CONSEQUENCES OF MORSE HOMOLOGY

Let $(M; M_0, M_1)$ be oriented & n-dim.

A) MORSE INEQUALITIES:

If $f: M \to [0,1]$ is Morse, then

$$C_k \geq \operatorname{rank} H_k(M, M_0)$$

B) LEFSCHETZ DUALITY:

$$H_k(M, M_0) \cong H^{n-k}(M, M_1)$$

C) SMALE'S THEOREM: $(n \geq 6)$

If M, M_0, M_1 have $\pi_1 = 0$ and $H_k(M, M_0)$ is free $\forall k$, we can get equality in (A).

$\cong$ diffeo $\simeq$ homeo $\sim$ hom equiv

IMPORTANT LEMMAS

D) SPHERE LEMMA:

For any $f \in \mathrm{Homeo}(S^{n-1})$, $D^n \cup_f D^n \cong S^n$.

E) DISK LEMMA:

Any 2 orientation-preserving embeddings $D^n \hookrightarrow M$ are ambiently isotopic.

F) KERVAIRE, MILNOR, WALL: ($n = 4, 5, 6$)

If $\Sigma \sim S^n$, then Σ bounds a contractible manifold.

G) CONNECTED SUM:

- If $\Sigma \sim S^n$, then $\Sigma \# (-\Sigma)$ bounds a contractible manifold.
- If $M \# N \cong S^n$, then $M \cong N \cong S^n$.

IMMEDIATE RESULTS

H) H-COBORDISM THM: $(n \geq 6)$

If M, M_0, M_1 have $\pi_1 = 0$ and the inclusion $M_0 \hookrightarrow M$ is a hom equiv, then $M \cong M_0 \times [0,1]$.

I) CLASSIFICATION OF D^n: $(n \geq 6)$

If M is contractible and $\pi_1(\partial M) = 0$, then $M \cong D^n$.

J) GENERALIZED POINCARÉ CONJ: $(n \geq 5)$

If $\Sigma \sim S^n$, then $\Sigma \simeq S^n$.

For $n = 5$ or 6, we get $\Sigma \cong S^n$.

K) CHARACTERIZATION OF D^5:

If M is contractible and $\partial M \cong S^4$, then $M \cong D^5$.

GROUPS OF HOMOTOPY SPHERES

L) Closed, connected n-manifolds form a monoid under $\#$.
Let A^n be the group of invertibles.
For $n \geq 5$, $A^n = \{\text{smooth structures on } S^n\}$.

M) For any $f \in \mathrm{Diff}(S^{n-1})$, let $p(f) = D^n \cup_f D^n$.
If $f \sim g$, then $p(f) \cong p(g)$.

$$\begin{tikzcd}
& & \mathrm{Diff}(D^n) \arrow[d] & & \\
0 \arrow[r] & \mathrm{Diff}_0(S^{n-1}) \arrow[r] & \mathrm{Diff}(S^{n-1}) \arrow[r] \arrow[d, "p"] & \mathrm{MCG}(S^{n-1}) \arrow[r] \arrow[dl, dashed, "\bar{p}"] & 0 \\
& & A^n & &
\end{tikzcd}$$

The vertical maps are exact.

$n=5 \Rightarrow p$ is onto
$n=4 \Rightarrow p$ is 0 (Cerf)

Milnor and Kervaire relate these groups to stable homotopy groups of spheres.

VERSIONS OF SMOOTH SCHOENFLIES:

N) If $\Sigma \sim S^{n-1}$, then any embedding $(n \neq 4, 5)$
$\Sigma \hookrightarrow S^n$ cuts out two standard disks.

O) Any embedding $S^{n-1} \hookrightarrow S^n$ cuts out two standard disks. $(n \neq 4)$

P) If $\Sigma \sim S^{n-1}$ embeds into S^n, then $\Sigma \cong S^{n-1}$!
$(n \neq 5)$

LOW DIMENSIONS:

Q) If $\Sigma \sim S^n$, then $\Sigma \cong S^n$. $(n = 1, 2, 3, 5, 6)$

R) Any $\Sigma \sim S^n$ admits a Morse function with only two critical points. $(n \neq 4)$

Freedman: If $\Sigma \sim S^4$, then $\Sigma \cong S^4$.

Perelman: Everything works for $n = 3$!

But open questions remain...

EQUIVALENCE EXERCISES

1. Use $Q(3)$ to prove everything else for $n=3$.

2. Prove the following implications between open questions for $n=4,5$:

$$
\begin{tikzcd}
 & & I(4) \arrow[dl, Rightarrow] & & & \\
C(4) \arrow[r, Rightarrow] & H(4) \arrow[dr, Rightarrow] & & R(4) \arrow[ul, Rightarrow] & & O(4) \arrow[dd, Rightarrow] \\
 & & Q(4) \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & A^4=0 \arrow[ur, Rightarrow] & \\
 & P(5) \arrow[ur, Rightarrow] & & I(5) \arrow[dl, Rightarrow] & & N(4) \arrow[ul, Rightarrow] \\
 & & N(5) \arrow[ul, Rightarrow] & & &
\end{tikzcd}
$$