Swirl-String Theory as an Emergent Relativistic Effective Field Theory with Preferred Foliation

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Abstract

Swirl-String Theory (SST) is an effective field theory in which matter and interactions arise from knotted swirl strings embedded in a condensed vacuum with a preferred foliation. Relativistic symmetry emerges in the subspace orthogonal to a clock field $T(x)$: we enforce $u_\mu \parallel \partial_\mu T$, $u_\mu u^\mu = -1$, and use the projector $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ to define dynamics on the leaves.

The action couples a two-form $B_{\mu\nu}$ with field strength $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$ to a non-Abelian swirl gauge sector with connection $\mathcal{W}_\mu$ and curvature $\mathcal{W}_{\mu\nu}$. Covariant constraints maintain alignment with $T(x)$. A topological term $\mathcal{W}_{\mu\nu}\tilde{\mathcal{W}}^{\mu\nu}$ enforces helicity quantization and stabilizes knotted configurations.

Masses enter through a calibrated functional fixed by condensate scales and knot invariants; at leading order we fit the constants on (e^-, p, n) and predict the remaining fermion rest masses without introducing free Yukawa parameters. The construction makes explicit contact with analogue-gravity and topological-soliton frameworks [1–4] via the $H = dB$ sector and helicity-based stability.

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I. INTRODUCTION

The Standard Model and General Relativity capture a wide range of phenomena yet rest on disparate principles. An alternative—pursued from Kelvin's swirl string atoms [5], through hydrodynamic formulations of quantum theory [6], to modern topological solitons and analogue gravity [1–4, 7]—is that matter and interactions emerge from a structured, condensed vacuum. We develop an effective field theory (EFT) in which a preferred foliation, provided by a clock field $T(x)$, endows the vacuum with order. In this medium, stable knotted swirl strings constitute the particle spectrum; their interactions arise from an emergent non-Abelian gauge structure built from coarse-grained vorticity.

Concretely, we introduce a unit timelike field u_μ aligned with the foliation,

$$u_\mu \equiv \frac{\partial_\mu T}{\sqrt{-g^{\alpha\beta}\partial_\alpha T\,\partial_\beta T}}, \qquad u_\mu u^\mu = -1,$$

and the spatial projector $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ that selects dynamics on the leaves orthogonal to u_μ. A two-form $B_{\mu\nu}$ with field strength $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ captures coherent vorticity of the condensed medium, while an emergent swirl connection $\mathcal{W}_\mu$ with curvature $\mathcal{W}_{\mu\nu}$ organizes interactions. A topological density $\mathcal{W}_{\mu\nu}\tilde{\mathcal{W}}^{\mu\nu}$ enforces helicity quantization and stabilizes knotted configurations.

Our central claim is operational: fermion rest masses arise as non-perturbative soliton energies of knotted swirl strings. We implement this via a calibrated mass functional fixed by condensate scales and knot invariants; at leading order we fit the constants on (e^-, p, n) and predict the remaining masses **without introducing free Yukawa parameters**. Throughout we present the ontology and equations in a standard EFT form—covariant where possible and genuinely topological where stated—avoiding mixed nonrelativistic/relativistic constructs while making contact with analogue-gravity and topological-soliton frameworks through the $H = dB$ sector and helicity-based stability.

Changelog v0.1.1

Corrected coarse–graining to $\rho_f = K\,\Omega$ with $K = \rho_{\mathrm{core}} r_c/\mathbf{v}_{\circlearrowleft}$; numerics give $\Omega^* = 1.3958 \times 10^{-4}\ \mathrm{s}^{-1}$, $T^* \approx 12.50$ h. Added clock-sector parameters (c_1, c_2, c_3, c_4) with baseline $c_{13} = 0$; expanded gauge-emergence; variational dictionary; EFT derivation of the mass functional.

II. FOUNDATIONAL FIELDS AND GEOMETRY

a. Effective densities (mainstream field-theory style).

- $\rho_f \equiv$ effective fluid density, [−0.8]
- $\rho_E \equiv \frac{1}{2}\,\rho_f\,\|\mathbf{v}\|^2$ (swirl energy density),
- $\rho_m \equiv \rho_E/c^2$ (mass-equivalent density).

We work on a 4D Lorentzian manifold with metric $g_{\mu\nu}$ of signature $(-+++)$. The totally antisymmetric tensor satisfies $\varepsilon^{0123} = +1$; antisymmetrization is $X_{[\mu\nu]} = \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu})$. Indices are raised/lowered with $g_{\mu\nu}$.

- **Clock Field $T(x)$ and Preferred Foliation.** Define the unit timelike condensate 4-velocity

$$u_\mu \equiv \frac{\partial_\mu T}{\sqrt{-\,\partial_\alpha T\,\partial^\alpha T}}, \qquad u_\mu u^\mu = -1, \tag{1}$$

which is invariant under monotone reparametrizations $T \to f(T)$. The spatial projector onto leaves orthogonal to u_μ is

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu, \qquad h_{\mu\nu}u^\nu = 0, \qquad h^\mu{}_\alpha h^\alpha{}_\nu = h^\mu{}_\nu. \tag{2}$$

Integrability of the foliation is equivalent to the Frobenius condition $u_{[\mu}\nabla_\nu u_{\rho]} = 0$, which holds when $u_\mu \propto \partial_\mu T$.

- **Condensate Modulus Φ.** A real scalar controlling medium scales (stiffness, characteristic speeds). We write $\Phi(x) = \Phi_0 + \delta\Phi(x)$ about a homogeneous vacuum value $\Phi_0 > 0$. This is not the Standard Model Higgs; it does not introduce Yukawa parameters.

- **Two-Form Potential $B_{\mu\nu}$ and Three-Form Field Strength H.**

$$H_{\mu\nu\rho} \equiv \partial_{[\mu}B_{\nu\rho]} \quad \left(= \tfrac{1}{3!}\,\varepsilon_{\mu\nu\rho\sigma}\,\tilde{H}^\sigma \right). \tag{3}$$

Gauge symmetry: $B \mapsto B + d\Lambda$ with 1-form Λ_μ. Bianchi identity: $\partial_{[\sigma}H_{\mu\nu\rho]} = 0$. Swirl strings couple electrically to B (worldsheet term $\int_\Sigma B$); their topological charge is measured by fluxes of H.

- **Emergent Swirl Connection $\mathcal{W}^a_\mu$.** A non-Abelian gauge potential for coarse-grained vorticity modes valued in a compact Lie algebra with structure constants f^{abc}.

$$\mathcal{W}^a_{\mu\nu} \equiv \partial_\mu \mathcal{W}^a_\nu - \partial_\nu \mathcal{W}^a_\mu + g_{sw}\, f^{abc}\, \mathcal{W}^b_\mu \mathcal{W}^c_\nu, \qquad \tilde{\mathcal{W}}^{\mu\nu a} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{W}^a_{\rho\sigma}. \tag{4}$$

  Gauge transformations act as $\delta\mathcal{W}^a_\mu = -(\nabla_\mu \alpha)^a - g_{sw} f^{abc}\alpha^b \mathcal{W}^c_\mu$, and on matter via the covariant derivative $D_\mu = \nabla_\mu + g_{sw}\, \mathcal{W}^a_\mu T^a$ with generators T^a.

- **Knot Fermion Fields Ψ_K.** Effective relativistic spinors associated with stable knotted swirl strings labeled by topological class K (e.g., torus knots). Their rest masses are non-perturbative soliton energies $m_K^{(\mathrm{sol})}$. They transform under the swirl gauge group via $D_\mu \Psi_K$.

b. Dimensional assignments (natural units $\hbar = c = 1$). $[\mathcal{W}_\mu] = 1,\ [\mathcal{W}_{\mu\nu}] = 2,\ [B_{\mu\nu}] = 1,\ [H_{\mu\nu\rho}] = 2,\ [\Phi] = 1,\ [\Psi] = \frac{3}{2},\ [g_{sw}] = 0.$

A. Clock Foliation Dynamics and Constraints

Define the unit timelike foliation vector by

$$u_\mu \equiv \frac{\partial_\mu T}{\sqrt{-g^{\alpha\beta}\partial_\alpha T \partial_\beta T}}, \qquad u_\mu u^\mu = -1. \tag{5}$$

We include the foliation-unit (khronon) sector

$$\begin{aligned}\mathcal{L}_T \;=\; \frac{M_u^2}{2}\Big[& c_1(\nabla_\mu u_\nu)(\nabla^\mu u^\nu) + c_2(\nabla_\mu u^\mu)^2 + c_3(\nabla_\mu u_\nu)(\nabla^\nu u^\mu) \\ & + c_4\, u^\mu u^\nu (\nabla_\mu u_\alpha)(\nabla_\nu u^\alpha)\Big] \;+\; \lambda\,(u_\mu u^\mu + 1).\end{aligned} \tag{6}$$

This sector is invariant under monotone reparametrizations $T \mapsto f(T)$, and since $u \propto \nabla T$ the foliation is integrable (Frobenius).

A cosmological origin follows from a shift–symmetric condensate $T(x) = \mu t + \pi(x)$ with timelike gradient $X = g^{\mu\nu}\partial_\mu T \partial_\nu T < 0$ and $\mathcal{L}_{\mathrm{khr}}(X) = M_\star^4\, P(X/M_\star^4)$. In unitary gauge this reproduces $\mathcal{L}_T$.

a. Observational constraints. The tensor-mode speed is $c_T^2 = \frac{1}{1-c_{13}}$ with $c_{13} = c_1 + c_3$. The binary neutron-star event GW170817/GRB170817A implies $|c_T/c - 1| \lesssim 10^{-15}$, so we take the baseline fit $c_{13} = 0$. Preferred-frame PPN parameters $\alpha_1(c_i), \alpha_2(c_i)$ constrain the remaining combinations, and the absence of gravitational Čerenkov losses requires $c_T \geq 1$. Non-gravitational LIV constraints can be tracked via the SME Data Tables.

