EIGENTENSORS OF LINEAR ANISOTROPIC ELASTIC MATERIALS

By MORTEZA M. MEHRABADI

(*Department of Mechanical Engineering, Tulane University, New Orleans, Louisiana* 70118, *USA*)

and STEPHEN C. COWIN†

(*Department of Biomedical Engineering, Tulane University, New Orleans, Louisiana* 70118, *USA*)

[Received 1 September 1988. Revise 19 April 1989]

SUMMARY

There are two eigentensors for a linear isotropic elastic material; one is the deviatoric second-rank tensor, and the other is a second-rank tensor proportional to the unit tensor and often called the spherical or hydrostatic part of the tensor. The eigentensors of isotropic elasticity have many properties of physical and mathematical significance. In this paper a method of construction of the eigentensors for the anisotropic elastic-material symmetries is presented and applied to determine the eigentensors of each anisotropic elastic symmetry. The eigentensors for the anisotropic symmetries are shown to have the same important properties as those possessed by the eigentensors of isotropic elasticity.

1. Introduction

THE eigenvectors of the 3-dimensional fourth-rank anisotropic elasticity tensor, considered as a second-rank tensor in a 6-dimensional space, are called *eigentensors* when projected back into the 3-dimensional space. The maximum number of eigentensors for any elastic symmetry is therefore six. A precise definition of eigentensors is given in section 4. The eigentensors for a linear isotropic elastic material are familiar. They are the deviatoric second-rank tensor and a tensor proportional to the unit tensor, the spherical or hydrostatic or dilatational part of the tensor.

The concept of an eigentensor was introduced by Kelvin (**1**) who employed a different terminology. Kelvin determined the eigentensors for many elastic symmetries and gave a concise summary of his results in his *Encyclopaedia Britannica* article on Elasticity (**2**, section 41). Todhunter and Pearson (**3**, pp. 448–451) reviewed the contribution of Kelvin in this area. The present work extends and develops an approach and viewpoint initiated by Kelvin, although we only learned of Kelvin's work in this area from the remarks of a referee.

In order to explain our motivation for constructing the eigentensors associated with the anisotropic elastic symmetries we describe the properties

† Present Address: Department of Mechanical Engineering, City College of the City University of New York, New York 10031, USA.

of the isotropic elastic eigentensors that will be generalized to the eigentensors for the anisotropic elastic symmetries. The decomposition of the total stress $\mathbf{T}$ and the total strain $\mathbf{E}$ into deviatoric and spherical components is accomplished by the following formulae:

$$\mathbf{T} = \tilde{\mathbf{T}} + \tfrac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{1}, \tag{1.1}$$

$$\mathbf{E} = \tilde{\mathbf{E}} + \tfrac{1}{3}(\operatorname{tr}\mathbf{E})\mathbf{1}, \tag{1.2}$$

where $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{E}}$ are the deviatoric parts of the stress and strain tensor, respectively. The formulae (1.1) and (1.2) can be viewed as the definition of the deviatoric parts of the stress and strain tensors, given that tr indicates the trace. However, in the context of linear isotropic elasticity, these equations can be viewed as the decomposition of total stress and the total strain into a sum of 'eigentensors of identical form'. The eigentensors $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{E}}$, in (1.1) and (1.2), are of identical form as are the eigentensors $\frac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{1}$ and $\frac{1}{3}(\operatorname{tr}\mathbf{E})\mathbf{1}$.

The extension of the property mentioned above to include the eigentensors of elastic symmetries other than isotropy is stated as follows.

PROPERTY A. *For any elastic symmetry the total stress tensor and the total strain tensor can be additively decomposed into a sum of six or fewer eigentensors of identical form.*

In isotropic elastic materials the unique and complete decomposition into spherical and deviatoric modes can be extended to the stress–strain relations and to the strain energy. For an isotropic material, Hooke's law can be written in the form

$$\mathbf{T} = \lambda(\operatorname{tr}\mathbf{E})\mathbf{1} + 2\mu\mathbf{E}, \tag{1.3}$$

where λ and μ are the Lamé elastic moduli. When the decompositions (1.1) and (1.2) of the stress and strain, respectively, are substituted in (1.3), we can rewrite (1.3) as the two equations

$$\tfrac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{1} = \tfrac{1}{3}(3\lambda + 2\mu)(\operatorname{tr}\mathbf{E})\mathbf{1}, \qquad \tilde{\mathbf{T}} = 2\mu\tilde{\mathbf{E}}. \tag{1.4}$$

The first of these shows that the spherical eigentensor of stress is proportional to the spherical eigentensor of strain and that $3\lambda + 2\mu$ is the factor of proportionality. The second shows that the deviatoric eigentensor of stress is proportional to the deviatoric eigentensor of strain and that 2μ is the factor of proportionality. The generalization to the anisotropic elastic symmetries is summarized in the following property statement.

PROPERTY B. *For any elastic symmetry each stress eigentensor is directly proportional to its strain eigentensor of identical form.*

The strain energy Σ per unit volume is given by

$$2\Sigma = \operatorname{tr}(\mathbf{TE}). \tag{1.5}$$

When the decompositions (1.1) and (1.2) of the stress and strain, respectively, are substituted into (1.5), the strain energy can be written in the form

$$\Sigma = \Sigma_{\text{dev}} + \Sigma_{\text{sph}}, \tag{1.6}$$

where Σ_{dev} is the deviatoric strain energy, that is, the strain energy of the deviatoric stress and strain:

$$2\Sigma_{\text{dev}} = \operatorname{tr}(\tilde{\mathbf{T}}\tilde{\mathbf{E}}) = 2\mu \operatorname{tr} \tilde{\mathbf{E}}^2, \tag{1.7}$$

and Σ_{sph} is the spherical or hydrostatic strain energy, that is, the strain energy of the spherical stress and strain:

$$2\Sigma_{\text{sph}} = \tfrac{1}{3}(\operatorname{tr} \mathbf{T})(\operatorname{tr} \mathbf{E}) = \tfrac{1}{3}(3\lambda + 2\mu)(\operatorname{tr} \mathbf{E})^2. \tag{1.8}$$

The generalization of this property is summarized as follows.

PROPERTY C. *For any elastic symmetry there is an additive decomposition of the total strain energy into a sum of six or fewer terms, each term being a scalar-valued product of the stress and strain eigentensors of identical form. These terms represent energy modes which are not interactive.*†

The quantities $\operatorname{tr} \mathbf{T}$ and $\operatorname{tr} \tilde{\mathbf{T}}^2$ have been particularly useful in the analysis of experimental data in plasticity. The quantity $\operatorname{tr} \mathbf{T}$ divided by three is often called the mean pressure and the quantity $\operatorname{tr} \tilde{\mathbf{T}}^2$, or a constant times this quantity, has many names including the von Mises stress, the deviatoric strain energy, the octahedral shear stress, the root-mean-square stress, etc. From (1.4) it is easy to see that $\operatorname{tr} \mathbf{T}$ and $\operatorname{tr} \mathbf{E}$ are proportional as are $\operatorname{tr} \tilde{\mathbf{T}}^2$ and $\operatorname{tr} \tilde{\mathbf{E}}^2$; thus

$$\operatorname{tr} \mathbf{T} = (3\lambda + 2\mu) \operatorname{tr} \mathbf{E}, \qquad \operatorname{tr} \tilde{\mathbf{T}}^2 = 4\mu^2 \operatorname{tr} \tilde{\mathbf{E}}^2. \tag{1.9}$$

Both $\operatorname{tr} \mathbf{T}$ (or $\operatorname{tr} \mathbf{E}$) and $\operatorname{tr} \tilde{\mathbf{T}}^2$ (or $\operatorname{tr} \tilde{\mathbf{E}}^2$) are isotropic invariants of the stress (or strain) tensor; thus (1.9) represents a subset of the isotropic invariants of stress and strain characterized by direct proportionality between stress and strain invariants of eigentensors of identical form. We summarize the generalization of this property for all elastic symmetries as follows.

