Bayesian data analysis: Theory & practice

Part 4c: Multi-level models

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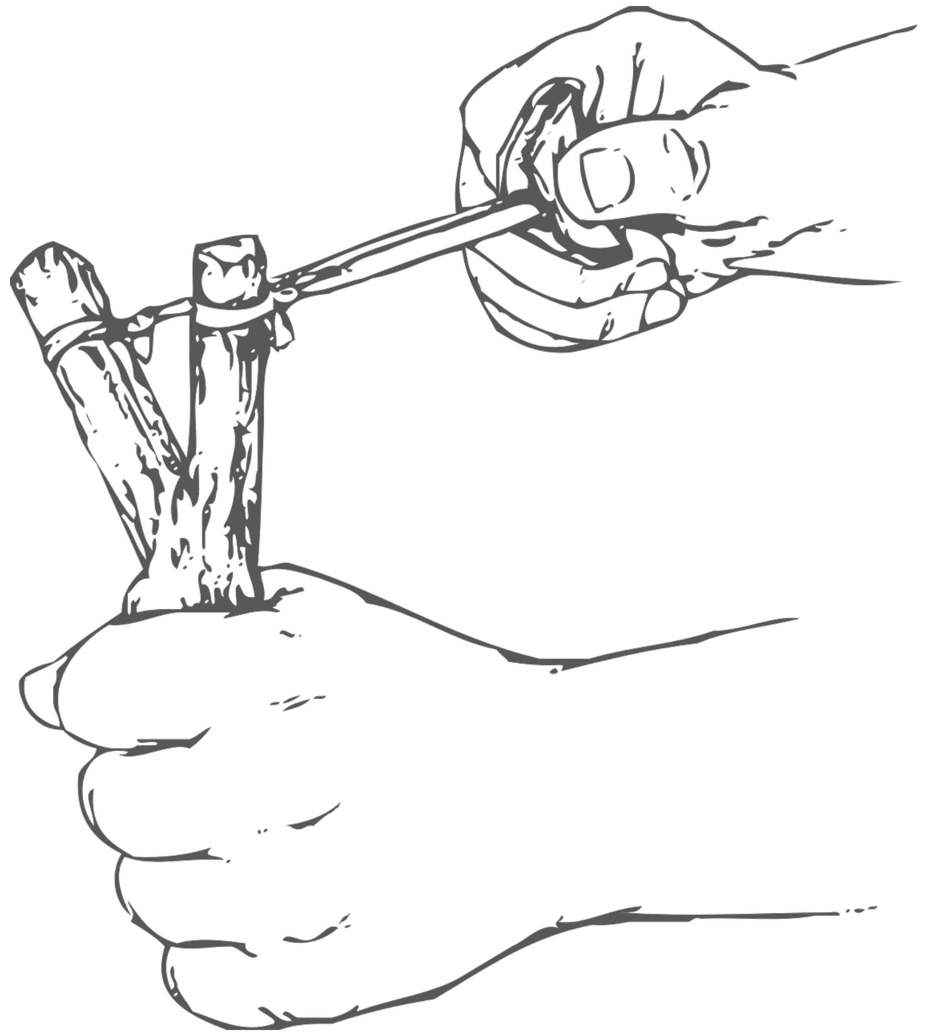
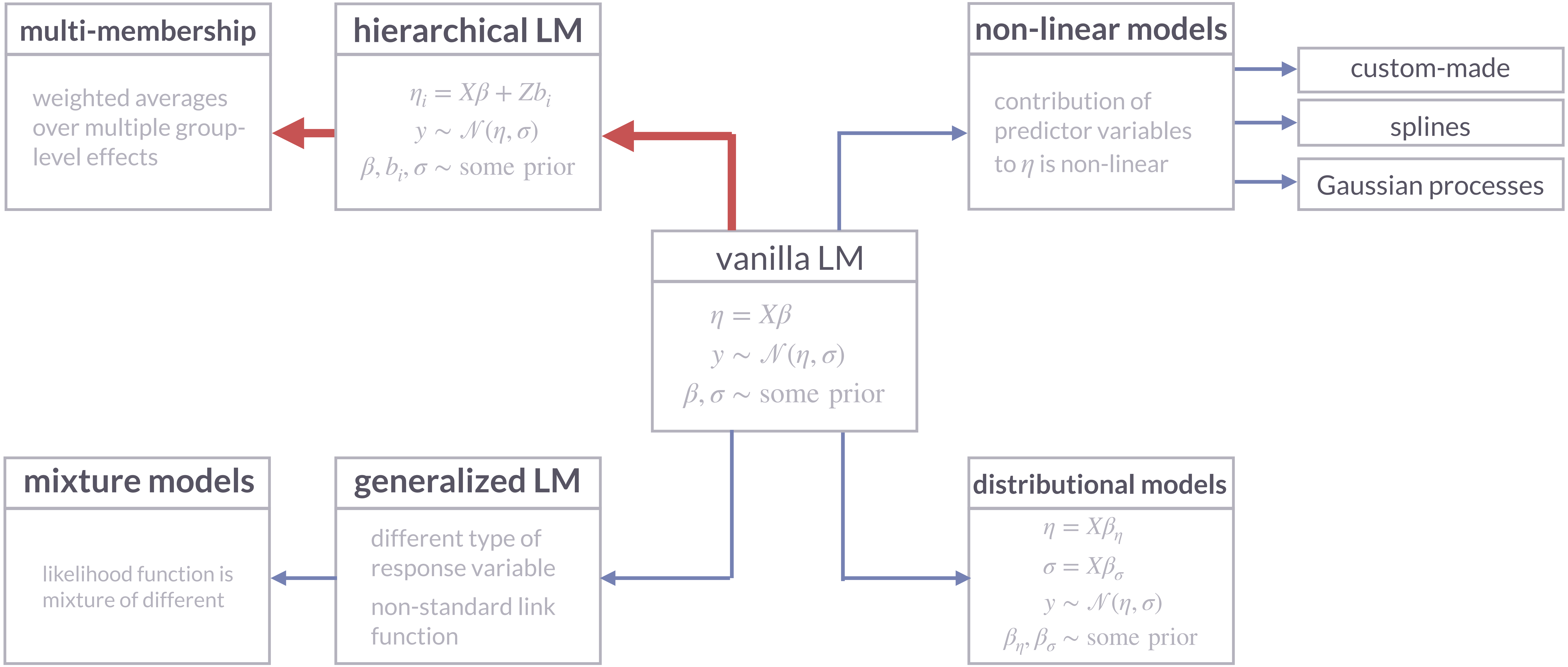
Main learning goals
for this part



