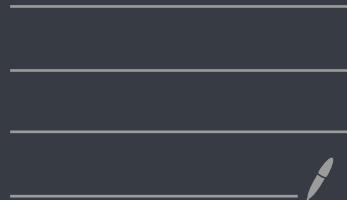


Homotopical Categories I

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- Some constructions

Preserve weak equivalences:

Eg $F: \text{Top} \rightarrow \text{Top}$

$$\begin{aligned} X &\mapsto X \times I \\ f &\mapsto f \times 1_I \end{aligned}$$

(clearly) $f_*: \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$

implies $F(f)_*: \pi_n(X \times I) \xrightarrow{\cong} \pi_n(Y \times I)$

Eg Let A, B be Abelian

categories, & $F: A \rightarrow B$

be additive.

Then F induces a functor

$$F_*: \text{Ch}(A) \rightarrow \text{Ch}(B).$$

If $f \in \text{Mor}(\text{Ch}(A))$ is a chain homotopy equiv, $F(f) \in \text{Mor}(\text{Ch}(B))$ too.

Def: (2 of 3) Model Category Axiom

A collection $W \subseteq \text{Mor}(e)$ satisfies the

two-of-three property if whenever any two of $\{f, g, gf\}$

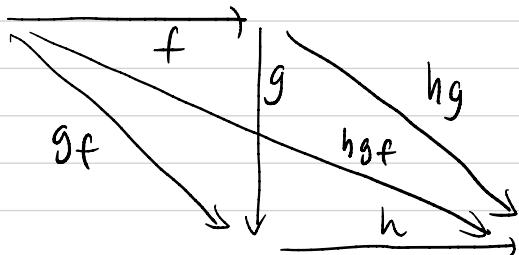
are in W , so is the third.

Eg The isomorphisms of any cat.

Eg Homotopy equivalences in Top.

Def (2 of 6) (Dwyer, Hirschhorn, Kan, Smith '04)

Let $W \subseteq \text{Mor}(\mathcal{C})$. If $\mathcal{H} f, g, h$,



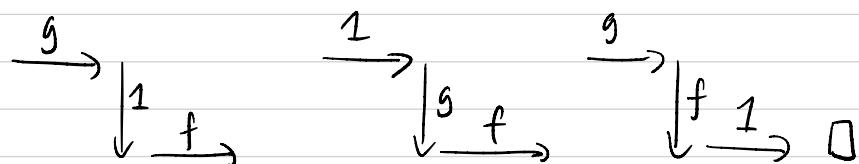
$gf, hg \in W \Rightarrow f, g, h, hgf \in W,$

then W satisfies the two-of-six property.

Rm Weak equivalences in a model cat. satisfy 2 of 6.

Exercise: 2 of 6 \Rightarrow 2 of 3.

Pf: Consider



(D+KS)

Def: Let M be a category, and

$W \subseteq M$ a subcategory. If:

- $\text{obj}(M) = \text{obj}(W)$
- W satisfies 2 of 6,

then (M, W) is called a homotopical category / homotopical.

Eg $(M, \text{Iso}(M))$ is homotopical.

Pf: WTS If hg, gf are ISOs, so are f, g, h, hgf .

Let $\overline{f_1 \dots f_n}$ denote $(f_1 \dots f_n)^{-1}$, and $g \in \text{Mor}(x, y)$. Then $g(h\overline{gh}) = 1_y$, $(h\overline{gh})g = (\overline{fg} + g)\overline{hgh}g = \overline{fg}f(g\overline{gh})g = \overline{fg}fg = 1_x$,

so $g \in \text{Iso}(e)$, and hence f, h , and gfh are too \square

Rm Since $1_x \in W \forall x \in \text{obj}(e)$,

2 of 6 applied to $\begin{array}{c} \overrightarrow{f} \\ \downarrow \overline{f} \\ \overrightarrow{f} \end{array}$
means $\text{Iso}(e) \subseteq W$.

Def: $(M, \text{Iso}(M))$ is called the minimal homotopical category of M .

Eg $(\text{Top}, \{\text{weak equiv}\})$ is homotopical.

Pf: Follows from the functoriality of π_n , and the previous example. \square

Eg $(\text{Top}, \{\text{homotopy equiv}\})$ is homotopical.

Pf: Same proof as with $(M, \text{Iso}(M))$. \square

Def A functor $F: (A, W_A) \rightarrow (B, W_B)$ is homotopical if $F(W_A) \subseteq F(W_B)$.

Eg π_n, h_n .

Non-Example

Let $F: A \rightarrow B$ be additive, A, B abelian, $F_{*}: \text{Ch}_{\geq 0}(A) \rightarrow \text{Ch}_{\geq 0}(B)$.

Let $W = \{\text{quasi-isomorphism}\}$
 $\{f: C \rightarrow D : f_{*}: h_n(C) \xrightarrow{\cong} h_n(D)\}$.

Let $f: C \rightarrow D$ be

$$\cdots \rightarrow \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4 \xrightarrow{\circ} \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4 \rightarrow 0$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z}/2 \rightarrow 0 \rightarrow \mathbb{Z}/2 \rightarrow 0$$

Let $F: Ab \rightarrow Ab$ be $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$.

Claim: $F(f)$ is not a quasi-isom.

Pf: Board exercise!

(Applying $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ to

$\mathbb{Z}/4 \rightarrow \mathbb{Z}/2$ gives

$\mathbb{Z}/2 \xrightarrow{\circ} \mathbb{Z}/2$, which after

homology does not give an iso.)

Non-Eg

Let Top^D be a homotopical category.

Let $\text{colim}: \text{Top}^D \rightarrow \text{Top}$.

Claim: colim is not homotopical.

Pf: Board exercise!

(Consider

$$D^{n+1} \hookleftarrow S^n \hookrightarrow D^{n+1}$$

$$\downarrow \simeq \quad \downarrow 1 \quad \downarrow \simeq$$

$$\star \leftarrow S^n \rightarrow \star$$

$$\text{colim}(D^{n+1} \hookleftarrow S^n \hookrightarrow D^{n+1}) = S^{n+1}, \text{ but}$$

$$\text{colim}(\star \leftarrow S^n \rightarrow \star) = \star, \& S^n \neq \star.$$

- We want to find the closest homotopical approximation to an arbitrary $F: (A, W_A) \rightarrow (B, W_B)$. How do we make one?

- Idea: Kan extensions.

Def Let $F: (A, M_A) \rightarrow (B, M_B)$ be a functor, & HoM , HoN be the localizations of M, N .

A total left derived functor LF of a functor F is a right Kan extension $\text{Ran}_F S$

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \tau \downarrow & \nearrow \pi & \downarrow s \\ \text{HoM} & \xrightarrow[\text{LF}]{} & \text{HoN} \end{array}$$

Def: A left derived functor of $F: M \rightarrow N$ is a homotopical functor $-LF: M \rightarrow N$ and a nt

$\lambda: \mathbb{L}F \rightarrow F$ s.t

$S\lambda: S\mathbb{L}F \Rightarrow SF$ is a total
left derived functor of F .
(after lifting).

References: Riehl, "Categorical homotopy theory" (2014)