A Hyperbolic Identity for the Golden Ratio

Let the golden ratio be

$$\varphi \equiv \frac{1+\sqrt{5}}{2}. \tag{1}$$

Recall the definition of the inverse hyperbolic sine [1]:

$$\operatorname{asinh}(x) = \ln\left(x + \sqrt{x^2+1}\right). \tag{2}$$

Substituting $x = \frac{1}{2}$ into (2) gives

$$\begin{aligned}
\operatorname{asinh}\left(\tfrac{1}{2}\right) &= \ln\left(\tfrac{1}{2} + \sqrt{\tfrac{1}{4}+1}\right) &(3)\\
&= \ln\left(\tfrac{1+\sqrt{5}}{2}\right) &(4)\\
&= \ln\varphi. &(5)
\end{aligned}$$

Exponentiating both sides yields the clean identity

$$\boxed{\varphi = \exp\left(\operatorname{asinh}\left(\tfrac{1}{2}\right)\right)}. \tag{6}$$

Numerical check. Using double precision, $\varphi \approx 1.618033988749895$ and $\exp(\operatorname{asinh}(1/2)) \approx 1.618033988749895$, matching to machine precision.

References

[1] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark. Nist digital library of mathematical functions. `https://dlmf.nist.gov/`, 2023. See §4.37 for inverse hyperbolic functions.