A multi-material HLLC Riemann solver with both elastic and plastic waves for 1D elastic-plastic flows

1. Introduction

In this paper, a multi-material elastic-plastic HLLC-type approximate Riemann solver(MHLLCEP) is developed, with the capability of resolving both elastic and plastic waves, to simulate one-dimensional multi-material elastic-plastic solid problems with the isotropic elastic-plastic model [?] and the von Mises' yielding condition in the framework of the high-order cell-centered Lagrangian scheme.

Generally, elastic-plastic flows can be mainly simulated in three ways, Eulerian methods [? ? ?], staggered Lagrangian schemes [?] and cell-centered Lagrangian schemes [? ? ? ?] that is considered in this paper. Comparing with the staggered Lagrangian scheme, cell-centered Lagrangian schemes have many advantages. Firstly, it's no necessary to add extra artificial viscosity which must be used in the staggered Lagrangian schemes; Secondly, it is easy to guarantee the total energy conservation; Besides, it can also be used to simulate the problems with both hyper-elastic and hypo-elastic models [? ? ? ?] as a Lagrangian scheme.

For a cell-centered Lagrangian scheme, a core process is to move the node of grids with the speed of fluids by solving a Riemann problem at each cell face. As the Riemann problem contains many physical structures, especially in elastic-plastic flows, such as elastic waves, plastic waves and contact waves , the property of the approximate Riemann solver is of great importance in the simulation. Recently, a lot of works have been done in this area. For example, Gavrilyuk et al. [?] analyzed the structure of the Riemann solution and construct a Riemann solver for the linear elastic system of hyperbolic non-conservative models for transverse waves, wherein, an extra evolution equation was added in order to make the elastic transformations reversible in the absence of shock waves. Despres [**?**] built a shock solution to a non-conservations reversible system of hypo-elasticity models and found that a sonic point is necessary to construct the compression solution that begins at a constrained compressed state. Cheng et al. [**?**] analyzed the wave structures of one-dimensional elastic-plastic flows and developed an effective two-rarefaction approximate Riemann solver with elastic waves (TRRSE) and built a second-order and a third-order cell-centered Lagrangian schemes based on the TRRSE, but the TRRSE is a little expensive as the iteration method is used. In [**?**], for one-dimensional elastic-plastic flows, Cheng introduced a HLLCE Riemann solver, which is fast and efficient in resolving elastic waves and plastic waves. In the HLLCE, Cheng evaluated the deviatoric stresses from the following *assumption: a pressure is continuous across the contact wave*. This assumption is valid for pure fluids, but in elastic-plastic flows, this assumption may lead to some errors. There are three cases we need to consider.

1. If states in the star regions on both sides of the contact wave are elastic, this assumption does not result in errors;
2. If both states reach the elastic limit, there are two cases need to be taken into account:
   (a) if both materials in both sides of the interface are same, this assumption does not result in errors either.
   (b) if materials are different, this assumption will result in big errors because the yielding strengths of different materials are different;
3. If one state in one side of the interface reaches the elastic limit, but another is not, this assumption will also result in big errors.

In this paper, in order to eliminate these errors, we want to construct a new HLLC-type Riemann solver for 1D multi-material elastic-plastic flows. In the new solver, both the elastic waves and plastic waves are correctly resolved and the assumption in [**?**] that the pressure is continuous across the interface is deleted; Correspondingly, the errors introduced by the assumption will also be eliminated.

Based on the MHLLCEP, we develop a high-order elastic-plastic cell-centered Lagrangian scheme for one-dimensional multi-material elastic-plastic flows. If we directly use the WENO reconstruction method [**?**] for multi-material elastic-plastic problems, the spacial stencil may cross the material interface and numerical oscillations may be caused near the interface. In order to delete the numerical oscillations neat the material interface, a ghost cell method is used Combined with an improved third-order WENO scheme[**?**] and the third-order Runge-Kutta scheme, a high-order cell-centered Lagrangian scheme is given in this paper for one-dimensional multi-material elastic-plastic flows.

This paper is organized as follows. In section 2, we briefly introduce the governing equations to be studied. In section 3, the MHLLCEP method is constructed. Then, high-order elastic-plastic cell-centered Lagrangian scheme is given in section 4. Some numerical examples are presented to validate the method. Conclusions are shown in section 5.

2. Governing equations

In this paper, the elastic energy is not included in the total energy. The exclution of the elastic energy is usual for practical engineering problems [**?**] and is different from that in Ref.[**?**].

The governing equations system is given as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p - s_{xx}) = 0, \\ \partial_t(\rho E) + \partial_x\left[(\rho E + p - s_{xx})u\right] = 0, \\ \partial_t s_{xx} + u\partial_x s_{xx} - \dfrac{4}{3}\partial_x u = 0, \\ |s_{xx}| \leq \dfrac{2}{3}Y_0, \end{cases} \tag{1}$$

It contains the following parts.

2.1. Conservation terms

For the continuous one-dimensional solid, the conservation terms in differential form can be given as

$$\partial_t \mathbf{U} + \partial_x \boldsymbol{F}(\mathbf{U}) = 0,\ x \in\ \Omega \subset \mathbf{R},\ t > 0,$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 - \sigma \\ (\rho E - \sigma)u \end{bmatrix}, \tag{2}$$

ρ, u, σ and E are the density, velocity in $x-$direction, Cauchy stress and total energy per unit volume, respectively, E has the relation with specific internal energy e as

$$E = e + \frac{1}{2}u^2, \tag{3}$$

$$\sigma = -p + s_{xx}, \tag{4}$$

where p and s_{xx} denote hydrostatic pressure and deviatoric stress in the $x-$direction, respectively.

2.2. The equation of state (EOS)

The relation of the pressure with the density and the specific internal energy is gotten from the equation of state (EOS). In this paper, we consider the Mie-Grüneisen EOS,

$$p(\rho, e) = \rho_0 a_0^2 f(\eta) + \rho_0 \Gamma_0 e, \tag{5}$$

where $f(\eta) = \frac{(\eta-1)(\eta-\Gamma_0(\eta-1)/2)}{(\eta-s(\eta-1))^2}$, $\eta = \frac{\rho}{\rho_0}$, ρ_0, a_0, s and Γ_0 are constant parameters of the Mie-Grüneisen EOS.