PROPERTY D. *For any elastic symmetry the traces of the stress and strain eigentensors of identical form, or the traces of the squares of the stress and strain eigentensors of identical form, are directly proportional.*

This concept identifies an important subset of the invariants associated with any elastic symmetry, the subset for which the invariant of stress is directly proportional to the invariant of strain that is of identical invariant form.

The problem we pose and solve is the construction of a method to obtain the eigentensors with the properties described above for each distinct

† Kelvin discusses properties A, B and C in (1).

anisotropic elastic symmetry. There are many possible applications of these results, of which we shall discuss two here.

First, consider the problem of graphical representation of the stress and strain fields that result from finite-element analyses of objects. Many commercial finite-element programs provide, as optional outputs, contour plots of the mean pressure $\frac{1}{3}\mathrm{tr}\,\mathbf{T}$ and the von Mises stress $\mathrm{tr}\,\tilde{\mathbf{T}}^2$. This is very useful information for isotropic elastic materials. However, many users of the finite-element method also use contour plots of $\frac{1}{3}\mathrm{tr}\,\mathbf{T}$ and $\mathrm{tr}\,\tilde{\mathbf{T}}^2$ when the material is anisotropic. It is easy to show that $\frac{1}{3}\mathrm{tr}\,\mathbf{T}$ and $\mathrm{tr}\,\tilde{\mathbf{T}}^2$ do not have the special properties that they have for isotropic materials when the material is anisotropic. In particular, except for isotropic and cubic symmetry, $\frac{1}{3}\mathrm{tr}\,\mathbf{T}$ and $\mathrm{tr}\,\tilde{\mathbf{T}}^2$ are coupled. For anisotropic elastic materials we suggest that contour plots of stress and strain fields use the special set of six or fewer invariants identified in property D above. Using this approach to the graphical representation of computed stress and strain fields in anisotropic elastic materials can be viewed as an appropriate generalization of the present practice for isotropic materials since the invariants identified in property D for isotropic materials are $\mathrm{tr}\,\mathbf{T}$ and $\mathrm{tr}\,\tilde{\mathbf{T}}^2$.

A second application of the properties listed above is in the formulation of fracture or strength criteria. For isotropic materials many of the widely-used fracture or strength criteria involve only the mean stress $\frac{1}{3}\mathrm{tr}\,\mathbf{T}$ and the von Mises stress $\mathrm{tr}\,\tilde{\mathbf{T}}^2$. In particular, the widely-used von Mises criterion for the yielding of metals involves only $\mathrm{tr}\,\tilde{\mathbf{T}}^2$. Since $\mathrm{tr}\,\mathbf{T}$ and $\mathrm{tr}\,\tilde{\mathbf{T}}^2$ are the invariants with the special property D for isotropic materials, we suggest that the invariants with the special property D for anisotropic materials should be considered in the formulation of fracture or strength theories for anisotropic elastic materials.

In the section that follows, three different notations for Hooke's law are described: the notations associated with its fourth-rank tensor formulation in 3-space, its second-rank tensor formulation in 6-space, and the non-tensorial notation due to Voigt. In section 3 the equivalence betwen the 3-space and the 6-space tensor formulations is established. In section 4 the concepts of eigentensors and eigenelastic constants associated with particular material symmetries are introduced in the 6-space formulation. The eigentensors and eigenelastic constants for cubic, hexagonal (7), hexagonal (6), transversely isotropic (hexagonal (5)), tetragonal (7), tetragonal (6) and orthotropic symmetries are obtained in section 5, and those for the monoclinic and triclinic symmetries are described therein. The invariants of the eigentensors described in property D above are calculated in section 6. Section 7 contains a discussion.

2. Three notations for the generalized Hooke's law

The generalized Hooke's law

$$T_{ij} = C_{ijkm}E_{km} \tag{2.1}$$

is the most general linear relation between the stress tensor whose components are T_{ij} and the linear strain tensor whose components are E_{ij}, where the strain has been assumed to be measured from an unstressed reference state. The coefficients of linearity, namely C_{ijkm}, are the components of the fourth-rank elasticity tensor. There are three important symmetry restrictions on the elasticity tensor C_{ijkm}; restrictions that are independent of those imposed by material symmetry. These are the symmetries

$$C_{ijkm} = C_{jikm}, \qquad C_{ijkm} = C_{ijmk}, \qquad C_{ijkm} = C_{kmij}, \tag{2.2}$$

which follow from the symmetry of the stress tensor, the symmetry of the strain tensor, and the thermodynamic requirement that no work be produced by the elastic material in a closed loading cycle, respectively. The number of independent components of a fourth-rank tensor in three dimensions is 81, but the restrictions (2.2) reduce the number of independent components of C_{ijkm} to 21.

The purpose of this section is to present the different notations for the generalized Hooke's law (2.1). An effort is made to present this material in a format that allows for comparison of the notations and for quick reference. Motivations or justifications for the notations will be given in the following section. These notations are called the *fourth-rank tensor* notation, the *Voigt* notation and the *second-rank tensor* notation. The fourth-rank tensor notation employs a fourth-rank Cartesian tensor in three dimensions. The Voigt notation is a non-tensorial notation that employs a 6×6 matrix. The second-rank tensor notation employs a second-rank Cartesian tensor in six dimensions.

The fourth-rank tensor notation for the generalized Hooke's law is represented in Cartesian-index notation by (2.1). It is easy to see that (2.1) can also be written in matrix notation as

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}. \tag{2.3}$$

The representation (2.3) is obtained from (2.1) by use of the index-notation conventions of the free and summation indices. The factor of 2 that premultiplies the shearing strains is significant because of the notational difficulties it causes. The Voigt notation for Hooke's law represents (2.3) in the form

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}, \tag{2.4}$$

where the equivalence between stress components and strain components for the two notations is given in columns 1 and 2 of Table 1 and the equivalence between elastic coefficients for the two notations is given in columns 1 and 2 of Table 2. The Voigt notation is important because it has become the standard in anisotropic elasticity (Voigt (4), Hearmon (5)).

The 6-dimensional second-rank tensor notation is introduced by rewriting (2.3) with factors of 2 and its square root as multipliers of various terms:

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ 2^{\frac12}T_{23} \\ 2^{\frac12}T_{13} \\ 2^{\frac12}T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2^{\frac12}C_{1123} & 2^{\frac12}C_{1113} & 2^{\frac12}C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & 2^{\frac12}C_{2223} & 2^{\frac12}C_{2213} & 2^{\frac12}C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & 2^{\frac12}C_{3323} & 2^{\frac12}C_{3313} & 2^{\frac12}C_{3312} \\ 2^{\frac12}C_{2311} & 2^{\frac12}C_{2322} & 2^{\frac12}C_{2333} & 2C_{2323} & 2C_{2313} & 2C_{2312} \\ 2^{\frac12}C_{1311} & 2^{\frac12}C_{1322} & 2^{\frac12}C_{1333} & 2C_{1323} & 2C_{1313} & 2C_{1312} \\ 2^{\frac12}C_{1211} & 2^{\frac12}C_{1222} & 2^{\frac12}C_{1233} & 2C_{1223} & 2C_{1213} & 2C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2^{\frac12}E_{23} \\ 2^{\frac12}E_{13} \\ 2^{\frac12}E_{12} \end{bmatrix}. \tag{2.5}$$

TABLE 1. *The components of stress and strain in different notations. Column 1 illustrates their customary notation as second-rank tensors in a 3-dimensional Cartesian space. Column 2 represents their customary notation as vectors in a 6-dimensional space. Column 3 is the new notation introduced here. In this notation they are vectors in a 6-dimensional Cartesian space*

1		2		3
T_{11}	=	σ_1	=	$\hat{T}_1$
T_{22}	=	σ_2	=	$\hat{T}_2$
T_{33}	=	σ_3	=	$\hat{T}_3$
T_{23}	=	σ_4	=	$2^{-\frac12}\hat{T}_4$
T_{13}	=	σ_5	=	$2^{-\frac12}\hat{T}_5$
T_{12}	=	σ_6	=	$2^{-\frac12}\hat{T}_6$
E_{11}	=	ε_1	=	$\hat{E}_1$
E_{22}	=	ε_2	=	$\hat{E}_2$
E_{33}	=	ε_3	=	$\hat{E}_3$
E_{23}	=	$\frac12\varepsilon_4$	=	$2^{-\frac12}\hat{E}_4$
E_{13}	=	$\frac12\varepsilon_5$	=	$2^{-\frac12}\hat{E}_5$
E_{12}	=	$\frac12\varepsilon_6$	=	$2^{-\frac12}\hat{E}_6$

