Integrasjonstriks

Finnes på en måte ④ derivasjonsregler:

① $\left(f(x)+g(x)\right)' = f'(x) + g'(x)$

② $\left(k \cdot f(x)\right)' = k\, f'(x)$

③ $\left(f(u(x))\right)' = f'(u) \cdot u'(x)$

④ $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Vi har allerede sett:

① $\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$

② $\int k\, f(x)\, dx = k \cdot \int f(x)\, dx$

Får nå en ny regel basert på ③.

Variabelskifte:

$$\boxed{\int f(u(x)) \cdot u'(x)\, dx} = \int f(u)\, du \qquad \text{hvor } u = u(x)$$

Idé: Velger noe som vår $u(x)$, og prøver kjenne igjen venstresida

Eks: $\int x \cdot \sqrt{x^2+1}\, dx$

Velg $u(x) = x^2+1 \quad \Big| \quad u'(x) = 2x$

$$\int \sqrt{u(x)} \cdot \frac{u'(x)}{2}\, dx = \frac{1}{2} \int \sqrt{u(x)} \cdot u'(x)\, dx = \frac{1}{2} \int \sqrt{u}\, du$$

$$= \frac{1}{2} \cdot \left(\frac{1}{\frac{1}{2}+1}\, u^{\frac{1}{2}+1} + C \right) = \frac{1}{3} \left(x^2+1\right)^{\frac{3}{2}} + C$$

Eks:

$$\int \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$$

Velger $u(x) = x^3 + 3x$ $\quad u'(x) = 3x^2 + 3 = 3\cdot(x^2+1)$

$$= \int \frac{x^2+1}{\sqrt{x^3+3x}}\,dx = \int \frac{1}{\sqrt{u}}\cdot(x^2+1)\,dx = \int \frac{1}{\sqrt{u}}\cdot\frac{1}{3}\,u'(x)\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{u}}\cdot u'(x)\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{3}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{3}\left(\frac{1}{1+-\frac{1}{2}}\,u^{1+-\frac{1}{2}} + C\right) = \frac{2}{3}\sqrt{u} + C$$

$$= \frac{2}{3}\sqrt{x^3+3x} + C$$

Bruk av notasjon for å forenkle:

To forskjellige skrivemåter for derivate:

$$f'(x) = \frac{df}{dx}$$

df ← Uendelig liten endring i f

dx ← Uendelig liten endring i x.

$$f'(x) = \lim_{x\to a} \frac{\overbrace{\boxed{f(x)-f(a)}}^{\Delta f}}{\underbrace{\boxed{x-a}}_{\Delta x}}$$

Jukser litt, og skriver

$$df = f'(x)\cdot dx \Rightarrow dx = \frac{df}{f'(x)}$$

For variabel skifte: $\cancel{du = \frac{dx}{u'(x)}}$ $\Rightarrow dx = \frac{du}{u(x)}$

Eks:

$$\int \boxed{x}\sqrt{x^2+1}\,\boxed{dx}$$

$u(x) = x^2+1$ $\qquad \frac{du}{dx} = 2x \Rightarrow \boxed{du = 2\boxed{x\,dx}} \rightarrow \frac{du}{2} = x\boxed{dx}$

$\Rightarrow \frac{du}{2x} = dx$

$$\int x\sqrt{u}\,\frac{du}{2x} = \int \frac{\cancel{x}\sqrt{u}}{2\cancel{x}}\,du = \int \frac{\sqrt{u}}{2}\,du$$

$$= \frac{1}{2}\left(\frac{1}{1+\frac{1}{2}}u^{1+\frac{1}{2}} + C\right) = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

En måte:
$(4x+1)^5 = 1024x^5 + 1280x^4 + \dots$

Eks:

$$\int (4x+1)^5\,dx$$

$u(x) = 4x+1$ $\qquad \frac{du}{dx} = 4 \Rightarrow \frac{du}{4} = dx$

$$\rightarrow \int u^5 \cdot \frac{du}{4} = \frac{1}{4}\int u^5\,du = \frac{1}{4}\left(\frac{u^6}{6} + C\right)$$

$$= \frac{(4x+1)^6}{24} + C$$

Delvis integrasjon:

Produktregel for derivasjon: $~~(u(x)\cdot v(x))' = u'(x)\cdot v~~$

$$(u(x)\cdot v(x))' = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

Integrere denne formelen,

$$u(x)\cdot v(x) = \int u'(x)\cdot v(x)\,dx + \int u(x)\cdot v'(x)\,dx$$

$$\int u(x)\, v'(x)\,dx = u(x)\cdot v(x) - \int u'(x)\cdot v(x)\,dx$$

Idéen/håpet er å velge u og v slik at integralet på høgre siden er enklere å løse enn integralet på venstre siden.