2.3. The constitutive relation

Hooke's law is used here to describe the relationship between the deviatoric stress and the strain,

$$\dot{s}_{xx} = 2\mu \left(\dot{\varepsilon}_x - \frac{1}{3}\frac{\dot{V}}{V} \right), \tag{6}$$

where μ is the shear modulus, V is the volume, and the dot means the material time derivative,

$$\dot{()} = \frac{\partial()}{\partial t} + u\frac{\partial()}{\partial t}, \tag{7}$$

and

$$\dot{\varepsilon}_x = \frac{\partial u}{\partial x}, \quad \frac{\dot{V}}{V} = \frac{\partial u}{\partial x}. \tag{8}$$

By using Eq.(8), Eq.(6) can be rewritten as

$$\frac{\partial s_{xx}}{\partial t} + u\frac{\partial s_{xx}}{\partial t} = \frac{4}{3}\mu\frac{\partial u}{\partial x}. \tag{9}$$

2.4. The yielding condition

The Von Mises' yielding condition is used here to describe the elastic limit. In one spatial dimension, the von Mises' yielding criterion is given by

$$|s_{xx}| \leq \frac{2}{3}Y_0, \tag{10}$$

where Y_0 is the yield strength of the material in simple tension.

3. The Riemann problem

The Riemann problem for the 1D time dependent elastic-plastic equations is given as follows:

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p - s_{xx}) = 0, \\ \partial_t(\rho E) + \partial_x\left[(\rho E + p - s_{xx})u\right] = 0, \\ \partial_t s_{xx} + u\partial_x s_{xx} - \dfrac{4}{3}\partial_x u = 0, \\ |s_{xx}| \leq \dfrac{2}{3}Y_0, \\ Q(x, t=0) = \begin{cases} Q_L, \text{ if } x < 0, \\ Q_R, \text{ if } x \geq 0, \end{cases} \end{cases} \tag{11}$$

where $Q = (\rho, \rho u, \rho E, s_{xx})^T$.

According to the yielding of the material, the euqations may have different Jacobian matrix and different sonic velocities. We will discuss them seprately.

3.1. Jacobian matrix in elastic regions

For the Mie-Grüneisen EOS, if the material is not yielding and

$$|s_{xx}| < \frac{2}{3}Y_0, \tag{12}$$

the system (11) can be written as

$$\partial_t \mathbf{Q} + \mathbf{J}(\mathbf{Q})\partial_x \mathbf{Q} = 0, \tag{13}$$

where $Q = (\rho, \rho u, \rho E, s_{xx})$, and the Jacobian matrix is

$$J(Q) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 + \frac{\partial p}{\partial \rho} + \Gamma(\frac{u^2}{2} - e) & u(2-\Gamma) & \Gamma & -1 \\ (\Gamma(\frac{u^2}{2} - e) - e - \frac{u^2}{2} + \frac{\sigma}{\rho} + \frac{\partial p}{\partial \rho})u & -\Gamma u^2 - \frac{\sigma}{\rho} + \frac{u^2}{2} + e & (1+\Gamma)u & -u \\ \frac{4}{3}\mu\frac{u}{\rho} & -\frac{4}{3}\mu\frac{1}{\rho} & 0 & u \end{bmatrix}, \tag{14}$$

where $\Gamma = \frac{\Gamma_0 \rho_0}{\rho}$.

The eigenvalues of the coefficient matrix $\mathbf{J}(\mathbf{Q})$ are given as

$$\lambda_1 = \lambda_2 = u, \quad \lambda_3 = u - c, \quad \lambda_4 = u + c, \tag{15}$$

where

$$\begin{cases} c = \sqrt{a^2 - \dfrac{\rho_0}{\rho^2}\Gamma_0 s_{xx} + \dfrac{4}{3}\dfrac{\mu}{\rho}}, \\ a^2 = \dfrac{\partial p}{\partial \rho} + \dfrac{p}{\rho^2}\dfrac{\partial p}{\partial e} = a_0^2 \dfrac{\partial f}{\partial \eta} + \dfrac{p}{\rho^2}\rho_0 \Gamma_0. \end{cases} \tag{16}$$

The corresponding right eigenvectors are

$$r_1 = \begin{bmatrix} \frac{1}{b_1} \\ \frac{u}{b_1} \\ 0 \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} -\frac{\Gamma}{b_1} \\ -\frac{\Gamma u}{b_1} \\ 1 \\ 0 \end{bmatrix}, \quad r_3 = \frac{1}{\phi^2}\begin{bmatrix} 1 \\ u-c \\ h-uc \\ \phi^2 \end{bmatrix}, \quad r_4 = \frac{1}{\phi^2}\begin{bmatrix} 1 \\ u+c \\ h+uc \\ \phi^2 \end{bmatrix}, \tag{17}$$

where

$$b_1 = \frac{\partial p}{\partial \rho} - \Gamma E, \quad h = E + \frac{p - s_{xx}}{\rho}, \tag{18}$$

and

$$\phi^2 = a^2 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} - c^2 = -\frac{4\mu}{3}\frac{1}{\rho}. \tag{19}$$

3.2. Jacobian matrix in plastic regions

When the material is yielding,

$$|s_{xx}| = \frac{2}{3}Y_0, \tag{20}$$

the equations will turn into a more simple system with only constitutive terms as

$$\partial_t \mathbf{U} + \mathbf{J}_p(\mathbf{U})\partial_x \mathbf{U} = 0, \tag{21}$$

where $\mathbf{U} = (\rho, \rho u, \rho E)$, and the Jacobian matrix is

$$\mathbf{J}_p(\mathbf{U}) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + \frac{\partial p}{\partial \rho} + \Gamma(\frac{u^2}{2} - e) & u(2-\Gamma) & \Gamma \\ (\Gamma(\frac{u^2}{2} - e) - e - \frac{u^2}{2} + \frac{\sigma}{\rho} + \frac{\partial p}{\partial \rho})u + \frac{u^2}{2} & -\Gamma u^2 - \frac{\sigma}{\rho} + e & (1+\Gamma)u \end{bmatrix}. \tag{22}$$

