\frac{\sigma + \rho u c_e}{\rho^2} d\rho. \tag{37}$$

$$\mathrm{d}s_{xx} = -\frac{4}{3}\frac{\mu}{\rho} d\rho, \tag{38}$$

Take the relation of EOS (5) in, we can get the pressure as

$$dp = \left(a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0\Gamma_0 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} \right) d\rho, \tag{39}$$

it can be written as a differential equation of $p(\rho)$

$$p'(\rho) - \lambda_1 \frac{p}{\rho^2} = f_2(\rho), \tag{40}$$

where

$$\lambda_1 = \rho_0 \Gamma_0 \quad f_2(\rho) = a_0^2 \frac{\partial f}{\partial \eta} - \lambda_1 \frac{s_{xx}(\rho)}{\rho^2}. \tag{41}$$

The pressure can be solved out as

$$p e^{\frac{\lambda_1}{\rho}} - \int f_2(\rho) e^{\frac{\lambda_1}{\rho}} d\rho = \text{constant}. \tag{42}$$

Then for a given density ρ, we can get the sonic speed c_e and velocity

$$u + \int \frac{c_e}{\rho} d\rho = \text{constant}. \tag{43}$$

Right-going rarefaction wave Across the right wave associated with λ_4-wave, ($\lambda_3 = u + c_e$), we have

$$\frac{d\rho}{1} = \frac{d(\rho u)}{u + c_e} = \frac{d(\rho E)}{h + u c_e} = \frac{d s_{xx}}{-\frac{4\mu}{3}\frac{1}{\rho}}. \tag{44}$$

which lead to

$$\mathrm{d}u = \frac{c_e}{\rho} \mathrm{d}\rho, \tag{45}$$

$$\mathrm{d}E = -\frac{\sigma + \rho u c_e}{\rho^2} d\rho. \tag{46}$$

$$\mathrm{d}s_{xx} = -\frac{4}{3}\frac{\mu}{\rho} d\rho, \tag{47}$$

We can get similar relations as the left-going wave as

$$p e^{\frac{\lambda_1}{\rho}} - \int f_2(\rho) e^{\frac{\lambda_1}{\rho}} d\rho = \text{constant}. \tag{48}$$

$$u - \int \frac{c_e}{\rho} d\rho = \text{constant}. \tag{49}$$

3.1.5. Relations across shock waves

Now we consider a shock wave with a speed of s, suppose the state in front of the shock is known as $(\rho_1, u_1, p_1, s_{xx1})$ and the state after the shock is unknown as $(\rho_2, u_2, p_2, s_{xx2})$.

Then use the conservation relation across the wave, which is also known as the Rankine-Hugoniot relation for a shock

$$\rho_2 u_2 = \rho_1 u_1 + s(\rho_2 - \rho_1), \tag{50}$$

$$\rho_2 u_2^2 - \sigma_2 = \rho_1 u_1^2 - \sigma_1 + s(\rho_2 u_2 - \rho_1 u_1), \tag{51}$$

$$(\rho_2 E_2 - \sigma_2) u_2 = (\rho_1 E_1 - \sigma_1) u_1 + s(\rho_2 E_2 - \rho_1 E_1), \tag{52}$$

above relations also can be written as

$$\rho_2(u_2 - s) = \rho_1(u_1 - s), \tag{53}$$

$$\rho_2 u_2(u_2 - s) = \rho_1 u_1(u_1 - s) + \sigma_2 - \sigma_1, \tag{54}$$

$$\rho_2 E_2(u_2 - s) = \rho_1 E_1(u_1 - s) + \sigma_2 u_2 - \sigma_1 u_1, \tag{55}$$

substituting (53) into (54) yields

$$\rho_1(u_2 - u_1)(u_1 - s) = \sigma_2 - \sigma_1, \tag{56}$$

also according to (53), we have

$$u_1 - s = \frac{(u_1 - u_2)\rho_2}{\rho_2 - \rho_1}, \tag{57}$$

then subtituting it into (56)

$$-t(u_2 - u_1)^2 = \sigma_2 - \sigma_1, \tag{58}$$

where $t = \frac{\rho_1 \rho_2}{\rho_2 - \rho_1}$.

Simimar to (56), (55) can be changed into

$$t(u_1 - u_2)(E_2 - E_1) = \sigma_2 u_2 - \sigma_1 u_1. \tag{59}$$

Because of $E = e + \frac{1}{2}u^2$, we can get

$$e_2 - e_1 = -\frac{\sigma_1 + \sigma_2}{2t}. \tag{60}$$

Using the EOS of Mie-Grüneisen (5), can get

$$e = c_0 p - c_1 f(\rho/\rho_0), \tag{61}$$

where $c_0 = \frac{1}{\rho_0 \Gamma_0}$ and $c_1 = \frac{a_0^2}{\Gamma_0}$. Put above equation into (60), we can solve the pressure p_2 out as a function of ρ_2.

$$p_2 = \frac{2t(c_1 f(\rho_2/\rho_0) + e_1) - (\sigma_1 + s_{xx2})}{2tc_0 - 1}, \tag{62}$$

The derivative stress σ_{xx2} also is a function of ρ_2 only. So we can solve the Cauchy stress out as

$$\sigma_2 = -p_2 + s_{xx2}. \tag{63}$$

Then we can use (58) to solve the velocity after the shock

$$u_2 = \begin{cases} u_1 - \sqrt{\dfrac{\sigma_1 - \sigma_2}{t}} & \text{Left-going,} \\ u_1 + \sqrt{\dfrac{\sigma_1 - \sigma_2}{t}} & \text{Right-going.} \end{cases} \tag{64}$$

And the shock speed is given as

$$s = \frac{\rho_2 u_2 - \rho_1 u_1}{\rho_2 - \rho_1}. \tag{65}$$

By the above deductions of the shock wave, we can get that, if the density after the shock is known, all the unknowns can be solved out.