1. multi-level models (MLMs)
   a. varying intercepts & slopes
   b. Bayesian priors for random-effect structure
   c. `BRMS` syntax for specifying MLMs
2. multi-membership models
3. two views on MLMs
   a. breaking stochastic independence
   b. adaptive regularizing priors

Roadmap “beyond vanilla”

common extensions of linear regression modeling



Prediction, please!

participant	condition	response
Alex	A	1
Alex	B	3
Alex	C	0
Bo	A	2
Bo	B	5
Bo	C	???

Group-level effects
motivation

Two ways to motivate group-level effects:

1. breaking incorrect independence assumptions
2. “adaptive, regularizing priors”

These are not in competition, just two ways of looking at the same thing.

Case study: MLMs

Case study: processing relative clauses

motivation

Subject relative clause

The senator who interrogated the journalist …

Object relative clause

The senator who the journalist interrogated …

English: SRCs $\ll$ ORCs ✅

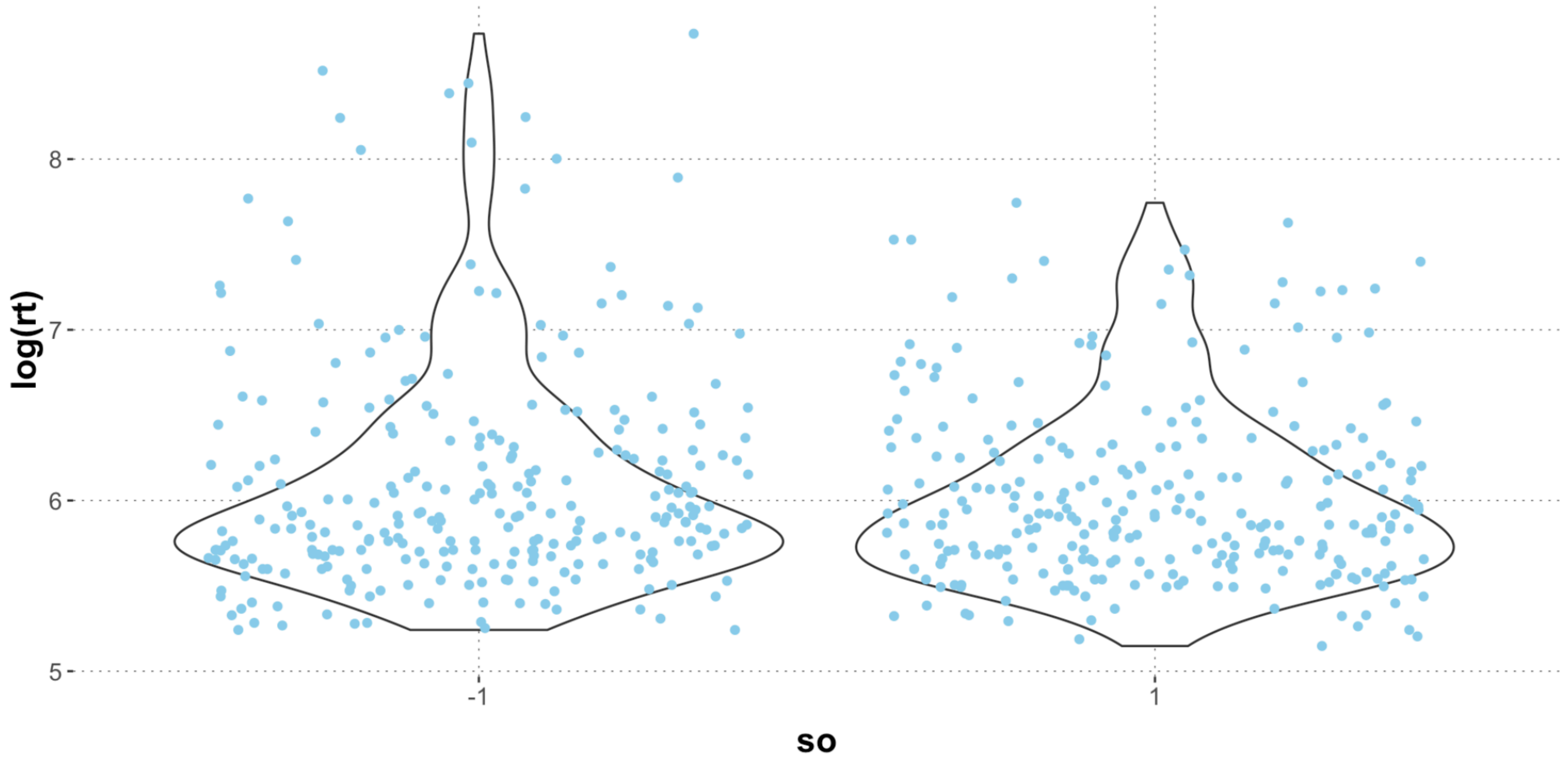
Chinese: ORCs $\ll$ SRCs ❓

example based on tutorial by Sorensen et al. (2016)

Self-paced reading times

37 subjects read 15 sentences either with an SRC or an ORC

```
     subj  item so       rt
    <dbl> <dbl> <chr> <dbl>
 1      1    13 1      1561
 2      1     6 -1      959
 3      1     5 1       582
 4      1     9 1       294
 5      1    14 -1      438
 6      1     4 -1      286
 7      1     8 -1      438
 8      1    10 -1      278
 9      1     2 -1      542
