Matrix Algebra

Definition:

A matrix is a set of numbers arranged in rows and columns to form a rectangular array

Matrices provide a convenient notation for displaying and manipulating data in tabular form.

The basis for spreadsheets
Arrays are extensive in programming

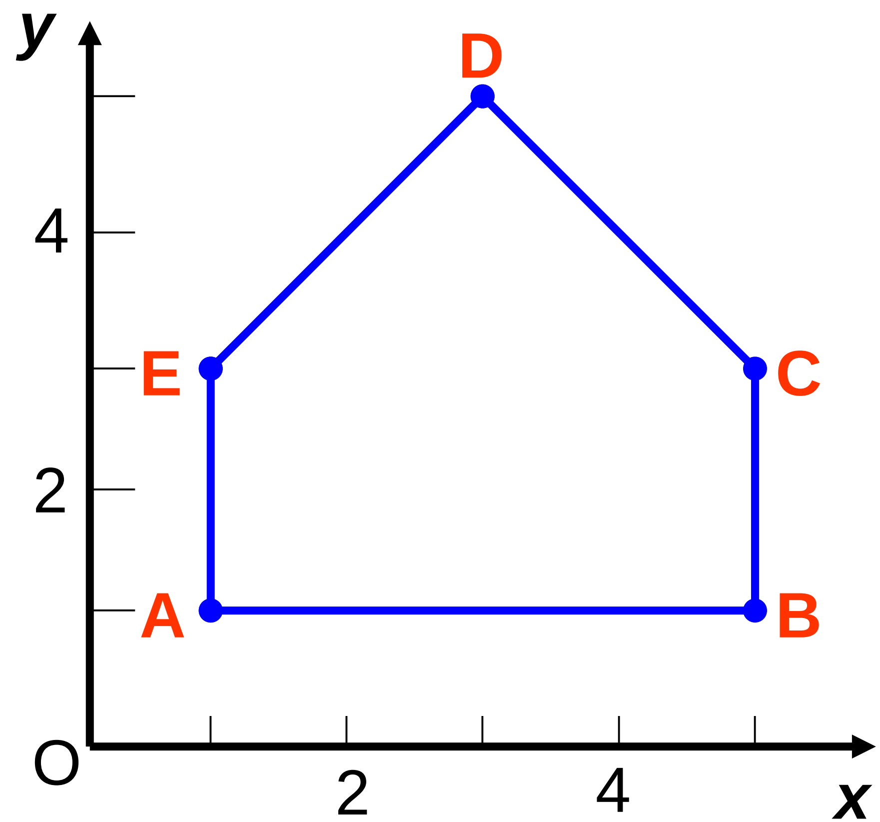
Example 1 **Co-ordinates of vertices:** $A(1,1),\ B(5,1),\ C(5,3)$

$D(3,5),\ E(1,3)$

Points Matrix:

