THE FITTING OF A GENERALIZATION OF THE LOGISTIC CURVE

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This paper describes the fitting of the four-parameter family of curves defined by the differential equation.

$$\frac{dW}{dt} = \kappa W\left[1 - \left(\frac{W}{A}\right)^{1/\theta}\right] \tag{1}$$

Special cases were originally proposed by Pütter [1920] for various types of animal growth (see e.g. von Bertalanffy [1957]), and recently Richards [1959] has exemplified the general form of the curves and suggested that they may be useful for the empirical description of plant growth. For further details of the history of these curves and their mathematical properties reference should be made to Richards' paper. It suffices to say here that the family defined by (1) includes as special cases several curves which have been used empirically for the description of growth, including the 'monomolecular' (diminishing returns) curve ($\theta = -1$), the exponential curve ($\theta \rightarrow 0$ through positive values), the logistic curve ($\theta = 1$), and the Gompertz curve ($\theta \rightarrow \infty$ with A fixed and κ a linear function of θ).

MODEL AND NOTATION

In using (1) in the description of growth, W will usually denote a weight of some kind, while t is usually chronological time, but may be a suitable 'time scale' (Skellam *et al.* [1959], Nelder *et al.* [1960]) which can replace chronological time when the environment is variable. Since it seems to be characteristic of many growth phenomena that the relative growth rate ($W^{-1}\, dW/dt$) is almost constant when the weight is small compared to the final weight, we shall restrict θ to be positive in what follows. For if $\theta < 0$, then, for W sufficiently small, $W^{-1}\, dW/dt$ becomes as large as we please in absolute value. When $\theta > 0$, the solution of (1) can be written in the form

$$W = A/\{1 + e^{-(\lambda+\kappa t)/\theta}\}^{\theta} \tag{2}$$

where λ is the constant of integration. A and κ are taken as positive since W is positive and is assumed to increase with time. Thus $W^{1/\theta}$ satisfies the logistic equation with upper asymptote $A^{1/\theta}$ as $t \geq \infty$, while for t large and negative

$$W \sim Ae^{\lambda+\kappa t}.$$

We now write $Y = \ln W$ and $\alpha = \ln A$ so that

$$Y = \alpha - \theta \ln [1 + e^{(\lambda+\kappa t)/\theta}] \tag{3}$$

where ln denotes the natural logarithm. (3) is the form of the equation that will be used for fitting, and we shall assume that if $y = \ln w$ is a sample value of $Y = \ln W$ at time t, then $E(y) = Y$, and var $y = \sigma^2$ independently of Y. We shall also assume that the y_i ($i = 1, \cdots, n$) obtained at time t_i, i.e. the sample points to which the curve is to be fitted, are independent. The independence condition is satisfied for instance in plant growth analysis when a randomized field layout is used, and the sampling is destructive, i.e., a different set of plants is taken on each occasion. When the same set of plants is used on each occasion (when w might denote leaf area measured non-destructively) it is still possible to use the method proposed for estimation, though of course sampling variances cannot be obtained from residual mean squares 'within curves' if the condition of independence is not satisfied. This situation is further discussed later in the paper.

The assumption that $y = \ln w$ has constant variance has been found to be reasonable from an examination of data on the weight of part or the whole of several crops. In fact it is usually found that the distribution of y does not differ significantly from a normal distribution.

For the purposes of this paper, (2) will be described as the *generalized logistic equation*, though it is not of course the only generalization that could or has been made from the logistic equation.

FITTING THE LOGISTIC CURVE

We consider first the case where θ is known. This is equivalent to fitting the logistic equation, since by putting $W' = W^{1/\theta}$, $Y' = Y/\theta$, $\alpha' = \alpha/\theta$, $\lambda' = \lambda/\theta$ and $\kappa' = \kappa/\theta$ we can convert (2) into the logistic function. We may, therefore, put $\theta = 1$ for this case without loss of generality. Let us write

$$\tau_i = \lambda + \kappa t_i \quad \text{and} \quad \xi_i = \frac{e^{-\tau_i}}{1 + e^{-\tau_i}} = \frac{1}{1 + e^{\tau_i}} = \frac{A - W_i}{A};$$

then from (3)

$$Y_i = \alpha + \ln(1 - \xi_i), \quad \frac{\partial Y_i}{\partial \alpha} = 1, \quad \frac{\partial Y_i}{\partial \lambda} = \xi_i, \quad \text{and} \quad \frac{\partial Y_i}{\partial \kappa} = \xi_i t_i,$$

Thus the least-square equations are given by

$$\begin{aligned} \sum_i (y_i - Y_i) &= 0, \\ \sum_i (y_i - Y_i)\xi_i &= 0, \\ \sum_i (y_i - Y_i)\xi_i t_i &= 0. \end{aligned} \tag{4}$$

These equations have no explicit solution in general, so that they must be solved by iteration. The usual method, using the expected values of the information matrix (see e.g. Bailey [1951]), gives the following

TABLE I

Extension of Berkson's Table of Anti-Logits $[1/1 + e^{-\tau})]$

τ	0	1	2	3	4	5	6	7	8	9
					0.99					
5.0	331	337	344	350	357	363	369	376	382	388
5.1	394	400	406	412	418	423	429	435	440	446
5.2	451	457	462	467	473	478	483	488	493	498
5.3	503	508	513	518	523	527	532	537	541	546
5.4	550	555	559	564	568	572	576	581	585	589
5.5	593	597	601	605	609	613	617	620	624	628
5.6	632	635	639	642	646	649	653	656	660	663
5.7	667	670	673	676	680	683	686	689	692	695
5.8	698	701	704	707	710	713	716	719	721	724
5.9	727	730	732	735	737	740	743	745	748	750
6.0	753	755	758	760	762	765	767	769	772	774
6.1	776	778	781	783	785	787	789	791	793	795
6.2	797	799	801	803	805	807	809	811	813	815
6.3	817	819	820	822	824	826	827	829	831	832
6.4	834	836	837	839	841	842	844	845	847	848
6.5	850	851	853	854	856	857	859	860	861	863
6.6	864	865	867	868	869	871	872	873	875	876
6.7	877	878	879	881	882	883	884	885	887	888
6.8	889	890	891	892	893	894	895	896	897	898
6.9	899	900	901	902	903	904	905	906	907	908

To obtain ξ, from this table and Berkson's table in Berkson [1953], use the formula, ξ = antilogit $(-\tau)$ = 1 − antilogit (τ).