3.2. Plastic state

3.2.1. Jacobian matrix in plastic regions[1]

When the material is yielding,

$$|s_{xx}| = \frac{2}{3}Y_0, \tag{66}$$

the equations will turn into a more simple system with only constitutive terms as

$$\partial_t \mathbf{U} + \mathbf{J}_p(\mathbf{U})\partial_x \mathbf{U} = 0, \tag{67}$$

where $\mathbf{U} = (\rho, \rho u, \rho E)$, and the Jacobian matrix is

$$\mathbf{J}_p(\mathbf{U}) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + \frac{\partial p}{\partial \rho} + \Gamma(\frac{u^2}{2} - e) & u(2-\Gamma) & \Gamma \\ (\Gamma(\frac{u^2}{2} - e) - e - \frac{u^2}{2} + \frac{\sigma}{\rho} + \frac{\partial p}{\partial \rho})u + \frac{u^2}{2} & -\Gamma u^2 - \frac{\sigma}{\rho} + e & (1+\Gamma)u \end{bmatrix}. \tag{68}$$

The eigenvalues of The eigenvalues of the coefficient matrix $\mathbf{J}_p(\mathbf{Q})$ are given as

$$\lambda_1 = u, \quad \lambda_2 = u - c, \quad \lambda_3 = u + c,$$

where

$$\begin{cases} c_p = \sqrt{a^2 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx}}, \\ a^2 = \frac{\partial p}{\partial \rho} + \frac{p}{\rho^2}\frac{\partial p}{\partial e} = a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0 \Gamma_0. \end{cases} \tag{69}$$

The corresponding right eigenvectors are

$$r_1 = \begin{bmatrix} -\frac{\Gamma}{b_1} \\ -\frac{\Gamma u}{b_1} \\ 1 \end{bmatrix}, \quad r_2 = \frac{1}{h - uc_p}\begin{bmatrix} 1 \\ u - c_p \\ h - uc_p \end{bmatrix}, \quad r_3 = \frac{1}{h + uc_p}\begin{bmatrix} 1 \\ u + c_p \\ h + uc_p \end{bmatrix}. \tag{70}$$

[1] Code site link

where

$$b_1 = \frac{\partial p}{\partial \rho} - \Gamma E, \quad h = E + \frac{p - s_{xx}}{\rho}. \tag{71}$$

Take a comparason of Eq.(16) and Eq.(69), we notice that the sonic speed is not continuous between the states of elastic and plastic. As the shear modulus μ is always positive, so the elastic wave is always faster than the plastic wave. This is very important and may cause wrong results if ignoring it.

3.2.2. Relations across the contact wave

According to the eigenvectors in Eq.(70), for the λ_1-wave ($\lambda_1 = u$), we have

$$\frac{d\rho}{\frac{-\Gamma}{b_1}} = \frac{d(\rho u)}{\frac{-u\Gamma}{b_1}} = \frac{d(\rho E)}{1}. \tag{72}$$

From the above equations, we can easily deduce that

$$du = 0, \quad dp = 0. \tag{73}$$

Samilar to that in section 3.2.2, we can also get the relations

$$s^* = u_L^* = u_R^*, \quad \sigma_L^* = \sigma_R^*. \tag{74}$$

3.2.3. Relations across rarefaction waves

Left-going rarefaction wave Across the left wave associated with λ_2-wave, ($\lambda_2 = u - c_p$), we have

$$\frac{d\rho}{1} = \frac{d(\rho u)}{u - c_p} = \frac{d(\rho E)}{h - uc_p}. \tag{75}$$

Samilar to section 3.1.4, we can get the relations

$$pe^{\frac{\lambda_1}{\rho}} - \int f_2(\rho) e^{\frac{\lambda_1}{\rho}} d\rho = \text{constant}. \tag{76}$$

and

$$u + \int \frac{c_p}{\rho} d\rho = \text{constant}, \tag{77}$$

where

$$\lambda_1 = \rho_0 \Gamma_0 \quad f_2(\rho) = a_0^2 \frac{\partial f}{\partial \eta} - \lambda_1 \frac{s_{xx}(\rho)}{\rho^2}. \tag{78}$$

Right-going rarefaction wave Across the right wave associated with λ_3-wave, ($\lambda_3 = u + c_e$), we have

$$\frac{d\rho}{1} = \frac{d(\rho u)}{u + c_p} = \frac{d(\rho E)}{h + uc_p}. \tag{79}$$

We can get similar relations as the left-going wave as

$$pe^{\frac{\lambda_1}{\rho}} - \int f_2(\rho) e^{\frac{\lambda_1}{\rho}} d\rho = \text{constant}. \tag{80}$$

$$u - \int \frac{c_p}{\rho} d\rho = \text{constant}. \tag{81}$$

3.2.4. Relations across shock waves

By a same deducing process with Section 3.1.5, we can get the state after a shock as

$$s_{xx2} = s_{xx1}, \tag{82}$$

$$p_2 = \frac{2t(c_1 f(\rho_2/\rho_0) + e_1) - (\sigma_1 + s_{xx2})}{2tc_0 - 1}, \tag{83}$$

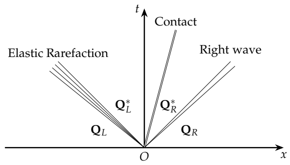
where $c_0 = \frac{1}{\rho_0 \Gamma_0}$ and $c_1 = \frac{a_0^2}{\Gamma_0}$.