TABLE 2. *The elasticity and compliance in different notations. Column 1 illustrates the notation of these quantities as fourth-rank tensor components in a 3-dimensional Cartesian space. Column 2 represents the traditional or Voigt double-index notation. Column 3 illustrates the notation for these quantities as second-rank tensor components in a 6-dimensional Cartesian space*

Elasticity 1		2		3	Compliance 1		2		3
C_{1111}	=	c_{11}	=	$\hat{c}_{11}$	K_{1111}	=	s_{11}	=	$\hat{s}_{11}$
C_{2222}	=	c_{22}	=	$\hat{c}_{22}$	K_{2222}	=	s_{22}	=	$\hat{s}_{22}$
C_{3333}	=	c_{33}	=	$\hat{c}_{33}$	K_{3333}	=	s_{33}	=	$\hat{s}_{33}$
C_{1122}	=	c_{12}	=	$\hat{c}_{12}$	K_{1122}	=	s_{12}	=	$\hat{s}_{12}$
C_{1133}	=	c_{13}	=	$\hat{c}_{13}$	K_{1133}	=	s_{13}	=	$\hat{s}_{13}$
C_{2233}	=	c_{23}	=	$\hat{c}_{23}$	K_{2233}	=	s_{23}	=	$\hat{s}_{23}$
C_{2323}	=	c_{44}	=	$\frac{1}{2}\hat{c}_{44}$	K_{2323}	=	$\frac{1}{4}s_{44}$	=	$\frac{1}{2}\hat{s}_{44}$
C_{1313}	=	c_{55}	=	$\frac{1}{2}\hat{c}_{55}$	K_{1313}	=	$\frac{1}{4}s_{55}$	=	$\frac{1}{2}\hat{s}_{55}$
C_{1212}	=	c_{66}	=	$\frac{1}{2}\hat{c}_{66}$	K_{1212}	=	$\frac{1}{4}s_{66}$	=	$\frac{1}{2}\hat{s}_{66}$
C_{1323}	=	c_{54}	=	$\frac{1}{2}\hat{c}_{54}$	K_{1323}	=	$\frac{1}{4}s_{54}$	=	$\frac{1}{2}\hat{s}_{54}$
C_{1312}	=	c_{56}	=	$\frac{1}{2}\hat{c}_{56}$	K_{1312}	=	$\frac{1}{4}s_{56}$	=	$\frac{1}{2}\hat{s}_{56}$
C_{1223}	=	c_{64}	=	$\frac{1}{2}\hat{c}_{64}$	K_{1223}	=	$\frac{1}{4}s_{64}$	=	$\frac{1}{2}\hat{s}_{64}$
C_{2311}	=	c_{41}	=	$2^{-\frac{1}{2}}\hat{c}_{41}$	K_{2311}	=	$\frac{1}{2}s_{41}$	=	$2^{-\frac{1}{2}}\hat{s}_{41}$
C_{1311}	=	c_{51}	=	$2^{-\frac{1}{2}}\hat{c}_{51}$	K_{1311}	=	$\frac{1}{2}s_{51}$	=	$2^{-\frac{1}{2}}\hat{s}_{51}$
C_{1211}	=	c_{61}	=	$2^{-\frac{1}{2}}\hat{c}_{61}$	K_{1211}	=	$\frac{1}{2}s_{61}$	=	$2^{-\frac{1}{2}}\hat{s}_{61}$
C_{2322}	=	c_{42}	=	$2^{-\frac{1}{2}}\hat{c}_{42}$	K_{2322}	=	$\frac{1}{2}s_{42}$	=	$2^{-\frac{1}{2}}\hat{s}_{42}$
C_{1322}	=	c_{52}	=	$2^{-\frac{1}{2}}\hat{c}_{52}$	K_{1322}	=	$\frac{1}{2}s_{52}$	=	$2^{-\frac{1}{2}}\hat{s}_{52}$
C_{1222}	=	c_{62}	=	$2^{-\frac{1}{2}}\hat{c}_{62}$	K_{1222}	=	$\frac{1}{2}s_{62}$	=	$2^{-\frac{1}{2}}\hat{s}_{62}$
C_{2333}	=	c_{43}	=	$2^{-\frac{1}{2}}\hat{c}_{43}$	K_{2333}	=	$\frac{1}{2}s_{43}$	=	$2^{-\frac{1}{2}}\hat{s}_{43}$
C_{1333}	=	c_{53}	=	$2^{-\frac{1}{2}}\hat{c}_{53}$	K_{1333}	=	$\frac{1}{2}s_{53}$	=	$2^{-\frac{1}{2}}\hat{s}_{53}$
C_{1233}	=	c_{63}	=	$2^{-\frac{1}{2}}\hat{c}_{63}$	K_{1233}	=	$\frac{1}{2}s_{63}$	=	$2^{-\frac{1}{2}}\hat{s}_{63}$

It is easy to verify that (2.5) is equivalent to (2.3) by performing the indicated matrix multiplications. The representation (2.5) of (2.1) appears in Walpole (6). In the next section we shall show that the 6×6 matrix in (2.5) represents the components of a second-rank tensor. Based on (2.5) the second-rank tensor notation for Hooke's law is written in the form

$$\begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \\ \hat{T}_6 \end{bmatrix} = \begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & \hat{c}_{14} & \hat{c}_{15} & \hat{c}_{16} \\ \hat{c}_{12} & \hat{c}_{22} & \hat{c}_{23} & \hat{c}_{24} & \hat{c}_{25} & \hat{c}_{26} \\ \hat{c}_{13} & \hat{c}_{23} & \hat{c}_{33} & \hat{c}_{34} & \hat{c}_{35} & \hat{c}_{36} \\ \hat{c}_{14} & \hat{c}_{24} & \hat{c}_{34} & \hat{c}_{44} & \hat{c}_{45} & \hat{c}_{46} \\ \hat{c}_{15} & \hat{c}_{25} & \hat{c}_{35} & \hat{c}_{45} & \hat{c}_{55} & \hat{c}_{56} \\ \hat{c}_{16} & \hat{c}_{26} & \hat{c}_{36} & \hat{c}_{46} & \hat{c}_{56} & \hat{c}_{66} \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \\ \hat{E}_5 \\ \hat{E}_6 \end{bmatrix}, \tag{2.6}$$

where the equivalence between the notations for stress components and strain components is given in Table 1 and the equivalence between the notations for the elastic coefficients is given in Table 2.

Tables 1 and 2 describe notational changes from an index system with a range of three $(i, j = 1, 2, 3)$ to one with a range of six $(\alpha, \beta = 1, 2, 3, 4, 5, 6)$ by the following rules for replacing a pair of lower case italic indices by a single Greek index: $11 \to 1$, $22 \to 2$, $33 \to 3$, $23 \to 4$, $13 \to 5$ and $12 \to 6$. Hooke's law in the Voigt notation (2.4) can be written in the Greek index and matrix notation as

$$\sigma_\alpha = c_{\alpha\beta}\varepsilon_\beta, \qquad \boldsymbol{\sigma} = \mathbf{c}\boldsymbol{\varepsilon}, \tag{2.7}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are 1×6 column matrices and $\mathbf{c}$ is the 6×6 matrix of (2.4). Hooke's law in the second-rank tensor notation (2.6) can be written similarly as

$$\hat{T}_\alpha = \hat{c}_{\alpha\beta}\hat{E}_\beta, \qquad \hat{\mathbf{T}} = \hat{\mathbf{c}}\hat{\mathbf{E}}, \tag{2.8}$$

where $\hat{\mathbf{T}}$ and $\hat{\mathbf{E}}$ are 1×6 column matrices and $\hat{\mathbf{c}}$ is the 6×6 matrix of (2.6). There appears to be little superficial difference between these notations. One might say that they only differ by factors of 2 and its square root. However, it is well known (Nye (7), Hearmon (5)) that the components of the Voigt matrix $\mathbf{c}$ are not the components of a second-rank tensor. In the next section we show that the matrix components of $\hat{\mathbf{c}}$ are the components of a second-rank tensor in six dimensions.

