Foundations of Velocity Fields and Energies in a Vortex System: A Brief Article

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Abstract

This article outlines theoretical foundations of vortical velocity fields and their associated energies, including a distinction between self- and cross-energies, in the context of a generic vortex-based model. We close with a derivation outline for the cross-energy term, highlighting its application in vortex dynamics and fluid–structure interactions.

1 Introduction

Vortex dynamics are a core component of many fluid and plasma systems, including tornado-like flows, knotted vortices in classical or superfluid turbulence, and various complex topological fluid systems. A deeper understanding of the energy budgets associated with these flows can shed light on processes like vortex stability, reconnection, and global flow organization. We begin by motivating how velocity fields can be decomposed so as to capture the total energy (i.e. self- plus cross-energy), and how this approach helps track flows in both 2D and 3D.

2 Foundations: Velocity Fields and Total (Self + Cross) Energy

In an incompressible fluid, the velocity field $\mathbf{u}(\mathbf{x}, t)$ is typically governed by the Navier–Stokes or Euler equations. For inviscid analyses, the Euler equations for incompressible flow read

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p, \quad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

We also consider the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, which can be used to characterize vortex structures.

To understand the *total* kinetic energy, we can split it as follows:

$$E_{\text{total}} \; = \; E_{\text{self}} \; + \; E_{\text{cross}}. \tag{2}$$

Here, E_{self} is that portion of energy which each vortex or partial flow element contributes independently (for instance, from local swirling motions), while E_{cross} encodes the contributions that arise from the interaction of different vortical elements. In a multi-vortex scenario, such a decomposition helps isolate the direct interaction between two (or more) vortex filaments or sheets.

3 Momentum and Self-Energy Considerations

A starting point is to recall that for a single vortex of circulation Γ, with an azimuthally symmetric core, the induced velocity is sometimes approximated by classical results such as

$$V = \frac{\Gamma}{4\pi R}\big(\ln \tfrac{8R}{a} - \beta\big), \tag{3}$$

where R is the main vortex loop radius, $a \ll R$ is a measure of core thickness, and β depends on details of the core model [1]. The *self-energy* associated with that vortex, E_{self}, can be cast in a similar form that depends on $\ln(R/a)$, exemplifying how thin-core vortices' energies scale with geometry.

In more general fluid or vortex-lattice models, we can track E_{self} as the sum of individual core energies. Further, the presence of multiple filaments modifies the total energy by cross-terms of the velocity fields (the cross-energy). This cross-energy often drives key phenomena such as vortex merging or the 'recoil' effects in wave–vortex interactions.

4 Defining and Tracking Cross-Energy

When multiple vortices (or partial velocity distributions) co-exist, the total velocity field $\mathbf{u}$ can be superposed:

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \tag{4}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ come from distinct sub-systems. In that scenario, the kinetic energy for a fluid volume V is

$$E_{\text{total}} = \frac{\rho}{2}\int_V \mathbf{u}^2\, dV = \frac{\rho}{2}\int_V \big(\mathbf{u}_1 + \mathbf{u}_2\big)^2\, dV \tag{5}$$

$$= \frac{\rho}{2}\int_V \mathbf{u}_1^2\, dV \;+\; \frac{\rho}{2}\int_V \mathbf{u}_2^2\, dV \;+\; \rho\int_V \mathbf{u}_1\cdot\mathbf{u}_2\, dV, \tag{6}$$

revealing an interaction or *cross-energy* term

$$E_{\text{cross}} = \rho\int_V \mathbf{u}_1\cdot\mathbf{u}_2\, dV. \tag{7}$$

Much of the interesting physics arises from (7), because it grows or shrinks depending on the vortex geometry and distance between them. Its dynamical evolution can lead to, e.g., merging or rebound. A main point is that each vortex's self-velocity can significantly affect the mutual velocities and thus create net forces or torque.

5 Applications to Helicity and Topological Flows

A related concept is helicity, measuring the topological complexity (knotting or linking) of vortex tubes. Classically, helicity H is given by

$$H = \int_V \mathbf{u}\cdot\boldsymbol{\omega}\, dV, \tag{8}$$

which can remain constant or be partially lost during reconnection events. In certain dissipative flows, the cross-energy terms in (7) can influence the effective rate of helicity change. Understanding E_{cross} is important for analyzing reconnection pathways in classical or superfluid turbulence.

6 Derivation Outline for Cross-Energy

Finally, we provide a succinct outline for deriving the cross-energy expression. Starting with the total velocity field $\mathbf{u} = \sum_{n=1}^{N} \mathbf{u}_n$ for N vortex or partial velocity fields, the total kinetic energy is:

$$E_{\text{total}} = \frac{\rho}{2} \int_V \left(\sum_{n=1}^{N} \mathbf{u}_n \right)^2 dV = \frac{\rho}{2} \sum_{n=1}^{N} \int_V \mathbf{u}_n^2 \, dV \; + \; \rho \sum_{n<m} \int_V \mathbf{u}_n \cdot \mathbf{u}_m \, dV. \tag{9}$$

One obtains N self-energy terms plus pairwise cross-energy integrals. The cross-energy for a pair (i, j) is:

$$E_{\text{cross}}^{(ij)} \; = \; \rho \int_V \mathbf{u}_i \cdot \mathbf{u}_j \, dV. \tag{10}$$

In practice, each $\mathbf{u}_n$ may be represented by known solutions of the Stokes or potential flow equations, or from approximate solutions for vortex loops. Then, either analytically or numerically, one obtains approximate cross-energies that can be used in reduced models describing the evolution of multi-vortex systems.

Conclusion

We have surveyed how the total fluid kinetic energy in the presence of multiple vortices can be split into self- and cross-energy terms. These cross-energy contributions are crucial for understanding vortex merging, knotted vortex untangling, or vortex–wave interactions in classical, superfluid, and plasma flows. In addition, we have sketched a systematic derivation of cross-energy and highlighted key aspects in discussing momentum and helicity. Future directions include refining these expressions for axisymmetric or knotted vortices and integrating them into large-scale models or computational frameworks.

References

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