TABLE II

VALUES OF $10^4 \ln (1 + e^{-\tau})$

For τ positive $-\ln (1 - \xi)$ = tabulated value.
For τ negative $-\ln (1 - \xi)$ = $|\tau|$ + tabulated value for $|\tau|$.

τ	0	1	2	3	4	5	6	7	8	9
0.0	6931	6882	6832	6783	6733	6685	6636	6588	6539	6492
0.1	6444	6397	6349	6303	6256	6210	6163	6118	6072	6027
0.2	5981	5936	5892	5847	5803	5759	5716	5672	5629	5586
0.3	5544	5501	5459	5417	5375	5334	5293	5252	5211	5170
0.4	5130	5090	5050	5011	4972	4932	4894	4855	4817	4779
0.5	4741	4703	4666	4629	4592	4555	4518	4482	4446	4410
0.6	4375	4340	4304	4270	4235	4201	4166	4132	4099	4065
0.7	4032	3999	3966	3933	3901	3869	3837	3805	3773	3742
0.8	3711	3680	3649	3619	3589	3559	3529	3499	3470	3441
0.9	3412	3383	3354	3326	3298	3270	3242	3214	3187	3160
1.0	3133	3106	3079	3053	3027	3001	2975	2949	2924	2898
1.1	2873	2848	2824	2799	2775	2751	2727	2703	2679	2656
1.2	2633	2610	2587	2564	2542	2519	2497	2475	2453	2432
1.3	2410	2389	2368	2347	2326	2305	2285	2264	2244	2224
1.4	2204	2184	2165	2146	2126	2107	2088	2070	2051	2032
1.5	2014	1996	1978	1960	1942	1925	1907	1890	1873	1856
1.6	1839	1822	1806	1789	1773	1757	1741	1725	1709	1693
1.7	1678	1662	1647	1632	1617	1602	1588	1573	1558	1544
1.8	1530	1516	1502	1488	1474	1460	1447	1433	1420	1407
1.9	1394	1381	1368	1355	1343	1330	1318	1306	1293	1281
2.0	1269	1257	1246	1234	1222	1211	1200	1188	1177	1166
2.1	1155	1144	1134	1123	1112	1102	1091	1081	1071	1061
2.2	1051	1041	1031	1021	1012	1002	993	983	974	965
2.3	955	946	937	928	920	911	902	894	885	877
2.4	868	860	852	844	836	828	820	812	804	796
2.5	789	781	774	766	759	752	745	737	730	723
2.6	716	710	703	696	689	683	676	670	663	657
2.7	650	644	638	632	626	620	614	608	602	596
2.8	590	585	579	573	568	562	557	551	546	541
2.9	536	530	525	520	515	510	505	500	495	491
3.0	486	481	476	472	467	463	458	454	449	445
3.1	441	436	432	428	424	420	415	411	407	403
3.2	399	396	392	388	384	380	377	373	369	366
3.3	362	359	355	352	348	345	341	338	335	331
3.4	328	325	322	319	316	312	309	306	303	300
3.5	297	295	292	289	286	283	280	278	275	272
3.6	270	267	264	262	259	257	254	252	249	247
3.7	244	242	239	237	235	232	230	228	226	223
3.8	221	219	217	215	213	211	208	206	204	202
3.9	200	198	196	195	193	191	189	187	185	183
4.0	181	180	178	176	174	173	171	169	168	166
4.1	164	163	161	160	158	156	155	153	152	150
4.2	149	147	146	144	143	142	140	139	137	136
4.3	135	133	132	131	130	128	127	126	124	123
4.4	122	121	120	118	117	116	115	114	113	112
4.5	110	109	108	107	106	105	104	103	102	101

TABLE II—(*Continued*)

τ		τ		τ		τ		τ	
4.60	100	5.10	61	5.60	37	6.10	22	6.60	14
4.70	91	5.20	55	5.70	33	6.20	20	6.70	12
4.80	82	5.30	50	5.80	30	6.30	18	6.80	11
4.90	74	5.40	45	5.90	27	6.40	17	6.90	10
5.00	67	5.50	41	6.00	25	6.50	15	—	—

adjustments $\delta\alpha_0$, $\delta\lambda_0$, $\delta\kappa_0$ to initial 'guesses' α_0 , λ_0 , κ_0 for the parameters:—

$$\begin{bmatrix} n & \sum \xi_i & \sum \xi_i t_i \\ & \sum \xi_i^2 & \sum \xi_i^2 t_i \\ & & \sum \xi_i^2 t_i^2 \end{bmatrix} \begin{bmatrix} \delta\alpha_0 \\ \delta\lambda_0 \\ \delta\kappa_0 \end{bmatrix} = \begin{bmatrix} \sum (y_i - Y_i) \\ \sum \xi_i(y_i - Y_i) \\ \sum \xi_i t_i (y_i - Y_i) \end{bmatrix} \tag{5}$$

where ξ_i is evaluated with $\lambda = \lambda_0$ and $\kappa = \kappa_0$. New values of ξ_i and Y_i are then recalculated using $\alpha_1 = \alpha_0 + \delta\alpha_0$ etc. and further adjustments obtained, the process being repeated until whatever accuracy required is achieved. Note that the information matrix depends on the parameters only through τ_i ; the process of solution of (5) would obviously be considerably speeded up if tables of ξ_i and ln $(1 - \xi_i)$ as functions of τ_i were available.

Now ξ = antilogit $(-\tau)$ = 1 − antilogit (τ) in Berkson's [1953] notation; hence ξ can be obtained for the range $-4.99 \leq \tau \leq 4.99$ from the table of antilogits given in that paper. An extension of this table for $5 \leq |\tau| \leq 6.99$ which covers the region down to $W/A = 0.001$ and up to $W/A = 0.999$ is given in Table I. Values of ln $(1 + e^{-\tau})$ to four decimal places are given in Table II for $-6.99(0.01)6.99$ from which ln $(1 - \xi)$ can be quickly derived. These tables have been computed from the tables of e^x and ln x given in Comrie [1949] and should prove sufficiently accurate for most purposes. If linear interpolation is used from the nearest tabulated value, using the tabulated first difference, rounding-off errors will rarely exceed 1 unit in the last place.