10      1    11 1       494
11      1     7 1       270
12      1     3 1       406
13      1    16 -1      374
14      1    15 1       286
15      1     1 1       246
```



NB: deviation coding

Gibson & Wu (2013)

Fixed-effects model

model specs

- predict log `rt` in terms of factor `so`
- improper prior on all parameters

$$
\begin{aligned}
\log(\mathtt{rt}_i) &\sim \mathcal{N}(\eta_i, \sigma_{err}) \\
\eta_i &= \beta_0 + \beta_1 \mathtt{so}_i
\end{aligned}
\qquad\qquad
\begin{aligned}
\sigma_{err} &\sim \mathcal{U}(0, \infty) \\
\beta_0, \beta_1 &\sim \mathcal{U}(-\infty, \infty)
\end{aligned}
$$

Fixed-effects model

results

```
 Family: gaussian
  Links: mu = identity; sigma = identity
Formula: log(rt) ~ so
   Data: rt_data (Number of observations: 547)
  Draws: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
         total post-warmup draws = 4000

Population-Level Effects:
          Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
Intercept     6.10      0.04     6.03     6.17 1.00     4030     2692
so1          -0.08      0.05    -0.17     0.02 1.00     4150     2620

Family Specific Parameters:
      Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sigma     0.60      0.02     0.57     0.64 1.00     3898     2821

Draws were sampled using sampling(NUTS). For each parameter, Bulk_ESS
and Tail_ESS are effective sample size measures, and Rhat is the potential
scale reduction factor on split chains (at convergence, Rhat = 1).
```