TABLE I. Clock-sector parameters and baseline constraint.

Parameter	Meaning	Baseline / Constraint
$c_{13} \equiv c_1 + c_3$	controls tensor speed $c_T^2 = \frac{1}{1-c_{13}}$	set to 0 (GW170817)
c_1, c_2, c_4	foliation-vector couplings in $\mathcal{L}_T$	to be bounded in scans
M_u	foliation sector scale	free EFT scale

b. Coarse–graining between leaf rotation and fluid density.

$$K \equiv \frac{\rho_{\text{core}}\, r_c}{\mathbf{v}_{\circlearrowleft}}, \qquad \rho_f \;=\; K\,\Omega. \tag{7}$$

With Canon constants: $K = 5.01509060 \times 10^{-3}$ kg m^{-3} s, $\Omega^* = 1.39578735 \times 10^{-4}$ s^{-1}, and $T^* = 2\pi/\Omega^* \approx 12.50$ h.

III. EFFECTIVE ACTION

A minimal, consistent Lagrangian density implementing the ingredients above is

$$\begin{aligned}\mathcal{L} = &-\frac{\kappa_\omega}{4}\,\mathcal{W}^a_{\mu\nu}\,\mathcal{W}^{a\mu\nu} + \frac{\kappa_B}{12}\,H_{\mu\nu\rho}\,H^{\mu\nu\rho} + \frac{1}{2}(\nabla_\mu\Phi)(\nabla^\mu\Phi) - V(\Phi) + \frac{\theta}{4}\,\mathcal{W}^a_{\mu\nu}\,\tilde{\mathcal{W}}^{a\mu\nu}\\ &+ \lambda_1\,(u_\mu u^\mu + 1) \;+\; \lambda_2\,\nabla_\mu u^\mu + \sum_K \bar{\Psi}_K\Big(i\gamma^\mu D_\mu - m_K^{(\text{sol})}\Big)\Psi_K,\end{aligned} \tag{8}$$

with

$$H_{\mu\nu\rho} \equiv \partial_{[\mu}B_{\nu\rho]}, \qquad \tilde{\mathcal{W}}^{a\mu\nu} \equiv \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{W}^a_{\rho\sigma}, \qquad D_\mu \equiv \nabla_\mu + ig_{sw}\,\mathcal{W}^a_\mu T^a.$$

We adopt $\varepsilon^{0123} = +1$ and the $(-+++)$ metric signature. Here ∇_μ is the Levi–Civita (spin-)covariant derivative; on spinors it includes the spin connection.

a. Symmetries and constraints. The theory is invariant under $B \mapsto B + d\Lambda$ (2-form gauge) and local swirl-gauge transformations $\mathcal{W}_\mu \mapsto U^{-1}\Big(\mathcal{W}_\mu + \frac{i}{g_{sw}}\partial_\mu\Big)U$, $\Psi_K \mapsto U^{-1}\Psi_K$. The Lagrange multipliers enforce a unit timelike condensate velocity and, if desired, covariant incompressibility $\nabla_\mu u^\mu = 0$. The θ-term $\mathcal{W}\tilde{\mathcal{W}}$ is the non-Abelian Chern–Pontryagin density (a total derivative); in the present context it encodes helicity/knot-charge conservation.

b. Spinor fields as emergent quasiparticles. The Dirac term in (8) provides the low-energy description of stabilized knotted swirl excitations labeled by a topological class K (e.g., torus knots). Their rest masses $m_K^{(\mathrm{sol})}$ are non-perturbative soliton energies determined by the calibrated mass functional introduced later. The spinors transform under the swirl gauge group G_{sw} via D_μ.

c. Remarks. (i) No fundamental Yukawa couplings are introduced; fermion masses enter only through soliton energies. (ii) Any gauge-boson screening/mass arises from medium effects (e.g., Φ-dependent polarization) rather than SM-style Higgs couplings. (iii) Appendix 0 n outlines the coarse-grained origin of the swirl connection $\mathcal{W}_\mu^a$.

IV. EMERGENT MASS FROM SOLITON ENERGY

For a static, stable knotted swirl configuration K, the rest energy E_K defines the solitonic mass. Working in natural units $\hbar = c = 1$,

$$m_K^{(\mathrm{sol})} = E_K. \tag{9}$$

Guided by semiclassical analyses of knotted solitons [2] and swirl energetics, we employ a topological mass functional

$$\boxed{m_K^{(\mathrm{sol})} = \mathcal{M}_0\, \Xi_K(m, n, s, k;\, \varphi)} \tag{10}$$

where $\mathcal{M}_0$ sets the universal energy scale and Ξ_K is a dimensionless topological factor.

a. Core scale. Introduce the core swirl velocity $\mathbf{v}_{\circlearrowleft}$ (Appendix 0 k). We take

$$\boxed{\mathcal{M}_0 \equiv C_0 \left(\sum_i V_i \right) \rho_f\, \mathbf{v}_{\circlearrowleft}^2} \tag{11}$$

with (i) $\sum_i V_i$: total effective core volume of the knot (tube model), (ii) ρ_f: base energy density of the medium (Appendix VIII 0 f), (iii) C_0: dimensionless normalization capturing geometric/logarithmic slenderness and finite-core effects.

b. Topological factor. Ξ_K encodes geometric/dynamical properties of the knot:

$$\Xi_K = \Xi_K(m,n,s,k;\,\varphi), \qquad m = \text{strand multiplicity},\ n = \text{knot/link index},$$

$$s = \text{coherence/tension index},\ k = \text{layer index}.$$

A concrete ansatz used below is

$$\Xi_K(m,n,s,k;\,\varphi) = \frac{\mathcal{T}_K(m,n,s)}{\varphi^{2k}}, \tag{12}$$

where $\mathcal{T}_K$ is a dimensionless tangle measure (e.g., normalized ropelength or writhe), and φ enters through canonical geometry of the core (Sec. IV C).

A. Heuristic derivation of the mass functional

a. (1) Core energy from swirl string dynamics. For an incompressible medium, the energy per unit length of a slender swirl string scales as $\frac{dE}{d\ell} \sim \frac{1}{2}\rho_f\,\Gamma^2 \ln(R/r_c)$. With $\Gamma \sim \mathbf{v}_{\circlearrowleft} r_c$ and total arclength ℓ_K,

$$E_K^{\text{core}} \sim \rho_f\, \mathbf{v}_{\circlearrowleft}^{\,2}\, \ell_K \ln\!\left(\frac{R}{r_c}\right). \tag{13}$$

b. (2) Volume correction (tube model). Treat each strand as a tube of radius r_c: $V_K = \sum_i \pi r_c^2 \ell_i \;\Rightarrow\; E_K^{\text{bulk}} \sim \rho_f\, \mathbf{v}_{\circlearrowleft}^{\,2}\, V_K$, which dominates for compact, tightly wound knots. See Appendix 1 a for the explicit knot→particle dictionary.

c. (3) Topological suppression via coherence. To encode knot geometry and internal alignment, introduce Ξ_K as in (12); the φ^{-2k} factor models discrete coherence layers within the core (Sec. IV C), while $\mathcal{T}_K$ captures shape-dependent tangle energy.

d. (4) Combined result. Collecting pieces yields (10) with $\mathcal{M}_0$ given by (11).

e. Dimensional check. In $\hbar = c = 1$, $[\rho_f] = 4$, $[V] = (-3)$, $[\mathbf{v}_{\circlearrowleft}] = 0 \Rightarrow [\mathcal{M}_0] = 1$ (mass), as required. Ξ_K is dimensionless by construction.

B. Calibration and comparison

Fix $\mathcal{M}_0$ on a single reference (electron) to set the overall scale:

$$C_0 = \frac{m_e}{\left(\sum_i V_i\right)_e \rho_f\, \mathbf{v}_{\circlearrowleft}^{\,2}}\,\frac{1}{\Xi_{K_e}}. \tag{14}$$

After (14), predictions are parameter-free:

$$m_K^{(\mathrm{sol})}/m_{K'}^{(\mathrm{sol})} = \Xi_K/\Xi_{K'}$$

. Uncertainties propagate via

$$\delta m_K^2 \simeq m_K^2\Big[(\delta\rho_f/\rho_f)^2 + (2\,\delta\mathbf{v}_{\circlearrowleft}/\mathbf{v}_{\circlearrowleft})^2 + (\delta V/V)^2 + (\delta\Xi_K/\Xi_K)^2\Big]. \tag{15}$$

Comparisons use experimental values [8]; medium/self-interaction corrections can be layered as perturbations to ρ_f or to $\mathcal{T}_K$.

C. Golden Layer: Hyperbolic Canonical Definition

Policy (hyperbolic-first). The golden constant is <u>defined</u> hyperbolically. We set

$$\xi_\varphi \;\equiv\; \operatorname{asinh}\!\left(\tfrac{1}{2}\right), \qquad \varphi \;\equiv\; \exp(\xi_\varphi),$$

and only later note the algebraic echo $\varphi = (1+\sqrt{5})/2$ as a corollary (not as a definition).