Because Y_i is linear in α, knowledge of its approximate value is not necessary for the iterative process using the expected information matrix, and we can replace $\delta\alpha_0$ in (5) by α_1 , and Y by ln $(1 - \xi)$, solving directly for α_1 , $\delta\lambda_0$ and $\delta\kappa_0$. However when we come to consider the exact equations for the iteration, using the empirical information matrix instead of the expected one, we shall find no advantage in suppressing α_0, so that it will be retained in what follows. The calculations can be conveniently laid out as follows:

TABLE III

Computing Layout for Fitting the Logistic Curve

(1)	(2)	(3)	(4)	(5)	(6)	(7)
t_i	$\tau_i = \lambda + \kappa t_i$	ξ_i	$\xi_i t_i$	$\ln(1 - \xi_i)$	y_i	$y_i - Y_i$

The t_i are entered in column 1, whence the τ_i are calculated using the starting values of λ and κ, and entered in column 2. Column 3 is filled up from Table I, and then column 4 calculated. Column 5 is filled up from Table II, and the y's entered in column 6. α is estimated by $\bar{y} - \sum \ln(1 - \xi_i)/n$ from the sums of columns 5 and 6, and hence $y_i - Y_i = y_i - \alpha - \ln(1 - \xi_i)$ is calculated. The coefficients on the left-hand side of (5) can now be calculated from columns 3 and 4, by summing the columns and from sums of squares and products of ξ_i and $\xi_i t_i$. Finally the right-hand side of (4) may be calculated from $\sum (y_i - Y_i)$, which should be zero, apart from rounding errors, and from products of columns 3 and 4 with column 7. The equations (4) may now be solved using any of the standard techniques (see e.g., Dwyer [1951]).

The exact iterative equations

If we now denote the parameters to be fitted by θ_i, then the above method uses the expected information matrix whose general term is

$$\sum_{\kappa} \frac{\partial Y_\kappa}{\partial \theta_i} \frac{\partial Y_\kappa}{\partial \theta_j}.$$

The exact equations for the generalized Newton-Raphson process are those using the empirical information matrix with general term

$$\sum_{\kappa} \frac{\partial Y_\kappa}{\partial \theta_i} \frac{\partial Y_\kappa}{\partial \theta_j} - \sum_{\kappa} (y_\kappa - Y_\kappa) \frac{\partial^2 Y_\kappa}{\partial \theta_i \, \partial \theta_j}.$$

Cornfield and Mantel [1950] for the analogous situation in probit analysis found a substantial improvement in the speed of convergence using this latter form, so that it may be worth considering the appropriate modification here. This necessitates adding to the left-hand-side of (5) the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ & \sum (y_i - Y_i)\xi_i(1 - \xi_i) & \sum (y_i - Y_i)\xi_i(1 - \xi_i)t_i \\ & & \sum (y_i - Y_i)\xi_i(1 - \xi_i)t_i^2 \end{bmatrix}$$

and two more columns need to be added to Table III, namely $(y - Y)(1 - \xi)$ and ξt^2 in order to compute the elements of this matrix. The extra calculation is not large but the increase in speed of convergence achieved may justify the extra labour only when σ^2 is large so that the $(y_i - Y_i)$ are appreciable.

The dispersion matrix of the estimates of the parameters may be obtained in the usual way by inverting the information matrix on the left-hand side of (5) and then multiplying it by an estimate of σ^2. For methods of matrix inversion, and the combination of the inversion with the solution of an associated set of equations, see e.g., Dwyer [1951].

If a variance for $\hat{A}$ rather than $\hat{\alpha}$ is wanted, then we may use the approximate formula

$$\text{var } \hat{A} = \text{var } e^{\hat{\alpha}} \sim e^{2\hat{\alpha}} \text{ var } \hat{\alpha} = \hat{A}^2 \text{ var } \hat{\alpha}$$

provided that var $\hat{A}$ is small compared to $\hat{A}^2$.

Obtaining the starting values

The starting values may be obtained in various ways, some wholly graphical and some partly so. Some compromise must be made between the desirability of having them as accurate as possible so that iterations are reduced to a minimum, and the undesirability of spending an excessive amount of time on a preliminary stage of the computations.

A semi-graphical method which I have found useful is as follows. First plot the logistic variable $w^{1/\theta}$, henceforth denoted by x, against t, and estimate by eye the value of A; call this estimate A'. Then plot $\ln [x/(A' - x)]$ against t. If the logistic function is the correct one and A' is also correct, this should give, apart from random errors, a straight line. Now the variance of $\ln [x/(A - x)]$ using the first-order approximation, is given by $[A/(A - x)]^2\sigma^2 = \xi^2\sigma^2$ in the notation already established. This variance increases as x increases, tending to infinity as x tends to A. Thus when the variance of x is constant, the graph of $\ln [x/(A' - x)]$ against t will show increasing scatter as x approaches A. In addition, if x has a symmetrical distribution, that of $\ln [x/(A - x)]$ will be positively skewed for x near A and positive deviations will be in mean value much greater than negative ones. If A' is too high, then the graph of $\ln [x/(A' - x)]$ against t will show a tendency to level off at $\ln [A/(A' - A)]$, while if A' is too low, the graph will show an increasing slope as x approaches A. If the graph shows a sigmoid shape, the logistic equation is inappropriate. Assuming that no sigmoid shape has been found, we may obtain an improved estimate of A as follows. Divide the points into two sets, with the dividing line roughly where $x = A'/2$; then points with $x < A'/2$ are relatively slightly affected by errors in A, and have the smallest variances. Fit by eye a straight line to the points on the $(\ln [x/(A' - x)], t)$-graph using the points where $x < A'/2$ and giving most weight to the smaller values of x. Read off the fitted values of $x/(A - x)$ on this line corresponding to the values of t for points having $x > A'/2$. If z is such a fitted value, then, ignoring errors, $z = x/(A - x)$ and so $A = x(1 + z)/z$. Hence if we write $u = x(1 + z)/z$, then u is an estimate of A. To obtain an improved value for A, calculate u for each point having $x > A'/2$ and use $\bar{u}$, the simple mean of the u's. The method is quite rapid in practice if a logarithmic graph paper is used, since then $\ln [x/(A' - x)]$ can be plotted directly from a knowledge of $x/(A' - x)$ and z can also be read off directly from the fitted line. If the improved value of A still gives curvature, the process may be repeated. Having obtained a good value for A_0, the starting value for A, we may now calculate λ_0 and κ_0 either by a graphical fitting of a straight line to the graph of $\ln [x/(A_0 - x)]$ against t, using the relation $\ln [X/(A_0 - X)] = \lambda + \kappa t$ or by a weighted regression of $\ln [x/(A_0 - x)]$ on t using as weights $\xi_0^2 = [(A_0 - x)/A_0]^2$. When

TABLE IV

Calculations to Obtain Starting Values for Fitting the Logistic Curve

t	x	$x/(71 - x)$	$u = x(1 + z)/z$	$x/(73.2 - x)$
−2.15	3.57	0.0529		0.0513
−1.50	6.25	0.0965		0.0934
−0.85	9.54	0.155		0.150
−0.08	16.91	0.313		0.300
+0.52	24.51	0.510		0.503
1.10	33.78	0.908		0.857
2.28	50.00	2.38	71.9	2.16
3.23	62.05	6.93	74.0	5.57
4.00	69.34	41.8	76.5	18.0
4.65	67.09	17.2	71.1	11.0
5.00	69.34	41.8	72.4	18.0
			Mean 73.2	

σ^2 is small, the fit by eye should be quite adequate, but the weighted regression may have some advantage when the scatter about the line is considerable. If logarithmic paper is used, it must be remembered that one cycle corresponds to $\log_e 10 = 2.30259$ units when the slope is to be measured.

