Close Form Pricing of Plain and Partial Outside Double Barrier Options

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Abstract
Outside Double Barrier Options are two-asset options where the payoff is defined on one asset and the barrier is defined on another asset. This paper gives the formulas for Outside Double Barrier Options where the barrier is either plain or partially monitored at the front, rear and middle. Since the corresponding Outside Single Barrier Options prices can be written down by taking the corresponding upper (lower) barrier to infinity (zero), the formulas in this paper can be also used as a reference for Outside Single Barrier Options. Crude approximations for the discretely observed barrier cases are also discussed.

I. Introduction

Barrier options are among the most liquid and popularly traded exotic options in the FX market. There are many exotic varieties of barrier options that also get traded. One such sub-class of barrier options is **outside barrier options**, first introduced in Heynan and Kat (1994) who defined the option such that the payoff (call or put) is defined on one asset and the barrier is defined on the other and the two assets are correlated. Within this sub-class of outside barrier options, one can define barriers that are double in nature, more specifically there is an upper barrier and a lower barrier and the option becomes extinguished or not extinguished contingent on whether the "barrier" asset breaches the barriers. The barriers can be monitored during the entire period of existance or can be monitored during a part of the period.

In the case where there is a **single** barrier only, close-form solutions for the case where the barrier is monitored through-out the entire period of existance was given by Heynan and Kat (1994). The formulas for partial single outside barrier options have been given by Bermin (1996) (see also Huag (1997)) and Hui (1997). Another relevant academic paper is Kwok, Wu and Tu (1998), which also discusses pricing of such barriers using finite difference techniques. This work is the extension of the formulas presented in the above papers to the case of double barriers. The options whose formulas are given in this paper are very much traded in the OTC (Over-the-Counter) FX market.

We give the close-form prices of double outside barrier options for the cases where the barriers are monitored continuously during the entire period of existance, as well as during a partial period of existance (front/rear/middle). Section II gives the notations and the functional form of the basic formulas. Section III gives the formulas for some terms of the basic formulas in Section II. Section IV discusses crude approximations for the discrete case.

II. Notation

The 'payoff' asset is given by

$$d\ln(S_t/S_0) = \mu_1 dt + \sigma_1 dW_t^1$$

and the 'barrier' asset is given by

$$d\ln(R_t/R_0) = \mu_2 dt + \sigma_2 dW_t^2$$

where $\mu_1 = r - q_1 - \frac{1}{2}\sigma_1^2$ and $\mu_2 = r - q_2 - \frac{1}{2}\sigma_2^2$ and $dW_t^1.dW_t^2 = dt\,\rho$.

From put-call parity relationships one gets that

$$\textit{knock-out} + \textit{knock-in} = \textit{vanilla}$$

which holds both for single and double barrier outside options.

Hence, one needs to concentrate only on 'out' options and the 'in' options can be got by subtracting the prices from the vanillas. Denote the upper barrier as H and the lower barrier as L.

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If t is the life-time of the option, consider t_1, t_2 satisfying $0 < t_1 < t_2 < t$. Define $\tau_1, \tau_2, \tau_3, \tau_4$ as the event that the double barrier is NOT hit when the monitoring interval is $[0, t]$, $[0, t_1]$, $[t_2, t]$, $[t_1, t_2]$ respectively and $\mathbf{1}_{\{\tau_j\}}$ as the indicator function corresponding to the τ_j.

Here, τ_1 refers to the case of **Plain** Outside Double Barrier Option, τ_2 refers to the case of **Partial Front** Outside Double Barrier Option, τ_3 refers to the case of **Partial Rear** Outside Double Barrier Option, τ_4 refers to the case of **Partial Middle** Outside Double Barrier Option.

Define

$$X_t = \ln(S_t/S_0), a = \ln(K/S_0), Y_t = \ln(R_t/R_0),$$
$$b^H = \ln(H/R_0), b^L = \ln(L/R_0).$$

Also define

$$b'_n = 2n(b^H - b^L), b''_n = 2b^H - b'_n$$

where $n = 0, \pm 1, \pm 2, \ldots$

The symbol ν is will be used to differentiate the payoffs, i.e. $(\nu(S_t - K))^+$, where $\nu = 1$ for a call and $\nu = -1$ for a put.