Varying-intercepts model

model specs

- predict log `rt` in terms of factor `so`
- improper prior on all parameters
- different subjects / items could be faster or slower *tout court*

$$\log(\texttt{rt}_i) \sim \mathcal{N}(\eta_i, \sigma_{err})$$

$$\eta_i = \beta_0 + \boxed{\underbrace{u_{0,\texttt{subj}_i} + w_{0,\texttt{item}_i}}_{\text{varying intercepts}}} + \beta_1 \texttt{so}_i$$

$$\boxed{\begin{aligned} u_{0,\texttt{subj}_i} &\sim \mathcal{N}(0, \sigma_{u_0}) \\ w_{0,\texttt{subj}_i} &\sim \mathcal{N}(0, \sigma_{w_0}) \end{aligned}}$$

Varying-intercepts model

results

```
 Family: gaussian
  Links: mu = identity; sigma = identity
Formula: log(rt) ~ (1 | subj + item) + so
   Data: rt_data (Number of observations: 547)
  Draws: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
         total post-warmup draws = 4000

Group-Level Effects:
~item (Number of levels: 15)
              Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sd(Intercept)     0.20      0.05     0.12     0.32 1.00     1475     2392

~subj (Number of levels: 37)
              Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sd(Intercept)     0.25      0.04     0.18     0.34 1.00     1593     2503

Population-Level Effects:
          Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
Intercept     6.10      0.08     5.95     6.24 1.00     1145     1379
so1          -0.07      0.04    -0.16     0.01 1.00     7191     2986

Family Specific Parameters:
      Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sigma     0.52      0.02     0.49     0.55 1.00     5466     3115
```

Varying-intercepts & varying slopes model

model specs

- predict log `rt` in terms of factor `so`
- improper prior on all parameters
- different subjects / items could be faster or slower *tout court*
- different subjects / items could be more or less sensitive to factor `so`

$$\log(\mathtt{rt}_i) \sim \mathcal{N}(\eta_i, \sigma_{err})$$

$$\eta_i = \beta_0 + \underbrace{u_{0,\mathtt{subj}_i} + w_{0,\mathtt{item}_i}}_{\text{varying intercepts}} + (\beta_1 + \boxed{\underbrace{u_{1,\mathtt{subj}_i} + w_{1,\mathtt{item}_i}}_{\text{varying slopes}})\mathtt{so}_i}$$

$$u_{0,\mathtt{subj}_i} \sim \mathcal{N}(0,\sigma_{u_0}) \qquad w_{0,\mathtt{subj}_i} \sim \mathcal{N}(0,\sigma_{w_0})$$

$$\boxed{u_{1,\mathtt{subj}_i} \sim \mathcal{N}(0,\sigma_{u_1}) \qquad w_{1,\mathtt{subj}_i} \sim \mathcal{N}(0,\sigma_{w_1})}$$

Varying-intercepts & varying slopes model

results

```
Group-Level Effects:
~item (Number of levels: 15)
              Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sd(Intercept)     0.20      0.05     0.12     0.31 1.00     1514     2007
sd(so1)           0.07      0.05     0.00     0.20 1.00     1653     2508

~subj (Number of levels: 37)
              Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sd(Intercept)     0.26      0.04     0.19     0.35 1.00     1743     2015
sd(so1)           0.07      0.05     0.00     0.17 1.00     1288     1966

Population-Level Effects:
          Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
Intercept     6.10      0.07     5.95     6.24 1.00     1565     2375
so1          -0.07      0.05    -0.17     0.03 1.00     4754     2930

Family Specific Parameters:
      Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sigma     0.52      0.02     0.49     0.55 1.00     5597     2604
```

Varying-intercepts & varying slopes model w/ correlation

model specs

- predict log `rt` in terms of factor `so`
- improper prior on all parameters
- different subjects / items could be faster or slower *tout court*
- different subjects / items could be more or less sensitive to factor `so`

$$\eta_i = \beta_0 + u_{0,\texttt{subj}_i} + w_{0,\texttt{item}_i} + \left(\beta_1 + u_{1,\texttt{subj}_i} + w_{1,\texttt{item}_i}\right) \texttt{so}_i$$

$$\boxed{\begin{pmatrix} u_{0,\texttt{subj}_i} \\ u_{1,\texttt{subj}_i} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_u\right)}$$

$$\Sigma_u = \begin{pmatrix} \sigma_{u0}^2 & \rho_u \sigma_{u0} \sigma_{u1} \\ \rho_u \sigma_{u0} \sigma_{u1} & \sigma_{u1}^2 \end{pmatrix} \quad \text{same for } \texttt{item}$$