Worked example

The data in Table IV relate to the growth of carrot tops in a field experiment, and were obtained by my colleague Mr. R. B. Austin. θ has been taken as known and equal to 2, so that x, the logistic variable, equals $W^{1/2}$; t is a time scale based on total incoming radiation (negative values occur because an origin had been taken with a zero between

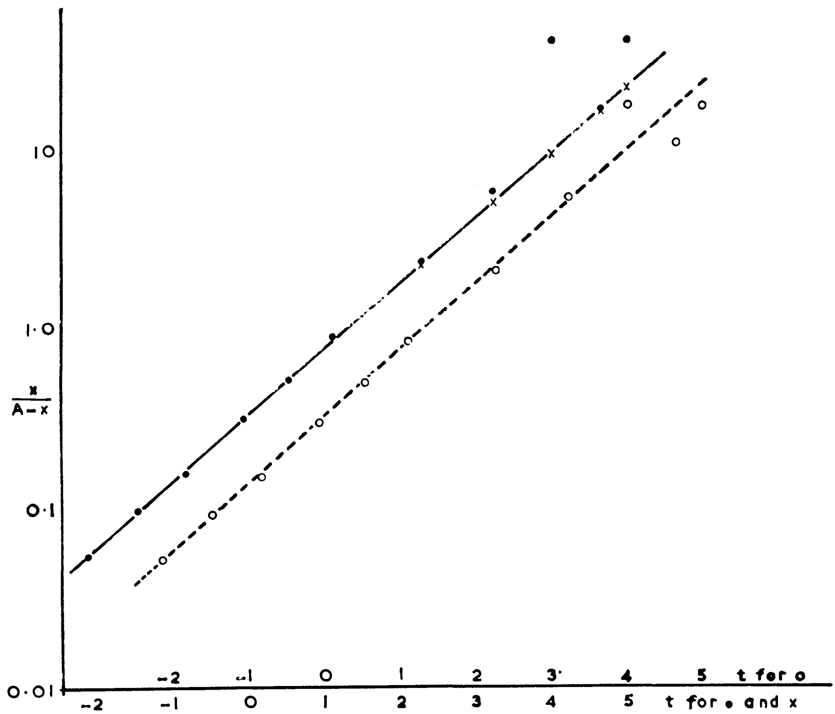

FIGURE 1

METHOD OF OBTAINING STARTING VALUES FOR α, λ, κ, FROM DATA OF TABLE IV.

● Values of $x/(71 - x)$ plotted on logarithmic scale against t.
× Fitted values of $Z = (x/A - x)$ obtained from line drawn by eye through points having $x < 35.5$.
○ Values of $x/(73.2 - x)$ plotted on logarithmic scale against t.
[Note: t — scale displaced to avoid overlapping]
- - - Line fitted by eye through points ○.

the fourth and fifth reading; the reasons for this have no relevance for this example). Table IV also contains the preliminary calculations necessary to arrive at A_0, the starting value for A. An initial plot of x against t gave an A', as judged by eye, of 71. $x/(71 - x)$ was then calculated and recorded in the third column of Table IV. Figure 1 shows the graph of ln $[x/(71 - x)]$ against t. From the solid line fitted by eye to the points having $x < A'/2 = 35.5$, the fitted values of $x/(A - x)$ for the last 5 points were read off, with u calculated for each, and recorded in the fourth column of Table IV. The revised value of A' is $\bar{u} = 73.2$. Values of $x/(73.2 - x)$ were then calculated (fifth column of Table IV) and plotted against t (Figure 1); the broken line was fitted by eye, giving most weight to the small values of x, and gave $\lambda_0 = -1.109$, $\kappa_0 = 0.861$. The calculations for the exact iterative method discussed above can now be started. Filling up Table III we have the following:

t_i	τ_i	ξ_i	$\xi_i t_i$	$\ln(1 - \xi_i)$	y_i	$y_i - Y_i$
−2.15	−2.960	0.9507	−2.0440	−3.0105	1.272	−0.0020
−1.50	−2.400	0.9168	−1.3752	−2.4868	1.832	+0.0343
−0.85	−1.841	0.8631	−0.7336	−1.9883	2.255	−0.0412
−0.08	−1.178	0.7646	−0.0612	−1.4464	2.828	−0.0101
+0.52	−0.661	0.6595	+0.3429	−1.0773	3.199	−0.0082
+1.10	−0.162	0.5404	0.5944	−0.7774	3.520	+0.0129
+2.28	+0.854	0.2986	0.6808	−0.3547	3.912	−0.0178
+3.23	+1.672	0.1582	0.5110	−0.1722	4.128	+0.0157
+4.00	+2.335	0.0883	0.3532	−0.0924	4.239	+0.0469
+4.65	+2.895	0.0524	0.2437	−0.0538	4.206	−0.0247
+5.00	+3.196	0.0393	0.1965	−0.0401	4.239	−0.0054

$\bar{y} = 3.2391, \sum(1 - \xi_i)/n = -1.0454, \alpha_0 = 4.2845, S.S. = 0.00661578.$

Adjustment equations are

$$\begin{bmatrix} 11.0000 & 5.3319 & -1.2915 \\ & 3.9272 & -3.0008 \\ & & 8.0293 \end{bmatrix} \begin{bmatrix} \delta\alpha_0 \\ \delta\lambda_0 \\ \delta\kappa_0 \end{bmatrix} = \begin{bmatrix} 0.0004 \\ -0.0124 \\ -0.0020 \end{bmatrix}.$$

Solution is given by

$$\begin{aligned} \delta\alpha_0 &= 0.010025, & & \alpha_1 = 4.2940, \\ \delta\lambda_0 &= -0.021566, & \text{from whence} \quad & \lambda_1 = -1.1300, \\ \delta\kappa_0 &= -0.006746, & & \kappa_1 = 0.8544. \end{aligned}$$

If a further iteration is required then α_1 should be estimated in the same way as α_0, i.e. using $\bar{y} - \sum \ln(1 - \xi_i)/n$, and the value obtained above ignored. However the adjustments obtained from the first cycle are sufficiently small to justify our stopping in this case.