$$\sigma_2 = -p_2 + s_{xx2}. \tag{84}$$

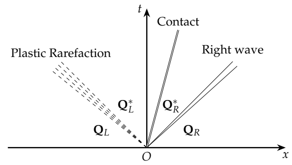
$$u_2 = \begin{cases} u_1 - \sqrt{\dfrac{\sigma_1 - \sigma_2}{t}} & \text{Left-going,} \\ u_1 + \sqrt{\dfrac{\sigma_1 - \sigma_2}{t}} & \text{Right-going.} \end{cases} \tag{85}$$

And the shock speed is given as

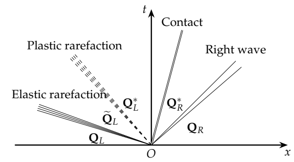
$$s = \frac{\rho_2 u_2 - \rho_1 u_1}{\rho_2 - \rho_1}. \tag{86}$$



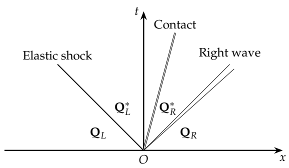
(a) Elastic rarefaction wave



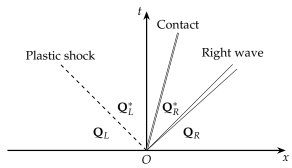
(b) Plastic rarefaction wave



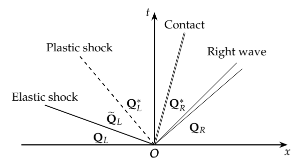
(c) Both elastic and plastic rarefaction waves



(d) Elastic shock wave



(e) Plastic shock wave



(f) Both elastic and plastic shock waves

Figure 1: The possible cases of Riemann solution structures in the left side.

4. Exact Riemann solver [2]

Now we consider the constructing details of the exact Riemann solver. For the Riemann problem in Section 4, there are 6×6 possible cases in the Riemann solution with different wave structures. Now we consider the left six cases as shown in Fig.1.

There are mainly five steps in the soluting process.

1. In section (4.1), a pre-evaluation is done with a shock-contact-shock waves assumption to give an initial values of ρ_L^* and ρ_R^*.
2. With given ρ_L^* and ρ^{*R}, we need to detect the left cases in Fig.1 and the corresponding right cases, this is done in section 4.1.
3. We solve functions $s_{xx}(\rho)$, $p(\rho)$ and $u(\rho)$ for different cases in section 4.3.
4. At last, with the informations in Section 3.2.2,

$$
\begin{aligned}
f_u(\rho_L^*, \rho_R^*) &= u_L^* - u_R^* = 0, \\
f_\sigma(\rho_L^*, \rho_R^*) &= \sigma_L^* - \sigma_R^* = 0,
\end{aligned}
\tag{87}
$$

[2] Code site link

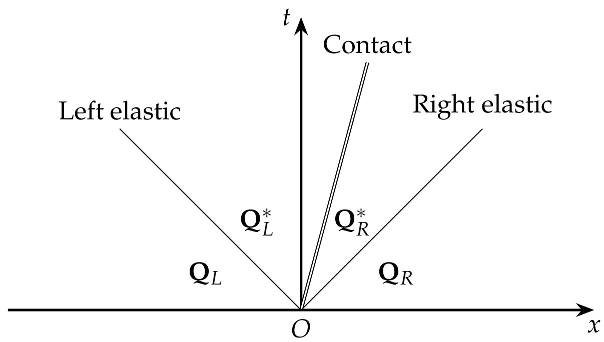


Figure 2: Pre-evaluation with a three-waves structure.

use a Newton iteration method to get new ρ_L^* and ρ_R^*, then do steps 2-5 again until the result convergents. this process is given in Section 4.5.

4.1. Pre-evaluation and case classfication

At first, we assume that there are only one elastic shock wave in the left side and another one elastic shock in the right side as shown in Fig.2. According to the conservation relations across the shocks, we have

$$\begin{cases} \hat{\rho}_L \hat{u}_L^* = \rho_L u_L + s_L(\hat{\rho}_L^* - \rho_L), \\ \hat{\rho}_L \hat{u}_L^{*2} - \sigma_L^* = \rho_L u_L^2 - \sigma_L + s_L(\hat{\rho}_L^* \hat{u}_L^* - \rho_L u_L), \end{cases} \tag{88}$$

and

$$\begin{cases} \hat{\rho}_R \hat{u}_R^* = \rho_R u_R + s_R(\hat{\rho}_R^* - \rho_R), \\ \hat{\rho}_R \hat{u}_R^{*2} - \sigma_R^* = \rho_R u_R^2 - \sigma_R + s_R(\hat{\rho}_R^* \hat{u}_R^* - \rho_R u_R). \end{cases} \tag{89}$$

Using the relation across the interface,

$$\hat{u}_L^* = \hat{u}_R^* = \hat{s}^*, \quad \hat{\sigma}_L^* = \hat{\sigma}_R^*. \tag{90}$$

the speed of contact wave can be evaluated as

$$\hat{s}^* = \frac{\sigma_L - \sigma_R + \rho_L u_L(s_L - u_L) - \rho_R u_R(s_R - u_R)}{\rho_L(s_L - u_L) - \rho_R(s_R - u_R)}, \tag{91}$$

Table 1: The condition of cases classification.