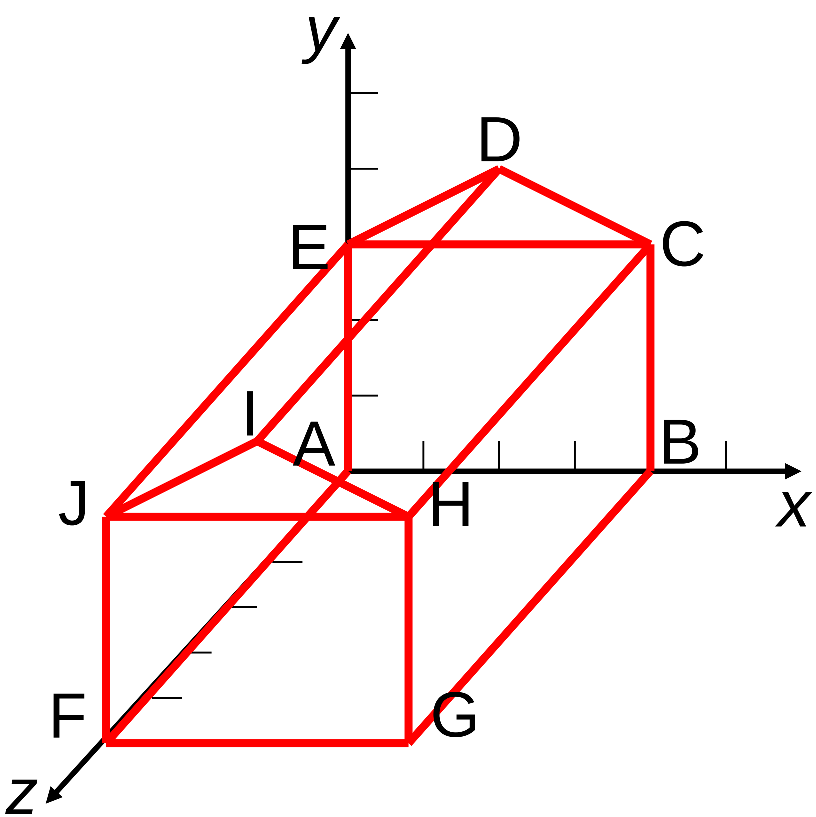
$$\underline{P} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{l}\left[\begin{array}{cc} 1 & 1 \\ 5 & 1 \\ 5 & 3 \\ 3 & 5 \\ 1 & 3 \end{array}\right] \\ \;\;\; x \;\;\; y \end{array}\begin{array}{l} \leftarrow \textbf{Vertex A} \\ \leftarrow \textbf{Vertex B} \\ \leftarrow \textbf{Vertex C} \\ \leftarrow \textbf{Vertex D} \\ \leftarrow \textbf{Vertex E} \\ \\ \end{array}$$

5 *rows* and 2 columns



10 *rows* and 3 columns

Example 2



$$\underline{P} = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 3 & 0 \\ 2 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \\ 4 & 0 & 6 \\ 4 & 3 & 6 \\ 2 & 4 & 6 \\ 0 & 3 & 6 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \end{array} \end{array}$$

Special Matrices

When the rows and columns of a matrix $\underline{A}$ are interchanged the resulting matrix is called the *transpose*

The transpose of $\underline{A}$ is denoted by $\underline{A'}$ or $\underline{A}^T$ or $\underline{A}^t$

Example 3

$$\underline{A} = \begin{bmatrix} 1 & 6 & 3 \\ -1 & 2 & 4 \end{bmatrix} \quad \text{then} \quad \underline{A}^T = \begin{bmatrix} 1 & -1 \\ 6 & 2 \\ 3 & 4 \end{bmatrix}$$

$$(2 \times 3) \qquad\qquad (3 \times 2)$$

The transpose of a symmetric matrix is the same matrix

Matrix Operations

Scalar Multiplication:

If $\underline{A}$ is a matrix and k is a scalar, then the matrix $k\underline{A}$ is obtained by multiplying each element of $\underline{A}$ by k

Example

$$\underline{A} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -5 \\ 0 & 4 & 6 \end{bmatrix}$$

$$3\underline{A} = \begin{bmatrix} 3\times 1 & 3\times 0 & 3\times(-2) \\ 3\times 2 & 3\times 3 & 3(-5) \\ 3\times 0 & 3\times 4 & 3\times 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -6 \\ 6 & 9 & -15 \\ 0 & 12 & 18 \end{bmatrix}$$

Addition and Subtraction:

To ***add*** or ***subtract*** two matrices - must have the same order

$$\underline{A} = \begin{bmatrix} 2 & 5 & -1 & 4 \\ 1 & -3 & 3 & 2 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 3 & -2 & 7 & 0 \\ 0 & 6 & 1 & 0 \end{bmatrix}$$

$\underline{A}$ and $\underline{B}$ are of the same order, i.e. (2×4)

$\underline{A}$ + $\underline{B}$, add corresponding elements of $\underline{A}$ and $\underline{B}$

$$\underline{A} + \underline{B} = \begin{bmatrix} 2+3 & 5+(-2) & -1+7 & 4+0 \\ 1+0 & -3+6 & 3+1 & 2+0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 6 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}$$