Define the option prices as

$$V_{\nu,j} = e^{-rt}\mathbf{E}[(\nu(S_t - K))^+ \mathbf{1}_{\{\tau_j\}}] \quad (1)$$

These option prices can be rewritten as

$$V_{\nu,j} = \nu(S_0 e^{-q_1 t}.w_s^j - Ke^{-rt}.w_k^j), j = 1, \ldots, 4 \quad (2)$$

Section II gives the formulas for expression $w_s^j, w_k^j, j = 1, \ldots, 4$.

Finally note that the single partial outside barrier option prices can be written down from the double barrier cases by simply taking $H \to \infty$ or $L \to 0$ as the case may be.

III. Formulas for W_s^j and W_k^j

Theorem 1 (Expressions for the case τ_1)

$$w_s^1 = \sum_{n=-\infty}^{\infty} \Big[e^{(\mu_2+\rho\sigma_1\sigma_2)b'_n/\sigma_2^2}.\left(\Phi_2\left(\nu d'_1, e'_1; \Gamma\right) - \Phi_2\left(\nu d'_1, f'_1; \Gamma\right)\right)$$
$$- e^{(\mu_2+\rho\sigma_1\sigma_2)b''_n/\sigma_2^2}.\left(\Phi_2\left(\nu d''_1, e''_1; \Gamma\right) - \Phi_2\left(\nu d''_1, f''_1; \Gamma\right)\right)\Big]$$

$$w_k^1 = \sum_{n=-\infty}^{\infty} \Big[e^{\mu_2 b'_n/\sigma_2^2}.\left(\Phi_2\left(\nu d'_2, e'_2; \Gamma\right) - \Phi_2\left(\nu d'_2, f'_2; \Gamma\right)\right)$$
$$- e^{\mu_2 b''_n/\sigma_2^2}.\left(\Phi_2\left(\nu d''_2, e''_2; \Gamma\right) - \Phi_2\left(\nu d''_2, f''_2; \Gamma\right)\right)\Big]$$

$$d'_2 = \frac{-a + \mu_1 t + \rho b'_n \sigma_1/\sigma_2}{\sigma_1\sqrt{t}}, d'_1 = d'_2 + \sigma_1\sqrt{t}$$
$$d''_2 = \frac{-a + \mu_1 t + \rho b''_n \sigma_1/\sigma_2}{\sigma_1\sqrt{t}}, d''_1 = d''_2 + \sigma_1\sqrt{t}$$

$$e'_2 = \frac{b^H - \mu_2 t - b'_n}{\sigma_2\sqrt{t}}, e'_1 = e'_2 - \rho\sigma_1\sqrt{t}$$
$$f'_2 = \frac{b^L - \mu_2 t - b'_n}{\sigma_2\sqrt{t}}, f'_1 = f'_2 - \rho\sigma_1\sqrt{t}$$
$$e''_2 = \frac{b^H - \mu_2 t - b''_n}{\sigma_2\sqrt{t}}, e''_1 = e''_2 - \rho\sigma_1\sqrt{t}$$
$$f''_2 = \frac{b^L - \mu_2 t - b''_n}{\sigma_2\sqrt{t}}, f''_1 = f''_2 - \rho\sigma_1\sqrt{t}$$

where Γ is the correlation matrix

$$\begin{pmatrix} 1 & -\nu\rho \\ -\nu\rho & 1 \end{pmatrix}$$

Theorem 2 (Expressions for the case τ_2)

$$w_s^2 = \sum_{n=-\infty}^{\infty} \Big[e^{(\mu_2+\rho\sigma_1\sigma_2)b'_n/\sigma_2^2}.\left(\Phi_2\left(\nu d'_1, e'_1; \Gamma\right) - \Phi_2\left(\nu d'_1, f'_1; \Gamma\right)\right)$$
$$- e^{(\mu_2+\rho\sigma_1\sigma_2)b''_n/\sigma_2^2}.\left(\Phi_2\left(\nu d''_1, e''_1; \Gamma\right) - \Phi_2\left(\nu d''_1, f''_1; \Gamma\right)\right)\Big]$$