The eigenvalues of The eigenvalues of the coefficient matrix $\mathbf{J}_p(\mathbf{Q})$ are given as

$$\lambda_1 = u, \quad \lambda_2 = u - c, \quad \lambda_3 = u + c,$$

where

$$\begin{cases} c = \sqrt{a^2 - \dfrac{\rho_0}{\rho^2}\Gamma_0 s_{xx}}, \\ a^2 = \dfrac{\partial p}{\partial \rho} + \dfrac{p}{\rho^2}\dfrac{\partial p}{\partial e} = a_0^2 \dfrac{\partial f}{\partial \eta} + \dfrac{p}{\rho^2}\rho_0\Gamma_0. \end{cases} \tag{23}$$

The corresponding right eigenvectors are

$$r_1 = \begin{bmatrix} 1 \\ u \\ E - \frac{c^2}{\Gamma} - \frac{\sigma}{\rho} \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ u - c \\ E - uc - \frac{\sigma}{\rho} \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 \\ u + c \\ E + uc - \frac{\sigma}{\rho} \end{bmatrix}. \tag{24}$$

Take a comparason of Eq.(16) and Eq.(23), we notice that the sonic speed is not continuous between the states of elastic and plastic. This is very important and may cause wrong results if ignoring it.

3.3. Formulations across the contact wave

For a system without molecular diffusion, there is no materials convecting across the contact wave or interface, so the velocities on two sides of the discontinuity are always equal. This can also be verified by the eigenvectors in Eq.(17) and Eq.(24).

Using $\mathbf{W}_L^*$ and $\mathbf{W}_R^*$ to denote the two data states connected the contact wave, where $\mathbf{W} = (\rho, u, p, s_{xx})$.

According to the eigenvectors in Eq.(17), for the λ_1-wave and λ_2 wave, we have

$$\frac{d\rho}{1} = \frac{d\rho u}{u}. \tag{25}$$

From the above equations, we can easily deduce that

$$du = 0, \tag{26}$$

Samilarly, using the eigenvectors in Eq.(24), for the λ_1-wave, we have

$$\frac{d\rho}{1} = \frac{d\rho u}{u}, \tag{27}$$

we also can get

$$du = 0, \tag{28}$$

which means

$$u_L^* = u_R^*, \tag{29}$$

For convenience, we define

$$s^* = u_L^* = u_R^*. \tag{30}$$

where s^* denotes the velocity of the contact wave.

Then using the conservation relations between the contact wave

$$\boldsymbol{F}_R^* = \boldsymbol{F}_L^* + s^*(\boldsymbol{U}_R^* - \boldsymbol{U}_L^*), \tag{31}$$

From the momentum term in Eq.(31),

$$\rho_R^* u_R^{*2} - \sigma_R^* = \rho_L^* u_L^{*2} - \sigma_L^* + s^*(\rho_R^* u_R^* - \rho_L^* u_L^*). \tag{32}$$

and taking $s^* = u_L^* = u_R^*$ in, we can get the relation of Cauchy stress

$$\sigma_L^* = \sigma_R^*, \tag{33}$$

is always satisfied.

3.4. A relation between ρ and s_{xx} in 1D elastic-plastic equation system

Thanks to (7), the equations of the density and the deviatoric stress in Eq.(11) can be written as

$$\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{d\rho}{dt}, \tag{34}$$

and

$$\frac{ds_{xx}}{dt} = \frac{4}{3}\mu\frac{\partial u}{\partial x}. \tag{35}$$

Substituting (34) into (35) yields

$$\frac{ds_{xx}}{dt} = -\frac{4}{3}\mu\frac{1}{\rho}\frac{d\rho}{dt}. \tag{36}$$

Integrate the above equation from the data in front of a wave to the data behind the wave and perform some simple algebraic manipulations, one can get

$$s_{xx2} = -\frac{4}{3}\mu\ln(\frac{\rho_2}{\rho_1}) + s_{xx1}. \tag{37}$$

The subscripts 2 and 1 mean the states in front of and behind the wave, respectively. This relation always hold if there is no yielding in the integration path.

3.5. Relations across rarefaction waves

A Rarefaction wave contains a continuous of waves with the speed of $u \pm c$. Now we consider a left-going rarefaction wave as shown in Fig.**??** as an example, the state in front of the wave is known as $(\rho_1, u_1, p_1, s_{xx1})$ and after the wave the state changes to $(\rho_2, u_2, p_2, s_{xx2})$ which is unknown.

We random choose a particle point inside the expansion region with the state of (ρ, u, p, s_{xx}), after a little expansion with the wave speed of $u - c$, the state changes to$(\rho + d\rho, u + du, p + dp, s_{xx} + ds_{xx})$. According to the conservation relations we can get

$$\begin{aligned}
&(\rho + \mathrm{d}\rho)(u + \mathrm{d}u) = \rho u + (u - c)\mathrm{d}\rho \\
&(\rho + \mathrm{d}\rho)(u + \mathrm{d}u)^2 - (\sigma + \mathrm{d}\sigma) = \rho u^2 - \sigma \\
&+ (u - c)[(\rho + \mathrm{d}\rho)(u + \mathrm{d}u) - \rho u] \\
&[(\rho + \mathrm{d}\rho)(E + \mathrm{d}E) - (\sigma + \mathrm{d}\sigma)](u + \mathrm{d}u) = \\
&(\rho E - \sigma)u + (u - c)[(\rho + \mathrm{d}\rho)(E + \mathrm{d}E) - \rho E]
\end{aligned} \tag{38}$$

Ignoring high-order infinite small terms, we can get

$$\mathrm{d}u = -\frac{c}{\rho}\mathrm{d}\rho, \tag{39}$$

$$\mathrm{d}\sigma = -c^2 d\rho, \tag{40}$$

$$\mathrm{d}E = -\frac{\sigma + \rho u c}{\rho^2} d\rho. \tag{41}$$

Also we know the deviatoric stress is a function of the density by Eq.(37)

$$ds_{xx} = \begin{cases} -\frac{4}{3}\frac{\mu}{\rho} d\rho & |s_{xx}| < \frac{2}{3}Y_0, \\ 0 & |s_{xx}| \geq \frac{2}{3}Y_0. \end{cases} \tag{42}$$