$\underline{A}$ - $\underline{B}$, subtract corresponding elements of $\underline{B}$ from $\underline{A}$

$$\underline{A} - \underline{B} = \begin{bmatrix} 2-3 & 5-(-2) & -1-7 & 4-0 \\ 1-0 & -3-6 & 3-1 & 2-0 \end{bmatrix} = \begin{bmatrix} -1 & 7 & -8 & 4 \\ 1 & -9 & 2 & 2 \end{bmatrix}$$

Matrix Multiplication

Not all combinations of matrices can be multiplied
Check the dimensions before attempting multiplication

Number of columns of $\underline{A}$ must equal number of rows of $\underline{B}$
(otherwise $\underline{A} \times \underline{B}$ is not defined)

Write down dimensions, inner numbers <u>must</u> match

$\underline{A}$	$\underline{B}$	$\underline{A} \times \underline{B}$
$(m \times n)$	$(n \times p)$	$(m \times p)$

The element in *row i* and *column j* of $\underline{AB}$ is obtained by forming the product of *row i* of $\underline{A}$ and *column j* of $\underline{B}$

Note: $\underline{A}\underline{B} = \underline{B}\underline{A}$ is **not** usually true

Don't usually include multiplication symbol

Matrix Multiplication

$$\begin{bmatrix} 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\times(-2) + (-2)\times 1 + 4\times 3 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$$

$$(1\times 3) \quad (3\times 1) \qquad\qquad (1\times 1)$$

$$\begin{bmatrix} 1 & -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

Undefined!!!

4 Cols

3 Rows

$$(1\times 4) \quad (3\times 1)$$

Inner numbers don't match

Matrix Multiplication

RC

Rows in first matrix multiply columns in second

$$\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} . & . \end{bmatrix}$$

$$(1 \times 3) \quad (3 \times 2) \quad (1 \times 2)$$

Inner numbers same

$$\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times (-2) + (-2) \times 1 + 4 \times 3 & 1 \times 1 + (-2) \times 0 + 4 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 9 \end{bmatrix}$$

Matrix Multiplication

RC

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$(2 \times 3) \quad (3 \times 2) \qquad (2 \times 2)$$

$$= \begin{bmatrix} 1 \times (-2) + (-2) \times 1 + 4 \times 3 & 1 \times 1 + (-2) \times 0 + 4 \times 2 \\ 0 \times (-2) + 3 \times 1 + 1 \times 3 & 0 \times 1 + 3 \times 0 + 1 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 9 \\ 6 & 2 \end{bmatrix}$$

To support earlier statement that

$\underline{A} \times \underline{B} = \underline{B} \times \underline{A}$ is <u>not</u> usually true........

Matrix Multiplication

RC

Change order of previous multiplication:

$$\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} . & . & . \\ . & . & . \\ . & . & . \end{bmatrix}$$

$$(3\times {\color{red}2}) \qquad ({\color{red}2}\times 3) \qquad (3\times 3)$$

$$= \begin{bmatrix} (-2)\times 1 + 1\times 0 & (-2)\times(-2) + 1\times 3 & (-2)\times 4 + 1\times 1 \\ 1\times 1 + 0\times 0 & 1\times(-2) + 0\times 3 & 1\times 4 + 0\times 1 \\ 3\times 1 + 2\times 0 & 3\times(-2) + 2\times 3 & 3\times 4 + 2\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 7 & -7 \\ 1 & -2 & 4 \\ 3 & 0 & 14 \end{bmatrix}$$

Usually $\underline{A}\,\underline{B} \neq \underline{B}\,\underline{A}$

often different dimensions

Linear Equations & Matrices

$$\begin{cases} x + 2y = 3 \\ -2x + 4y = 1 \end{cases}$$

Pair of ***linear***, ***simultaneous*** equations in ***two*** unknowns

Can be expressed as a ***single*** matrix equation

$$\underline{\mathrm{A}}\,\underline{x} = \underline{\mathrm{b}}$$

$$\underline{\mathrm{A}} = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}, \ \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \ \underline{\mathrm{b}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Check:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 \times x + 2 \times y \\ -2 \times x + 4 \times y \end{bmatrix} = \begin{bmatrix} x + 2y \\ -2x + 4y \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\mathbf{b}}$$