$$w_k^2 = \sum_{n=-\infty}^{\infty} \Big[e^{\mu_2 b'_n/\sigma_2^2}.\left(\Phi_2\left(\nu d'_2, e'_2; \Gamma\right) - \Phi_2\left(\nu d'_2, f'_2; \Gamma\right)\right)$$
$$- e^{\mu_2 b''_n/\sigma_2^2}.\left(\Phi_2\left(\nu d''_2, e''_2; \Gamma\right) - \Phi_2\left(\nu d''_2, f''_2; \Gamma\right)\right)\Big]$$

$$d'_2 = \frac{-a + \mu_1 t + \rho b'_n \sigma_1/\sigma_2}{\sigma_1\sqrt{t}}, d'_1 = d'_2 + \sigma_1\sqrt{t}$$

$$d''_2 = \frac{-a + \mu_1 t + \rho b''_n \sigma_1/\sigma_2}{\sigma_1\sqrt{t}}, d''_1 = d''_2 + \sigma_1\sqrt{t}$$

$$e'_2 = \frac{b^H - \mu_2 t_1 - b'_n}{\sigma_2\sqrt{t_1}}, e'_1 = e'_2 - \rho\sigma_1\sqrt{t_1}$$
$$f'_2 = \frac{b^L - \mu_2 t_1 - b'_n}{\sigma_2\sqrt{t_1}}, f'_1 = f'_2 - \rho\sigma_1\sqrt{t_1}$$
$$e''_2 = \frac{b^H - \mu_2 t_1 - b''_n}{\sigma_2\sqrt{t_1}}, e''_1 = e''_2 - \rho\sigma_1\sqrt{t_1}$$
$$f''_2 = \frac{b^L - \mu_2 t_1 - b''_n}{\sigma_2\sqrt{t_1}}, f''_1 = f''_2 - \rho\sigma_1\sqrt{t_1}$$

where Γ is the correlation matrix

$$\begin{pmatrix} 1 & -\nu\rho\sqrt{\frac{t_1}{t}} \\ -\nu\rho\sqrt{\frac{t_1}{t}} & 1 \end{pmatrix}$$

Theorem 3 (Expressions for the case τ_3)

$$w_s^3 = \sum_{n=-\infty}^{\infty} \Big[e^{(\mu_2+\rho\sigma_1\sigma_2)b_n'/\sigma_2^2} . \big(\Phi_3\left(\nu d_1', e_1', g_1; \Gamma\right) - \Phi_3\left(\nu d_1', f_1', g_1; \Gamma\right)$$
$$-\Phi_3\left(\nu d_1', e_1', h_1; \Gamma\right) + \Phi_3\left(\nu d_1', f_1', h_1; \Gamma\right)\big)$$
$$- e^{(\mu_2+\rho\sigma_1\sigma_2)b_n''/\sigma_2^2} . \big(\Phi_3\left(\nu d_1'', e_1'', g_1; \Gamma\right) - \Phi_3\left(\nu d_1'', f_1'', g_1; \Gamma\right)$$
$$-\Phi_3\left(\nu d_1'', e_1'', h_1; \Gamma\right) + \Phi_3\left(\nu d_1'', f_1'', h_1; \Gamma\right)\big) \Big]$$

$$w_k^3 = \sum_{n=-\infty}^{\infty} \Big[e^{\mu_2 b_n'/\sigma_2^2} . \big(\Phi_3\left(\nu d_2', e_2', g_2; \Gamma\right) - \Phi_3\left(\nu d_2', f_2', g_2; \Gamma\right)$$
$$-\Phi_3\left(\nu d_2', e_2', h_2; \Gamma\right) + \Phi_3\left(\nu d_2', f_2', h_2; \Gamma\right)\big)$$
$$- e^{\mu_2 b_n''/\sigma_2^2} . \big(\Phi_3\left(\nu d_2'', e_2'', g_2; \Gamma\right) - \Phi_3\left(\nu d_2'', f_2'', g_2; \Gamma\right)$$
$$-\Phi_3\left(\nu d_2'', e_2'', h_2; \Gamma\right) + \Phi_3\left(\nu d_2'', f_2'', h_2; \Gamma\right)\big) \Big]$$