And from Eq.(16) and Eq.(23) we can get

$$c^2 = \begin{cases} a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0\Gamma_0 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} + \frac{4}{3}\frac{\mu}{\rho} & |s_{xx}| < \frac{2}{3}Y_0, \\ a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0\Gamma_0 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} & |s_{xx}| \geq \frac{2}{3}Y_0. \end{cases} \tag{43}$$

Then the differential of pressure is given as

$$dp = ds_{xx} - d\sigma = ds_{xx} + c^2 d\rho = \left(a_0^2 \frac{\partial f}{\partial \eta} + \frac{p}{\rho^2}\rho_0\Gamma_0 - \frac{\rho_0}{\rho^2}\Gamma_0 s_{xx} \right) d\rho, \tag{44}$$

it can be written as a differential equation of $p(\rho)$

$$p'(\rho) - \lambda_1 \frac{p}{\rho^2} = f_2(\rho), \quad p(\rho_1) = p_1, \tag{45}$$

where

$$\lambda_1 = \rho_0\Gamma_0 \quad f_2(\rho) = a_0^2 \frac{\partial f}{\partial \eta} - \lambda_1 \frac{s_{xx}(\rho)}{\rho^2}. \tag{46}$$

The pressure can be solved out as

$$p(\rho) = p_1 e^{\frac{\lambda_1}{\rho_1} - \frac{\lambda_1}{\rho}} + e^{-\frac{\lambda_1}{\rho}} \int_{\rho_1}^{\rho} f_2(x) e^{\frac{\lambda_1}{x}} dx. \tag{47}$$

By the above equations of Eq.(39-41,43,47), we know that the state (ρ, u, p, s_{xx})

is only a function of the density ρ no matter within or after the rarefaction wave.

3.6. Relations across shock waves

Now we consider a shock wave with a speed of s, suppose the state in front of the shock is known as $(\rho_1, u_1, p_1, s_{xx1})$ and the state after the shock is unknown as $(\rho_2, u_2, p_2, s_{xx2})$.

Then use the conservation relation across the wave, which is also known as the Rankine-Hugoniot relation for a shock

$$\rho_2 u_2 = \rho_1 u_1 + s(\rho_2 - \rho_1), \tag{48}$$

$$\rho_2 u_2^2 - \sigma_2 = \rho_1 u_1^2 - \sigma_1 + s(\rho_2 u_2 - \rho_1 u_1), \tag{49}$$

$$(\rho_2 E_2 - \sigma_2)u_2 = (\rho_1 E_1 - \sigma_1)u_1 + s(\rho_2 E_2 - \rho_1 E_1), \tag{50}$$

also can be written as

$$\rho_2(u_2 - s) = \rho_1(u_1 - s), \tag{51}$$

$$\rho_2 u_2(u_2 - s) = \rho_1 u_1(u_1 - s) + \sigma_2 - \sigma_1, \tag{52}$$

$$\rho_2 E_2(u_2 - s) = \rho_1 E_1(u_1 - s) + \sigma_2 u_2 - \sigma_1 u_1, \tag{53}$$

substituting (51) into (52) yields

$$\rho_1(u_2 - u_1)(u_1 - s) = \sigma_2 - \sigma_1, \tag{54}$$

also according to (51), we have

$$u_1 - s = \frac{(u_1 - u_2)\rho_2}{\rho_2 - \rho_1}, \tag{55}$$

then subtituting it into (54)

$$-t(u_2 - u_1)^2 = \sigma_2 - \sigma_1, \tag{56}$$

where $t = \frac{\rho_1\rho_2}{\rho_2-\rho_1}$.

Simimar to (54), (53) can be changed into

$$t(u_1 - u_2)(E_2 - E_1) = \sigma_2 u_2 - \sigma_1 u_1. \tag{57}$$

Because of $E = e + \frac{1}{2}u^2$, we can get

$$e_2 - e_1 = -\frac{\sigma_1 + \sigma_2}{2t}. \tag{58}$$

Using the EOS of Mie-Grüneisen (5), can get

$$e = c_0 p - c_1 f(\rho/\rho_0), \tag{59}$$

where $c_0 = \frac{1}{\rho_0\Gamma_0}$ and $c_1 = \frac{a_0^2}{\Gamma_0}$. Put above equation into (58), we can solve the pressure p_2 out as a function of ρ_2.

$$p_2 = \frac{2t(c_1 f(\rho_2/\rho_0) + e_1) - (\sigma_1 + s_{xx2})}{2tc_0 - 1}, \tag{60}$$

The derivative stress σ_{xx2} is also a function of rho_2 only. So we can solve the Cauchy stress out as

$$\sigma_2 = -p_2 + s_{xx2}. \tag{61}$$

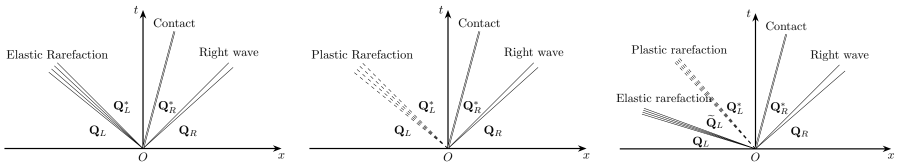
Then we can use (56) to solve the velocity after the shock

$$u_2 = \begin{cases} u_1 - \sqrt{\dfrac{\sigma_1 - \sigma_2}{t}} & \text{Left-going,} \\ u_1 + \sqrt{\dfrac{\sigma_1 - \sigma_2}{t}} & \text{Right-going.} \end{cases} \tag{62}$$

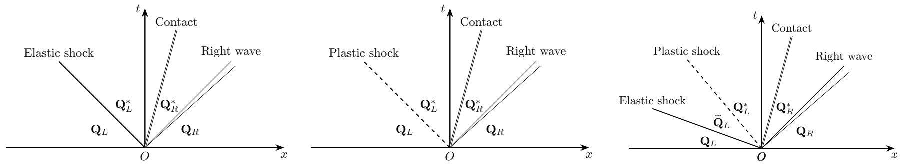
And the shock speed is given as

$$s = \frac{\rho_2 u_2 - \rho_1 u_1}{\rho_2 - \rho_1}. \tag{63}$$

By the above deductions of the shock wave, we can get that, if the density



(a) Elastic rarefaction wave (b) Plastic rarefaction wave (c) Both elastic and plastic rarefaction waves



(d) Elastic shock wave (e) Plastic shock wave (f) Both elastic and plastic shock waves

Figure 1: The possible cases of Riemann solution structures in the left side.

after the shock is known, all the unknowns can be solved out.