The goodness of fit of the logistic curve can now be checked by comparing the residual mean square after fitting with the residual error obtained from replication. Using the estimates α_1, λ_1, κ_1, we have a residual sum of squares after fitting of 0.00642 giving a mean square with $11 - 3 = 8$ d.f. of 0.00080. The error mean square was 0.0033 so that the fit is satisfactory. We may now obtain the dispersion matrix by inverting the information matrix and multiplying it by the variance of a single y_i, as estimated by the error mean square. Thus the dispersion matrix is given by

$$0.0033 \times \begin{bmatrix} 11.0000 & 5.3319 & -1.2915 \\ & 3.9272 & -3.0008 \\ & & 8.0293 \end{bmatrix}^{-1} = \begin{bmatrix} 0.00137 & -0.00237 & 0.00066 \\ & 0.00526 & 0.00159 \\ & & 0.00090 \end{bmatrix}.$$

The standard errors of the estimates affect the second place of decimals, thus justifying the stopping of the iterative process after one cycle.

FITTING THE GENERALIZED LOGISTIC CURVE

If we now define τ_i as $(\lambda + \kappa t_i)/\theta$, then from (2)

$$Y = \alpha + \theta \ln(1 - \xi)$$

where $\xi = e^{-\tau}/(1 + e^{-\tau})$ as before, and

$$\frac{\partial Y_i}{\partial \alpha} = 1, \quad \frac{\partial Y_i}{\partial \lambda} = \xi_i, \quad \frac{\partial Y_i}{\partial \kappa} = \xi_i t_i, \quad \frac{\partial Y_i}{\partial \theta} = \ln(1 - \xi_i) - \tau_i \xi_i = \beta_i, \text{ say.}$$

The least-square equations, therefore, are given by

$$\begin{aligned} &\sum (y_i - Y_i) = 0, \qquad &\sum (y_i - Y_i)\xi_i t_i = 0, \\ &\sum (y_i - Y_i)\xi_i = 0, \qquad &\sum (y_i - Y_i)\beta_i = 0 \end{aligned} \tag{6}$$

and the iterative solution from starting values α_0, λ_0, κ_0, θ_0, gives adjustments $\delta\alpha_0$ etc. from the solution of

$$\begin{bmatrix} n & \sum \xi_i & \sum \xi_i t_i & \sum \beta_i \\ & \sum \xi_i^2 & \sum \xi_i^2 t_i & \sum \beta_i \xi_i \\ & & \sum \xi_i^2 t_i^2 & \sum \beta_i \xi_i t_i \\ & & & \sum \beta_i^2 \end{bmatrix} \begin{bmatrix} \delta\alpha_0 \\ \delta\lambda_0 \\ \delta\kappa_0 \\ \delta\theta_0 \end{bmatrix} = \begin{bmatrix} \sum (y_i - Y_i) \\ \sum (y_i - Y_i)\xi_i \\ \sum (y_i - Y_i)\xi_i t_i \\ \sum (y_i - Y_i)\beta_i \end{bmatrix}. \tag{7}$$

TABLE V

VALUES OF $-10^4\beta$, $-\beta = \tau/(1 + e^{-\tau}) - \ln(1 + e^{\tau})$

τ	0	1	2	3	4	5	6	7	8	9
0.0	6931	6931	6931	6930	6929	6928	6927	6925	6923	6921
0.1	6919	6916	6913	6910	6907	6903	6900	6895	6891	6887
0.2	6882	6877	6871	6866	6860	6854	6848	6841	6834	6827
0.3	6820	6813	6805	6797	6789	6781	6772	6763	6754	6745
0.4	6735	6726	6716	6706	6695	6685	6674	6663	6652	6640
0.5	6628	6617	6605	6592	6580	6567	6554	6541	6528	6515
0.6	6501	6487	6473	6459	6445	6430	6415	6400	6385	6370
0.7	6355	6339	6323	6307	6291	6275	6258	6242	6225	6208
0.8	6191	6174	6157	6139	6122	6104	6086	6068	6050	6031
0.9	6013	5994	5976	5957	5938	5919	5900	5881	5861	5842
1.0	5822	5802	5782	5763	5743	5722	5702	5682	5662	5641
1.1	5620	5600	5579	5558	5537	5516	5495	5474	5453	5432
1.2	5411	5389	5368	5346	5325	5303	5281	5260	5238	5216
1.3	5194	5172	5150	5128	5106	5084	5062	5040	5018	4996
1.4	4974	4951	4929	4907	4885	4862	4840	4818	4795	4773
1.5	4751	4728	4706	4683	4661	4639	4616	4594	4571	4549
1.6	4527	4504	4482	4460	4437	4415	4393	4370	4348	4326
1.7	4304	4282	4259	4237	4215	4193	4171	4149	4127	4105
1.8	4083	4061	4039	4018	3996	3974	3952	3931	3909	3887
1.9	3866	3844	3823	3802	3780	3759	3738	3717	3695	3674
2.0	3653	3632	3611	3591	3570	3549	3528	3508	3487	3467
2.1	3446	3426	3406	3385	3365	3345	3325	3305	3285	3265
2.2	3245	3226	3206	3186	3167	3147	3128	3109	3090	3070
2.3	3051	3032	3013	2994	2976	2957	2938	2920	2901	2883
2.4	2864	2846	2828	2810	2792	2774	2756	2738	2721	2703
2.5	2685	2668	2650	2633	2616	2599	2582	2565	2548	2531
2.6	2514	2497	2481	2464	2448	2431	2415	2399	2383	2367
2.7	2351	2335	2319	2303	2288	2272	2257	2241	2226	2211
2.8	2195	2180	2165	2150	2136	2121	2106	2091	2077	2062
2.9	2048	2034	2020	2005	1991	1977	1964	1950	1936	1922
3.0	1909	1895	1882	1868	1855	1842	1829	1816	1803	1790
3.1	1777	1764	1752	1739	1726	1714	1702	1689	1677	1665
3.2	1653	1641	1629	1617	1605	1594	1582	1570	1559	1547
3.3	1536	1525	1514	1502	1491	1480	1469	1459	1448	1437
3.4	1426	1416	1405	1395	1384	1374	1364	1354	1344	1333
3.5	1323	1314	1304	1294	1284	1274	1265	1255	1246	1236
3.6	1227	1218	1208	1199	1190	1181	1172	1163	1154	1146
3.7	1137	1128	1120	1111	1102	1094	1086	1077	1069	1061
3.8	1053	1045	1037	1029	1021	1013	1005	997	989	982
3.9	974	967	959	952	944	937	930	922	915	908
4.0	901	894	887	880	873	866	859	853	846	839
4.1	833	826	820	813	807	800	794	788	782	775
4.2	769	763	757	751	745	739	733	728	722	716