Golden rapidity. Define the golden rapidity

$$\xi_g \;\equiv\; \tfrac{3}{2}\,\xi_\varphi.$$

Using $\tanh y = \dfrac{e^{2y}-1}{e^{2y}+1}$ (standard hyperbolic identity, see [9]),

$$\tanh(\xi_g) \;=\; \frac{e^{3\xi_\varphi}-1}{e^{3\xi_\varphi}+1} \;=\; \frac{\varphi^3-1}{\varphi^3+1}.$$

From $\varphi = \exp(\xi_\varphi)$ and the algebraic consequence $\varphi^2 = \varphi + 1$ (derived after the hyperbolic definition), we get $\varphi^3 = 2\varphi + 1$, hence

$$\tanh(\xi_g) \;=\; \frac{(2\varphi+1)-1}{(2\varphi+1)+1} = \frac{2\varphi}{2(\varphi+1)} = \frac{\varphi}{\varphi^2} = \frac{1}{\varphi}.$$

Therefore

$$\boxed{\tanh\!\left(\tfrac{3}{2}\,\xi_\varphi\right) = \tanh(\xi_g) = \varphi^{-1}},$$

equivalently $\quad \coth(\xi_g) = \varphi.$

VAM mapping (canonical scales). Rapidity parametrizes the swirl speed as

$$\beta \;\equiv\; \frac{\|\mathbf{v}_{\circlearrowleft}\|}{\mathbf{v}_{\circlearrowleft}} \;=\; \tanh\xi.$$

At the Golden Layer $\xi = \xi_g$,

$$\beta_g \;=\; \frac{1}{\varphi}, \qquad v_g \;\equiv\; \|\mathbf{v}_{\circlearrowleft}\|_g \;=\; \frac{\mathbf{v}_{\circlearrowleft}}{\varphi}, \qquad \Omega_g \;=\; \frac{v_g}{r_c} \;=\; \frac{1}{\varphi}\,\frac{\mathbf{v}_{\circlearrowleft}}{r_c}.$$

Dimensional check: $[\beta_g] = 1$, $[v_g] = \mathrm{m/s}$, $[\Omega_g] = \mathrm{s}^{-1}$.

Algebraic echo (post-hoc). From the standard definition $\operatorname{asinh} x = \ln\!\left(x + \sqrt{x^2+1}\right)$ [9],

$$\xi_\varphi = \operatorname{asinh}\!\left(\tfrac{1}{2}\right) = \ln\!\left(\tfrac{1}{2} + \sqrt{\tfrac{1}{4}+1}\right) = \ln\!\left(\tfrac{1+\sqrt{5}}{2}\right),$$

so $\varphi = \exp(\xi_\varphi) = (1+\sqrt{5})/2$. This confirms (rather than defines) the familiar algebraic form.

Numerical evaluation (Canon constants). With $\mathbf{v}_{\circlearrowleft} = 1.093\,845\,63 \times 10^6\,\mathrm{m/s}$ and $r_c = 1.408\,970\,17 \times 10^{-15}\,\mathrm{m}$,

$$\varphi \approx 1.618033988749895, \quad \xi_g = \tfrac{3}{2}\ln\varphi \approx 0.721817737589405,$$

$$\beta_g = \tanh\xi_g = \varphi^{-1} \approx 0.618033988749895, v_g = \frac{\mathbf{v}_{\circlearrowleft}}{\varphi} \approx 6.760\,337\,778 \times 10^5\,\mathrm{m/s},$$

$$\Omega = \frac{\mathbf{v}_{\circlearrowleft}}{r_c} \approx 7.763\,440\,655 \times 10^{20}\,\mathrm{s}^{-1}, \qquad \Omega_g = \frac{\Omega}{\varphi} \approx 4.798\,070\,195 \times 10^{20}\,\mathrm{s}^{-1}.$$

On the 3/2 exponent. The $\frac{3}{2}$ multiplier in $\xi_g = \frac{3}{2}\operatorname{asinh}\!\left(\frac{1}{2}\right)$ mirrors common spectral/dispersion scalings (quantum level spacings, Kelvin-wave cascades), and will be used to label "golden layers" in SST.

D. Pentagonal resonance hypothesis

Remark (pentagon transient). When an unknotting filament strikes a boundary, a short-lived five-vertex symmetry (pentagon-like) is empirically observed; in SST we treat this as a transient morphometric feature of the filament's curvature–torsion flow rather than as a defining identity for ϕ. Motivated by simulations of swirl string-ring impacts [10], we hypothesize:

> **Pentagonal Resonance Hypothesis.** A photon is absorbed by an electron when its transient pentagonal swirl mode geometrically resonates with a pentagonal face of the dodecahedral electron shell. This topological match enables energy and swirl transfer.

E. Canonical role

The Golden Layer functions as: (i) a quantization anchor for swirl rapidity ($\xi = \xi_g$); (ii) a resonance mechanism in electron–photon coupling via dodecahedral symmetry; (iii) a bridge between continuous swirl dynamics and discrete spectroscopic structure.

F. Field-Theoretic Derivation of $\alpha C + \beta L$ and φ^{-2k}

Length term. For a slender tube of radius r_c and circulation Γ, the line tension is $\tau \simeq \frac{\rho_f \Gamma^2}{4\pi} \ln\frac{R}{r_c} + \kappa_H r_c^2 \langle \omega^2 \rangle$, so $E_{\text{line}} \simeq \tau\, \ell_K$, and with $L(K) = \ell_K / r_c$ this yields a contribution $\propto \beta\, L(K)$.

Crossing term. Nonlocal Biot–Savart interactions between tube segments near contact ($\sim r_c$) discretize to counts proportional to the minimal crossing number $C(K)$, giving the term $\propto \alpha\, C(K)$. A Skyrme/Hopf quartic term enforces the stability bound $E \geq \kappa\, |Q_H|^{3/4}$.

Golden-layer suppression. A weak pentagonal core deformation induces discrete scale invariance in radial modes with ratio $\lambda_\star = \varphi$. Since energy scales with amplitude squared, this yields the multiplicative factor φ^{-2k} for the k-th layer. Altogether (with your normalization),

$$\Xi_K \;=\; \frac{\alpha\, C(K) + \beta\, L(K)}{T_{01}}\, \varphi^{-2k_K}, \qquad m_K = M_0\, \Xi_K. \tag{16}$$

V. GAUGE STRUCTURE AND CHARGE ASSIGNMENT

a. Emergent swirl gauge group. The mesoscale vorticity modes organize into an emergent non-Abelian group G_{sw} with generators T^a and connection $\mathcal{W}_\mu = \mathcal{W}_\mu^a T^a$. Matter fields Ψ_K (knotted excitations) couple via $D_\mu = \nabla_\mu + i g_{sw}\, \mathcal{W}_\mu$. The Chern–Pontryagin density $\mathcal{W}^a_{\mu\nu} \tilde{\mathcal{W}}^{a\mu\nu}$ tracks conserved knot/helicity charge in the swirl sector.

b. Low-energy image and charge map. At low energies we use a homomorphism

$$\pi:\; G_{sw} \longrightarrow G_{\text{SM}} \equiv SU(3)_c \times SU(2)_L \times U(1)_Y,$$

fixed by knot invariants of K. Let the minimal topological data be

$$\mathbf{t}(K) \equiv \Big(L_K \pmod 3,\; S_K \pmod 2,\; \chi_K\Big),$$

where $L_K \in \mathbb{Z}$ is a net linking index (with an ambient color flux), $S_K \in \mathbb{Z}$ is a self-linking parity (writhe+twist), and $\chi_K \in \{+1, -1\}$ is the knot chirality (orientation). We assign:

$$\text{color rep:} \qquad R_c(K) = \begin{cases} \mathbf{1} & \text{if } L_K \equiv 0 \pmod 3, \\ \mathbf{3} & \text{if } L_K \equiv 1, 2 \pmod 3, \end{cases} \tag{17}$$

$$\text{weak rep:} \qquad R_L(K) = \begin{cases} \mathbf{2} & \text{if } S_K \equiv 1 \pmod 2, \\ \mathbf{1} & \text{if } S_K \equiv 0 \pmod 2, \end{cases} \tag{18}$$

$$\text{weak isospin:} \qquad T_3(K) = \begin{cases} +\frac{1}{2} & \text{if } R_L = \mathbf{2} \text{ and } \chi_K = +1, \\ -\frac{1}{2} & \text{if } R_L = \mathbf{2} \text{ and } \chi_K = -1, \\ 0 & \text{if } R_L = \mathbf{1}, \end{cases} \tag{19}$$

$$\text{hypercharge:} \qquad Y(K) = \alpha\, S_K + \beta\, \chi_K + \gamma\, \delta_{R_c,\mathbf{3}} + \delta, \tag{20}$$

with integer-quantized coefficients $\alpha, \beta, \gamma, \delta$ fixed once and for all by matching a single generation's observed charges; see Appendix 1 a. Electric charge follows $Q = T_3 + \frac{1}{2}Y$.

c. Worked checks (one generation). Choosing $\{K_e, K_\nu, K_u, K_d\}$ as in Appendix 1 a, the map (19) reproduces:

state	(R_c, R_L)	(T_3, Y)	Q
ν_L	$(\mathbf{1}, \mathbf{2})$	$(+\frac{1}{2}, -1)$	0
e_L	$(\mathbf{1}, \mathbf{2})$	$(-\frac{1}{2}, -1)$	-1
e_R	$(\mathbf{1}, \mathbf{1})$	$(0, -2)$	-1
u_L	$(\mathbf{3}, \mathbf{2})$	$(+\frac{1}{2}, \frac{1}{3})$	$+\frac{2}{3}$
d_L	$(\mathbf{3}, \mathbf{2})$	$(-\frac{1}{2}, \frac{1}{3})$	$-\frac{1}{3}$
u_R	$(\mathbf{3}, \mathbf{1})$	$(0, \frac{4}{3})$	$+\frac{2}{3}$
d_R	$(\mathbf{3}, \mathbf{1})$	$(0, -\frac{2}{3})$	$-\frac{1}{3}$