Conditions	$\lvert s_{xx}\rvert < \frac{2}{3}Y_0$ and $\lvert \hat{s}_{xx}\rvert < \frac{2}{3}Y_0$	$s_{xx} = \frac{2}{3}Y_0$	other
$\hat{\rho}^* < \rho$	case a	case b	case c
$\hat{\rho}^* > \rho$	case d	case e	case f

the density is solved as

$$\hat{\rho}_L^* = \frac{\rho_L(u_L - s_L)}{\hat{s}^* - s_L}, \quad \hat{\rho}_R^* = \frac{\rho_R(u_R - s_R)}{\hat{s}^* - s_R}. \tag{92}$$

4.2. Determination of the structure case

Using a given density ρ_L^*, we can distinguish the shock and rarefaction in the left side. This is done easily by comparing ρ_L^* with ρ_L,

$$\begin{cases} \text{Rarefaction wave:} & \rho_L > \rho_L^* \\ \text{Shock wave:} & \rho_L < \rho_L^* \end{cases} \tag{93}$$

The devaitoric stress is evaluated as

$$\hat{s}_{xxL}^* = -\frac{4}{3}\mu\ln(\frac{\hat{\rho}_L^*}{\rho_L}) + s_{xxL}, \quad \hat{s}_{xxR}^* = -\frac{4}{3}\mu\ln(\frac{\hat{\rho}_R^*}{\rho_R}) + s_{xxR}. \tag{94}$$

Here we define the speeds of left and right going waves as

$$s_L = \min(u_L - c_L, u_R - c_R, 0), \quad s_R = \max(u_L + c_L, u_R + c_R, 0). \tag{95}$$

By the relation of $\hat{\rho}_{L(R)}$ and $\hat{s}_{xxL(R)}$ we can classify every side into six cases, and conditions for the classfication are shown in Table 1, the subscripts $_L$ and $_R$ are omitted for simplication.

4.3. States for one wave cases (a,b, d,e)

For cases (a,b,d,e) in Fig.1, the material is totally yielding or totally not yielding, if we give a density ρ_L^* in the region $\mathbf{Q}_L^*$, by the deductions in Section 3.1.4 and Section 3.1.5, all the unknowns in region $\mathbf{Q}_L^*$ can be solved out.

4.3.1. Rarefaction wave cases (a,b)

For Rarefaction wave, we not only need to solve the state after the wave in region $\mathbf{Q}^*_{L(R)}$, but also need to know the states inside the expansion region.

First we give the function of s_{xx},

$$s_{xx}(\rho) = \begin{cases} -\dfrac{4}{3}\mu \ln\left(\dfrac{\rho}{\rho_{L(R)}}\right) + s_{xxL(R)}, & \rho_{L(R)} \geq \rho \geq \rho^*_{L(R)}, \quad \text{case (a)}, \\ \dfrac{2}{3}Y_0, & \rho_{L(R)} \geq \rho \geq \rho^*_{L(R)}, \quad \text{case (b)}. \end{cases} \tag{96}$$

Then we give the pressure,

$$p(\rho) = p_{L(R)} e^{\frac{\lambda_1}{\rho_{L(R)}} - \frac{\lambda_1}{\rho}} + e^{-\frac{\lambda_1}{\rho}} \int_{\rho_{L(R)}}^{\rho} f_2(x) e^{\frac{\lambda_1}{x}} dx. \quad \text{case (a,b)}. \tag{97}$$

where

$$\lambda_1 = \rho_0 \Gamma_0 \quad f_2(\rho) = a_0^2 \frac{\partial f}{\partial \eta} - \lambda_1 \frac{s_{xx}(\rho)}{\rho^2}. \tag{98}$$

and sonic speed,

$$c(\rho) = \begin{cases} \sqrt{a_0^2 \dfrac{\partial f}{\partial \eta} + \dfrac{p(\rho)}{\rho^2}\rho_0 \Gamma_0 - \dfrac{\rho_0}{\rho^2}\Gamma_0 s_{xx}(\rho) + \dfrac{4}{3}\dfrac{\mu}{\rho}} & \text{case (a)}, \\ \sqrt{a_0^2 \dfrac{\partial f}{\partial \eta} + \dfrac{p(\rho)}{\rho^2}\rho_0 \Gamma_0 - \dfrac{\rho_0}{\rho^2}\Gamma_0 s_{xx}(\rho)} & \text{case (b)}. \end{cases} \tag{99}$$

Then we can get the function of velocity

$$u(\rho) = \begin{cases} u_L - \displaystyle\int_{\rho_L}^{\rho} \frac{c(x)}{x} dx, & \rho_L \geq \rho \geq \rho^*_L, \quad \text{case (a,b)}, \\ u_R + \displaystyle\int_{\rho_R}^{\rho} \frac{c(x)}{x} dx, & \rho_R \geq \rho \geq \rho^*_R, \quad \text{case (a,b)}, \end{cases} \tag{100}$$

4.3.2. Shock wave cases (d,e)

For shock waves, the function of deviatoric is given as

$$s_{xx}(\rho) = \begin{cases} -\dfrac{4}{3}\mu\ln\left(\dfrac{\rho}{\rho_{L(R)}}\right) + s_{xxL(R)}, & \rho = \rho_{L(R)}, \rho^*_{L(R)}, \quad \text{case (d)}, \\ -\dfrac{2}{3}Y_0, & \rho = \rho_{L(R)}, \rho^*_{L(R)}, \quad \text{case (e)}. \end{cases} \tag{101}$$

