# Equilibrium position

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The piston under a gravitational field with pressure satisfies the equation

$$\frac{Nk_BT}{Mg} = x$$

where $M$ is the mass of the piston, $x$ is the piston's position and $N$ is the number of particles. Boltzmann's constant is related to the ideal gas constant through $nR = Nk_B$ where $n$ is the number of mols of gas. For a 1D gas, the average kinetic energy is the thermal energy:

$$k_BT = m\left\langle v^2\right\rangle$$

(note the lack of a factor of 3 because we're in 1D) so

$$\frac{Nm\left\langle v^2\right\rangle}{Mg} = x$$

is the equilibrium position of the piston in terms of our variables. This only gives a relation between $x$ and $\left\langle v^2\right\rangle$ but their concrete values depend on the initial conditions.

To find it, we can use conservation of energy: we know that

$$E_0 = \frac{1}{2}Mu_0^2 + Mgx_0 + \sum_{n=1}^{N}\frac{1}{2}mv_{n,0}^2 = \frac{1}{2}Mu_f^2 + Mgx_f + \sum_{n=1}^{N}\frac{1}{2}mv_{n,f}^2 = E_f.$$

where $u$ is the piston's velocity. This expression can be cast in a way compatible with our notation, by noting that $\sum_{n=1}^{N} v_n^2 = N\left\langle v^2\right\rangle$. Then,

$$\begin{aligned}\frac{2E_0}{M} = u_0^2 + 2gx_0 + \frac{m}{M}\sum_{n=1}^{N} v_{n,0}^2 &= u_f^2 + 2gx_f + \frac{m}{M}\sum_{n=1}^{N} v_{n,f}^2 \\ u_0^2 + 2gx_0 + \frac{m}{M}N\left\langle v_0^2\right\rangle &= u_f^2 + 2gx_f + \frac{m}{M}N\left\langle v_f^2\right\rangle\end{aligned}$$

In equilibrium, we expect that $u_f = 0$, so the initial energy is related to the final position through

$$\frac{2E_0}{M} = 2gx_f + \frac{m}{M}N\left\langle v_f^2\right\rangle$$

and now we can use the previous expression to eliminate $v_f$

$$\frac{2E_0}{3Mg} = x_f$$