Question 1

One of the most important inequalities in mathematics in the Cauchy–Schwarz inequality. In the Euclidean setting, it states that for all vectors $a, b \in R^n$,

$$|a \cdot b| \leq ||a|| \cdot ||b||$$

where the dot on the left represents the dot product and the dot on the right is multiplication of real numbers.

1. Using elementary algebra, prove the Cauchy–Schwarz inequality for vectors in R^2.
2. Using the Cauchy–Schwarz inequality, prove that when $x_1 + x_2 + \ldots + x_n \geq 0$,
$$\sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \ldots x_n}{n}.$$
3. Prove that if $a + b + c = 1$, then $a^2 + b^2 + c^3 \geq \frac{1}{3}$.
4. Using the Cauchy–Schwarz inequality, prove that
$$\frac{a+b}{2} \geq \sqrt{ab}.$$
5. Prove that for any $a \geq 0$,
$$\frac{1+a}{2\sqrt{a}} > 1.$$

Question 2

For what values of a, b, are the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix},$$

linearly independent?