And the pressure is given as

$$p(\rho) = \frac{2t(c_1 f(\rho/\rho_0) + e_L) - (\sigma_{L(R)} + s_{xx}(\rho))}{2tc_0 - 1}, \quad \rho = \rho_{L(R)}, \rho^*_{L(R)}, \quad \text{case (d,e)}, \tag{102}$$

where $c_0 = \frac{1}{\rho_0\Gamma_0}$ and $c_1 = \frac{a_0^2}{\Gamma_0}$ and $t = \frac{\rho_{L(R)}\rho^*_{L(R)}}{\rho^*_{L(R)} - \rho_{L(R)}}$. The velocity is given as

$$u(\rho) = \begin{cases} u_L - \sqrt{\dfrac{\sigma_L - \sigma(\rho)}{t}} & \rho = \rho_L, \rho^*_L \\ u_R + \sqrt{\dfrac{\sigma_R - \sigma(\rho)}{t}} & \rho = \rho_R, \rho^*_R \end{cases} \quad \text{case (d,e)} \tag{103}$$

4.4. States for two wave cases (c,f)

For cases (c,f), the meterial periods a yielding process from elastic to plastic. There are two waves and one more state $\tilde{\mathbf{Q}}_{L(R)}$ exists. In state $\tilde{\mathbf{Q}}_{L(R)}$, the derivative stress achieves the elastic limit.

$$\tilde{s}_{xxL(R)} = \begin{cases} \dfrac{2}{3}Y_0 & \text{Case (c)}, \\ -\dfrac{2}{3}Y_0 & \text{Case (f)}, \end{cases} \tag{104}$$

By (23),at state $\tilde{\mathbf{Q}}_{L(R)}$ we can solve the density out as

$$\tilde{\rho}_{L(R)} = \begin{cases} \widetilde{\rho}_{L(R)} = \rho_{L(R)}\exp\left(-\dfrac{Y_0}{2\mu} + \dfrac{3s_{xxL(R)}}{4\mu}\right), & \text{Case (c)}, \\ \widetilde{\rho}_{L(R)} = \rho_{L(R)}\exp\left(\dfrac{Y_0}{2\mu} + \dfrac{3s_{xxL(R)}}{4\mu}\right), & \text{Case (f)}. \end{cases} \tag{105}$$

4.4.1. Rarefaction wave case (c)

For rarefaction wave case, we give the function of s_{xx} at first,

$$s_{xx}(\rho) = \begin{cases} -\dfrac{4}{3}\mu \ln\left(\dfrac{\rho}{\rho_{L(R)}}\right) + s_{xxL(R)} & \rho_{L(R)} \geq \rho \geq \tilde{\rho}_{L(R)}, \\ \dfrac{2}{3}Y_0, & \tilde{\rho}_{L(R)} \geq \rho \geq \rho^*_{L(R)}. \end{cases} \quad \text{case (c).} \tag{106}$$

The pressure is given as

$$p(\rho) = \begin{cases} p_{L(R)} e^{\frac{\lambda_1}{\rho_{L(R)}} - \frac{\lambda_1}{\rho}} + e^{-\frac{\lambda_1}{\rho}} \displaystyle\int_{\rho_{L(R)}}^{\rho} f_2(x) e^{\frac{\lambda_1}{x}} dx, & \rho_{L(R)} \geq \rho \geq \tilde{\rho}_{L(R)}, \\ \tilde{p}_{L(R)} e^{\frac{\lambda_1}{\tilde{\rho}_{L(R)}} - \frac{\lambda_1}{\rho}} + e^{-\frac{\lambda_1}{\rho}} \displaystyle\int_{\tilde{\rho}_{L(R)}}^{\rho} f_2(x) e^{\frac{\lambda_1}{x}} dx, & \tilde{\rho}_{L(R)} \geq \rho \geq \rho^*_{L(R)}, \end{cases} \quad \text{case (c),} \tag{107}$$

where

$$\lambda_1 = \rho_0 \Gamma_0 \quad f_2(\rho) = a_0^2 \frac{\partial f}{\partial \eta} - \lambda_1 \frac{s_{xx}(\rho)}{\rho^2}. \tag{108}$$

And sonic speed,

$$c(\rho) = \begin{cases} \sqrt{a_0^2 \dfrac{\partial f}{\partial \eta} + \dfrac{p(\rho)}{\rho^2}\rho_0\Gamma_0 - \dfrac{\rho_0}{\rho^2}\Gamma_0 s_{xx}(\rho) + \dfrac{4}{3}\dfrac{\mu}{\rho}} & \rho_{L(R)} \geq \rho \geq \tilde{\rho}_{L(R)} \\ \sqrt{a_0^2 \dfrac{\partial f}{\partial \eta} + \dfrac{p(\rho)}{\rho^2}\rho_0\Gamma_0 - \dfrac{\rho_0}{\rho^2}\Gamma_0 s_{xx}(\rho)} & \tilde{\rho}_{L(R)} \geq \rho \geq \rho^*_{L(R)}. \end{cases} \quad \text{case (c).} \tag{109}$$

Then we can get the function of velocity

$$u(\rho) = \begin{cases} u_L - \displaystyle\int_{\rho_L}^{\rho} \frac{c(x)}{x} dx, & \rho_L \geq \rho \geq \rho^*_L, \\ u_R + \displaystyle\int_{\rho_R}^{\rho} \frac{c(x)}{x} dx, & \rho_L \geq \rho \geq \rho^*_R, \end{cases} \quad \text{case (c).} \tag{110}$$

4.4.2. Shock wave case (f)

For shock waves the function of deviatoric is given as

$$s_{xx}(\rho)=\begin{cases} -\dfrac{4}{3}\mu\ln\left(\dfrac{\rho}{\rho_{L(R)}}\right)+s_{xxL(R)}, & \rho=\rho_{L(R)},\tilde{\rho}_{L(R)}, \\ -\dfrac{2}{3}Y_0, & \rho=\rho^*_{L(R)}, \end{cases} \quad \text{case (f)}. \tag{111}$$