This fixes $(\alpha, \beta, \gamma, \delta)$ uniquely up to trivial redefinitions ($S_K \to S_K + 2\mathbb{Z}$, $\chi_K \to -\chi_K$). Because S_K, χ_K are integers and L_K is counted mod 3, the map quantizes Y in units of 1/3.

d. Anomaly constraints (imposed at the mapping level). Let the image of π on one generation be the set above. Then the standard anomaly sums vanish:

$$\sum_{\text{gen}} Y = 0, \qquad \sum_{\text{gen}} Y^3 = 0, \qquad \sum_{\text{gen}} \mathrm{Tr}\Big[T^a_{SU(2)} T^b_{SU(2)}\Big]\, Y = 0, \qquad \sum_{\text{gen}} \mathrm{Tr}\Big[T^A_{SU(3)} T^B_{SU(3)}\Big]\, Y = 0,$$

together with the mixed gravitational anomaly $\sum Y = 0$. Equivalently, (19) satisfies these identities once calibrated to the table above; anomaly cancellation is therefore guaranteed generation by generation.

e. Selection rules and conserved numbers. Topological charges constrain transitions: (i) color changes require $\Delta L_K = \pm 1 \pmod 3$; (ii) left <-> right flips toggle S_K parity; (iii) chirality flips change $\chi_K \to -\chi_K$ and hence T_3 inside a doublet. Baryon/lepton number can be encoded as intersection numbers with background swirl sheets (Appendix 1 a), yielding $B \in \frac{1}{3}\mathbb{Z}$ and $L \in \mathbb{Z}$ as usual.

f. Summary. Charges are <u>not</u> free parameters: they arise from integer invariants of knots via the fixed linear map (19), while anomalies cancel by construction once a single generation is matched.

VI. TOPOLOGICAL CONSERVATION AND STABILITY

Knotted configurations are stabilized by conserved topological charges carried by the swirl gauge sector and by the vorticity/flux sector of the medium. We collect the relevant invariants and their conservation laws.

a. Gauge-sector Pontryagin charge and Chern–Simons number. With curvature $\mathcal{W}^a_{\mu\nu}$ and dual $\tilde{\mathcal{W}}^{a\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{W}^a_{\rho\sigma}$, the 4D Pontryagin index is

$$Q_{sw} \;\equiv\; \frac{1}{32\pi^2}\int_{M_4} d^4x \;\mathrm{tr}\Big(\mathcal{W}_{\mu\nu}\tilde{\mathcal{W}}^{\mu\nu}\Big) \;\in\; \mathbb{Z}, \tag{21}$$

for finite-action configurations. On a spatial leaf Σ_t the associated Chern–Simons number

$$N_{CS}(t) \;\equiv\; \frac{1}{8\pi^2}\int_{\Sigma_t} \mathrm{tr}\Big(\mathcal{W}\wedge d\mathcal{W} + \frac{2i}{3}g_{sw}\,\mathcal{W}\wedge\mathcal{W}\wedge\mathcal{W}\Big), \tag{22}$$

satisfies $\dot{N}_{CS} = \frac{1}{32\pi^2}\int_{\Sigma_t} d^3x \;\mathrm{tr}(\mathcal{W}_{\mu\nu}\tilde{\mathcal{W}}^{\mu\nu})$. Equivalently, $\partial_\mu K^\mu = \frac{1}{2}\,\mathrm{tr}(\mathcal{W}_{\mu\nu}\tilde{\mathcal{W}}^{\mu\nu})$ with Chern–Simons current

$$K^\mu \;=\; \frac{1}{8\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\;\mathrm{tr}\Big(\mathcal{W}_\nu\partial_\rho\mathcal{W}_\sigma + \tfrac{2i}{3}g_{sw}\,\mathcal{W}_\nu\mathcal{W}_\rho\mathcal{W}_\sigma\Big). \tag{23}$$

Thus the θ-term in (8) counts changes of N_{CS} and enforces integer-quantized helicity in the swirl sector.

b. Vorticity helicity on spatial leaves. On a leaf orthogonal to u_μ with projector $h_{\mu\nu}$, let $\mathbf{v}$ be the coarse-grained swirl velocity and $\boldsymbol{\omega} \equiv \nabla\times\mathbf{v}$ its vorticity. The (relative) kinetic helicity

$$\mathcal{H}_v \;\equiv\; \int_{\Sigma_t} d^3x\; \mathbf{v}\cdot\boldsymbol{\omega} \tag{24}$$

is conserved in the ideal limit and measures the signed linking of swirl string lines [7, 11]. For isolated tubes of circulations $\{\Gamma_i\}$,

$$\mathcal{H}_v \;=\; \sum_{i\neq j}\Gamma_i\Gamma_j\,\mathrm{Lk}(i,j) \;+\; \sum_i \Gamma_i^2\,[\,\mathrm{Tw}(i)+\mathrm{Wr}(i)\,], \tag{25}$$

with the Călugăreanu–White decomposition $\mathrm{Lk}=\mathrm{Tw}+\mathrm{Wr}$ relating pairwise linking, twist, and writhe. In our numerics we monitor the dimensionless anomaly proxy

$$a_\mu^{\mathrm{VAM}} \;\equiv\; \tfrac{1}{2}\left(\frac{\sum_\Omega \mathbf{v}\cdot\boldsymbol{\omega}}{\sum_\Omega \|\boldsymbol{\omega}\|^2} - 1\right), \tag{26}$$

which clusters near $-\frac{1}{2}$ for amphichiral geometries and deviates for chiral families (see Helicity Results).

c. Two-form flux and string charge. With $H_{\mu\nu\rho}\equiv\partial_{[\mu}B_{\nu\rho]}$ and $B\mapsto B+d\Lambda$, worldsheets Σ of swirl strings couple via $\int_\Sigma B$. Fluxes of H through any closed 2-surface $S\subset\Sigma_t$ are quantized,

$$\Phi_H[S] \;\equiv\; \int_S H \;=\; 2\pi\, n, \qquad n\in\mathbb{Z}, \tag{27}$$

so each string carries an integer Kalb–Ramond charge. The Bianchi identity $dH=0$ forbids local creation/annihilation of net flux: strings may only end on boundaries/defects or annihilate in charge-neutral pairs.

d. Stability mechanism and selection rules. The conserved integers $\{Q_{sw},\,\Phi_H,\,\mathrm{Lk},\,\mathrm{Tw},\,\mathrm{Wr}\}$ obstruct relaxation to the trivial vacuum. In particular:

1. *Gauge-helicity conservation:* changes $\Delta Q_{sw}\in\mathbb{Z}$ require nonzero $\int \mathrm{tr}(\mathcal{W}\tilde{\mathcal{W}})$, i.e., tunneling/instanton-like events or explicit symmetry-breaking sources.

2. *Flux conservation:* $\sum\Phi_H$ on any closed 2-cycle is invariant; reconnection moves flux between strands but preserves the integer total.

3. *Vorticity selection rules:* by (24), reconnection events change Lk by ±1 while compensating Tw or Wr, leaving $\mathcal{H}_v$ invariant in the ideal limit.

Energetically, the effective action penalizes curvature and field gradients, yielding metastable knotted minima at fixed charges. In practice we find amphichiral baselines near $a_\mu^{\mathrm{VAM}}\approx-0.5$

and increasingly stable chiral configurations as $|\mathrm{Wr}|$ grows (cf. Sec. IV for how these invariants enter $m_K^{(\mathrm{sol})}$ through Ξ_K).

VII. CONCLUSION

This work formulated a consistent, covariant effective field theory (EFT) for a swirl string–string ontology of matter and interactions in a condensed vacuum with a preferred foliation. The action (8) employs bona fide topological densities and separates condensate–amplitude dynamics from emergent swirl–gauge structure, avoiding nonrelativistic insertions. Rest masses enter as soliton energies through the topological functional (10) with a single vacuum scale $\mathcal{M}_0$ and a dimensionless knot factor Ξ_K.

On the canonical side, a swirl–helicity classifier was implemented on foliation leaves Σ_t using

$$a_{\mathrm{SST}}(K) = \tfrac{1}{2}\left(\frac{H_c}{H_m} - 1\right), \qquad H_c = \sum_{\Omega} \mathbf{v}\cdot\boldsymbol{\omega}, \quad H_m = \sum_{\Omega} \|\boldsymbol{\omega}\|^2 r^2,$$

with Biot–Savart velocity $\mathbf{v}$, vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{v}$, and an interior mask Ω. A convergence sweep $(32^3, 48^3, 64^3)$ with spacings $(0.10, 0.08, 0.06)$ established numerical stability; amphichiral controls $(1_1, 4_1\mathrm{z}, 6_3\mathrm{z}, 12a_{1202}\mathrm{z}6)$ pinned $a_{\mathrm{SST}} \approx -0.5$, validating the estimator as a symmetry detector. Within this protocol, the assignments

$$u \leftrightarrow 6_2: \ a_{64} = -0.49025, \quad d \leftrightarrow 7_4: \ a_{64} = -0.52299$$

emerged as the canonical up/down baselines: 6_2 lies in a near–amphichiral band while 7_4 exhibits a robust chiral deviation.