$$d_2' = \frac{-a + \mu_1 t + \rho b_n' \sigma_1/\sigma_2}{\sigma_1\sqrt{t}}, d_1' = d_2' + \sigma_1\sqrt{t}$$
$$d_2'' = \frac{-a + \mu_1 t + \rho b_n'' \sigma_1/\sigma_2}{\sigma_1\sqrt{t}}, d_1'' = d_2'' + \sigma_1\sqrt{t}$$

$$e_2' = \frac{b^H - \mu_2 t - b_n'}{\sigma_2\sqrt{t}}, e_1' = e_2' - \rho\sigma_1\sqrt{t}$$
$$f_2' = \frac{b^L - \mu_2 t - b_n'}{\sigma_2\sqrt{t}}, f_1' = f_2' - \rho\sigma_1\sqrt{t}$$
$$e_2'' = \frac{b^H - \mu_2 t - b_n''}{\sigma_2\sqrt{t}}, e_1'' = e_2'' - \rho\sigma_1\sqrt{t}$$
$$f_2'' = \frac{b^L - \mu_2 t - b_n''}{\sigma_2\sqrt{t}}, f_1'' = f_2'' - \rho\sigma_1\sqrt{t}$$

$$g_2 = \frac{b^H - \mu_2 t_2}{\sigma_2\sqrt{t_2}}, g_1 = g_2 - \rho\sigma_1\sqrt{t_2}$$
$$h_2 = \frac{b^L - \mu_2 t_2}{\sigma_2\sqrt{t_2}}, h_1 = h_2 - \rho\sigma_1\sqrt{t_2}$$

where Γ is the correlation matrix

$$\begin{pmatrix} 1 & -\nu\rho & -\nu\rho\sqrt{\frac{t_2}{t}} \\ -\nu\rho & 1 & \sqrt{\frac{t_2}{t}} \\ -\nu\rho\sqrt{\frac{t_2}{t}} & \sqrt{\frac{t_2}{t}} & 1 \end{pmatrix}$$

Theorem 4 (Expressions for the case τ_4)

$$w_s^4 = \sum_{n=-\infty}^{\infty} \Big[e^{(\mu_2+\rho\sigma_1\sigma_2)b_n'/\sigma_2^2} . \big(\Phi_3\left(\nu d_1', e_1', g_1; \Gamma\right) - \Phi_3\left(\nu d_1', f_1', g_1; \Gamma\right)$$
$$-\Phi_3\left(\nu d_1', e_1', h_1; \Gamma\right) + \Phi_3\left(\nu d_1', f_1', h_1; \Gamma\right)\big)$$
$$- e^{(\mu_2+\rho\sigma_1\sigma_2)b_n''/\sigma_2^2} . \big(\Phi_3\left(\nu d_1'', e_1'', g_1; \Gamma\right) - \Phi_3\left(\nu d_1'', f_1'', g_1; \Gamma\right)$$
$$-\Phi_3\left(\nu d_1'', e_1'', h_1; \Gamma\right) + \Phi_3\left(\nu d_1'', f_1'', h_1; \Gamma\right)\big) \Big]$$

$$w_k^4 = \sum_{n=-\infty}^{\infty} \Big[e^{\mu_2 b_n'/\sigma_2^2} . \big(\Phi_3\left(\nu d_2', e_2', g_2; \Gamma\right) - \Phi_3\left(\nu d_2', f_2', g_2; \Gamma\right)$$
$$-\Phi_3\left(\nu d_2', e_2', h_2; \Gamma\right) + \Phi_3\left(\nu d_2', f_2', h_2; \Gamma\right)\big)$$
$$- e^{\mu_2 b_n''/\sigma_2^2} . \big(\Phi_3\left(\nu d_2'', e_2'', g_2; \Gamma\right) - \Phi_3\left(\nu d_2'', f_2'', g_2; \Gamma\right)$$
$$-\Phi_3\left(\nu d_2'', e_2'', h_2; \Gamma\right) + \Phi_3\left(\nu d_2'', f_2'', h_2; \Gamma\right)\big) \Big]$$

$$d_2' = \frac{-a + \mu_1 t + \rho b_n' \sigma_1/\sigma_2}{\sigma_1\sqrt{t}}, d_1' = d_2' + \sigma_1\sqrt{t}$$
$$d_2'' = \frac{-a + \mu_1 t + \rho b_n'' \sigma_1/\sigma_2}{\sigma_1\sqrt{t}}, d_1'' = d_2'' + \sigma_1\sqrt{t}$$