And the pressure is given as

$$p(\rho)=\begin{cases} \dfrac{2t(c_1f(\rho/\rho_0)+e_L)-(\sigma_{L(R)}+s_{xx}(\rho))}{2t_1c_0-1}, & \rho=\rho_{L(R)},\tilde{\rho}_{L(R)}, \\ \dfrac{2t(c_1f(\rho/\rho_0)+\tilde{e}_L)-(\tilde{\sigma}_{L(R)}+s_{xx}(\rho))}{2t_2c_0-1}, & \rho=\rho^*_{L(R)}, \end{cases} \quad \text{case (f)}, \tag{112}$$

where $c_0=\frac{1}{\rho_0\Gamma_0}$ and $c_1=\frac{a_0^2}{\Gamma_0}$ and $t_1=\frac{\rho_{L(R)}\tilde{\rho}_{L(R)}}{\tilde{\rho}_{L(R)}-\rho_{L(R)}}$, $t_2=\frac{\tilde{\rho}_{L(R)}\rho^*_{L(R)}}{\rho^*_{L(R)}-\tilde{\rho}_{L(R)}}$.

The velocity is given as

$$u(\rho)=\begin{cases} u_L-\sqrt{\frac{\sigma_L-\sigma(\rho)}{t_1}}, & \rho=\rho_L,\tilde{\rho}_L, \\ \tilde{u}_L-\sqrt{\frac{\tilde{\sigma}_L-\sigma(\rho)}{t_2}}, & \rho=\rho^*_L, \\ u_R+\sqrt{\frac{\sigma_R-\sigma(\rho)}{t_1}}, & \rho=\rho_R,\tilde{\rho}_R, \\ \tilde{u}_R+\sqrt{\frac{\tilde{\sigma}_R-\sigma(\rho)}{t_2}}, & \rho=\rho^*_R, \end{cases} \quad \text{case (f)}. \tag{113}$$

4.5. An iteration process of ρ^*_L and ρ^*_R

For a given ρ^*_L and a given ρ^*_R,

$$\begin{aligned} &u^*_L=u(\rho^*_L), \quad u^*_R=u(\rho^*_R), \\ &\sigma^*_L=-p(\rho^*_L)+s_{xx}(\rho^*_L), \quad \sigma^*_R=-p(\rho^*_R)+s_{xx}(\rho^*_R), \end{aligned} \tag{114}$$

by relations of

$$\begin{aligned} &f_u(\rho^*_L,\rho^*_R)_{=}u^*_L-u^*_R=0, \\ &f_\sigma(\rho^*_L,\rho^*_R)_{=}\sigma^*_L-\sigma^*_R=0, \end{aligned} \tag{115}$$

and used a Newton iteration method, we can solve the densities ρ^*_L and ρ^*_R out.

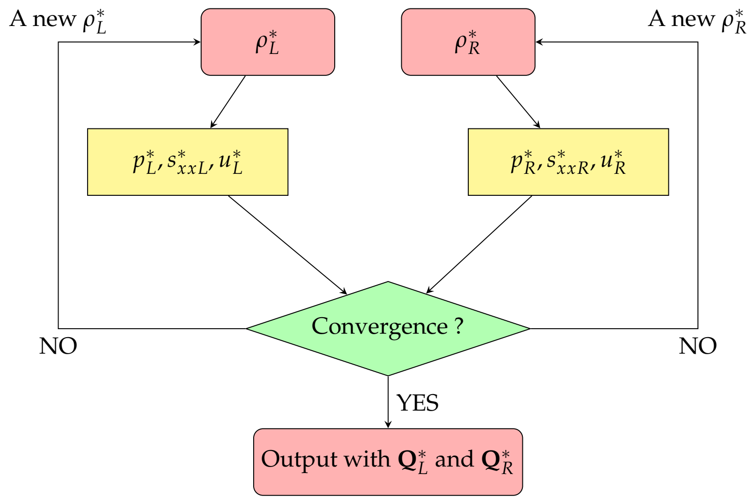


Figure 3: A flow chat of the Newton iteration process.

The Newton iteration to evaluate ρ_L^* and ρ_R^* is given as

$$\begin{bmatrix} \rho^*_{L,(k+1)} \\ \rho^*_{R,(k+1)} \end{bmatrix} = \begin{bmatrix} \rho^*_{L,(k)} \\ \rho^*_{R,(k)} \end{bmatrix} - \begin{bmatrix} \frac{\partial f_{u(k)}}{\partial \rho_L^*} & \frac{\partial f_{u(k)}}{\partial \rho_R^*} \\ \frac{\partial f_{\sigma(k)}}{\partial \rho_L^*} & \frac{\partial f_{\sigma(k)}}{\partial \rho_R^*} \end{bmatrix}^{-1} \begin{bmatrix} f_{u(k)} \\ f_{\sigma(k)} \end{bmatrix} \tag{116}$$

The initial of densities are given as the pre-evalation values,

$$\rho^*_{L(0)} = \hat{\rho}_L \quad \rho^*_{R(0)} = \hat{\rho}_R. \tag{117}$$

The convergence is meassured by

$$\text{CHA} = \max\left[\frac{|\rho^*_{L(k+1)} - \rho^*_{L,(k)}|}{\frac{1}{2}|\rho^*_{L(k+1)} + \rho^*_{L(k)}|}, \frac{|\rho^*_{R(k+1)} - \rho^*_{R,(k)}|}{\frac{1}{2}|\rho^*_{R(k+1)} + \rho^*_{R(k)}|}, |f_u|, |f_\sigma|\right]. \tag{118}$$

and the tolerance is taken as $\text{TOL} = 10^{-4}$. It usually takes 3-4 step to get a convergence result.

