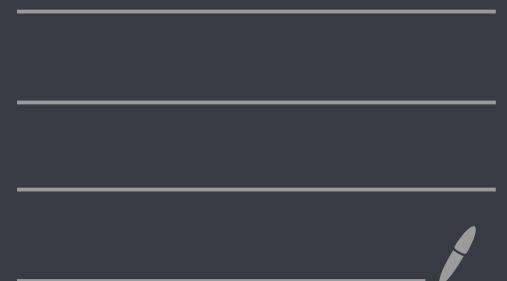


Homotopical Categories II

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Homotopical Categories II

Last time:

homotopical category (M, W) where

$W \subseteq M$ is a subcat. st:

- $\text{obj } W = \text{obj } M$

- W satisfies the 2 of 6 condition.

 $(gf, hg \in W \Rightarrow f, g, h, hg \in W)$

Eg $(C, \text{from}(C))$ (minimal homotopical)

Eg $(\text{Top}, \{\text{hpt}\text{y equiv}\})$

Homotopical functor := preserves weak equiv.

Eg π_n, h_n

Non-Eg $\text{colim}: \text{Top}^D \rightarrow \text{Top}$

Notes induced $F_*: \text{Ch}(Ab) \rightarrow \text{Ch}(Ab)$

where $W = \{\text{quasi-isomorphisms}\}$

Idea: Use Kan extensions to approximate non-homotopical functors.

Def: Given $F: C \rightarrow E$, $K: C \rightarrow D$, a left Kan extension is a pair $(\text{Lan}_K F: D \rightarrow E, \eta: F \rightarrow \text{Lan}_K F \circ K)$ s.t. any (L, γ) satisfying

$$\begin{array}{ccc} C & \xrightarrow{F} & E \\ K \searrow & \Downarrow \gamma & \nearrow L \\ & D & \end{array}$$

factors uniquely

$$\begin{array}{ccc} C & \xrightarrow{F} & E \\ K \searrow & \Downarrow \eta \Downarrow & \nearrow \text{Lan}_K F \\ & D & \end{array}$$

Right Kan extension:

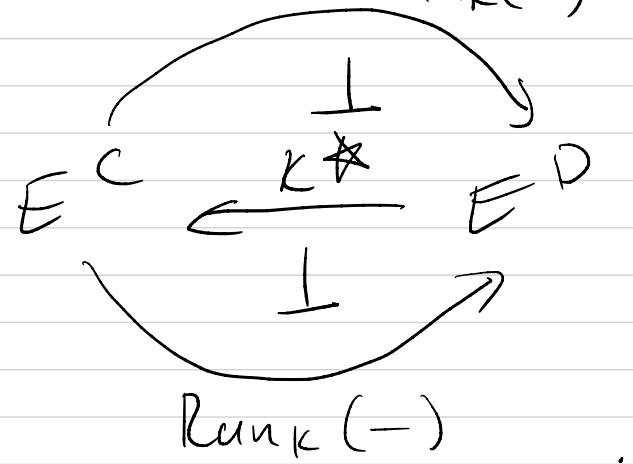
natural trans. go in other directions

Prop: Let $K: C \rightarrow D$ be fixed.

If $\text{Lan}_K F$ & $\text{Ran}_K F$ exist
for every $F \in E^C$ then:

(1) $\text{Lan}_K(-): E^C \rightarrow E^D$ is
a functor.

(2) We have $\text{Lan}_K(-)$



Pf: (1)

Let

$$C \xrightarrow{\gamma} \text{Lan}_K F \rightarrow E, \quad \& \quad G$$

$$C \xrightarrow{F} E \xrightarrow{\eta} \text{Lan}_K F$$

$$C \xrightarrow{G} E \xrightarrow{\chi} \text{Lan}_K G$$

Define $\text{Lan}_K(\gamma)$ by:

$$\begin{array}{ccc} F & \xrightarrow{\gamma} & G \\ \downarrow \eta & \swarrow \gamma & \downarrow \chi \\ \text{Lan}_K F \circ K & \dashrightarrow & \text{Lan}_K G \circ K \\ \text{Lan}_K(\gamma) \cdot K & & (\exists!) \end{array}$$

To check $\text{Lan}_K(-)$ is a functor:

$$\begin{array}{ccc} F & \xrightarrow{\quad \gamma_F \quad} & F \\ n \downarrow & \searrow \gamma \downarrow n & \\ \text{Lan}_K F \cdot K & \longrightarrow & \text{Lan}_K F \cdot K \\ & \gamma_{\text{Lan}_K F \cdot K} & \end{array}$$

is a commutative diagram.

$$\text{So uniqueness } \Rightarrow \text{Lan}_K(\gamma_F) = \gamma_{\text{Lan}_K F}.$$

$$\begin{array}{ccccc} F & \xrightarrow{\quad \alpha \quad} & G & \xrightarrow{\quad \beta \quad} & H \\ n \downarrow & \chi \downarrow & & \phi \downarrow & \\ \text{Lan}_K F \cdot K & \rightarrow & \text{Lan}_K G \cdot K & \rightarrow & \text{Lan}_K H \cdot K \\ \text{Lan}_K d \cdot K & & \text{Lan}_K \beta \cdot K & & \end{array}$$

$\text{Lan}_K \beta d \cdot K$

A diagram chase shows

$$\text{Lan}_K \beta d = \text{Lan}_K \beta \cdot \text{Lan}_K d.$$

(2): See ^{Riehl,} Category Theory in Context, p.192. \square

Eg let $H \leq G$ be groups.

$$\begin{array}{ccc} BH & \xrightarrow{\quad \phi \quad} & \underline{\text{Vect}}_K \\ i \downarrow & \nearrow \text{Ind}_H^G \phi & \downarrow \text{Ind}_H^G \phi \\ BG & & \end{array}$$

$\text{Ind}_H^G \phi$ is a $\text{Lan}_i \phi$

Similarly $\text{coind}_H^G \phi$ is a $\text{Ran}_i \phi$.

hence

$$\begin{array}{ccc} & \text{Ind}_H^G \phi & \\ & \perp & \\ \underline{\text{Vect}}_n & \xleftarrow{\quad i^* \quad} & \underline{\text{Vect}}_n \\ & \perp & \\ & \text{coind}_H^G \phi & \end{array}$$

Def: Let M, N be homotopical.