TABLE V—(*Continued*)

τ	0	1	2	3	4	5	6	7	8	9
4.3	710	705	699	694	688	682	677	672	666	661
4.4	656	650	645	640	635	630	625	620	615	610
4.5	605	600	595	590	586	581	576	572	567	562
4.6	558	553	549	544	540	536	531	527	523	518
4.7	514	510	506	502	498	494	490	486	482	478
4.8	474	470	466	462	458	455	451	447	444	440
4.9	436	433	429	426	422	419	415	412	408	405
5.0	402	398	395	392	389	386	382	379	376	373
5.1	370	367	364	361	358	355	352	349	346	343
5.2	340	337	335	332	329	326	324	321	318	316
5.3	313	310	308	305	303	300	298	295	293	290
5.4	288	285	283	281	278	276	274	271	269	267
5.5	265	262	260	258	256	254	252	249	247	245
5.6	243	241	239	237	235	233	231	229	227	225
5.7	223	222	220	218	216	214	212	211	209	207
5.8	205	204	202	200	198	197	195	193	192	190
5.9	188	187	185	184	182	181	179	178	176	175
6.0	173	172	170	169	167	166	164	163	162	160
6.1	159	158	156	155	154	152	151	150	148	147
6.2	146	145	143	142	141	140	138	137	136	135
6.3	134	133	132	130	129	128	127	126	125	124
6.4	123	122	121	120	119	118	116	116	114	114
6.5	113	112	111	110	109	108	107	106	105	104
6.6	103	102	101	101	100	99	98	97	96	95
6.7	95	94	93	92	91	91	90	89	88	88
6.8	87	86	85	85	84	83	82	82	81	80
6.9	80	79	78	77	77	76	75	75	74	73
7.0	73	72	72	71	70	70	69	69	68	67

Little amendment is needed to Table III for the solution of these equations. A column for β must be inserted between columns 4 and 5, and β can be obtained rapidly from Table V. α_0 is estimated as $\bar{y} - \theta_0 \cdot \sum \ln (1 - \xi_i)/n$, and $y_i - Y_i = y_i - \alpha_0 - \theta_0 \ln (1 - \xi_i)$. All the elements in (7) can then be calculated.

Obtaining the starting values

A sign that θ is not equal to unity is the appearance of a trend in the values of u during the graphical process for finding A_0. If $\theta > 1$, then values of u show a decreasing trend as w departs from A, and the lowest of them are often below the highest values of w. Conversely if $\theta < 1$, then u shows an increasing trend as w falls. According to the trend shown by the u's, θ can be changed in the appropriate direction, and when θ_0 has been obtained, with the corresponding $A^{1/\theta}$, we can use the relation

$$E\left[\ln \frac{x^{1/\theta}}{A^{1/\theta} - x^{1/\theta}}\right] \sim \frac{1}{\theta}(\lambda + \kappa t) = \tau$$

to get estimate of λ_0/θ_0 and κ_0/θ_0 and hence of λ_0 and κ_0 . For the calculations of coefficients in (7), however, only t and τ are needed.

Worked example

In the example above of fitting the logistic curve to $W^{1/2}$, we were in fact fitting the generalized logistic curve taking θ as known and equal to 2. We now consider the fitting when θ is not assumed known, using the value 2 as a starting point. Columns 1–5 of Table III are unchanged but y in column 6 is now twice the previous value since we were working before with $\ln x^{1/2} = \frac{1}{2} \ln x$. Column 7 will also be doubled and a column of β_i must be added. The revised and added columns are given together with the iterative equations in Table VI.

TABLE VI

REVISED AND ADDITIONAL COLUMNS FOR FITTING THE GENERALIZED LOGISTIC CURVE

β_i	y_i	$y_i - Y_i$
−0.1964	2.544	−0.0038
−0.2865	3.664	+0.0688
−0.3992	4.510	−0.0824
−0.5457	5.656	−0.0200
−0.6414	6.398	−0.0162
−0.6895	7.040	+0.0252
−0.6096	7.824	+0.0356
−0.4366	8.256	+0.0314
−0.2987	8.478	+0.0942
−0.2054	8.412	−0.0494
−0.1657	8.478	−0.0106

Adjustment equations are

$$\begin{bmatrix} 11.0000 & 5.3319 & -1.2915 & -4.4747 \\ & 3.9273 & -3.0008 & -2.3015 \\ & & 8.0293 & -0.3343 \\ & & & 2.1857 \end{bmatrix} \begin{bmatrix} \delta\alpha_0 \\ \delta\lambda_0 \\ \delta\kappa_0 \\ \delta\theta_0 \end{bmatrix} = \begin{bmatrix} 0.0016 \\ -0.0244 \\ -0.0048 \\ 0.0096 \end{bmatrix}.$$

The solution of these equations is given by

$$\delta\alpha_0 = -0.016240,$$
$$\delta\lambda_0 = -0.201431,$$
$$\delta\kappa_0 = -0.089091,$$
$$\delta\theta_0 = -0.254585,$$

whence

$$\lambda_1 = \theta_0\left(\frac{\lambda_0}{\theta_0}\right) + \delta\lambda_0 = 2(-1.109) - 0.2014 = -2.4194,$$

$$\kappa_1 = \theta_0\left(\frac{\kappa_0}{\theta_0}\right) + \delta\kappa_1 = 2(0.861) - 0.0891 = +1.6329,$$

$$\theta_1 = 2 - 0.2546 = 1.7454.$$

α_1 is not given because it is better estimated from $\bar{y} - \theta_1 \sum \ln(1 - \xi_i)/n$ in the next iteration, as described above.

The rather large and approximately equal adjustments to λ_0 and θ_0 should be noted; the estimates of these parameters are in fact highly correlated and have relatively large variances; hence the reduction of the *S. S.* of residuals (from 0.0265 to 0.0239) following the adjustment is rather small.