The derivatives of f_u and f_σ are given by

$$\frac{\partial f_{u,(k+1)}}{\partial \rho^*_{L(R)}} = \frac{f_{u,(k+1)} - f_{u,(k)}}{\rho^*_{L(R),(k+1)} - \rho_{L(R),(k)}}, \quad \frac{\partial f_{\sigma,(k+1)}}{\partial \rho^*_{L(R)}} = \frac{f_{\sigma,(k+1)} - f_{\sigma,(k)}}{\rho^*_{L(R),(k+1)} - \rho_{L(R),(k)}}, \tag{119}$$

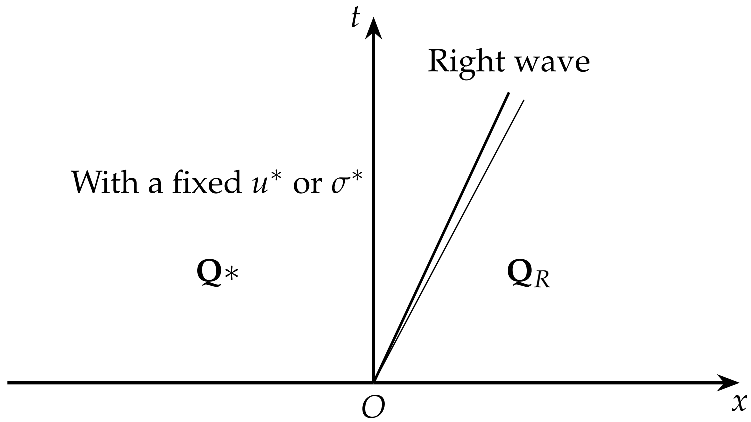


Figure 4: Half Riemann problem with a given left velocity or Cauchy stress..

At the first step, we use a simple numerical difference method

$$\frac{\partial f_{u,(1)}}{\partial \rho^*_{L(R)}} = \frac{f_u(\rho^*_{L(R)} + \Delta\rho) - f_u(\rho^*_{L(R)})}{\Delta\rho_{L(R)}}, \quad \frac{\partial f_{u,(1)}}{\partial \rho^*_{L(R)}} = \frac{f_u(\rho^*_{L(R)} + \Delta\rho) - f_u(\rho^*_{L(R)})}{\Delta\rho_{L(R)}}, \tag{120}$$

where $\Delta\rho$ is a little quatity, we can choose it as

$$\Delta\rho_{L(R)} = \frac{\rho^*_{L(R),(0)}}{100}. \tag{121}$$

A flow chat of this process is shown in Fig.3.

5. Half Riemann problem and solver

Some times we need to analyse a half Riemann problem with a given velocity or Cauchy stress in one side. Shown in Fig.5 , in these cases, we only need to solve the states in another side. There are six possible cases just like them in Section 4.

The process of solving a half Riemann problem is in three step:

First we evaluate the density in the star region $\mathbf{Q}^*$ with a elastic shock assumption. Then using the relation of $\sigma(\rho^*) = \sigma^*$. We can solve the evaluated density $\hat{\rho}$ out. A Newton iteration process will be used.

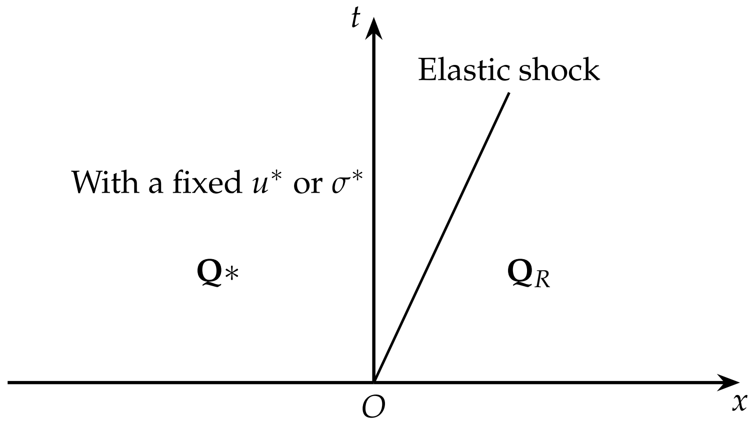


Figure 5: Pre-evaluation of the half Riemann problem with a elastic shock assumeption.

Second, we can determine the real structure case by Table 1.

At last, as we know the type of the structures, we can use the iteration process again to solve the real density in region $\mathbf{Q}^*$ out. With the density, the pressure and the devaitoric stress are also can be solved.

5.1. *Evaluate of the density $\hat{\rho}$ in region $\mathbf{Q}^*$*

As shown in Fig.**??**, we assume there is only one elastic shock wave between region $\mathbf{Q}$ and region $\mathbf{Q}^*$.

Then use the shock relation in Section 3.1.5, and by the Newton iteration process we can solve the density $\hat{\rho}$ out.