4. Exact Riemann solver

Now we consider the constructing details of the exact Riemann solver. For the Riemann problem in Section **??**, there are 6×6 possible cases in the Riemann solution with different wave structures. Now we consider the left six cases as shown in Fig.**??**.

There are mainly three parts in the soluting process.

Firstly, a pre-evaluation need to be done to detect the left cases in Fig.**??** and the corresponding right cases. This is done in Section (**??**) with shock-contact-shock waves assumption.

Secondly, for the cases with two waves in one side (i.e. case c and case f in Fig.**??**),we need to solve the states in the region $\tilde{\mathbf{Q}}_L$ and $\tilde{\mathbf{Q}}_R$. This is done in Section **??**.

At last, we know all the states besides Q_L^* and Q_R^*. According to the knowledge in Section 3.5 and Section 3.6, if we know the densities ρ_L^* and ρ_R^*, we can solve all other unknowns out. Also, by the informations in Section 3.3, the velocities and the Cauchy stresses in the two sides of the contact must be equal.

So there are two equations

$$\begin{aligned} f_u(\rho_L^*, \rho_R^*) &= u_L^* - u_R^* = 0, \\ f_\sigma(\rho_L^*, \rho_R^*) &= \sigma_L^* - \sigma_R^* = 0, \end{aligned} \tag{64}$$

and two unknowns ρ_L^* and ρ_R^*. Equation (**??**) is implicit, we have use a Newton iteration method to solve it, this process is given in Section **??**. The details of f_u and f_σ are listed in Section **??**.

4.1. Pre-evaluation and case classfication

At first, we assume that there are only one shock wave in the left side and one shock in the right side as shown in Fig.**??**. According to the conservation relations across the shocks, we have

$$\begin{cases} \hat{\rho}_L \hat{u}_L^* = \rho_L u_L + s_L(\hat{\rho}_L^* - \rho_L), \\ \hat{\rho}_L \hat{u}_L^{*2} - \sigma_L^* = \rho_L u_L^2 - \sigma_L + s_L(\hat{\rho}_L^* \hat{u}_L^* - \rho_L u_L), \end{cases} \tag{65}$$

and

$$\begin{cases} \hat{\rho}_R \hat{u}_R^* = \rho_R u_R + s_R(\hat{\rho}_R^* - \rho_R), \\ \hat{\rho}_R \hat{u}_R^{*2} - \sigma_R^* = \rho_R u_R^2 - \sigma_R + s_R(\hat{\rho}_R^* \hat{u}_R^* - \rho_R u_R). \end{cases} \tag{66}$$

Using the relation across the interface,

$$\hat{u}_L^* = \hat{u}_R^* = \hat{s}^*, \quad \hat{\sigma}_L^* = \hat{\sigma}_R^*. \tag{67}$$

the speed of contact wave can be evaluated as

$$\hat{s}^* = \frac{\sigma_L - \sigma_R + \rho_L u_L(s_L - u_L) - \rho_R u_R(s_R - u_R)}{\rho_L(s_L - u_L) - \rho_R(s_R - u_R)}, \tag{68}$$

the density is solved as

$$\hat{\rho}_L^* = \frac{\rho_L(u_L - s_L)}{\hat{s}^* - s_L}, \quad \hat{\rho}_R^* = \frac{\rho_R(u_R - s_R)}{\hat{s}^* - s_R}. \tag{69}$$

Table 1: The condition of cases classification.

Conditions	$\lvert s_{xx}\rvert < \frac{2}{3}Y_0$ and $\lvert\hat{s}_{xx}\rvert < \frac{2}{3}Y_0$	$s_{xx} = \frac{2}{3}Y_0$	other
$\hat{\rho}^* < \rho$	case a	case b	case c
$\hat{\rho}^* > \rho$	case d	case e	case f

The devaitoric stress is ecaluated as

$$\hat{s}^*_{xxL} = -\frac{4}{3}\mu \ln(\frac{\hat{\rho}^*_L}{\rho_L}) + s_{xxL}, \quad \hat{s}^*_{xxR} = -\frac{4}{3}\mu \ln(\frac{\hat{\rho}^*_R}{\rho_R}) + s_{xxR}. \tag{70}$$

Here we define the speeds of left and right going waves as

$$s_L = \min(u_L - c_L, u_R - c_R, 0), \quad s_R = \max(u_L + c_L, u_R + c_R, 0). \tag{71}$$

By the relation of $\hat{\rho}_{L(R)}$ and $\hat{s}_{xxL(R)}$ we can classify every side into six cases, and conditions for the classfication are shown in Table **??**, the subscripts $_L$ and $_R$ are omitted for simplication.

4.2. State in region $\tilde{\mathbf{Q}}_{L(R)}$

For cases (a,b,d,e) in Fig.**??**, the material is totally yielding or totally not yielding, if we give a density ρ^*_L in the region $\mathbf{Q}^*_L$, by the deductions in Section 3.5 and Section 3.6, all the unknowns in region $\mathbf{Q}^*_L$ can be solved out.