A second iteration using the values α_1, λ_1, κ_1, θ_1 gives

$$\begin{aligned} \delta\alpha_1 &= -0.00446770, & \qquad \alpha_2 &= 8.5538,\\ \delta\lambda_1 &= +0.02565976, \quad \text{and} & \lambda_2 &= 2.3937,\\ \delta\kappa_1 &= +0.00782844, & \kappa_2 &= 1.6407,\\ \delta\theta_1 &= +0.02010622, & \theta_2 &= 1.7655. \end{aligned}$$

and a third and final iteration gives

$$\alpha_3 = 8.5539,$$
$$\lambda_3 = 2.3917,$$
$$\kappa_3 = 1.6415,$$
$$\theta_3 = 1.7676$$

with dispersion matrix

$$\begin{bmatrix} 0.01419 & 0.02379 & 0.01526 & 0.06075\\ & 0.13220 & 0.06751 & 0.21618\\ & & 0.03705 & 0.11775\\ & & & 0.40847 \end{bmatrix}.$$

In comparing the estimates and their variances and covariances with those obtained for θ assumed known and equal to 2, it should be remembered that the effective parameters estimated in the latter case were $\alpha/\vartheta = \alpha/2$ etc. Thus parameter values for the logistic case should be multiplied by 2 and their variances and covariances by 4 for comparison with those of the four-parameter curve. As was to be expected, the variances of $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\kappa}$ have been considerably increased by the inclusion of θ, also $\hat{\theta}$ itself is poorly determined with a standard error of ± 0.639, so that it does not differ significantly from the value of 2 originally used.

ALLOCATION OF SAMPLE POINTS

This section is concerned with the effect on the efficiency of estimation of different allocations of position of sample points on the curve. Obviously points on the curve where $x^{1/\theta}$ is small compared to $A^{1/\theta}$ give almost no information about A, while points where $x^{1/\theta}$ is close to $A^{1/\theta}$ give almost no information about λ and κ, the parameters defining the exponential part of growth. To reduce the range of possibilities of the distribution of the sample points to manageable proportions, attention will be confined to samples at points equally spaced with regard to t and symmetrically arranged about $\tau = 0$. [In practice, a rough idea of the values of the parameters will enable observations to be taken at times similar to those considered here]. The points of the arrangements considered are spaced along the t scale in units of τ or $\frac{1}{2}\tau$, while the ranges spanned are 6τ, 10τ, and 14τ. Table VII gives the relative sampling variances per point for the parameters, and also the generalized information per point using $|\ \mathbf{Q}\ |/n$ where n is the number of points and $\mathbf{Q}$ the information matrix for the samples. Two cases are considered: θ known (3 parameters to be estimated) and θ unknown (4 parameters to be estimated).

The table shows that more intensive sampling of a given range [e.g. $-3(\frac{1}{2})3$ instead of $-3(1)3$] slightly increases the sampling variance per point while increasing the information per point. The increased information implies that some or all of the correlations between the estimates have been reduced by the increased density of points, although the arrangement $3(\frac{1}{2})3$ is less efficient than $3(1)3$ when measured in terms of sampling variances only. Increasing the range reduces the sampling variances per point as one might expect, and so yields more information than more intensive sampling of the same range. The sampling variances of $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\kappa}$ are considerably increased when θ is unknown, particularly when the range of τ is small. This is of course a reflection of the fact that curves for neighbouring values of

TABLE VII

SAMPLING VARIANCES PER POINT OF THE PARAMETER ESTIMATES AND THE GENERALIZED INFORMATION PER POINT FOR VARIOUS DISTRIBUTIONS OF THE OBSERVATIONS

θ KNOWN, α, λ, κ UNKNOWN, $\sigma^2 = 1$

No. of points	Allocation of τ	Sampling variance per point			Generalized information per point
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\kappa}$	
7	$-3(1)3$	5.39	39.2	3.60	1.57
11	$-5(1)5$	3.11	23.1	1.14	16.93
15	$-7(1)7$	2.59	18.5	0.56	73.16
13	$-3(\frac{1}{2})3$	6.05	42.9	4.21	4.32
21	$-5(\frac{1}{2})5$	3.22	23.8	1.25	54.80
29	$-7(\frac{1}{2})7$	2.63	18.8	0.60	252.6

α, λ, κ, θ UNKNOWN, $\sigma^2 = 1$

No. of points	Allocation of τ	Sampling variance per point				Generalized information per point
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\kappa}$	$\hat{\theta}$	
7	$-3(1)3$	14.26	361.2	41.50	339.3	0.0324
11	$-5(1)5$	4.99	84.0	5.28	89.4	2.008
15	$-7(1)7$	3.50	53.4	1.62	54.4	20.16
13	$-3(\frac{1}{2})3$	17.20	420.8	51.57	398.4	0.2655
21	$-5(\frac{1}{2})5$	5.34	91.5	6.13	96.3	11.95
29	$-7(\frac{1}{2})7$	3.60	45.5	1.82	56.2	130.4

θ can be made to agree very closely over this range of τ by suitable changes in the other parameters. $\hat{\lambda}$ and $\hat{\theta}$ have the largest sampling variances in all the arrangements and hence are the most difficult to estimate accurately. For instance, if an individual value of w_i in an experiment had a percentage standard error of 5, then var y = var ($\ln w$) ~ 0.0025, so that with the allocation $-3(1)3$ we should have var $\hat{\theta} = 0.0025 \times 48.47$, whence s.e. $\hat{\theta} = \pm 0.35$. Since θ is often in the range 1–2, the allocation $\tau = -3(1)3$ would be inadequate to determine this parameter at all accurately. Similar considerations apply to $\hat{\lambda}$. By such arguments one can find what range w must cover if adequate estimates of the parameters are to be obtained for

$$W = A/(1 + e^{-\tau})^{\theta}$$

so that $\tau = \ln [W^{1/\theta}/(A^{1/\theta} - W^{1/\theta})] = \text{logit } (W/A)^{1/\theta}$. Thus if θ is, say, equal to 2 then $\tau = 3$ corresponds to $(W/A)^{1/\theta} = 0.95257$ whence $W/A = 0.976$; similarly $\tau = -3$ gives $W/A = 0.024$. Thus when $\theta = 2$ the range $\tau = -3$ to 3 implies that W ranges from 2.4 to 97.6 percent of A, and the argument above shows that this range will often be insufficient for adequate estimates of some of the parameters. Note that the larger the value of θ the greater the range of W/A required to cover the same range of τ.