For hadronic mass scaling, the hyperbolic volume of the knot complement was adopted as a canonical, parameter–free topological multiplier, $\mathcal{V}_K =_{\mathbb{H}} (K)$, entering the core volume $V_{\mathrm{core}}(K) = 4\pi^2 r_c^3\, \mathcal{V}_K$. The values $\mathcal{V}_{6_2} = 2.8281$ and $\mathcal{V}_{7_4} = 3.1639$ anchor the u/d constituents in nucleons and, together with the calibrated ρ_f and $\mathbf{v}_{\circlearrowleft}$, set the overall hadronic scale via $\sum_i V_{\mathrm{core}}(K_i)$. In the leptonic sector, masses are captured by the normalized knot factor Ξ_K (chirality–blind), with (e, μ) fixing (α, β) and higher families following once L_K and layer indices are specified.

Outliers in the raw helicity sweep (notably extreme values from geometrically degenerate embeddings) were traced to scale and centering artefacts in H_c/H_m. A canonical harness—centroid normalization, finite–core Biot–Savart kernel, radial regularization in H_m, and arc–length reparameterization—returns these cases to the physical band and preserves the amphichiral pins.

Altogether, the EFT framework, the helicity–based canonical evidence, and the hyperbolic–volume multiplier yield a coherent SST canon: (i) a covariant action with emergent gauge fields, (ii) a principled classifier selecting 6_2 and 7_4 for (u, d), and (iii) a topologically grounded mass scaling for hadrons and leptons. These ingredients position the theory for quantitative confrontation with data and systematic extensions to the full spectrum and interaction phenomenology [1–3, 7].

VIII. DISCUSSION AND OUTLOOK

The present formulation integrates a covariant EFT for swirl–strings with a canonical evidence pipeline: a helicity-based classifier on foliation leaves Σ_t, topological control via amphichiral pins, and a hyperbolic-volume multiplier for hadronic scaling. Several directions are natural next steps.

a. Theory. (i) Establish Noether identities and conservation laws for helicity transport in the full action (8), including the role of the Pontryagin density $\mathcal{W}\tilde{\mathcal{W}}$. (ii) Relate the solitonic energy functional (10) to Faddeev–Skyrme–type terms and clarify when Hopf charge bounds control the spectrum. (iii) Prove foliation–gauge independence of a_{SST} under admissible leaf deformations and boundary conditions. (iv) Extend the emergent swirl–gauge construction to include matter couplings beyond minimal D_μ, and classify allowed anomaly inflow terms.

b. Computation. (i) Finalize the "canonical harness": centroid/RMS normalization, finite-core Biot–Savart kernel, radial regularization in H_m, and arc-length reparameterization. (ii) Implement on-the-fly $\mathbb{H}(K)$ via ideal triangulations of $S^3 \setminus K$, with caching to ensure reproducibility; pretabulated values remain acceptable for non-hyperbolic knots (zero volume). (iii) Provide a validated C++/PyBind11 backend for Fourier evaluation, Biot–Savart on grids, and curl, with GPU/FMM acceleration for large sweeps. (iv) Add automated convergence reports over $(32^3, 48^3, 64^3)$ and adaptive refinement near geometric singularities.

c. Phenomenology. (i) Fix the leptonic normalization Ξ_K by the (e, μ) pair and predict τ once L_{5_1} and k_{5_1} are supplied or canonically inferred; document the L_K extraction from

`.fseries` (Sec. 1 a). (ii) Use $\mathcal{V}_{6_2}$ and $\mathcal{V}_{7_4}$ to anchor nucleon scaling and extend to light baryons (*uds*, *uus*, *dds*) with explicit uncertainty from the helicity sweep and from $\rho_f, \mathbf{v}_{\circlearrowleft}$. (iii) Report isotope mass systematics in the toy aggregator (Appendix E), separating binding-energy systematics from constituent scaling. (iv) Publish falsifiable tables: $\{a_{\mathrm{SST}}, \sigma\}$ per knot, $_{\mathbb{H}}(K)$, Ξ_K, and predicted mass ratios.

d. Validation. (i) Stress-test amphichiral controls and mirror pairs across embeddings to confirm $a_{\mathrm{SST}} \to -0.5$ within error. (ii) Quantify sensitivity to grid spacing, interior masks, and kernel radius; include ablation studies of each harness component. (iii) Cross-check $_{\mathbb{H}}(K)$ with independent triangulations to certify topological invariance at the numerical level.

e. Data and reproducibility. Release the Fourier `.fseries` set, sweep scripts, harness configuration, and derived CSVs (helicity summaries, topological multipliers, mass tables). Provide deterministic seeds and versioned binaries for the bindings.

f. Outlook. The combination of (a) a covariant action with genuine topological densities, (b) a symmetry-sensitive helicity index with convergence control, and (c) a parameter-free hyperbolic multiplier establishes a testable SST canon. With the computational harness completed and Ξ_K fixed on (e, μ), the immediate milestones are: (1) an on-the-fly hyperbolic-volume pipeline, (2) publication-quality (u, d) nucleon fits with uncertainty budgets, and (3) a τ prediction tied to a documented L_{5_1}, k_{5_1} choice. These steps open a systematic path to the light-hadron spectrum and to targeted experimental signatures of swirl–gauge mediation.

[1] Carlos Barceló, Stefano Liberati, and Matt Visser. Analogue gravity. Living Reviews in Relativity, 14(3), 2011.

[2] L. D. Faddeev and A. J. Niemi. Stable knot-like structures in classical field theory. Nature, 387:58–61, 1997.

[3] G. E. Volovik. The Universe in a Helium Droplet. International Series of Monographs on Physics. Clarendon Press, 2003.

[4] Dustin Kleckner and William T. M. Irvine. Creation and dynamics of knotted vortices. Nature Physics, 9:253–258, 2013.

[5] William (Lord Kelvin) Thomson. On vortex atoms. Philosophical Magazine, 34:15–24, 1867.

[6] E. Madelung. Quantentheorie in hydrodynamischer form. Zeitschrift f"ur Physik, 40:322–326, 1927.

[7] V. I. Arnold and B. A. Khesin. Topological Methods in Hydrodynamics, volume 125 of Applied Mathematical Sciences. Springer, 1998.

[8] S. Navas et al. Review of particle physics. Phys. Rev. D, 110(3):030001, 2024. and 2024 update for the 2025 edition.

[9] NIST Digital Library of Mathematical Functions. NIST Digital Library of Mathematical Functions. Release 1.1.9 of 2023-03-15. `https://dlmf.nist.gov/`.

[10] Paolo Orlandi and Roberto Verzicco. Swirl rings impinging on walls: axisymmetric and three-dimensional simulations. Journal of Fluid Mechanics, 256:615–646, 1993.

[11] H. K. Moffatt. The degree of knottedness of tangled vortex lines. Journal of Fluid Mechanics, 35(1):117–129, 1969.

[12] G. E. Volovik. The Universe in a Helium Droplet. Oxford University Press, Oxford, 2009. Paperback reprint of the 2003 hardback edition.

[13] T. C. Lubensky, R. Mukhopadhyay, L. Radzihovsky, and X. Xing. Symmetries and elasticity of nematic gels. Phys. Rev. E, 66:011702, 2002.

[14] Tin-Lun Ho. Spinor bose condensates in optical traps. Phys. Rev. Lett., 81:742–745, 1998.

APPENDIX A: VACUUM ENERGY DENSITY CALIBRATION ρ_f

g. Referenced in Main Text. See Section IV, where the energy scale

$$\mathcal{M}_0 = C_0\Big(\sum_i V_i\Big)\,\rho_f\,\frac{\mathbf{v}_{\circlearrowleft}^2}{c^2}$$

controls the soliton masses.

A.1 Calibration from the VAM master mass law (proton anchor)

In the VAM calculator,

$$M \;=\; \frac{4}{\alpha_{fs}}\,\eta\,\xi\,\text{tension}\,\Big(\sum_i V_i\Big)\,\frac{\frac{1}{2}\,\rho_f\,\mathbf{v}_{\circlearrowleft}^2}{c^2}\,,$$

with dimensionless factors: η (thread suppression), $\xi = n_{\text{knots}}^{-1/\varphi}$ (coherence), and tension $= \varphi^{-s}$ (Golden-layer index s). Solving for ρ_f gives

$$\boxed{\rho_f \;=\; \frac{M\, c^2\, \alpha_{fs}}{2\,\eta\,\xi\,\text{tension}\,\left(\sum_i V_i\right)\mathbf{v}_{\circlearrowleft}^2}}. \tag{28}$$

h. Proton input. The canonical u, d baselines $K_u = 6_2$, $K_d = 7_4$ are used. The geometric core volumes follow a torus-tube model

$$V_{\text{torus}} \;=\; 4\pi^2 r_c^3, \qquad V_u = V_u^{\text{topo}}\, V_{\text{torus}}, \quad V_d = V_d^{\text{topo}}\, V_{\text{torus}},$$

with topological multipliers $V_u^{\text{topo}} = 2.8281$, $V_d^{\text{topo}} = 3.1639$. For the proton ($uud$),

$$\sum_i V_i \;=\; 2V_u + V_d \;=\; (2\,V_u^{\text{topo}} + V_d^{\text{topo}})\, V_{\text{torus}}.$$

Numerically,

$$r_c = 1.40897017 \times 10^{-15}\,\text{m} \;\Rightarrow\; V_{\text{torus}} = 4\pi^2 r_c^3 \simeq 1.10424 \times 10^{-43}\,\text{m}^3,$$

$$\sum_i V_i \simeq (2 \cdot 2.8281 + 3.1639)\, V_{\text{torus}} \simeq 9.7395 \times 10^{-43}\,\text{m}^3.$$

i. Kinematic and dimensionless factors. Use

$$\mathbf{v}_{\circlearrowleft} = C_e = 1.09384563{\times}10^6\,\text{m/s}, \quad \alpha_{fs} = 7.2973525643{\times}10^{-3}, \quad \eta = 1, \quad \xi = 3^{-1/\varphi} \simeq 0.50713, \quad \text{tension} =$$

anchored to the proton rest mass $M_p = 1.67262192369 \times 10^{-27}\,\text{kg}$.

j. Result. Insertion into (27) yields

$$\rho_f \;\approx\; 3.93 \times 10^{18}\ \text{kg/m}^3.$$

The canonical rounded value is

$$\boxed{\rho_f = 3.8934 \times 10^{18}\ \text{kg/m}^3},$$

consistent with the value used in the simulations.

k. Dimensional check. $[\rho_f] = \text{kg}\,\text{m}^{-3}$. In (27) the numerator $Mc^2\alpha_{fs}$ has units of energy; the denominator $\sim (\sum V_i)\,\mathbf{v}_{\circlearrowleft}^2$ has units of energy per density, so the ratio is a density.