For cases (c,f), the meterial periods a yielding process from elastic to plastic. There is one more state $\tilde{\mathbf{Q}}_{L(R)}$ exists. In state $\tilde{\mathbf{Q}}_{L(R)}$, the derivative stress achieves the elastic limit

$$\tilde{s}_{xxL(R)} = \begin{cases} \frac{2}{3}Y_0 & \text{Case c,} \\ -\frac{2}{3}Y_0 & \text{Case f,} \end{cases} \tag{72}$$

By (37), we can solve the density out as

$$\tilde{\rho}_{L(R)} = \begin{cases} \widetilde{\rho}_{L(R)} = \rho_{L(R)} \exp\left(-\frac{Y_0}{2\mu} + \frac{3s_{xxL(R)}}{4\mu}\right), & \text{Case c,} \\ \widetilde{\rho}_{L(R)} = \rho_{L(R)} \exp\left(\frac{Y_0}{2\mu} + \frac{3s_{xxL(R)}}{4\mu}\right), & \text{Case f.} \end{cases} \tag{73}$$

As the density in region $\mathbf{Q}^*_{L(R)}$ is known, all the pressure and velocity can be solved by equations (47,43, 39) for a rarefaction wave or by equations (60, 62) for a shock wave.

4.2.1. Elastic rarefaction wave in case c

For case c, the pressure within the elastic rarefaction wave is

$$p(\rho) = p_{L(R)} e^{\frac{\lambda_1}{\rho_{L(R)}} - \frac{\lambda_1}{\rho}} + e^{-\frac{\lambda_1}{\rho}} \int_{\rho_{L(R)}}^{\rho} f_2(x) e^{\frac{\lambda_1}{x}} dx, \quad \tilde{\rho}_{L(R)} \leq \rho \leq \rho_{L(R)}. \tag{74}$$

The sonic speed is

$$c(\rho) = \sqrt{a_0^2 \frac{\partial f}{\partial \eta} + \frac{p(\rho)}{\rho^2} \rho_0 \Gamma_0 - \frac{\rho_0}{\rho^2} \Gamma_0 s_{xx} + \frac{4}{3} \frac{\mu}{\rho}} \quad \tilde{\rho}_{L(R)} \leq \rho \leq \rho_{L(R)}. \tag{75}$$

With the sonic speed, the velocity can be solved as

$$u(\rho) = \begin{cases} u_L - \displaystyle\int_{\rho_L}^{\rho} \frac{c(x)}{x} dx, & \tilde{\rho}_L \leq \rho \leq \rho_L \\ u_R + \displaystyle\int_{\rho_R}^{\rho} \frac{c(x)}{x} dx, & \tilde{\rho}_R \leq \rho \leq \rho_R \end{cases} \tag{76}$$

So the pressure and velocity after the elastic wave are

$$\tilde{p}_{L(R)} = p(\tilde{\rho}_{L(R)}), \quad \tilde{u}_{L(R)} = u(\tilde{\rho}_{L(R)}). \tag{77}$$

4.2.2. Elastic shock wave in case f

For a shock wave (case f), the pressure is

$$\tilde{p}_{L(R)} = \frac{2t(c_1 f(\tilde{\rho}_{L(R)}/\rho_0) + e_L) - (\sigma_{L(R)} + \tilde{s}_{xxL(R)})}{2tc_0 - 1}, \tag{78}$$

where $c_0 = \frac{1}{\rho_0 \Gamma_0}$ and $c_1 = \frac{a_0^2}{\Gamma_0}$ and $t = \frac{\rho_{L(R)} \tilde{\rho}_{L(R)}}{\tilde{\rho}_{L(R)} - \rho_{L(R)}}$. And the velocity is

$$\tilde{u}_L = u_L - \sqrt{\frac{\sigma_L - \tilde{\sigma}_L}{t}}, \tag{79}$$

for the left, and

$$\tilde{u}_R = u_R + \sqrt{\frac{\sigma_R - \tilde{\sigma}_R}{t}}, \tag{80}$$

for the right-going wave.

4.3. Functions of velocity and Cauchy stress in region $\mathbf{Q}^*_{L(R)}$

The deductions of velocity and Cauchy stress for rarefaction waves and shocks are given in Section 3.5 and Section 3.6. For convenient, we list them in this section in different cases.

The deviatoric stress is

$$s^*_{xxL(R)} = \begin{cases} s_{xxL(R)} & \text{case (b, e)}, \\ -\dfrac{4}{3}\mu \ln\left(\dfrac{\rho^*_{L(R)}}{\rho_{L(R)}}\right) + s_{xxL(R)} & \text{case (a,d)}, \\ \dfrac{2}{3}Y_0 & \text{case (c)}, \\ -\dfrac{2}{3}Y_0 & \text{case (f)}. \end{cases} \tag{81}$$

The pressure for case (a,b,c) is

$$p^*_{L(R)} = \begin{cases} p_{L(R)} e^{\frac{\lambda_1}{\rho_{L(R)}} - \frac{\lambda_1}{\rho^*_{L(R)}}} + e^{-\frac{\lambda_1}{\rho^*_{L(R)}}} \displaystyle\int_{\rho_{L(R)}}^{\rho^*_{L(R)}} f_2(x) e^{\frac{\lambda_1}{x}} dx. & \text{case (a,b)}, \\ \tilde{p}_{L(R)} e^{\frac{\lambda_1}{\tilde{\rho}_{L(R)}} - \frac{\lambda_1}{\rho^*_{L(R)}}} + e^{-\frac{\lambda_1}{\rho^*_{L(R)}}} \displaystyle\int_{\tilde{\rho}_{L(R)}}^{\rho^*_{L(R)}} f_2(x) e^{\frac{\lambda_1}{x}} dx. & \text{case (c)}, \end{cases} \tag{82}$$

after get the pressure we can give the sonic speed as

$$c^{*2}_{L(R)} = \begin{cases} a_0^2 \dfrac{\partial f}{\partial \eta} + \dfrac{p^*_{L(R)}}{\rho^{*2}_{L(R)}} \rho_0 \Gamma_0 - \dfrac{\rho_0}{\rho^{*2}_{L(R)}} \Gamma_0 s_{xx} + \dfrac{4}{3} \dfrac{\mu}{\rho^*_{L(R)}} & \text{case a}, \\ a_0^2 \dfrac{\partial f}{\partial \eta} + \dfrac{p}{\rho^{*2}_{L(R)}} \rho_0 \Gamma_0 - \dfrac{\rho_0}{\rho^{*2}_{L(R)}} \Gamma_0 s^*_{xxL(R)} & \text{case (b,c)}. \end{cases} \tag{83}$$

then the velocity can be solved as

$$u_L^* = \begin{cases} u_L - \displaystyle\int_{\rho_L}^{\rho_L^*} \frac{c}{\rho} d\rho & \text{case a,b,} \\ \tilde{u}_L - \displaystyle\int_{\tilde{\rho}_L}^{\rho_L^*} \frac{c}{\rho} d\rho & \text{case c,} \end{cases} \quad u_R^* = \begin{cases} u_R + \displaystyle\int_{\rho_R}^{\rho_R^*} \frac{c}{\rho} d\rho & \text{case a,b,} \\ \tilde{u}_R + \displaystyle\int_{\tilde{\rho}_R}^{\rho_R^*} \frac{c}{\rho} d\rho & \text{case c.} \end{cases} \tag{84}$$