DISCUSSION

This paper is not the place for a discussion of the place of mathematical functions in the description and analysis of growth. A great deal has been written on the subject both for and against the utility of this approach, and it suffices to say that I assume in this paper that differential equations and their associated integrals *have* some usefulness, and that methods are required for the efficient estimation of the parameters. For an approach to the problem of interpreting growth curves which involves replacing chronological time t by some function of meteorological observations see Nelder *et al.* [1960]. [In the example above t is in fact a scale based on total incoming radiation].

The method of estimation used in this paper is based on three assumptions: (i) that the w_i are independent, (ii) that $E(y_i) = Y_i$ and (iii) that var y_i is independent of Y_i. It may reasonably be asked how suitable the procedure is likely to be if some or all of the assumptions are not true, and this involves us in an analysis of the errors in the Y_i. Errors may be due to variation in the parameters from one individual to another, for either genetic or non-genetic reasons; or they may arise from errors in t_i, caused by using the wrong time-scale. In addition we have the possibility of systematic errors arising from deviations from the form of the curve actually used in the fitting. It will certainly not be exactly true that if, for instance, κ has a normal distribution in the population of individuals whose growth is being studied, then random sampling will enable us to set $E(y_i) = Y_i$, where Y_i is given by (3) with the mean of κ used in the formula. But such biases will not be serious if the variance of κ is small, i.e. if we control the genetic and environmental uniformity of the population well. If the w_i are from the same experimental units throughout, which is the commonest cause of non-independence, the method will remain satisfactory if var (y_i) remains constant, where the variance is now measured *within* individual growth curves. However sampling variances of the estimates of parameters must now be calculated from the variation in the parameter values *between* individuals. If the variation between individual curves is much greater than the variation within them, then the efficiency of the fitting method becomes correspondingly less important, and even graphical methods might suffice. A frequent cause of deviations of var (y_i) from constancy is the occurrence of constant-variance weighing errors on the w scale. On the log. scale these become more serious as the weight falls, and lead to var (y_i) increasing rapidly at very low values of w, where the weights are not large compared to the weighing errors. Difficulties in measuring very small distances on a plant cause the same effect. In practice the change in var (y) can often be approximately eliminated by a suitable increase in replication at very low levels of w.

The model employed here for fitting the logistic curve may be contrasted with a method suggested by Stevens [1951], who proposed writing it in the form

$$\frac{1}{W} = \frac{1}{A}(1 + \beta e^{-\kappa t})$$

and using his method for fitting curves of the type

$$z = \alpha + \beta\rho^t, \quad \text{with} \quad z = \frac{1}{W}, \quad \text{and} \quad \rho = e^{-\kappa}.$$

Since he gave equal weight to his z's in the fitting, this is equivalent to assuming that $1/w$ has constant variance and hence that var $(\ln w) \propto W^2$ to the first order. Though conditions may exist where var $(\ln w) \propto W^2$, in my experience it is usually much closer to the truth to take var $(\ln w)$ constant. Several papers concerned with fitting the curve $z = \alpha + \beta\rho^t$ have appeared (e.g. Nair [1954], Patterson [1956], Finney [1958], Patterson and Lipton [1959]), but these are almost entirely concerned with the case where z has constant variance and where the t_i are equally spaced. Where time scales based on meteorological measurements (such as day-degrees) are used for field crops, it is usually impossible or impracticable to arrange for equally spaced t's, so that methods used in the above mentioned papers are no longer available, even if it could be assumed that var z was constant. In addition, none of the methods is immediately applicable to the general case where θ has to be estimated.

The reasons behind the particular choice of the parameters in (3) should perhaps be mentioned. There are, of course, an infinite number of equivalent ways of writing (3), in so far as the structural parameters are concerned. The particular form used has been chosen to reduce the computing labour of the iterative solution as much as possible, while retaining a set of parameters with fairly clear meanings as far as growth is concerned. Thus α = ln (asymptotic value of W), while λ and κ define the exponential growth stage for large negative t when $W \sim Ae^{\lambda+\kappa t}$. A possible interpretation of θ for a field crop in terms of the spatial distribution of plant material has been given by Nelder *et al.* [1960]. The form used may be capable of further improvement, especially from the point of view of the convergence of the iterative process. It is an interesting point to what extent approximately orthogonal parameters can be found for the general non-linear function, that is parameters such that the cross terms in the dispersion matrix are small compared to the diagonal terms, and whether the use of such parameters would speed the convergence of the iterative process. While a linear transformation of the parameters leads to equivalent iterative equations, non-linear transformations in general do not, and possibly the speed of convergence would be affected by the particular specification of the parameters.

A natural extension of the generalized logistic equation as defined here is the five-parameter equation $dW/dt = \alpha W^{\xi} - \beta W^{\eta}$ discussed by von Bertalanffy [1957]. The estimation of the parameters of this equation, subject to the same assumptions as those considered above, is greatly complicated by the lack of any general explicit solution of the equation. Consequently expected values would have to be computed by numerical integration which makes the labour involved excessive except to those having electronic computers available. A general method for the fitting of differential equations has been given by Box [1956], and this could be applied to the general equation. However we may expect that the introduction of the fifth parameter would entail the existence of very extensive data if reasonably accurate estimates of the parameters were to be obtained. The data discussed above cover a 380-fold range of weights, and the standard error of a single weight was about 5.5 percent, yet the s.e. of $\hat{\theta}$ was as high as ± 0.64. The introduction of the fifth parameter might well require a much larger range of weights to give reasonably accurate estimates. One consequence of this situation is that it becomes difficult to test models based on *a priori* values of the exponents ξ and η, unless very extensive data are available, since the fits of curves having ξ and η differing quite widely from the *a priori* values would be hardly distinguishable.

SUMMARY

The least-squares fit of the curves defined by

$$\frac{dW}{dt} = \kappa W\left[1 - \left(\frac{W}{A}\right)^{1/\theta}\right]$$

is derived for the case when the sample values of ln W are independent, unbiased, and of constant variance.

Tables are provided to assist the computing of the iterative process used for estimation, and a method is given for obtaining starting values for the parameters.

Evaluation of the sampling variances of the estimates of the parameters and the generalized information shows that the range which the observations cover is more important than the density of sample points within that range if minimum sampling variance or maximum information per point is required. The introduction of θ as an unknown increases the sampling variances of the other parameters, the increase being especially marked when the range of $(W/A)^{1/\theta}$ is small.

Some possible sources and consequences of deviations from the assumptions underlying the fitting are discussed, and also the difficulties involved in extending the fitting to the curves defined by

$$dW/dt = \alpha W^{\xi} - \beta W^{\eta}.$$

It is stressed that sensitive tests of hypotheses involving *a priori* values of parameters may demand very extensive data.

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