Numerical Solutions of Ordinary Differential Equations (II): Multi-Step Methods

Multi-Step Methods

A multi-step method is defined as:

> A q-step method ($q \geq 1$) is one which, $\forall n \geq q-1$, u_{n+1} depends on $u_{n+1}-q$, but not on the values u_k with $k < n+1-q$.

Some examples include the *Midpoint Method* and the *Simpson Method*.

> A general $(p+1)$-step method can be expressed as:
>
> $$u_{n+1} = \sum_{j=0}^{p} a_j u_{n-j} + h \sum_{j=0}^{p} b_j f_{n-j} + h b_{-1} f_{n+1} \text{ for } n = p, p+1, \ldots$$
>
> which means the we need to know the values of the first $p+1$ steps $(u_0, u_1, \ldots, u_p)$ to start with, which we can get using a one-step method.

The multistep method is completely described by the real coefficients a_j and b_j with a_p or b_p different from 0. Also, if $b_{-1} = 0$, the method is *explicit*, and *implicit* otherwise.

Adams Methods

Adams methods are derived from the integral form of the Cauchy problem (or IVP):

$$y(t) - y_0 = \int_{t_0}^{t} f(\tau, y(\tau)) d\tau$$

using a numerical approximation of the integral between t_n and t_{n+1}, using also u_{n-1}, u_{n-2}, ... This approximation is obtained supposing equispaced nodes, and integrating integrate the interpolating polynomial of f on $p+1$ distinct nodes. The resulting schemes are consistent by construction and have the following form:

$$u_{n+1} = u_n + h \sum_{j=-1}^{p} b_j f_{n-j},\ n \geq p$$

The interpolation nodes can be either:

- t_n, t_{n-1}, ... , t_{n-p} (in this case $b_{-1} = 0$ and the resulting method is **explicit**).
- t_{n+1}, t_n, ..., t_{n-p+1} (in this case $b_{-1} \neq 0$ and the scheme is **implicit**).

The **implicit** schemes are called **Adams-Moulton methods**, while the **explicit** ones are called **Adams-Bashforth methods**.

Adams-Bashford Methods

For this subfamily, $b_{-1} = 0$. Depending on the value of p, we obtain different methods:

- With $p = 0$, we obtain the **Explicit Euler Method**:

$$u_{n+1} = u_n + hf(t_n, u_n)$$

- With $p = 1$, we obtain the **Two-Step Adams-Bashford Method**:

$$u_{n+1} = u_n + \frac{h}{2}(3f_n - f_{n-1})$$

- With $p = 2$, we obtain the **Three-Step Adams-Bashford Method**:

$$u_{n+1} = u_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$$

- With $p = 3$, we obtain the **Four-Step Adams-Bashford Method**:

$$u_{n+1} = u_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

Adams-Moulton Methods

For this subfamily, $b_{-1} \neq 0$. Depending on the value of p, we obtain different methods:

- With $p = -1$, we obtain the **Implicit Euler Method**:

$$u_{n+1} = u_n + hf(t_{n+1}, u_{n+1})$$

- With $p = 0$, we obtain the **Cranck-Nicolson Method**:

$$u_{n+1} = u_n + \frac{h}{2}(f_n + f_{n+1})$$

- With $p = 1$, we obtain the **Two-Step Adams-Moulton Method**:

$$u_{n+1} = u_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})$$

- With $p = 2$, we obtain the **Three-Step Adams-Moulton Method**:

$$u_{n+1} = u_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})$$

- With $p = 2$, we obtain the **Three-Step Adams-Moulton Method**:

$$u_{n+1} = u_n + \frac{h}{720}(251f_{n+1} + 646f_n - 264f_{n-1} + 106f_{n-2} - 19f_{n-3})$$

The *q*-step *Adams-Moulton* methods have order $q + 1$.

BFD Methods

The **Backwards Differentiation Formulas** (BDFs) are *implicit multi-step methods* derived using the interpolation polynomial for y for the points t_{n+1}, t_n , ..., t_{n-k}, differentiating it and asserting that it is equal to f_{n+1}.

- For $k = 0$, we obtain:

  $$y(t) = y_{n+1} + (t - t_{n+1})\frac{y_{n+1} - y_n}{t_{n+1} - t_n}$$

  and

  $$y'(t) \approx \frac{y_{n+1} - y_n}{t_{n+1} - t_n} = f(t_{n+1}, y_{n+1})$$

  so, we obtain the **Implicit Euler Method**:

  $$u_{n+1} = u_n + hf(t_{n+1}, u_{n+1})$$

- For $k = 1$, we obtain:

  $$u_{n+1} = \frac{4}{3}u_n - \frac{1}{3}u_{n-1} + \frac{2}{3}hf_{n+1}$$

Only the first 6 BDF formulas are zero-stable, so they are the only ones used. The first two are *A-stable*, which means that they are a very good option for stiff problems.

Properties of Multi-Step Methods

> **First Dahlquist Barrier**: There isn't any zero-stable, p-step linear multistep method with order greater than $p + 1$ if p is odd, $p + 2$ if p is even.

> **Second Dahlquist barrier**: A linear explicit multistep method can be neither *A-stable*, nor θ-stable. Moreover, there is no *A-stable* linear multistep method with order greater than 2. Finally, for any

$\theta \in (0, \pi/2)$, there only exist θ-stable p-step linear multistep methods of order p for $p = 3$ and $p = 4$. The theorem states that the maximum order of a multi-step method that is *A-stable* is 2, and that among all order 2 multi-step methods, the one with the smallest local error is the Crank-Nicolson.

Why not using always Crank-Nicolson?

- Sometimes order 2 is not enough.
- It has the draw back that it doesn't dump errors for some (stiff) problems.