Varying-intercepts & varying slopes model w/ correlation

results

```
Group-Level Effects:
~item (Number of levels: 15)
                   Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sd(Intercept)          0.21      0.06     0.12     0.35 1.00     1683     2489
sd(so1)                0.07      0.05     0.00     0.20 1.00     1344     2006
cor(Intercept,so1)    -0.06      0.53    -0.94     0.92 1.00     4241     2651

~subj (Number of levels: 37)
                   Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sd(Intercept)          0.30      0.05     0.21     0.41 1.00     1435     2261
sd(so1)                0.13      0.07     0.01     0.26 1.00     1066     1030
cor(Intercept,so1)    -0.67      0.33    -0.99     0.28 1.00     2084     1626

Population-Level Effects:
          Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
Intercept     6.10      0.08     5.94     6.25 1.00     1484     2214
so1          -0.07      0.06    -0.18     0.04 1.00     4210     2934

Family Specific Parameters:
      Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
sigma     0.51      0.02     0.48     0.55 1.00     4003     3093
```

How to choose RE-structure?

How to choose multi-level architecture?

two approaches

(1) keep it maximal

- include maximum RE structure that makes sense
  - what “makes sense” may depend on *a priori* considerations
  - data might not be sufficient to estimate otherwise conceivable MLMs

(2) let the data decide

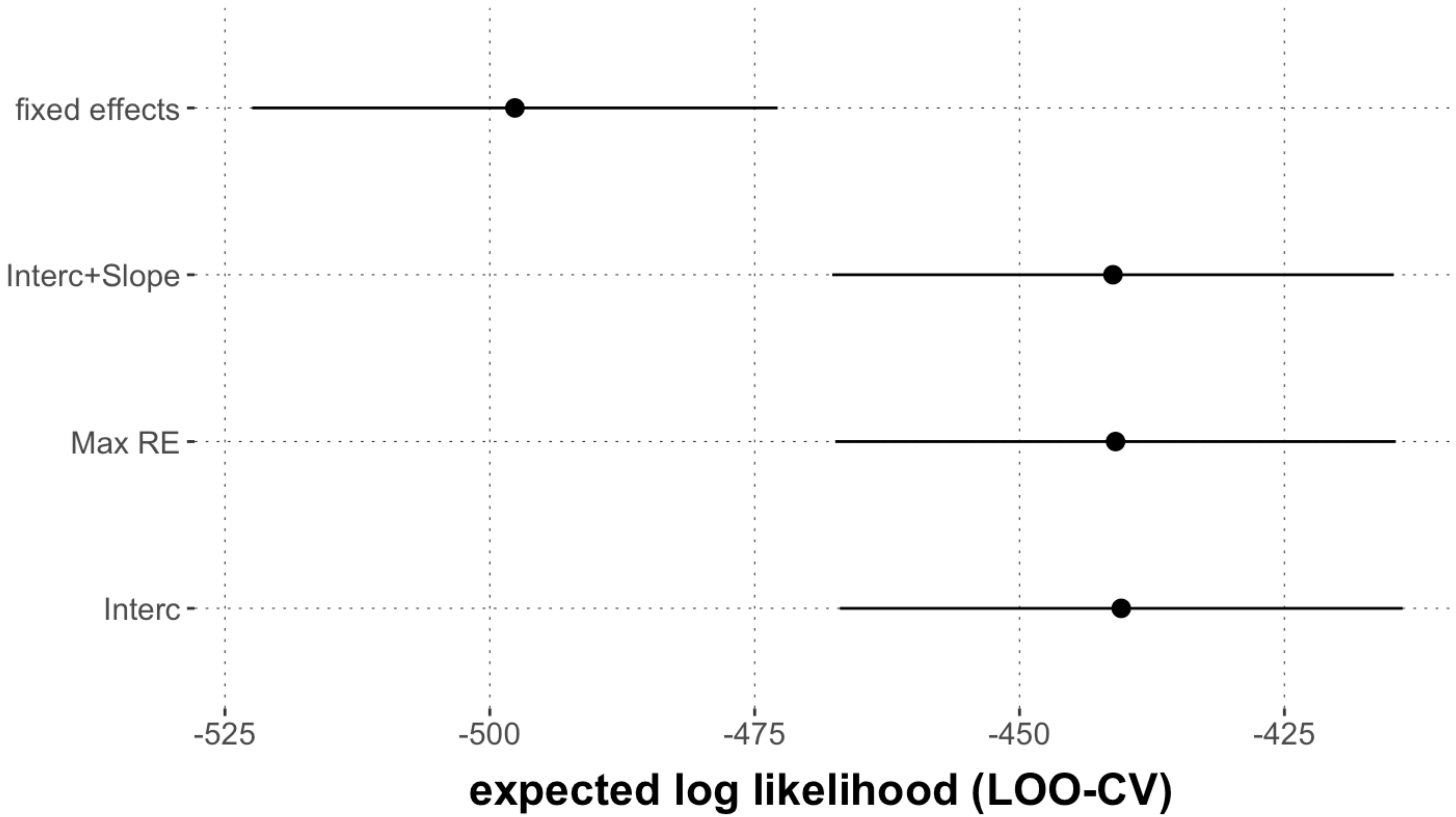
- compare different MLMs and choose the best