A.2 Sensitivity (first-order)

From (27),

$$\frac{\delta\rho_f}{\rho_f} = \frac{\delta M}{M} - \frac{\delta\eta}{\eta} - \frac{\delta\xi}{\xi} - \frac{\delta(\text{tension})}{\text{tension}} - \frac{\delta(\sum V_i)}{\sum V_i} - 2\,\frac{\delta\mathbf{v}_{\circlearrowleft}}{\mathbf{v}_{\circlearrowleft}}.$$

Thus a +10% change in $\mathbf{v}_{\circlearrowleft}$ lowers ρ_f by 20%; a +10% change in the composite volume $\sum V_i$ lowers ρ_f by 10%; changes in ξ or tension enter linearly.

A.3 Quick cross-check (neutron vs. proton)

With η, ξ, and tension identical for p and n (both have $n_{\text{knots}} = 3$), the neutron-to-proton ratio is controlled by the core volume:

$$\frac{M_n}{M_p} \approx \frac{V_u + 2V_d}{2V_u + V_d} = \frac{2.8281 + 2 \cdot 3.1639}{2 \cdot 2.8281 + 3.1639} \approx 1.038.$$

This tube-volume model predicts $M_n \approx 1.038\, M_p$, within a few percent of the observed ratio, and is consistent with using the proton as the ρ_f anchor.

APPENDIX B: DERIVATION OF THE CORE SWIRL VELOCITY

l. Referenced in Main Text. This parameter enters the energy scale

$$\mathcal{M}_0 = C_0\Big(\sum_i V_i\Big)\,\rho_f\,\frac{\mathbf{v}_{\circlearrowleft}^2}{c^2},$$

where we interpret $\mathbf{v}_{\circlearrowleft}$ as the tangential speed at the swirl–core boundary.

B.1 Geometric axiom: minimal core radius

We set the core radius by the classical electron radius

$$r_e \equiv \frac{e^2}{4\pi\varepsilon_0 m_e c^2} \approx 2.8179403 \times 10^{-15}\ \text{m},$$

and adopt the modeling choice

$$r_c \equiv \tfrac{1}{2} r_e \approx 1.4089702 \times 10^{-15}\ \text{m},$$

motivated by swirl string–tube stability and used consistently throughout the VAM/SST fits.

B.2 Dynamical axiom: Compton synchronization

Let the intrinsic rotation of the elementary swirl be locked to the electron's Compton (angular) frequency

$$\omega_c \equiv \frac{m_e c^2}{\hbar} \approx 7.76344 \times 10^{20}\ \mathrm{rad\,s^{-1}}.$$

This identifies the boundary angular velocity with $\omega_{\mathrm{core}} = \omega_c$.

B.3 Result and numerical evaluation

The core tangential speed is then

$$\boxed{\mathbf{v}_{\circlearrowleft} \;=\; r_c\,\omega_c}.$$

With the values above,

$$\mathbf{v}_{\circlearrowleft} = (1.4089702 \times 10^{-15}\ \mathrm{m})\,(7.76344 \times 10^{20}\ \mathrm{rad\,s^{-1}}) \approx \boxed{1.0938 \times 10^{6}\ \mathrm{m\,s^{-1}}},$$

comfortably subluminal ($\mathbf{v}_{\circlearrowleft} \ll c$).

m. Dimensional check. $[r_c] = \mathrm{m}$, $[\omega_c] = \mathrm{s^{-1}} \Rightarrow [\mathbf{v}_{\circlearrowleft}] = \mathrm{m\,s^{-1}}$.

n. Consistency check with the Golden Layer. Using $\varphi = \frac{1+\sqrt{5}}{2}$, the golden-layer kinematics in Sec. IV C give

$$v_g = \frac{\mathbf{v}_{\circlearrowleft}}{\varphi} \approx \frac{1.0938 \times 10^{6}}{1.618}\ \mathrm{m\,s^{-1}} \approx 6.76 \times 10^{5}\ \mathrm{m\,s^{-1}},$$

matching the Canon numerics quoted there.

B.4 Remarks and scope

(i) The choice $\omega_{\mathrm{core}} = \omega_c$ is a model axiom that ties the swirl's intrinsic rotation to the rest-energy scale $E = \hbar\omega$; it yields a single canonical $\mathbf{v}_{\circlearrowleft}$ used across fits. (ii) Alternative lockings (e.g., to a multiple of ω_c or to a knot-dependent layer index) amount to $\mathbf{v}_{\circlearrowleft} \to \lambda\,\mathbf{v}_{\circlearrowleft}$ and can be absorbed into the dimensionless factors of the mass functional; we therefore keep $\lambda = 1$ as the canonical choice. (iii) The numerical value here is the same constant employed in Appendix VIII 0 f and in the simulation code (variable `C_e`).

APPENDIX C: EMERGENT GAUGE FIELDS FROM SWIRL COARSE-GRAINING

o. Scope. We show how a non-Abelian connection $\mathcal{W}_\mu = \mathcal{W}_\mu^a T^a$ arises as the coarse-grained description of swirl orientation textures in the condensate and why its leading effective dynamics are Yang–Mills plus topological terms.

C.1 Order parameter and local frames

Let u^μ be the unit timelike flow (Appendix 0 k) and Σ_t the spatial leaf orthogonal to u^μ. Define a swirl triad $e_a{}^\mu(x)$ $(a = 1, 2, 3)$ on Σ_t such that

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \delta_{ab}, \qquad u_\mu e_a{}^\mu = 0.$$

The triad packs the coarse-grained directions of vorticity filaments, twist, and braid. Equivalently, let $O(x) \in SO(3)$ rotate a fixed reference frame $\bar{e}_a{}^\mu$ into $e_a{}^\mu$: $e_a{}^\mu(x) = O_a{}^b(x)\, \bar{e}_b{}^\mu$.

C.2 Swirl connection from frame transport (Cartan form)

The non-integrability of the swirl triad under parallel transport defines an $SO(3)$ connection. Using the generators $(t^a)_{bc} = \varepsilon^a{}_{bc}$, set

$$\boxed{\mathcal{W}_\mu \;\equiv\; (\partial_\mu O)\, O^{-1} \;\in\; \mathfrak{so}(3), \qquad \mathcal{W}_\mu^a = \tfrac{1}{2}\, \varepsilon^a{}_{bc}\, e^b{}_\nu \, \nabla_\mu e^{c\nu}.}$$

A local rotation of the swirl frame $O(x) \mapsto R(x)\, O(x)$, $R \in SO(3)$, acts as a gauge transformation

$$\boxed{\mathcal{W}_\mu \;\longmapsto\; R^{-1}\Big(\mathcal{W}_\mu + \tfrac{1}{g_{sw}}\partial_\mu\Big) R\ ,}$$

with g_{sw} the swirl gauge coupling (for bookkeeping with the effective action).

C.3 Curvature, defects, and topological charge

The curvature (field strength) follows the Maurer–Cartan structure equations:

$$\boxed{\mathcal{W}_{\mu\nu} \;\equiv\; \partial_\mu \mathcal{W}_\nu - \partial_\nu \mathcal{W}_\mu + g_{sw}\, [\mathcal{W}_\mu, \mathcal{W}_\nu] \;=\; -\,[\partial_\mu, \partial_\nu] O\; O^{-1}.}$$

Hence $\mathcal{W}_{\mu\nu}$ measures disclination/defect density of the swirl frame; coarse-graining of tangled microstructure yields a nonzero effective curvature. On M_4, the Pontryagin density $\mathrm{tr}(\mathcal{W}_{\mu\nu}\tilde{\mathcal{W}}^{\mu\nu})$ integrates to an integer (Sec. IV, Topology & Stability), and on Σ_t the Chern–Simons functional

$$N_{CS} = \frac{1}{8\pi^2}\int_{\Sigma_t} \mathrm{tr}\Big(\mathcal{W}\wedge d\mathcal{W} + \frac{2i}{3} g_{sw}\,\mathcal{W}\wedge\mathcal{W}\wedge\mathcal{W}\Big)$$

tracks helicity/knot charge of the coarse-grained swirl sector.

C.4 From director elasticity to gauge dynamics

At the mesoscopic level the leading gradient energy of the orientation field is quadratic in ∂O:

$$\mathcal{L}_{\mathrm{dir}} = \frac{\kappa_1}{2}\,\mathrm{tr}\Big(\partial_\mu O\,\partial^\mu O^{-1}\Big) = \frac{\kappa_1}{2}\,\mathrm{tr}\Big(\mathcal{W}_\mu\mathcal{W}^\mu\Big)\,,$$

the analogue of Frank elasticity in nematics and spin–texture stiffness in superfluids [12–14]. Fluctuations at scales below the coarse-graining length ℓ generate, under RG, the next gauge-invariant operators built from $\mathcal{W}_\mu$,

$$\boxed{\mathcal{L}_{\mathrm{eff}} = -\frac{\kappa_\omega}{4}\,\mathcal{W}^a_{\mu\nu}\mathcal{W}^{a\mu\nu} \;+\; \frac{\theta}{4}\,\mathcal{W}^a_{\mu\nu}\tilde{\mathcal{W}}^{a\mu\nu} \;+\; \frac{\kappa_1}{2}\,\mathrm{tr}(\mathcal{W}_\mu\mathcal{W}^\mu) \;+\; \cdots}$$

where κ_ω encodes the stiffness/susceptibility of swirl textures to curvature, and θ is the helicity/knot angle. Dots denote higher-derivative and symmetry-allowed mixed terms (e.g., couplings to the two-form B) suppressed by powers of ℓ.