Sa i stad $dv = v'(x)\,dx$ $\qquad du = u'(x)\,dx$

Amerikansk skrivemåte:

$$\boxed{\int u\,dv = u\cdot v - \int v\,du}$$

Eks:

$$\int 3x\cdot(x-1)^3\,dx = 3x\frac{(x-1)^4}{4} - \int \frac{3\cdot(x-1)^4}{4}\,dx$$

$$= \frac{3}{4}x(x-1)^4 - \frac{3}{4}\frac{1}{5}(x-1)^5 + C$$

$u = 3x \quad v = \frac{(x-1)^4}{4}$

$u' = 3 \quad v' = (x-1)^3$

$$v = \int (x-1)^3\,dx = \int w^3\,dw = \frac{w^4}{4} = \frac{(x-1)^4}{4}$$

$w = (x-1)$

$\frac{dw}{dx} = 1$

Eks:

$$\int 3x\,(x-1)^3\,dx = \frac{3}{2}x^2(x-1)^3 - \int 3(x-1)^2\cdot\frac{3}{2}x^2\,dx$$

$$= \frac{3}{2}x^2(x-1)^3 - \frac{9}{2}\int \boxed{(x-1)^2}\,\boxed{x^2}\,dx$$

$$= \frac{3}{2}x^2(x-1)^3 - \frac{9}{2}\left(\frac{1}{3}x^3\cdot(x-1)^2 - \frac{2}{3}\int(x-1)x^3dx\right)$$

$$= \frac{3}{2}x^2(x-1)^3 - \frac{3}{2}x^3(x-1)^2 + 3\int(x-1)x^3dx$$

$$= \frac{3}{2}x^2(x-1)^3 - \frac{3}{2}x^3(x-1)^2 + 3\left(\frac{(x-1)x^4}{4} - \int\frac{x^4}{4}dx\right)$$

$$= \frac{3}{2}x^2(x-1)^3 - \frac{3}{2}x^3(x-1)^2 + \frac{3}{4}x^4(x-1) - \frac{3}{20}x^5 + C$$

$u_1 = (x-1)^3 \qquad v_1 = \frac{3}{2}x^2$

$u_1' = 3(x-1)^2 \qquad v_1' = 3x$

$u_2 = (x-1)^2 \qquad v_2 = \frac{1}{3}x^3$

$u_2' = 2(x-1) \qquad v_2' = x^2$

$u_3 = x-1 \qquad v_3 = \frac{x^4}{4}$

$u_3' = 1 \qquad v_3' = x^3$

Eks: Smakebit:

$\sin' x = \cos x$

$\cos' x = -\sin x$

} Uten noen betingelser.

1 veien

$$\int \boxed{x^2}\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

1 veien

$$= -x^2\cos x + 2\left(x\sin x - \int\sin x\,dx\right)$$

$$= -x^2\cos x + 2x\sin x - 2\int\sin x\,dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

$$= -x^2\cos x + 2x\sin x$$

$u_1 = x^2 \qquad v_1 = -\cos x$

$u_1' = 2x \qquad v_1' = \sin x$

$u_2 = x \qquad v_2 = \sin x$

$u_2' = 1 \qquad v_2' = \cos x$

Eks:

$$\int x^{10}\sin x\,dx$$

må delvis integreres 10 ganger, for å komme i mål.

Eks:

$$\int x^2 \cdot \sin(x^3)\,dx$$

Her må vi bruke variabelskifte:

$$u = x^3 \qquad u'(x) = 3x^2 = \frac{du}{dx} \Rightarrow \frac{du}{3x^2} = dx$$

$$\int \cancel{x^2} \cdot \sin(u) \cdot \frac{du}{3\cancel{x^2}} = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos x^3 + C$$