(1) is more careful / prudent in a science context (learning about the world from the model and the data)

(2) may be more adequate for an engineering context (predicting well enough with efficient models)

LOO-based model comparison

results



priors for MLMs

Priors in MLMs

```r
# define the model as a "brmsformula" object
myFormula <- brms::bf(RT ~ 1 + condition + (1 + condition | submission_id))

# get prior information
brms::get_prior(
  formula = myFormula,
  data    = data_MC,
  family  = gaussian()
  )
```

```
                prior     class                    coef         group resp dpar nlpar lb ub       source
               (flat)         b                                                                   default
               (flat)         b conditiondiscrimination                                      (vectorized)
               (flat)         b       conditionreaction                                      (vectorized)
               lkj(1)       cor                                                                   default
               lkj(1)       cor                          submission_id                       (vectorized)
student_t(3, 385, 133.4) Intercept                                                                default
  student_t(3, 0, 133.4)        sd                                                     0          default
  student_t(3, 0, 133.4)        sd                          submission_id              0     (vectorized)
  student_t(3, 0, 133.4)        sd conditiondiscrimination submission_id              0     (vectorized)
  student_t(3, 0, 133.4)        sd       conditionreaction submission_id              0     (vectorized)
  student_t(3, 0, 133.4)        sd               Intercept submission_id              0     (vectorized)
  student_t(3, 0, 133.4)     sigma                                                     0          default
```


more on LKJ distribution here & here

MLM notation & formula syntax

Multi-level models
notation

Cumbersome: $$\eta_i = \beta_0 + u_{0,\texttt{subj}_i} + w_{0,\texttt{item}_i} + \left(\beta_1 + u_{1,\texttt{subj}_i} + w_{1,\texttt{item}_i}\right) \texttt{so}_i$$

Compact: $$\eta = X\beta + Z\gamma$$

$$X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ \vdots & \vdots \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 1 & \dots \\ 0 & \dots & 0 & 1 & \dots & 0 & \dots \\ 0 & \dots & 1 & 0 & \dots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

random effects matrix

	subj	item	so	rt
	<dbl>	<dbl>	<chr>	<dbl>
1	1	13	1	1561
2	1	6	-1	959
3	1	5	1	582
4	1	9	1	294
5	1	14	-1	438
6	1	4	-1	286
7	1	8	-1	438
8	1	10	-1	278
9	1	2	-1	542
10	1	11	1	494

BRMS formula syntax for MLMs

in a nutshel

basic form:

`response ~ pterms + (gterms | group)`

variations:

- `(gterms || group)` : suppress correlation between `gterms`
- `(gterms | g1 + g2)` : syntactic sugar for `(gterms | g1) + (gterms | g2)`
- `(gterms | g1 : g2)` : all combinations of `g1` and `g2` (Cartesian product)
- `(gterms | g1 / g2)` : nesting `g2` within `g1`; equals `(gterms | g1) + (gterms | g1 : g2)`
- `(gterms | IDx | group)` : correlation for all group-level categories with `IDx`
  - useful for multi-formula models (e.g., non-linear models)

read more here

multi-membership & an alternative motivation for MLMs

demo

walkthrough for multi-membership models & an alternative view on group-level effects