3. Tensorial equivalents in three and six dimensions

In this section we construct a second-rank Cartesian-tensor formulation of Hooke's law in six dimensions. That such a formulation is possible appears to be obvious to some researchers and unknown to others. The formulation was suggested to us by J. L. Ericksen (in a private communication) as a means of obtaining results not discussed in this paper. Here, this tensor formulation is accomplished by introducing a Cartesian basis in three dimensions to construct a Cartesian basis in six dimensions. The Cartesian base vectors in three dimensions are denoted by $\mathbf{e}_i$ $(i = 1, 2, 3)$ and those in six dimensions by $\hat{\mathbf{e}}_\alpha$ $(\alpha = 1, \ldots, 6)$. The relationship between these two bases is as follows:

$$\left.\begin{aligned} \hat{\mathbf{e}}_1 &= \mathbf{e}_1 \otimes \mathbf{e}_1, & \hat{\mathbf{e}}_4 &= \frac{1}{2^{\frac{1}{2}}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2), \\ \hat{\mathbf{e}}_2 &= \mathbf{e}_2 \otimes \mathbf{e}_2, & \hat{\mathbf{e}}_5 &= \frac{1}{2^{\frac{1}{2}}}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1), \\ \hat{\mathbf{e}}_3 &= \mathbf{e}_3 \otimes \mathbf{e}_3, & \hat{\mathbf{e}}_6 &= \frac{1}{2^{\frac{1}{2}}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \end{aligned}\right\} \tag{3.1}$$

where $\otimes$ indicates the tensor or dyadic product. It is easy to verify that orthonormality of the 3-dimensional basis ensures orthonormality of the 6-dimensional basis, if one employs a double-scalar product operation with the 3-dimensional Cartesian basis. One can think of the 6-dimensional base vectors $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_6$ either as vectors in the 6-dimensional space $(1, 0, 0, 0, 0, 0), \ldots, (0, 0, 0, 0, 0, 1)$, or as special second-rank tensors in three dimensions,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \ldots, \quad \frac{1}{2^{\frac{1}{2}}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.2}$$

The components of stress relative to the two bases are related by

$$T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \hat{T}_\alpha \hat{\mathbf{e}}_\alpha, \tag{3.3}$$

an expression that can be verified from (3.1), Table 1, and the use of the summation convention over repeated indices noting the different ranges for the italic and Greek subscripts. The direct notation $\mathbf{T}$ is used for the components relative to the italic or 3-dimensional basis and $\hat{\mathbf{T}}$ for the Greek or 6-dimensional basis. A similar representation is possible for any symmetric second-rank tensor in three dimensions, including strain.

The components of the tensor of elastic constants with respect to the 3- and 6-dimensional bases are related by

$$C_{ijkm}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_m = \hat{c}_{\alpha\beta}\hat{\mathbf{e}}_\alpha \otimes \hat{\mathbf{e}}_\beta. \tag{3.4}$$

This invariance relation relative to a particular Cartesian basis is equivalent to the equality of the 6×6 matrix in (2.5) to the 6×6 matrix in (2.6).

We turn now to the question of Cartesian transformation laws of tensor components when orthonormal bases are changed. Let $\mathbf{e}'_i$ $(i = 1, 2, 3)$ denote a second 3-dimensional Cartesian basis and let $\hat{\mathbf{e}}'_\alpha$ $(\alpha = 1, \ldots, 6)$ denote a second 6-dimensional Cartesian basis. The bases $\mathbf{e}_i$ and $\mathbf{e}'_i$ are related by

$$\mathbf{e}'_i = Q_{ij}\mathbf{e}_j, \tag{3.5}$$

where $\mathbf{Q}$ is an orthogonal 3-dimensional tensor. The bases $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}'_\alpha$ are related by

$$\hat{\mathbf{e}}'_\alpha = \hat{Q}_{\alpha\beta}\hat{\mathbf{e}}_\beta, \tag{3.6}$$

where $\hat{\mathbf{Q}}$ is an orthogonal 6-dimensional tensor. To construct $\hat{\mathbf{Q}}$ given $\mathbf{Q}$ we note that

$$\mathbf{e}'_i \otimes \mathbf{e}'_j = Q_{ik}Q_{jm}\mathbf{e}_k \otimes \mathbf{e}_m, \tag{3.7}$$

and

$$\tfrac{1}{2}(\mathbf{e}'_i \otimes \mathbf{e}'_j + \mathbf{e}'_j \otimes \mathbf{e}'_i) = \tfrac{1}{2}(Q_{ik}Q_{jm} + Q_{im}Q_{jk})\mathbf{e}_k \otimes \mathbf{e}_m. \tag{3.8}$$

It then follows from (3.6) and the definition (3.1) of the 6-dimensional basis that

$$\tfrac{1}{2}(Q_{ik}Q_{jm} + Q_{im}Q_{jk})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_m = \hat{Q}_{\alpha\beta}\hat{\mathbf{e}}_\alpha \otimes \hat{\mathbf{e}}_\beta. \tag{3.9}$$

From this formula one can construct the following matrix expressing the relationship between the components of $\mathbf{Q}$ and $\hat{\mathbf{Q}}$,

$$\hat{\mathbf{Q}} = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 & 2^{\frac{1}{2}}Q_{12}Q_{13} & 2^{\frac{1}{2}}Q_{11}Q_{13} & 2^{\frac{1}{2}}Q_{11}Q_{12} \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 & 2^{\frac{1}{2}}Q_{22}Q_{23} & 2^{\frac{1}{2}}Q_{21}Q_{23} & 2^{\frac{1}{2}}Q_{22}Q_{21} \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 & 2^{\frac{1}{2}}Q_{33}Q_{32} & 2^{\frac{1}{2}}Q_{33}Q_{31} & 2^{\frac{1}{2}}Q_{31}Q_{32} \\ 2^{\frac{1}{2}}Q_{21}Q_{31} & 2^{\frac{1}{2}}Q_{22}Q_{32} & 2^{\frac{1}{2}}Q_{23}Q_{33} & Q_{22}Q_{33}+Q_{23}Q_{32} & Q_{21}Q_{33}+Q_{31}Q_{23} & Q_{21}Q_{32}+Q_{31}Q_{22} \\ 2^{\frac{1}{2}}Q_{11}Q_{31} & 2^{\frac{1}{2}}Q_{12}Q_{32} & 2^{\frac{1}{2}}Q_{13}Q_{33} & Q_{12}Q_{33}+Q_{32}Q_{13} & Q_{11}Q_{33}+Q_{13}Q_{31} & Q_{11}Q_{32}+Q_{31}Q_{12} \\ 2^{\frac{1}{2}}Q_{11}Q_{31} & 2^{\frac{1}{2}}Q_{12}Q_{22} & 2^{\frac{1}{2}}Q_{13}Q_{23} & Q_{12}Q_{23}+Q_{22}Q_{13} & Q_{11}Q_{23}+Q_{21}Q_{13} & Q_{11}Q_{22}+Q_{21}Q_{12} \end{bmatrix}. \tag{3.10}$$

This formula is very similar to one proposed by Bond (**8**); the differences between (3.10) and Bond's formula are only factors of the square root of 2. These slight differences are significant, however, because the definition of Bond does not produce a tensor. It is a straightforward but very lengthy calculation to show that the orthogonality of $\mathbf{Q}$ ensures the orthogonality of $\hat{\mathbf{Q}}$,

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1} \quad \text{implies that} \quad \hat{\mathbf{Q}}\hat{\mathbf{Q}}^T = \hat{\mathbf{Q}}^T\hat{\mathbf{Q}} = \mathbf{1}. \tag{3.11}$$