4.4. An iteration process of ρ_L^* and ρ_R^*

As the states in regions $\mathbf{Q}_L$ and $\mathbf{Q}_R$ are known, for cases (c,f) with two waves in one side, regions $\hat{\mathbf{Q}}_L$ and $\hat{\mathbf{Q}}_L$ are also solved in the above section, by the relaions in Section 3.5 and Section 3.6, if we know the densities ρ_L^* and ρ_R^*, states in regions $\mathbf{Q}_L^*$ and $\mathbf{Q}_R^*$ can be deduced out. Till then, the whole Riemann problem is solved.

The Newton iteration to evaluate ρ_L^* and ρ_R^* is given as

$$\begin{bmatrix} \rho^*_{L,(k+1)} \\ \rho^*_{R,(k+1)} \end{bmatrix} = \begin{bmatrix} \rho^*_{L,(k)} \\ \rho^*_{R,(k)} \end{bmatrix} - \begin{bmatrix} \frac{\partial f_{u(k)}}{\partial \rho_L^*} & \frac{\partial f_{u(k)}}{\partial \rho_R^*} \\ \frac{\partial f_{\sigma(k)}}{\partial \rho_L^*} & \frac{\partial f_{\sigma(k)}}{\partial \rho_R^*} \end{bmatrix}^{-1} \begin{bmatrix} f_{u(k)} \\ f_{\sigma(k)} \end{bmatrix} \tag{85}$$

The initial of densities are given as the pre-evalation values,

$$\rho^*_{L(0)} = \hat{\rho}_L \quad \rho^*_{R(0)} = \hat{\rho}_R. \tag{86}$$

The convergence is meassured by

$$\text{CHA} = \max\left[\frac{|\rho^*_{L(k+1)} - \rho^*_{L,(k)}|}{\frac{1}{2}|\rho^*_{L(k+1)} + \rho^*_{L(k)}|}, \frac{|\rho^*_{R(k+1)} - \rho^*_{R,(k)}|}{\frac{1}{2}|\rho^*_{R(k+1)} + \rho^*_{R(k)}|}, |f_u|, |f_\sigma|\right]. \tag{87}$$

and the tolerance is taken as TOL $= 10^{-4}$. It usually takes 3-4 step to get a convergence result.

The derivatives of f_u and f_σ are given by

$$\frac{\partial f_{u,(k+1)}}{\partial \rho^*_{L(R)}} = \frac{f_{u,(k+1)} - f_{u,(k)}}{\rho^*_{L(R),(k+1)} - \rho_{L(R),(k)}}, \quad \frac{\partial f_{\sigma,(k+1)}}{\partial \rho^*_{L(R)}} = \frac{f_{\sigma,(k+1)} - f_{\sigma,(k)}}{\rho^*_{L(R),(k+1)} - \rho_{L(R),(k)}}, \tag{88}$$

At the first step, we use a simple numerical difference method

$$\frac{\partial f_{u,(1)}}{\partial \rho^*_{L(R)}} = \frac{f_u(\rho^*_{L(R)} + \Delta\rho) - f_u(\rho^*_{L(R)})}{\Delta\rho_{L(R)}}, \quad \frac{\partial f_{u,(1)}}{\partial \rho^*_{L(R)}} = \frac{f_u(\rho^*_{L(R)} + \Delta\rho) - f_u(\rho^*_{L(R)})}{\Delta\rho_{L(R)}}, \tag{89}$$

where $\Delta\rho$ is a little quatity, we can choose it as

$$\Delta\rho_{L(R)} = \frac{\rho^*_{L(R),(0)}}{100}. \tag{90}$$

A flow chat of this process is shown in Fig.

4.5. Summary of MHLLCEP

Here, we present all the procedures of MHLLCEP in a more simple way.

1. Assume that there are three waves in the Riemann solver. Based on this assumption, we perform the following evaluations:
   (a) evaluate $\hat{s}^*$

   $$\hat{s}^* = \frac{\sigma_L - \sigma_R + \rho_L u_L(s_L - u_L) - \rho_R u_R(s_R - u_R)}{\rho_L(s_L - u_L) - \rho_R(s_R - u_R)}.$$

   (b) Evaluate $\hat{\rho}^*_L$ and $\hat{\rho}^*_R$

   $$\hat{\rho}^*_L = \frac{\rho_L(u_L - s_L)}{s^* - s_L}, \quad \hat{\rho}^*_R = \frac{\rho_R(u_R - s_R)}{s^* - s_R}.$$

   (c) Evaluate the deviatoric stress

   $$\hat{s}^*_{xxL} = -\frac{4}{3}\mu\ln(\frac{\hat{\rho}^*_L}{\rho_L}) + s_{xxL}, \quad \hat{s}^*_{xxR} = -\frac{4}{3}\mu\ln(\frac{\hat{\rho}^*_R}{\rho_R}) + s_{xxR}.$$

   (d) Evaluate the pressure
2. Decide whether the state reach the elastic limit or not
   (a) If $|s_{xxL}| < \frac{2}{3}Y_0 \leq |\hat{s}^*_{xxL}|$, there are two waves exist in the left, we need to evaluate the left yielding state.