A total left derived functor

$\widetilde{LF} : \text{Ho}M \rightarrow \text{Ho}N$ is a right Kan extension $\text{Ran}_{\delta F}$

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow r & \uparrow \gamma & \downarrow s \\ \text{Ho}M & \xrightarrow{\text{LF}} & \text{Ho}N \end{array}$$

Rm: \widetilde{LF} corresponds to a homotopical functor

$$\text{LF} : M \rightarrow \text{Ho}N.$$

Def: A left derived functor of $F : M \rightarrow N$ is a homotopical functor $\text{LF} : M \rightarrow N$ with a n.t. $\lambda : \text{LF} \rightarrow F$ s.t

How is this defined? $\widetilde{\delta X} : \widetilde{\delta LF} \rightarrow \delta F$ is a total left derived functor of F .

$$\begin{array}{ccc} & F & \\ & \uparrow \lambda & \\ M & \xrightarrow{\text{LF}} & N \\ \downarrow r & \swarrow \text{SLF} & \downarrow s \\ \text{Ho}M & \xrightarrow{\widetilde{\delta X}} & \text{Ho}N \\ & \searrow \widetilde{\delta LF} & \\ & \widetilde{\delta LF} & \end{array}$$

Def: A left deformation on a homotopical category M is an endofunctor Q with a natural weak equivalence

$$q: Q \xrightarrow{\sim} 1$$

Lemma: Left deformations are homotopical.

Pf: For a homotopical (M, W) let $x \xrightarrow{f} y \in W$.

Then by definition of q ,

$$\begin{array}{ccc} Qx & \xrightarrow{\quad} & Qy \\ q_x \downarrow & \text{Qf} & \downarrow q_y \\ x & \xrightarrow{f} & y \end{array} \quad f q_x \in W \Rightarrow q_y Qf \in W$$

2 of 3 $\Rightarrow Qf \in W$. \square

Def: A left deformation of $F: M \rightarrow N$ is a left deformation Q of M s.t. M is homotopical on a full subcategory containing the image of Q .

Thm (Dwyer, Hirschhorn, Kan, Smith '04)

If $F: M \rightarrow N$ has a left deformation Q , then FQ is a left derived functor of F .

Pf: We want to show (SFQ, SFq) satisfy the universal property of total left derived functors.

Let $G: M \rightarrow H\Omega N$ be homotopical, and

$$\gamma: G \rightarrow SF.$$

By definition of γ being natural,

$$\begin{array}{ccc} GQ_x & \xrightarrow{Gq_x} & Gx \\ \gamma_{Qx} \downarrow & \Downarrow \text{tx} & \downarrow \gamma_x \\ SFQ_x & \xrightarrow{\quad} & SFx \\ & \scriptstyle \delta Fq_x & \end{array}$$

Since the q_x are weak equivs,

Gq_x are TSOS.

$$\text{Hence } \gamma_x = \delta Fq_x \circ \gamma_{Qx} \circ (Gq_x)^{-1},$$

meaning γ factors through SFq .

To show the factoring is

$$\text{unique, suppose } \gamma = SFq \circ \gamma'.$$

$$\text{Then } \gamma_Q = \delta Fq_Q \circ \gamma'_Q.$$

Since F is a left deformation, F sends the weak equivalences of Q to weak equivalences, hence δFq_Q is a natural γ_{SO} . Hence $\gamma'_Q = (\delta Fq_Q)^{-1} \circ \gamma_Q$ is determined by γ_Q . Uniqueness in general follows from

$$\begin{array}{ccc} GQ & \xrightarrow{\gamma_Q} & SFQ^2 \\ Gq \downarrow & \Downarrow & \downarrow SFQq \\ G & \xrightarrow{\gamma'} & SFQ \end{array}$$

since both Gq and $SFQq$ natural ISOMORPHISMS. \square

Coro: Given an additive
 $F: \text{Mod}_R \rightarrow \text{Mod}_S$,
 $F_{\#}: \text{Ch}_{\geq 0}(R) \rightarrow \text{Ch}_{\geq 0}(S)$ with
weak equivalences quasi-isos
has a left derived functor.

Rm: The classical left
derived functor is the
composition
 $\text{Mod}_R \xrightarrow{\text{deg}} \text{Ch}_{\geq 0}(R) \xrightarrow{\perp F_{\#}} \text{Ch}_{\geq 0}(S) \xrightarrow{\text{Ho}} \text{Mod}_S$.

Pf. Quasi-isos on non-negatively
graded projective chain complexes
are homotopy equivalence.
i.e. $F_{\#}$ is homotopical on the
image of the functor Q
which takes projective
resolutions.

Hence by the Thm, $\perp F_{\#} = F_{\#} Q$
is a left derived functor.