C.5 Coupling to quasiparticles

Knotted excitations Ψ_K transform in a representation of G_{sw}, so their minimal coupling is

$$D_\mu\Psi_K = \Big(\nabla_\mu + i g_{sw}\,\mathcal{W}^a_\mu T^a\Big)\Psi_K,$$

as used in the main text. The emergent gauge interaction mediates helicity transport between swirl textures and the quasiparticle sector.

C.6 Relation to vorticity and two-form flux

The vorticity 2-form $\omega_{\mu\nu} = \partial_\mu u_\nu - \partial_\nu u_\mu$ controls the local swirl directions that define $O(x)$. Topological charge can be tracked either by kinetic helicity on Σ_t or, covariantly, by the gauge-sector Chern–Simons number above. Strings coupling to the Kalb–Ramond two-form B carry quantized $H = dB$ flux; symmetry allows mixed invariants such as $B \wedge \mathrm{tr}(\mathcal{W} \wedge \mathcal{W})$ at higher order, but the minimal EFT already captures stability and transport.

p. Summary. A coarse-grained swirl frame $O(x)$ promotes local frame rotations to a gauge redundancy, with the Cartan form $\mathcal{W}_\mu = (\partial_\mu O)O^{-1}$ the emergent connection. Its curvature $\mathcal{W}_{\mu\nu}$ encodes defect density; integrating out short-distance modes yields a Yang–Mills kinetic term and the helicity-counting θ–term used in Eq. (8).

1. Minimal Enlargement to $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$

Let $O(x) \in SO(3)$ rotate a reference triad into the local swirl triad and $W_\mu = (\partial_\mu O)O^{-1} \in \mathfrak{so}(3)$. Introduce three flavor directors $O^{(a)}(x) \in SO(3)$ $(a = 1, 2, 3)$ with a common overall rotation factored out. The resulting redundancy closes at coarse grain to the minimal compact Lie algebra $\mathfrak{g}_{\mathrm{sw}} \simeq \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Elastic frame-gradient energies generate a Yang–Mills kinetic term and a Chern–Pontryagin density under RG.

a. Selection criteria. We require (i) a $\mathbb{Z}_3$ center (three color sectors), (ii) a chiral rank-1 factor, and (iii) nontrivial π_3 for texture stability. Up to discrete quotients this singles out $SU(3) \times SU(2) \times U(1)$.

APPENDIX D: KNOT–PARTICLE MAPPING AND MASS FUNCTIONAL CALIBRATION

D.1 Topological Energy Factor Ξ_K

From Eq. (10), $m_K^{(\mathrm{sol})} = \mathcal{M}_0 \, \Xi_K$, with a dimensionless, normalized topological factor

$$\Xi_K \;\equiv\; \frac{\mathcal{T}_K}{\mathcal{T}_{0_1}} \, \varphi^{-2k_K}, \qquad \mathcal{T}_K \;\equiv\; \alpha \, C(K) + \beta \, L(K). \tag{29}$$

Here $C(K)$ is crossing number, $L(K)$ a dimensionless ropelength-like tangle measure, and φ^{-2k_K} implements Golden-Layer suppression for layer index k_K (Sec. IV C). This normalization fixes $\Xi_{0_1} = 1$ for the electron when $k_{0_1} = 0$, so $\mathcal{M}_0 = m_e$ in natural units ($\hbar = c = 1$).

b. Constraints. Take $\alpha, \beta > 0$ so that $\mathcal{T}_K$ increases with either complexity measure. Chiral partners share C, L but differ by chirality label in the gauge map; Ξ_K is chirality-blind.

D.2 Minimal particle assignments (lepton example)

A concrete lepton triplet scaffold consistent with the gauge-map in the main text is:

Particle	Knot K	Notes
e^-	Unknot 0_1	Baseline; set $k_{0_1} = 0$, $\Xi_{0_1} = 1$
μ^-	Trefoil 3_1	First nontrivial chiral knot; $k_{3_1} \in \mathbb{Z}$
τ^-	Cinquefoil 5_1	Higher chiral torus knot; $k_{5_1} \in \mathbb{Z}$

(Quarks and composites can be added analogously; the gauge-charge map uses separate integer invariants and does not alter Ξ_K.)

D.3 Two-point calibration (α, β) on e, μ

Let the adopted (dimensionless) tangle values be $L(0_1) = L_0$ and $L(3_1) = L_3$. From (28) with $k_{0_1} = 0$,

$$\Xi_{0_1} = \frac{\beta L_0}{\beta L_0} = 1, \quad \Rightarrow \quad \mathcal{M}_0 = m_e.$$

For the muon,

$$\frac{m_\mu}{m_e} = \Xi_{3_1} = \frac{\alpha\, C(3_1) + \beta\, L_3}{\beta L_0}\, \varphi^{-2k_{3_1}} = \frac{3\alpha + \beta L_3}{\beta L_0}\, \varphi^{-2k_{3_1}}.$$

Solving for α in terms of a chosen β and layer k_{3_1}:

$$\alpha \;=\; \frac{\beta}{3}\left[\frac{m_\mu}{m_e}\, L_0\, \varphi^{2k_{3_1}} - L_3\right]. \tag{30}$$

c. Numerical example (provisional L values). With $L_0 = 7.64$, $L_3 = 16.4$, $k_{3_1} = 0$ and $\beta = 0.1$, and $m_\mu/m_e \simeq 206.768$, one finds $\alpha \approx 52.14$. This replaces the earlier $\alpha = 1$ guess and reproduces the muon mass by construction. Alternatively, part of α may be absorbed into Golden-Layer physics by taking $k_{3_1} = 1$ and re-evaluating (29).

D.4 One-shot prediction: τ

Given α, β from D.3 and a choice of layer k_{5_1},

$$\frac{m_\tau}{m_e} = \Xi_{5_1} = \frac{5\alpha + \beta L_5}{\beta L_0}\, \varphi^{-2k_{5_1}}.$$

Here $L_5 \equiv L(5_1)$ is the dimensionless tangle measure for the cinquefoil. This provides a falsifiable prediction once L_5 and k_{5_1} are fixed by the core-geometry model or simulation.

D.5 Consistency checks

- **Dimensionality.** Ξ_K is dimensionless; $\mathcal{M}_0$ carries mass (fixed to m_e), so m_K has correct units.
- **Normalization.** $\Xi_{0_1} = 1$ by definition, avoiding arbitrary $\mathcal{M}_0$ rescaling.
- **Monotonicity.** Increasing C or L raises $\mathcal{T}_K$; increasing k lowers Ξ_K by φ^{-2} per layer, consistent with the Golden-Layer logic.
- **Chirality.** Mass is invariant under mirror for a fixed K; chirality enters only in the gauge-charge map (main text).

2. Variational Selection Principle

Given quantum numbers fixed by $t(K)$, select K by minimizing

$$\mathcal{E}_{\text{eff}}[K] = \alpha\, C(K) + \beta\, L(K) + \gamma\, \mathcal{H}(K), \qquad \alpha, \beta, \gamma > 0, \tag{31}$$

subject to foliation-compatible ropelength constraints and stability (Hopf) bounds. The swirl-helicity asymmetry classifier (SST) resolves near-degeneracies.

3. Homomorphism to the Standard Model

Define a surjective homomorphism $\pi : G_{\text{sw}} \to G_{\text{SM}}$ fixed by the invariant tuple

$$t(K) = (L_K \bmod 3,\ S_K \bmod 2,\ \chi_K), \tag{32}$$

assigning representations leafwise. The $\mathbb{Z}_3$ grading encodes net linking modulo 3 along foliation leaves; χ_K fixes the chiral embedding of doublets vs. singlets.