The deviatoric stress, density and pressure behind the left elastic wave are given as

$$\widetilde{s}_{xxL} = \begin{cases} -\dfrac{2}{3}Y_0, & \text{if} \quad \rho_L^* > \rho_L, \\ \dfrac{2}{3}Y_0, & \text{if} \quad \rho_L^* < \rho_L, \end{cases}$$

$$\widetilde{\rho}_L = \begin{cases} \rho_L \exp\left(\dfrac{Y_0}{2\mu} + \dfrac{3s_{xxL}}{4\mu}\right) & \text{if} \quad \rho_L^* > \rho_L, \\ \rho_L \exp\left(-\dfrac{Y_0}{2\mu} + \dfrac{3s_{xxL}}{4\mu}\right) & \text{if} \quad \rho_L^* < \rho_L, \end{cases}$$

$$\widetilde{p}_L = \frac{2t(c_1 f(\widetilde{\rho}_L) + e_L) - (\sigma_L + \widetilde{s}_{xxL})}{2tc_0 - 1}, \quad t = \frac{\rho_L \widetilde{\rho}_L}{\widetilde{\rho}_L - \rho_L},$$

and the Cauchy stress and velocity are

$$\widetilde{\sigma}_L = -\widetilde{p}_L + \widetilde{s}_{xxL},$$

$$\widetilde{u}_L = \begin{cases} u_L - \sqrt{\dfrac{\sigma_L - \widetilde{\sigma}_L}{t}} & \text{if} \ \rho_L^* > \rho_L, \\ u_L + \sqrt{\dfrac{\sigma_L - \widetilde{\sigma}_L}{t}} & \text{if} \ \rho_L^* < \rho_L. \end{cases}$$

If not,

$$\widetilde{\mathbf{Q}}_L = \mathbf{Q}_L.$$

(b) If $|s_{xxR}| < \frac{2}{3}Y_0 \le |\hat{s}_{xxR}^*|$, right elastic and plastic wave exist, we need to evaluate the right yielding state.
The deviatoric stress, density and pressure behind the right elastic wave are given as

$$\widetilde{s}_{xxR} = \begin{cases} -\dfrac{2}{3}Y_0, & \text{if} \quad \rho_R^* > \rho_R, \\ \dfrac{2}{3}Y_0, & \text{if} \quad \rho_R^* < \rho_R, \end{cases} \quad \widetilde{\rho}_R = \begin{cases} \rho_R \exp\left(\dfrac{Y_0}{2\mu} + \dfrac{3s_{xxR}}{4\mu}\right) & \text{if} \quad \rho_R^* > \rho_R, \\ \rho_R \exp\left(-\dfrac{Y_0}{2\mu} + \dfrac{3s_{xxR}}{4\mu}\right) & \text{if} \quad \rho_R^* < \rho_R, \end{cases}$$

$$\widetilde{p}_R = \frac{2t(c_1 f(\widetilde{\rho}_R) + e_R) - (\sigma_R + \widetilde{s}_{xxR})}{2tc_0 - 1}, \quad t = \frac{\rho_R \widetilde{\rho}_R}{\widetilde{\rho}_R - \rho_R},$$

and the Cauchy stress and velocity are

$$\widetilde{\sigma}_R = -\widetilde{p}_R + \widetilde{s}_{xxR},$$

$$\widetilde{u}_R = \begin{cases} u_R + \sqrt{\dfrac{\sigma_R - \widetilde{\sigma}_R}{t}} & \text{if } \rho_R^* > \rho_R, \\ u_R - \sqrt{\dfrac{\sigma_R - \widetilde{\sigma}_R}{t}} & \text{if } \rho_R^* < \rho_R. \end{cases}$$

If not,

$$\widetilde{\mathbf{Q}}_R = \mathbf{Q}_R.$$

3. Re-evaluate the states in the star regions on both sides of the contact wave
   (a) Evaluate the wave speeds:

$$s_L = \min(\widetilde{u}_L - \widetilde{c}_L, \widetilde{u}_R - \widetilde{c}_R, 0), \quad s_R = \max(\widetilde{u}_L + \widetilde{c}_L, \widetilde{u}_R + \widetilde{c}_R, 0),$$

$$s^* = \frac{\widetilde{\sigma}_L - \widetilde{\sigma}_R + \widetilde{\rho}_L \widetilde{u}_L (s_L - \widetilde{u}_L) - \widetilde{\rho}_R \widetilde{u}_R (s_R - \widetilde{u}_R)}{\widetilde{\rho}_L (s_L - \widetilde{u}_L) - \widetilde{\rho}_R (s_R - \widetilde{u}_R)}.$$

   (b) Evaluate the densities:

$$\rho_L^* = \frac{\widetilde{\rho}_L (\widetilde{u}_L - s_L)}{s^* - s_L}, \quad \rho_R^* = \frac{\widetilde{\rho}_R (\widetilde{u}_R - s_R)}{s^* - s_R},$$

   (c) Evaluate the deviatoric stresses:

$$\widetilde{s}_{xxL}^* = \begin{cases} \widetilde{s}_{xxL} & if\ |\hat{s}_{xxL}^*| \geq \frac{2}{3} Y_0 \\ -\frac{4}{3}\mu \ln\left(\frac{\rho_L^*}{\widetilde{\rho}_L}\right) + \widetilde{s}_{xxL} & otherwise \end{cases},$$

$$\widetilde{s}_{xxR}^* = \begin{cases} \widetilde{s}_{xxR} & if\ |\hat{s}_{xxR}^*| \geq \frac{2}{3} Y_0 \\ -\frac{4}{3}\mu \ln\left(\frac{\rho_R^*}{\widetilde{\rho}_R}\right) + \widetilde{s}_{xxR} & otherwise \end{cases},$$

   then using the von Mises' yielding condition

$$s_{xxL}^* = \Upsilon(\widetilde{s}_{xxL}^*), \quad s_{xxR}^* = \Upsilon(\widetilde{s}_{xxR}^*).$$

(d) Solving the Cauchy stresses:

$$\sigma_L^* = \sigma_R^* = \widetilde{\sigma}_L - \widetilde{\rho_L}(s_L - \widetilde{u}_L)(s^* - \widetilde{u}_L).$$

(e) The pressure is given by $p = s_{xx} - \sigma$.