This calculation reduces to showing that each of the scalar equations resulting from $\hat{\mathbf{Q}}\hat{\mathbf{Q}}^T$ or $\hat{\mathbf{Q}}^T\hat{\mathbf{Q}}$ reduces to a square or product of scalar equations resulting from $\mathbf{Q}\mathbf{Q}^T$ or $\mathbf{Q}^T\mathbf{Q}$. In the important special case when

$$\mathbf{Q} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.12}$$

we have

$$\hat{\mathbf{Q}} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 0 & 0 & 0 & +2^{\frac{1}{2}}\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & 0 & 0 & 0 & -2^{\frac{1}{2}}\sin\alpha\cos\alpha \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha & 0 \\ -2^{\frac{1}{2}}\sin\alpha\cos\alpha & +2^{\frac{1}{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 & \cos^2\alpha - \sin^2\alpha \end{bmatrix}. \tag{3.13}$$

It should be noted that, while it is always easy to find a $\hat{\mathbf{Q}}$ given a $\mathbf{Q}$ by use of the formula (3.10), the construction of $\mathbf{Q}$ given $\hat{\mathbf{Q}}$ is more difficult. There are nine components of $\mathbf{Q}$ that satisfy six conditions given by $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1}$. There are therefore only three independent components of $\mathbf{Q}$. However, there are 36 components of $\hat{\mathbf{Q}}$ that satisfy the 21 conditions given by $\hat{\mathbf{Q}}\hat{\mathbf{Q}}^T = \hat{\mathbf{Q}}^T\hat{\mathbf{Q}} = \hat{\mathbf{1}}$ and hence 15 independent components of $\hat{\mathbf{Q}}$ in 6-space. Thus, while (3.10) uniquely determines a $\hat{\mathbf{Q}}$ given a $\mathbf{Q}$, the components of $\hat{\mathbf{Q}}$ must be restricted in order to determine a $\mathbf{Q}$ given a $\hat{\mathbf{Q}}$. These restrictions on $\hat{\mathbf{Q}}$ and a method of construction of $\hat{\mathbf{Q}}$ are described in the Appendix.

The equivalence relations between the bases in three and six dimensions allow us to express the familiar tensor transformation laws in three dimensions,

$$T'_{ij} = Q_{ik}Q_{jm}T_{km} \tag{3.14}$$

and

$$C'_{ijkm} = Q_{ip}Q_{jq}Q_{kr}Q_{ms}C_{pqrs},$$

as

$$\hat{T}'_{\alpha} = \hat{Q}_{\alpha\beta}\hat{T}_{\beta} \tag{3.15}$$

and

$$\hat{c}'_{\alpha\beta} = \hat{Q}_{\alpha\gamma}\hat{Q}_{\beta\nu}\hat{c}_{\gamma\nu}, \tag{3.16}$$

respectively. The following relationship between the components of $\mathbf{T}$ and $\hat{\mathbf{T}}$ is easy to verify:

$$\hat{\mathbf{T}} \cdot \hat{\mathbf{T}} = |\hat{\mathbf{T}}|^2 = \operatorname{tr} \mathbf{T}^2. \tag{3.17}$$

In the 6-dimensional Cartesian notation introduced above, the generalized Hooke's law (1.1) can be written

$$\hat{\mathbf{T}} = \hat{\mathbf{c}}\hat{\mathbf{E}}. \tag{3.18}$$

The strain energy Σ has a number of equivalent representations in the various notations:

$$2\Sigma = \hat{\mathbf{T}} \cdot \hat{\mathbf{E}} = \operatorname{tr} \mathbf{TE} = \hat{c}_{\alpha\beta}\hat{E}_{\alpha}\hat{E}_{\beta} = C_{ijkm}E_{ij}E_{km}. \tag{3.19}$$

4. Eigentensors and eigenelastic constants

We consider now the eigenvalue problem obtained by seeking those strain states $\hat{\mathbf{E}}$ for which $\hat{\mathbf{E}}$ and $\hat{\mathbf{T}}$ are parallel in the 6-dimensional Cartesian space. Specifically, we seek the values of the number Λ and the strain states $\hat{\mathbf{E}}$ that satisfy the equation

$$\hat{\mathbf{T}} = \Lambda\hat{\mathbf{E}} \quad \text{or} \quad (\hat{\mathbf{c}} - \Lambda\hat{\mathbf{1}})\hat{\mathbf{E}} = \mathbf{0}. \tag{4.1}$$

Since $\hat{\mathbf{c}}$ is a positive definite symmetric second-rank tensor in six dimensions there will be a maximum of six positive values of Λ, Λ_N $(N = 1,\ldots, 6)$ satisfying (4.1) and a maximum of six associated values of $\hat{\mathbf{E}}$, denoted by the vector $\hat{\mathbf{E}}^{(N)}$ in 6-space, and by the second-order tensor $\mathbf{E}^{(N)}$ in 3-space. The values of Λ_N are called the eigenelastic constants and the values of $\mathbf{E}^{(N)}$ are called the strain eigentensors of the linear elastic material characterized by $\mathbf{c}$. The stresses $\hat{\mathbf{T}}^{(N)}$ obtained by multiplying $\hat{\mathbf{E}}^{(N)}$ by the eigenvalue Λ_N are called the stress eigentensors. The equation (4.1) was given by Kelvin (**1**). Kelvin called the constraints Λ_N the six *principal elasticities* of the material, and he called the stress and strain eigentensors the *principal types* of stress and strain. Pipkin (**9**), who introduces such stress states in connection with constraints in linearly elastic materials, calls $\hat{\mathbf{T}}^{(N)}$ a 'principal state' and Λ_N a 'principal compliance'. The strain eigentensors associated with distinct values of Λ_N are orthogonal. For example, for the strain eigentensors

$$\hat{\mathbf{E}}^{(N)} \cdot \hat{\mathbf{E}}^{(M)} = 0 \quad \text{or} \quad \operatorname{tr}(\mathbf{E}^{(N)}\mathbf{E}^{(M)}) = 0 \quad \text{for } N \neq M, \tag{4.2}$$

and similar formulae hold for the stress eigentensors. Note that $\hat{\mathbf{E}}^{(N)}$ is not required to be a unit vector in the 6-dimensional space. If an elastic body is subjected to a compatible deformation that has everywhere the same strain eigentensor, then it is easy to show that the displacement field is governed by the wave equation with a squared wave speed given by the associated eigenelastic constant divided by the material density. If the deformation is quasistatic, the displacement field is harmonic in the absence of body forces.

A Cartesian basis in the 6-dimensional space can be constructed from the normalized eigentensors. The normalization of $\mathbf{E}$ is denoted by $\hat{\mathbf{N}}$,

$$\hat{\mathbf{E}} = \hat{\mathbf{N}}\,|\hat{\mathbf{E}}|, \qquad |\hat{\mathbf{E}}|^2 = \hat{\mathbf{E}} \cdot \hat{\mathbf{E}}. \tag{4.3}$$

The stress eigentensor is written in terms of the normalized eigentensor, using (3.18), (4.1) and (4.3), as

$$\hat{\mathbf{T}} = \Lambda\,|\hat{\mathbf{E}}|\,\hat{\mathbf{N}}. \tag{4.4}$$

The set of six $\hat{\mathbf{N}}$ form a Cartesian basis in the 6-dimensional space. We have

$$\sum_{K=1}^{6} \hat{\mathbf{N}}^{(K)} \otimes \hat{\mathbf{N}}^{(K)} = \hat{\mathbf{1}}. \tag{4.5}$$

With respect to such a space, $\hat{\mathbf{T}}$, $\hat{\mathbf{E}}$, and $\hat{\mathbf{c}}$ have the representation

$$\hat{\mathbf{T}} = \sum_{K=1}^{6} \Lambda_K \hat{\mathbf{E}}^{(K)} = \sum_{K=1}^{6} \Lambda_K\,|\hat{\mathbf{E}}^{(K)}|\,\hat{\mathbf{N}}^{(K)}, \tag{4.6}$$

$$\hat{\mathbf{E}} = \sum_{K=1}^{6} \hat{\mathbf{E}}^{(K)} = \sum_{K=1}^{6} |\hat{\mathbf{E}}^{(K)}|\,\hat{\mathbf{N}}^{(K)}, \tag{4.7}$$

and

$$\hat{\mathbf{c}} = \sum_{K=1}^{6} \Lambda_K \hat{\mathbf{N}}^{(K)} \otimes \hat{\mathbf{N}}^{(K)}. \tag{4.8}$$

When all the six values of Λ_K ($K = 1, \dots, 6$) are distinct, which is the case for orthotropy and lesser degrees of symmetry, we have (Walpole (**6**, equation (13) with $n = 6$)),

$$\hat{\mathbf{N}}^{(K)} \otimes \hat{\mathbf{N}}^{(K)} = \prod_{\substack{N=1 \\ (N \neq K)}}^{6} \frac{\hat{\mathbf{c}} - \Lambda_N \hat{\mathbf{1}}}{\Lambda_K - \Lambda_N}. \tag{4.9}$$

In the next section the eigentensors for particular elastic symmetries are derived.