If the Cauchy stress in region $\mathbf{Q}^*$ is given, let

$$f(\rho) = \sigma^*(\rho) - \sigma_R, \tag{122}$$

if the velocity is given, let

$$f(\rho) = u^*(\rho) - u_R. \tag{123}$$

The Newton iteration is given as

$$\hat{\rho}_{(k+1)} = \hat{\rho}_{(k)} - f/\frac{\partial f_{(k)}}{\partial \rho}, \tag{124}$$

and the initial is given as

$$\hat{\rho}_{(0)} = \rho_R. \tag{125}$$

The congergence is meassured by

$$\mathrm{CHA} = \max\left[\frac{|\hat{\rho}_{(k+1)} - \hat{\rho}_{(k)}|}{\frac{1}{2}|\hat{\rho}_{(k+1)} + \hat{\rho}_{(k)|}}, |f_{(k+1)}|\right]. \tag{126}$$

5.2. Resolve the state in region $\mathbf{Q}^*$

After evaluate the density $\hat{\rho}$ in region $\mathbf{Q}^*$, the deviatoric stress also can be evaluated by Eq 23, and use Table 1, the real structure case is known. By an exact same iteration process in above

$$\rho^*_{(k+1)} = \rho^*_{(k)} - f/\frac{\partial f_{(k)}}{\partial \rho}, \tag{127}$$

with the initial condition of

$$\rho^*_{(0)} = \hat{\rho}, \tag{128}$$

we can solve the ρ^* out.

In the iteration process the Cauchy stress $\sigma^*(\rho)$ or the velocity $\sigma^*(\rho)$ is given by the relation in Section 4.3 or Section 4.4 for different cases.

6. Numerical

Conclusions

In this paper, the multi-material HLLC-type approximate Riemann solver with both the elastic and plastic waves (MHLLCEP) is constructed for 1D elastic-plastic flows with a hypo-elastic model and the von Mises yielding condition. During constructing the MHLLCEP, we do not use the assumption in which the pressure is continuous across the contact wave and so describing and evaluating the plastic waves are more accurate than that in the HLLCE. Based on the MHLLCEP, combining with the third-order ghost-cell reconstruction method and the third-order TVD-Runge-Kutta method in time, a high-order cell-centered Lagrangian scheme for 1D multi-material elastic flows is built. Verified by the numerical experiments, both the plastic waves and elastic waves can be resolved correctly, our scheme appears to be convergent, stable, essentially non-oscillatory and can reach third-order accuracy for smooth problems. Especially for multi-material elastic-plastic flows, the results solved by the scheme with HLLCE are with large errors, but our new scheme can eliminate these errors and leads to the reasonable numerical results.

Acknowledgement

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We random choose a particle point inside the expansion region with the state of (ρ, u, p, s_{xx}), after a little expansion with the wave speed of $u - c$, the state changes to$(\rho + d\rho, u + du, p + dp, s_{xx} + ds_{xx})$. According to the conservation relations we can get

$$\begin{aligned}
&(\rho + \mathrm{d}\rho)(u + \mathrm{d}u) = \rho u + (u - c)\mathrm{d}\rho \\
&(\rho + \mathrm{d}\rho)(u + \mathrm{d}u)^2 - (\sigma + \mathrm{d}\sigma) = \rho u^2 - \sigma \\
&+ (u - c)[(\rho + \mathrm{d}\rho)(u + \mathrm{d}u) - \rho u] \\
&[(\rho + \mathrm{d}\rho)(E + \mathrm{d}E) - (\sigma + \mathrm{d}\sigma)](u + \mathrm{d}u) = \\
&(\rho E - \sigma)u + (u - c)[(\rho + \mathrm{d}\rho)(E + \mathrm{d}E) - \rho E]
\end{aligned} \tag{129}$$

Ignoring high-order infinite small terms, we can get

$$\mathrm{d}u = -\frac{c}{\rho}\mathrm{d}\rho, \tag{130}$$

$$\mathrm{d}\sigma = -c^2 d\rho, \tag{131}$$

$$\mathrm{d}E = -\frac{\sigma + \rho u c}{\rho^2} d\rho. \tag{132}$$

Also we know the deviatoric stress is a function of the density by Eq.(23)

$$ds_{xx} = \begin{cases} -\frac{4}{3}\frac{\mu}{\rho} d\rho & |s_{xx}| < \frac{2}{3}Y_0, \\ 0 & |s_{xx}| \geq \frac{2}{3}Y_0. \end{cases} \tag{133}$$

And from Eq.(16) and Eq.(69) we can get

$$c^2 = \begin{cases} a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0\Gamma_0 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} + \frac{4}{3}\frac{\mu}{\rho} & |s_{xx}| < \frac{2}{3}Y_0, \\ a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0\Gamma_0 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} & |s_{xx}| \geq \frac{2}{3}Y_0. \end{cases} \tag{134}$$

Then the differential of pressure is given as

$$dp = ds_{xx} - d\sigma = ds_{xx} + c^2 d\rho = \left(a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0\Gamma_0 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} \right) d\rho, \tag{135}$$

it can be written as a differential equation of $p(\rho)$

$$p'(\rho) - \lambda_1 \frac{p}{\rho^2} = f_2(\rho), \quad p(\rho_1) = p_1, \tag{136}$$

where

$$\lambda_1 = \rho_0\Gamma_0 \quad f_2(\rho) = a_0^2 \frac{\partial f}{\partial \eta} - \lambda_1 \frac{s_{xx}(\rho)}{\rho^2}. \tag{137}$$

The pressure can be solved out as

$$p(\rho) = p_1 e^{\frac{\lambda_1}{\rho_1} - \frac{\lambda_1}{\rho}} + e^{-\frac{\lambda_1}{\rho}} \int_{\rho_1}^{\rho} f_2(x) e^{\frac{\lambda_1}{x}} dx. \tag{138}$$

By the above equations of Eq.(130-132,134,138), we know that the state

(ρ, u, p, s_{xx}) is only a function of the density ρ no matter within or after the rarefaction wave.

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