Linear Equations & Matrices

$$\begin{cases} 2x + 3y + z = 4 \\ x + 2y + z = 6 \\ 3x - 5y + 2z = -1 \end{cases}$$

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\underline{A} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & -5 & 2 \end{bmatrix},\ \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ \underline{b} = \begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix}$$

Elimination Methods

$$\begin{cases} x + 2y = 3 & \cdots (E1) \\ -2x + 4y = 1 & \cdots (E2) \end{cases}$$

As title suggests – we eliminate one of the unknowns

Straightforward method to program
Easily extendable to large systems of equations

Needs careful labelling when more than 2 equations

Objective: Convert the system of equations to *upper-triangular form* and then use *back-substitution*

Back substitution process

These equations are already in upper triangular form:

$$\begin{cases} x + 2y = 3 & \cdots(E1) \\ \quad\;\; 8y = 4 & \cdots(E2) \end{cases}$$

Easy to solve *E2* for *y*

$$8y = 4 \qquad \cdots(E2)$$

$$\text{so} \quad y = \frac{4}{8} = \frac{1}{2}$$

Now substitute *y* value back into *E1* and solve for *x*

$$x + 2y = 3 \qquad \cdots(E1)$$

$$x + 1 = 3$$

$$\text{i.e. } x = 2$$

So the solution is *x* = 2, *y* = ½

Back substitution process

$$\begin{cases} x + 2y + 4z = 11 & \cdots(E1) \\ \quad 8y + 3z = 10 & \cdots(E2) \\ \quad\quad 2z = 4 & \cdots(E3) \end{cases}$$

Solve *E3* for *z*

$$2z = 4 \qquad \cdots(E3)$$

$$\text{so} \quad z = 2$$

Substitute *z* value back into *E2* and solve for *y*

$$8y + 3z = 10 \quad \cdots(E2)$$

$$8y + 6 = 10 \qquad \text{so} \quad y = \frac{1}{2}$$

Substitute *y*, *z* values back into *E1* and solve for *x*

$$x + 2y + 4z = 11 \qquad \cdots(E1)$$

$$x + 1 + 8 = 11$$

$$\text{i.e. } x = 2$$

So the solution is $x = 2$, $y = ½$, $z = 2$

Gaussian Elimination

$$\begin{cases} \quad x + 2y = 3 & \cdots (E1) \\ -2x + 4y = 1 & \cdots (E2) \end{cases}$$

Objective: Convert the system of equations to *upper-triangular form* and then use *back-substitution*

Needs careful labelling when more than 2 equations

Rules for Gaussian Elimination:

Permissible operations are:

- Change the order in which the equations are written
- Multiply any equation by non-zero number
- Replace any equation by the sum/difference of a multiple of itself and a multiple of any other equation

Algorithm:

Equation ($E1$) used to eliminate first unknown from other equations

New second equation used to eliminate second unknown from the remaining equations.

Repeat until last equation involves only the last unknown

Elimination Methods

$$\begin{cases} x + 2y = 3 & \cdots (E1) \\ -2x + 4y = -2 & \cdots (E2) \end{cases}$$

Use *E1* to eliminate *x* from *E2*: $E2 + 2 \times E1$:

$$(-2x + 4y) + 2 \times (x + 2y) = -2 + 2 \times 3$$

$$\text{i.e. } 8y = 4$$

Equivalent pair of equations is:

$$\begin{cases} x + 2y = 3 & \cdots (E1) \\ 8y = 4 & \cdots (E2a) \end{cases}$$

The equations are now in *upper-triangular form* and may be solved by *back-substitution.*

$$\begin{cases} x + 2y = 3 & \cdots (E1) \\ \quad\;\; 8y = 4 & \cdots (E2a) \end{cases}$$

$(E2a)$: $8y = 4, \; \therefore y = \dfrac{1}{2}$

Substitute for y in $E1$:

$$x + 2y = 3$$

$$\therefore x + 2 \times \frac{1}{2} = 3$$

and $x = 3 - 1 = 2$

Solution: $x = 2, \; y = \dfrac{1}{2}$

In matrix form we reduced the equations from

$$\begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 2 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$