5. Eigentensors for particular elastic symmetries

In this section the eigentensors for particular linear anisotropic symmetries are determined. The approach is to solve the eigenvalue problem (4.1) for each tensor $\mathbf{c}$ associated with a distinct linear elastic anisotropy. The eigenvectors $\hat{\mathbf{E}}^{(N)}$ determined are then projected back into the 3-dimensional Cartesian space and their components are denoted by $E_{ij}^{(N)}$. Table 3 lists the eigenvalues for the isotropic, cubic, transversely isotropic, hexagonal and tetragonal symmetries. The associated eigentensors for these elastic symmetries, as well as for orthotropic symmetry, are given in the text.

We consider cubic symmetry first and obtain the results for isotropic symmetry as a special case of cubic symmetry. This is done in section 5.1. Then we consider hexagonal symmetry in section 5.2, where we specialize the results for hexagonal (7) to obtain those of hexagonal (6) and transversely isotropic symmetries. Next, we discuss the tetragonal (7) and the tetragonal (6) symmetries in section 5.3. In section 5.4, we derive the eigentensors of the orthotropic symmetry. Finally in section 5.5 we describe, but do not calculate, the eigentensors for the monoclinic and triclinic symmetries.

5.1. *Cubic and isotropic symmetry*

For cubic symmetry there are three distinct elastic constants, c_{11}, c_{12} and c_{44} in the Voigt notation and $\hat{c}_{11}$, $\hat{c}_{12}$, and $\hat{c}_{44}$ in the 6-space tensor notation. The relationship between these constants is

$$\hat{c}_{11} = c_{11}, \qquad \hat{c}_{12} = c_{12}, \quad \hat{c}_{44} = 2c_{44}. \tag{5.1}$$

In the case of cubic symmetry (4.1) takes the form

$$\begin{bmatrix} c_{11}-\Lambda & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11}-\Lambda & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11}-\Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44}-\Lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{44}-\Lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_{44}-\Lambda \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \\ \hat{E}_5 \\ \hat{E}_6 \end{bmatrix} = \mathbf{0}. \tag{5.2}$$

The eigenvalues for this matrix and their multiplicity are given in Table 3. The first eigenvalue, $c_{11} + 2c_{12}$, is associated with the only dilatational strain eigentensor. The strain eigentensor associated with $c_{11} + 2c_{12}$ is a dilatation characterized in 6-space by $\hat{E}_1 = \hat{E}_2 = \hat{E}_3$ and $\hat{E}_4 = \hat{E}_5 = \hat{E}_6 = 0$, and by

$$\mathbf{E}^{(1)} = \tfrac{1}{3}(\operatorname{tr}\mathbf{E})\mathbf{1} \tag{5.3}$$

TABLE 3. *Properties of the eigenvalues for isotropic, cubic, transversely isotropic, hexagonal and tetragonal symmetries.*

Symmetry	Eigenvalue	Multiplicity of eigenvalue	Mode	Remarks
Isotropy	$c_{11}+2c_{12}$	1	D†	Section 5.1
	$2c_{44}$	5	I	
Cubic	$c_{11}+2c_{12}$	1	D	Section 5.1
	$c_{11}-c_{12}$	2	I	
	$2c_{44}$	3	I	
Transverse isotropy	$c_{33}+2^{\frac{1}{2}}c_{13}(\tan\alpha+\sec\alpha)$	1	D	Section 5.2; α is defined by (5.11)
	$c_{33}-2^{\frac{1}{2}}c_{13}(\tan\alpha+\sec\alpha)$	1	D	
	$c_{11}-c_{12}$	2	I	
	$2c_{44}$	2	I	
Hexagaonal (7) and hexagonal (6)	$c_{33}+2^{\frac{1}{2}}c_{13}(\tan\alpha+\sec\alpha)$	1	D	Section 5.2; α is defined by (5.11). For hexagonal (7), β is defined by (5.15) and for hexagonal (6) by (5.22)
	$c_{33}-2^{\frac{1}{2}}c_{13}(\tan\alpha+\sec\alpha)$	1	D	
	$\frac{1}{2}(c_{11}-c_{12})(1+\sec\beta)+c_{44}(1-\sec\beta)$	2	I	
	$\frac{1}{2}(c_{11}-c_{12})(1-\sec\beta)+c_{44}(1+\sec\beta)$	2	I	
Tetragonal (7)	$c_{33}+2^{\frac{1}{2}}c_{13}(\tan\alpha+\sec\alpha)$	1	D	Section 5.3; α is defined by (5.11); γ is defined by (5.28)
	$c_{33}-2^{\frac{1}{2}}c_{13}(\tan\alpha+\sec\alpha)$	1	D	
	$\frac{1}{2}(c_{11}-c_{12})(1+\sec\gamma)+c_{66}(1-\sec\gamma)$	1	I	
	$\frac{1}{2}(c_{11}-c_{12})(1-\sec\gamma)+c_{66}(1+\sec\gamma)$	1	I	
	$2c_{44}$	2	I	
Tetragonal (6)	$c_{33}+2^{\frac{1}{2}}c_{13}(\tan\alpha+\sec\alpha)$	1	D	Section 5.3; α is defined by (5.11)
	$c_{33}-2^{\frac{1}{2}}c_{13}(\tan\alpha+\sec\alpha)$	1	D	
	$c_{11}-c_{12}$	1	I	
	$2c_{66}$	1	I	
	$2c_{44}$	2	I	

† D = dilatational, I = isochoric.

in 3-space. The second eigenvalue is $c_{11}-c_{12}$; it is of multiplicity 2 and it is associated with an isochoric extensional deformation. The strain eigentensor associated with $c_{11}-c_{12}$ is one of distortion; it has two degrees of freedom and is given by $\hat{E}_1+\hat{E}_2+\hat{E}_3=0$, $\hat{E}_4=\hat{E}_5=\hat{E}_6=0$ in 6-space and

$$\mathbf{E}^{(2,3)}=\begin{bmatrix} E_{11}-\frac{1}{3}\operatorname{tr}\mathbf{E} & 0 & 0 \\ 0 & E_{22}-\frac{1}{3}\operatorname{tr}\mathbf{E} & 0 \\ 0 & 0 & E_{33}-\frac{1}{3}\operatorname{tr}\mathbf{E} \end{bmatrix} \quad (5.4)$$

in 3-space. The strain eigentensor associated with the third eigenvalue $2c_{44}$, an eigenvalue with a multiplicity of 3, is given by

$$\mathbf{E}^{(4,5,6)}=\begin{bmatrix} 0 & E_{12} & E_{13} \\ E_{12} & 0 & E_{23} \\ E_{13} & E_{23} & 0 \end{bmatrix}. \quad (5.5)$$

This strain eigentensor is one of isochoric shearing. It is easy to verify that (4.7) is satisfied; that is,

$$\mathbf{E}=\sum_{K=1}^{6}\mathbf{E}^{(K)}=\mathbf{E}^{(1)}+\mathbf{E}^{(2,3)}+\mathbf{E}^{(4,5,6)}, \quad (5.6)$$

and that the strain eigentensors are orthogonal in 6-space:

$$\operatorname{tr}(\mathbf{E}^{(1)}\mathbf{E}^{(2,3)})=0, \qquad \operatorname{tr}(\mathbf{E}^{(1)}\mathbf{E}^{(4,5,6)})=0, \qquad \operatorname{tr}(\mathbf{E}^{(2,3)}\mathbf{E}^{(4,5,6)})=0. \quad (5.7)$$

Kelvin (**1**) summarized these results in the following sentence. 'There are only three distinct Principal Elasticities for such a body [that is, a material with cubic elastic symmetry], and these are (A) its cubic compressibility, (B) its rigidity against diagonal distortion in any of its principal planes (three equal elasticities), and (C) its rigidity against rectangular distortions of a cube of symmetry (two equal elasticities)'. These results for cubic symmetry appear to be closely related to results given by Walpole (**10**, p. 180). Walpole's results are expressed in terms of decomposition of C_{ijkm} rather than in terms of a decomposition of the stress and strain tensors.

