IN3050 Mathgroup, Derivatives

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March 9th, 2023

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Outline

1. Notation
2. Computations
3. How we do it (at blackboard)

What is the derivative

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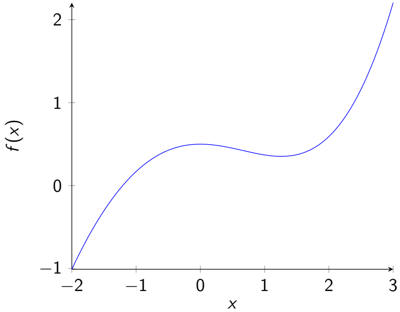
- This can be precisely formulated mathematically, and calculated many ways (analytically, numerically, automatic differentiation, etc).
- In machine learning, we are often interested in the gradient of the *loss*-functions with respect to its *weights*. That means, how much does the loss-function change when we change the weights a little bit.

The Derivative Graphically



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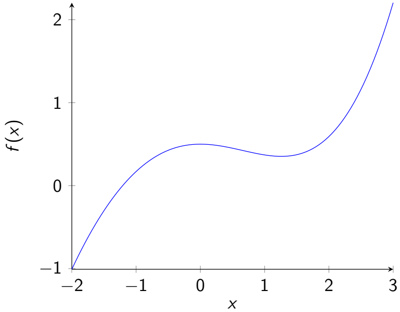

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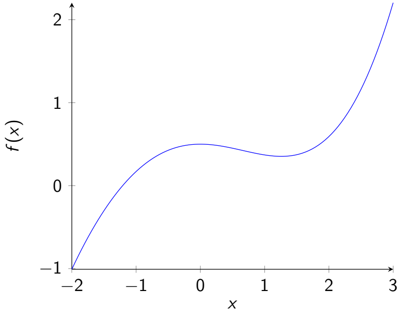

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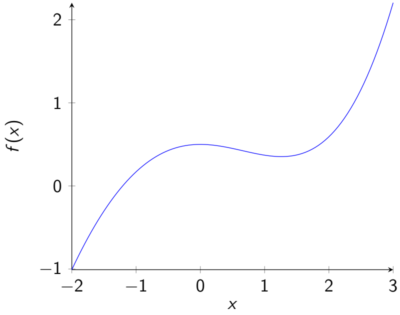

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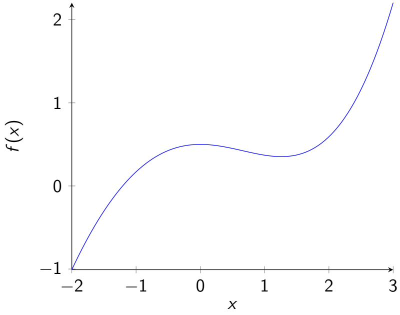

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- Around 2.5, the function changes *quickly*, so the absolute value of the derivative is high.

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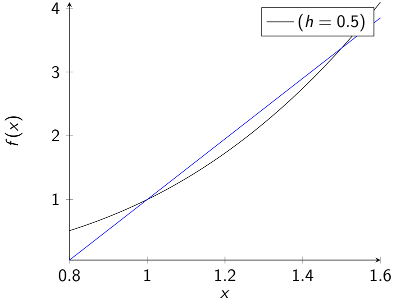
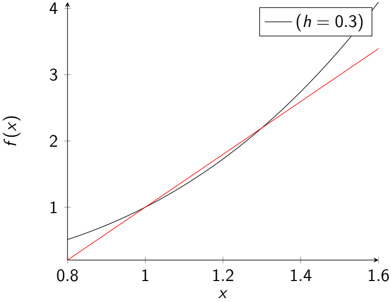
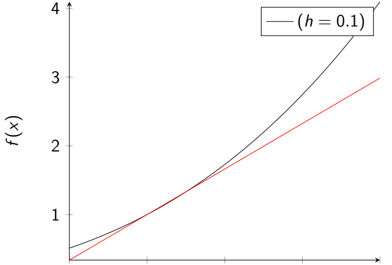
- Here h is "a little change in x, which we make go to 0 in *the limit*.

Vizulating h getting small

The lower the h the better approximation for the derivative.







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- Also note that deep-learning libraries use something called *automatic differentiation*, which makes gradient for neural networks extremely efficient and easy to use. We won't cover that here, but remember to look it up if you start to use PyTorch or TensorFlow.

The Chain Rule

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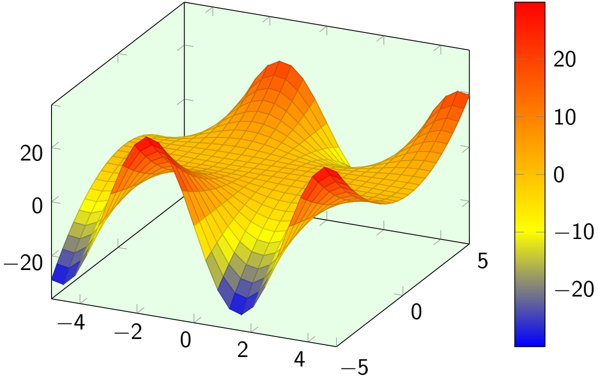
Note: We also have rules for exponents, multiplications and many other things that we won't cover here. If you do not remember them from school, watch the videos mentioned above.

Functions of more than one variable

We now need to generalize to function of many variables.



Think of the color-value, the value upwards, as the *loss-function*, as a function of two variables (left and right), which are our *weights*.

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- Similarly, we have $\frac{\partial f(w_1,w_2)}{\partial w_2} = \sin(w_1) \cdot ((1 - w_2) - w_2)$.

The Gradient

- The gradient can be viewed as a collection of the partial derivatives.
- We have $\nabla f(w_1, w_2) = (\frac{\partial f(w_1,w_2)}{\partial w_1}, \frac{\partial f(w_1,w_2)}{\partial w_2})$.
- Note that the gradient of a function of m variable becomes an *m-dimensional vector*.
- The gradient points in the direction where the function changes the fastest.

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- We then use *gradient descent* (or some version of it) to iterativly update the weights).