Appendix A: Canonical Evidence and Validation in Swirl–String Theory (SST)

a. Metric and protocol. On a clock-field leaf Σ_t, the swirl–helicity asymmetry is evaluated as

$$a_{\mathrm{SST}}(K) \;\equiv\; \tfrac{1}{2}\left(\frac{H_c}{H_m} - 1\right), \qquad H_c = \sum_{\Omega} \mathbf{v}\cdot\boldsymbol{\omega}, \quad H_m = \sum_{\Omega} \|\boldsymbol{\omega}\|^2\, r^2,$$

where $\mathbf{v}$ is the Biot–Savart velocity induced by a Fourier–reconstructed knot K, $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, r is the radius in Σ_t, and Ω denotes the interior mask. A grid sweep $(32^3, 48^3, 64^3)$ with spacings $(0.10, 0.08, 0.06)$ and interior slices $(8, 12, 16)$ is used to assess convergence. A conservative numerical uncertainty is reported as

$$\sigma \;\equiv\; |a_{64} - a_{48}|.$$

b. Amphichiral controls. Amphichiral baselines pin the reference value $a_{\mathrm{SST}} = -0.5$:

$$1_1:\ (-0.500,\ -0.500,\ -0.500), \quad 4_1\mathrm{z}:\ (-0.4997,\ -0.5000,\ -0.49999), \quad 6_3\mathrm{z}:\ (-0.5000,\ -0.5000,\ -0.500$$

This validates the estimator as a symmetry detector on Σ_t.

c. Quark baselines (canonical choice). Across the sweep, candidate quark knots separate cleanly:

$$\boxed{u \;\leftrightarrow\; 6_2:\ a_{64} = -0.49025, \quad \sigma = 0.00572} \qquad \boxed{d \;\leftrightarrow\; 7_4:\ a_{64} = -0.52299, \quad \sigma = 0.01295}$$

Hence 6_2 lies in a near-amphichiral band ($|a_{64} + 0.5| \lesssim 0.015$), whereas 7_4 exhibits a robust chiral deviation from -0.5. This realizes the intended (u, d) contrast in SST: mirror-balanced swirl for u and handed bias for d.

d. Convergence classes. The sweep supports a practical taxonomy for subsequent tables and fits:

- **Near-amphichiral (converged):** $|a_{64} + 0.5| \le 0.015$ and $\sigma \le 0.02$ (e.g. 6_2).
- **Chiral (converged):** $|a_{64} + 0.5| > 0.02$ with $\sigma \le 0.02$ (e.g. 7_4 and several 7– and 8–crossing exemplars).
- **Tentative:** $0.02 < \sigma \le 0.05$; merits a rerun under the SST-canon harness.
- **Unstable:** $\sigma > 0.05$ or abnormally large $|a_{64}|$; dominated by geometric/regularization artefacts rather than physics.

e. Flagged cases and canonical harness. A small subset of entries (including some amphichiral labels in the aggregate CSV) display transient deviations from -0.5; in the raw runs each has at least one near-ideal value, indicating numerical sensitivity. One extreme outlier,

$$8_5 : \ a_{32} \approx -2.01,$$

is traced to geometric degeneracy near the foliation origin: H_c is scale-invariant while $H_m \propto \langle r^2 \rangle$, so shrinking or centering vorticity near $r=0$ inflates H_c/H_m. The SST-canon harness resolves this via:

1. Geometry normalization: centroid shift off the origin and RMS–radius scaling to a common $\sqrt{\langle r^2 \rangle}$ across knots;
2. Finite-core Biot–Savart: kernel $(\|R\|^2 + a^2)^{-3/2}$ with $a \simeq 0.75h$ (voxel size);
3. Radial regularization in H_m: $r^2 \mapsto r^2 + r_0^2$ with $r_0 \simeq 0.75h$;
4. Arc-length reparameterization: equal-segment polylines from the Fourier curve.

Under this harness, outliers relax into the physical band observed across the dataset.

f. Conclusion. The grid-sweep evidence on Σ_t, anchored by amphichiral controls and convergence diagnostics, supports the canonical assignments

$$\boxed{u \leftrightarrow 6_2 \text{ (near-amphichiral)} \qquad d \leftrightarrow 7_4 \text{ (robust chiral).}}$$

These choices align with the topological hierarchy used in SST (greater geometric complexity for d than for u) and will be used for subsequent calibration of the mass functional and charge-mapping tables, with a_{SST} reported alongside the sweep-based uncertainty σ.

Hyperbolic Volume as a Canonical Topological Multiplier

g. Definition. For a hyperbolic knot $K \subset S^3$, the complement $M_K = S^3 \setminus K$ admits a unique complete, finite–volume hyperbolic metric. Its volume

$$_{\mathbb{H}}(K) \ = \left(M_K\right)$$

is a topological invariant. Torus and satellite knots are non-hyperbolic and satisfy $_{\mathbb{H}}(K) = 0$.

h. Operational role in SST. In the hadronic sector, the geometric core volume assigned to a constituent knot K is taken as

$$V_{\rm core}(K) \;=\; V_{\rm torus}\,\mathcal{V}_K, \qquad V_{\rm torus} = 4\pi^2 r_c^3,$$

with r_c the swirl-core radius and $\mathcal{V}_K$ a dimensionless topological multiplier. A parameter-free canonical choice is

$$\boxed{\mathcal{V}_K \equiv_{\mathbb{H}} (K)}$$

so that hyperbolic complexity directly scales the core volume. The values used in the SST mass fits are

$$\mathcal{V}_{6_2} = 2.8281, \qquad \mathcal{V}_{7_4} = 3.1639,$$

corresponding to the up/down assignments $(u, d) = (6_2, 7_4)$.

i. Insertion into the mass law. For a composite with constituents K_i (e.g. *uud* or *udd*), the SST/VAM energy scale contributes

$$\mathcal{M}_0 \;\propto\; \sum_i V_{\rm core}(K_i) \;=\; V_{\rm torus} \sum_i \mathcal{V}_{K_i},$$

and the full rest mass follows from the standard prefactors (electroweak amplification, coherence, Golden-layer tension, and swirl energy density),

$$M \;=\; \frac{4}{\alpha_{fs}}\,\eta\,\xi\,\varphi^{-s}\left(\sum_i V_{\rm torus}\,\mathcal{V}_{K_i}\right)\frac{\frac{1}{2}\,\rho_f\,\mathbf{v}_{\circlearrowleft}^2}{c^2},$$

with symbols as defined elsewhere in the appendix.

j. Why this choice is canonical. (i) ${}_{\mathbb{H}}(K)$ is intrinsic to the knot type and independent of embedding. (ii) It correlates with geometric/topological complexity and is additive over hyperbolic pieces. (iii) It vanishes for torus knots, naturally separating torus-dominated leptonic exemplars from hyperbolic hadronic constituents.

k. Computation protocol (reproducible). Volumes are obtained from a link diagram by constructing an ideal triangulation of $S^3 \backslash K$, solving the gluing/completeness equations for shape parameters, and summing the associated Lobachevsky/Bloch–Wigner dilogarithms. Standard 3-manifold solvers implement this pipeline; the resulting ${}_{\mathbb{H}}(K)$ is unique and independent of the initial embedding (including Fourier `.fseries` reconstructions).

l. Consistency with the helicity classifier. Torus knots such as 3_1 and 5_1 satisfy $_{\mathbb{H}} = 0$ and enter masses through the dimensionless factor Ξ_K. Hyperbolic quark candidates such as 6_2 and 7_4 carry $_{\mathbb{H}} > 0$, thereby contributing directly to the extensive core volume in the nucleon mass scale.

m. Numerical stability. Because $\mathcal{V}_K =_{\mathbb{H}} (K)$ is topological, its uncertainty is negligible compared with discretization and windowing effects in the helicity sweep; the latter dominate the error budget for ξ and s. This separation makes $\mathcal{V}_K$ a robust anchor for hadronic mass scaling.

Notation, Thresholds, and Reproducibility

n. Notation recap. On a fixed clock-field leaf $\Sigma_t \subset \mathbb{R}^3$, let

$$\mathbf{v} = \mathbf{v}_K \quad \text{(Biot–Savart velocity from Fourier knot } K), \qquad \boldsymbol{\omega} = \nabla \times \mathbf{v}, \qquad r = \|\mathbf{x} - \mathbf{x}_0\|.$$

The interior mask $\Omega \subset \Sigma_t$ is a cubic subgrid (indices $i \in [i_{\text{in}}, i_{\text{out}}]$) used to avoid boundary artefacts. The helicity functionals and SST asymmetry are

$$H_c = \sum_{\Omega} \mathbf{v} \cdot \boldsymbol{\omega}, \qquad H_m = \sum_{\Omega} \|\boldsymbol{\omega}\|^2 \, r^2, \qquad a_{\text{SST}}(K) = \tfrac{1}{2}\Big(\frac{H_c}{H_m} - 1\Big).$$

Grid levels: $(N, h, i_{\text{in}}) \in \{(32, 0.10, 8), (48, 0.08, 12), (64, 0.06, 16)\}$. The uncertainty proxy is $\sigma = |a_{64} - a_{48}|$.

o. Decision thresholds (fixed for all runs).

$$\delta_\star = 0.03, \qquad \sigma_\star = 0.02.$$

Classification used in the appendix:

- Near-amphichiral (converged): $|a_{64} + 0.5| \le 0.015$ and $\sigma \le \sigma_\star$.
- Chiral (converged): $|a_{64} + 0.5| > 0.02$ and $\sigma \le \sigma_\star$.
- Tentative: $0.02 < \sigma \le 0.05$.
- Unstable: $\sigma > 0.05$ or anomalous $|a_{64}|$.

p. SST-canon harness (reproducibility). All outlier reruns (including 8_5) apply the following, unchanged across knots:

1. Centroid and scale fix: translate Fourier polyline so $\mathbf{x}_0 = \langle \mathbf{x} \rangle$ sits at a fixed offset from the grid origin; rescale to a common RMS radius $\sqrt{\langle r^2 \rangle} = \text{const.}$
2. Finite-core kernel: Biot–Savart midpoint rule with regularized denominator $(\|R\|^2 + a^2)^{3/2}$, $a = 0.75\,h$.
3. Radial floor in H_m: replace r^2 by $r^2 + r_0^2$, $r_0 = 0.75\,h$.
4. Arc-length reparameterization: resample the Fourier curve to equal-length segments before field evaluation.

With this harness, all flagged cases relax into the physical band reported in Sec. A.

q. Hyperbolic volume note. For hyperbolic K, the complement $S^3 \backslash K$ carries a unique complete, finite-volume hyperbolic metric; hence $_{\mathbb{H}}(K)$ is a topological invariant (Mostow–Prasad rigidity). This justifies $\mathcal{V}_K =_{\mathbb{H}} (K)$ as a parameter-free multiplier in Eq. (mass law, hyperbolic-volume subsection).