The solution to the eigenvalue problem (4.1) for isotropic symmetry is the special case of the eigenvalue problem (5.2) for the cubic symmetry obtained by setting $2c_{44}=c_{11}-c_{12}$. From the information on cubic symmetry in Table 3, setting $2c_{44}=c_{11}-c_{12}$ equates the eigenvalues of multiplicity 2 and 3. Thus, for an isotropic material there is a first eigenvalue $c_{11}+2c_{12}$ ($3\lambda+2\mu$ in Lamé's notation) of multiplicity 1 and a second eigenvalue $2c_{44}$ (2μ in Lamé's notation) of multiplicity 5. The first eigenvalue is proportional to the bulk modulus of the material and the associated strain eigentensor is a pure dilatational deformation (5.3). The second eigenvalue is twice the shear modulus and the associated strain eigentensor is the deviatoric strain tensor, $\hat{\mathbf{E}}$:

$$\mathbf{E}^{(2,3,4,5,6)}=\bar{\mathbf{E}}=\mathbf{E}-\tfrac{1}{3}(\operatorname{tr}\mathbf{E})\mathbf{1}. \quad (5.8)$$

This eigentensor has five degrees of freedom.

5.2. *Hexagonal symmetry*

The eigenvalue problem (4.1) for hexagonal (7), hexagonal (6), and hexagonal (5) (transversely isotropic) symmetries are considered in this section. For hexagonal (7), the eigenvalue problem (4.1) has the following matrix:

$$\begin{bmatrix} c_{11}-\Lambda & c_{12} & c_{13} & 2^{\frac{1}{2}}c_{14} & 2^{\frac{1}{2}}c_{15} & 0 \\ c_{12} & c_{11}-\Lambda & c_{13} & -2^{\frac{1}{2}}c_{14} & -2^{\frac{1}{2}}c_{15} & 0 \\ c_{13} & c_{13} & c_{33}-\Lambda & 0 & 0 & 0 \\ 2^{\frac{1}{2}}c_{14} & -2^{\frac{1}{2}}c_{14} & 0 & 2c_{44}-\Lambda & 0 & -2c_{15} \\ 2^{\frac{1}{2}}c_{15} & -2^{\frac{1}{2}}c_{15} & 0 & 0 & 2c_{44}-\Lambda & 2c_{14} \\ 0 & 0 & 0 & -2c_{15} & 2c_{14} & c_{11}-c_{12}-\Lambda \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \\ \hat{E}_5 \\ \hat{E}_6 \end{bmatrix} = \mathbf{0}. \tag{5.9}$$

The eigenvalues for this matrix and their multiplicity are given in Table 3. There are four distinct eigenvalues, two associated with different dilatational modes and two of multiplicity 2 associated with different isochoric modes. The eigenvalues for the two dilatational modes are given by

$$c_{33} \pm 2^{\frac{1}{2}}c_{13}(\tan\alpha + \sec\alpha), \tag{5.10}$$

where

$$\tan\alpha = \frac{2^{\frac{1}{2}}}{4c_{13}}(c_{11}+c_{12}-c_{33}). \tag{5.11}$$

The associated strain eigentensors are

$$\mathbf{E}^{(1)} = \begin{bmatrix} E_{11}^{(1)} & 0 & 0 \\ 0 & E_{11}^{(1)} & 0 \\ 0 & 0 & E_{33}^{(1)} \end{bmatrix} \quad \text{and} \quad \mathbf{E}^{(2)} = \begin{bmatrix} E_{11}^{(2)} & 0 & 0 \\ 0 & E_{11}^{(2)} & 0 \\ 0 & 0 & E_{33}^{(2)} \end{bmatrix}, \tag{5.12}$$

where

$$\left.\begin{aligned} E_{11}^{(1)} &= \tfrac{1}{4}(1+\sin\alpha)(E_{11}+E_{22}) + \frac{2^{\frac{1}{2}}}{4}\cos\alpha E_{33}, \\ E_{33}^{(1)} &= \frac{2^{\frac{1}{2}}}{4}\cos\alpha(E_{11}+E_{22}) + \tfrac{1}{2}(1-\sin\alpha)E_{33}, \\ E_{11}^{(2)} &= \tfrac{1}{4}(1-\sin\alpha)(E_{11}+E_{22}) - \frac{2^{\frac{1}{2}}}{4}\cos\alpha E_{33}, \\ E_{33}^{(2)} &= -\frac{2^{\frac{1}{2}}}{4}\cos\alpha(E_{11}+E_{22}) + \tfrac{1}{2}(1+\sin\alpha)E_{33}. \end{aligned}\right\} \tag{5.13}$$

The eigenvalue

$$\tfrac{1}{2}(c_{11}-c_{12})(1+\sec\beta)+c_{44}(1-\sec\beta), \tag{5.14}$$

where

$$\tan\beta=\frac{4(c_{14}^2+c_{15}^2)^{\frac{1}{2}}}{c_{11}-c_{12}-2c_{44}}, \tag{5.15}$$

is of multiplicity 2 and is associated with an isochoric-shear–extensional-strain eigentensor

$$\mathbf{E}^{(3,4)}=\begin{bmatrix} E_{11}^{(3,4)} & E_{12}^{(3,4)} & E_{13}^{(3,4)} \\ E_{12}^{(3,4)} & -E_{11}^{(3,4)} & E_{23}^{(3,4)} \\ E_{13}^{(3,4)} & E_{23}^{(3,4)} & 0 \end{bmatrix}, \tag{5.16}$$

where

$$\left.\begin{aligned} E_{11}^{(3,4)} &= \tfrac{1}{4}(1+\cos\beta)(E_{11}-E_{22})+2\cos\beta(r_{14}E_{23}+r_{15}E_{13}), \\ E_{12}^{(3,4)} &= \tfrac{1}{2}(1+\cos\beta)E_{12}+2\cos\beta(r_{14}E_{13}-r_{15}E_{23}), \\ E_{13}^{(3,4)} &= \tfrac{1}{2}(1-\cos\beta)E_{13}+2r_{14}\cos\beta E_{12}+r_{15}\cos\beta(E_{11}-E_{22}), \\ E_{23}^{(3,4)} &= \tfrac{1}{2}(1-\cos\beta)E_{23}-2r_{15}\cos\beta E_{12}+r_{14}\cos\beta(E_{11}-E_{22}), \end{aligned}\right\} \tag{5.17}$$

and where

$$r_{\alpha\beta}=\frac{c_{\alpha\beta}}{c_{11}-c_{12}-2c_{44}} \qquad (\alpha=1,\ \beta=4,\ 5). \tag{5.18}$$

The fourth eigenvalue,

$$\tfrac{1}{2}(c_{11}-c_{12})(1-\sec\beta)+c_{44}(1+\sec\beta), \tag{5.19}$$

is also of multiplicity 2 and is associated with the isochoric-shear–extensional deformation characterized by the following strain eigentensor:

$$\mathbf{E}^{(5,6)}=\begin{bmatrix} E_{11}^{(5,6)} & E_{12}^{(5,6)} & E_{13}^{(5,6)} \\ E_{12}^{(5,6)} & -E_{11}^{(5,6)} & E_{23}^{(5,6)} \\ E_{13}^{(5,6)} & E_{23}^{(5,6)} & 0 \end{bmatrix}, \tag{5.20}$$

where

$$\left.\begin{aligned} E_{11}^{(5,6)} &= \tfrac{1}{4}(1-\cos\beta)(E_{11}-E_{22})-2\cos\beta(r_{14}E_{23}+r_{15}E_{13}), \\ E_{12}^{(5,6)} &= \tfrac{1}{2}(1-\cos\beta)E_{12}-2\cos\beta(r_{14}E_{13}-r_{15}E_{23}), \\ E_{13}^{(5,6)} &= \tfrac{1}{2}(1+\cos\beta)E_{13}-2r_{14}\cos\beta E_{12}-r_{15}\cos\beta(E_{11}-E_{22}), \\ E_{23}^{(5,6)} &= \tfrac{1}{2}(1+\cos\beta)E_{23}+2r_{15}\cos\beta E_{12}-r_{14}\cos\beta(E_{11}-E_{22}). \end{aligned}\right\} \tag{5.21}$$

The orthogonality of each of the strain eigentensors $\mathbf{E}^{(1)}$, $\mathbf{E}^{(2)}$, $\mathbf{E}^{(3,4)}$ and $\mathbf{E}^{(5,6)}$ to each of the other strain eigentensors in the set is easy to verify. It is also easy to verify that the sum of the four strain eigentensors is the strain tensor $\mathbf{E}$.