$$e_2' = \frac{b^H - \mu_2 t_2 - b_n'}{\sigma_2\sqrt{t_2}}, e_1' = e_2' - \rho\sigma_1\sqrt{t_2}$$
$$f_2' = \frac{b^L - \mu_2 t_2 - b_n'}{\sigma_2\sqrt{t_2}}, f_1' = f_2' - \rho\sigma_1\sqrt{t_2}$$
$$e_2'' = \frac{b^H - \mu_2 t_2 - b_n''}{\sigma_2\sqrt{t_2}}, e_1'' = e_2'' - \rho\sigma_1\sqrt{t_2}$$
$$f_2'' = \frac{b^L - \mu_2 t_2 - b_n''}{\sigma_2\sqrt{t_2}}, f_1'' = f_2'' - \rho\sigma_1\sqrt{t_2}$$

$$g_2 = \frac{b^H - \mu_2 t_1}{\sigma_2\sqrt{t_1}}, g_1 = g_2 - \rho\sigma_1\sqrt{t_1}$$
$$h_2 = \frac{b^L - \mu_2 t_1}{\sigma_2\sqrt{t_1}}, h_1 = h_2 - \rho\sigma_1\sqrt{t_1}$$

where Γ is the correlation matrix

$$\begin{pmatrix} 1 & -\nu\rho\sqrt{\frac{t_2}{t}} & -\nu\rho\sqrt{\frac{t_1}{t}} \\ -\nu\rho\sqrt{\frac{t_2}{t}} & 1 & \sqrt{\frac{t_1}{t_2}} \\ -\nu\rho\sqrt{\frac{t_1}{t}} & \sqrt{\frac{t_1}{t_2}} & 1 \end{pmatrix}$$

The formulas where the barriers are of the type τ_1 and τ_2 in the terms of bivariate normal distribution, while the formulas where the barriers are of the type τ_3 and τ_4 in the terms of trivariate normal distribution. Both types of probabilities can be calculated very fast by using suitable algorithms, see for instance Genz (2002) and the various references in that paper.

IV. Crude Approximations for the Discrete Case

In this paper, we have given the close-form valuations for four kinds of continuously monitored outside double barrier options. However, some of the options traded in the industry are discrete monitored versions of the barriers mentioned in this paper. As noted in Broadie et. al. (1997) and Kou (2003) for usual barrier options, the difference in the prices in the case of discrete monitoring to that of continuous monitoring can be significant even at a high number of observation points. Similar differences can be observed for the barriers mentioned in this paper if one carries out monte-carlo simulation or 3-D tree or finite difference methods.

Taking a clue from Broadie et. al. (1997) and Kou (2003), crude approximations can be given as follows:

Let $\beta = -\frac{\zeta(1/2)}{2\pi}$.

Let m be the number of observation periods.

- For the case τ_1, replace b^H by $b^H.e^{\beta\sigma\sqrt{t/m}}$ and b^L by $b^L.e^{-\beta\sigma\sqrt{t/m}}$ including wherever they are embedded in b'_n and b''_n.
- For the case τ_2, replace b^H by $b^H.e^{\beta\sigma\sqrt{t_1/m}}$ and b^L by $b^L.e^{-\beta\sigma\sqrt{t_1/m}}$ including wherever they are embedded in b'_n and b''_n.
- For the case τ_3, replace b^H by $b^H.e^{\beta\sigma\sqrt{(t-t_2)/m}}$ and b^L by $b^L.e^{-\beta\sigma\sqrt{(t-t_2)/m}}$ including wherever they are embedded in b'_n and b''_n, except in the terms g_1, g_2, h_1, h_2 where b^H and b^L should be kept as it is.
- For the case τ_4, replace b^H by $b^H.e^{\beta\sigma\sqrt{(t_2-t_1)/m}}$ and b^L by $b^L.e^{-\beta\sigma\sqrt{(t_2-t_1)/m}}$ including wherever they are embedded in b'_n and b''_n, except in the terms g_1, g_2, h_1, h_2 where b^H and b^L should be kept as it is.

These approximations tend to work well for general cases except where the barriers are too close to the spot prices.

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