The eigentensors and the eigenvalues for the hexagonal (6) symmetry are obtained from those of hexagonal (7) by allowing the elastic constant c_{15} to vanish. The eigenvalues and their multiplicity are given in Table 3. Again, there are four distinct eigenvalues, two associated with different dilatational modes and two of multiplicity 2 associated with different isochoric modes. The eigenvalues and the strain eigentensors for the two dilatational modes are identical to those of hexagonal (7) and are, hence, given by (5.10) to (5.13). The third eigenvalue and its associated isochoric-shear–extensional-strain eigentensor $\mathbf{E}^{(3,4)}$ are given by (5.14) and (5.16), respectively. However, since $c_{15} = 0$, it follows from (5.15) and (5,18) that

$$\tan\beta = \frac{4c_{14}}{c_{11} - c_{12} - 2c_{44}} \equiv 4r_{14} \quad \text{and} \quad r_{15} = 0. \tag{5.22}$$

On making use of (5.22), equations (5.17) reduce to the following:

$$\left.\begin{aligned} E_{11}^{(3,4)} &= \tfrac{1}{4}(1+\cos\beta)(E_{11} - E_{22}) + \tfrac{1}{2}\sin\beta E_{23}, \\ E_{12}^{(3,4)} &= \tfrac{1}{2}(1+\cos\beta)E_{12} + \tfrac{1}{2}\sin\beta E_{13}, \\ E_{13}^{(3,4)} &= \tfrac{1}{2}(1-\cos\beta)E_{13} + \tfrac{1}{2}\sin\beta E_{12}, \\ E_{23}^{(3,4)} &= \tfrac{1}{2}(1-\cos\beta)E_{23} + \tfrac{1}{4}\sin\beta(E_{11} - E_{22}). \end{aligned}\right\} \tag{5.23}$$

Similarly, the fourth eigenvalue and its associated isochoric-shear–extensional-strain eigentensor $\mathbf{E}^{(5,6)}$ are given by (5.19) and (5.20), respectively, where now

$$\left.\begin{aligned} E_{11}^{(5,6)} &= \tfrac{1}{4}(1-\cos\beta)(E_{11} - E_{22}) - \tfrac{1}{2}\sin\beta E_{23}, \\ E_{12}^{(5,6)} &= \tfrac{1}{2}(1-\cos\beta)E_{12} - \tfrac{1}{2}\sin\beta E_{13}, \\ E_{13}^{(5,6)} &= \tfrac{1}{2}(1+\cos\beta)E_{13} - \tfrac{1}{2}\sin\beta E_{12}, \\ E_{23}^{(5,6)} &= \tfrac{1}{2}(1+\cos\beta)E_{23} - \tfrac{1}{4}\sin\beta(E_{11} - E_{22}). \end{aligned}\right\} \tag{5.24}$$

These relations are obtained from (5.21) by employing (5.22).

The eigentensors and the eigenvalues for transversely isotropic or hexagonal (5) symmetry are obtained from those of hexagonal (6) by allowing the elastic constant c_{14} to vanish. The eigenvalues and their multiplicity for this case are given in Table 3. Similarly to the hexagonal (7) and hexagonal (6) symmetry, there are four distinct eigenvalues. Two are associated with different dilatational modes and two of multiplicity 2 are associated with different isochoric modes. The two dilatational modes, being identical to those of hexagonal (7) and hexagonal (6) symmetry, are defined by (5.10) to (5.13). However, since the elastic constant c_{14} and hence the angle β defined by (5.22) vanishes, the third eigenvalue (5.14) reduces to $c_{11} - c_{12}$ while the fourth eigenvalue (5.19) reduces to $2c_{44}$. The corresponding eigentensors are found from (5.16), (5.17), (5.20) and (5.21) by setting

$\beta = 0$. We obtain

$$\mathbf{E}^{(3,4)} = \begin{bmatrix} \frac{1}{2}(E_{11}-E_{22}) & E_{12} & 0 \\ E_{12} & -\frac{1}{2}(E_{11}-E_{22}) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{E}^{(5,6)} = \begin{bmatrix} 0 & 0 & E_{13} \\ 0 & 0 & E_{23} \\ E_{13} & E_{23} & 0 \end{bmatrix}. \tag{5.25}$$

5.3. *Tetragonal symmetry*

The eigenvalue problem (4.1) for tetragonal (7) and tetragonal (6) symmetry are considered in this section. For tetragonal (7) symmetry there are seven distinct elastic constants c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, c_{66} and c_{16}. Tetragonal (6) symmetry is obtained from tetragonal (7) by allowing the elastic constant c_{16} to vanish. For tetragonal (7), (4.1) takes the form

$$\begin{bmatrix} c_{11}-\Lambda & c_{12} & c_{13} & 0 & 0 & 2^{\frac{1}{2}}c_{16} \\ c_{12} & c_{11}-\Lambda & c_{13} & 0 & 0 & -2^{\frac{1}{2}}c_{16} \\ c_{13} & c_{13} & c_{33}-\Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44}-\Lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{44}-\Lambda & 0 \\ 2^{\frac{1}{2}}c_{16} & -2^{\frac{1}{2}}c_{16} & 0 & 0 & 0 & 2c_{66}-\Lambda \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \\ \hat{E}_5 \\ \hat{E}_6 \end{bmatrix} = \mathbf{0}. \tag{5.26}$$

The eigenvalues and their multiplicities are given in Table 3. There are five distinct eigenvalues, two corresponding to different dilatational modes, two corresponding to different isochoric-shear–extensional modes, and one of multiplicity two corresponding to an isochoric-shear mode. The eigenvalues and the strain eigentensors for the two dilatational modes are identical to those of hexagonal symmetry and are given by (5.10) to (5.13). The third and the fourth eigenvalues

$$\tfrac{1}{2}(c_{11}-c_{12})(1 \pm \sec\gamma) + c_{66}(1 \mp \sec\gamma), \tag{5.27}$$

where

$$\tan\gamma = \frac{4c_{16}}{c_{11}-c_{12}-2c_{66}}, \tag{5.28}$$

are associated with the strain eigentensors of the form

$$\mathbf{E}^{(K)} = \begin{bmatrix} E_{11}^{(K)} & E_{12}^{(K)} & 0 \\ E_{12}^{(K)} & -E_{11}^{(K)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (K = 3, 4), \tag{5.29}$$

where

$$\left.\begin{aligned} E_{11}^{(3)} &= \tfrac{1}{4}(1+\cos\gamma)(E_{11}-E_{22}) + \tfrac{1}{2}\sin\gamma E_{12}, \\ E_{12}^{(3)} &= \tfrac{1}{4}\sin\gamma(E_{11}-E_{22}) + \tfrac{1}{2}(1-\cos\gamma)E_{12}, \\ E_{11}^{(4)} &= \tfrac{1}{4}(1-\cos\gamma)(E_{11}-E_{22}) - \tfrac{1}{2}\sin\gamma E_{12}, \\ E_{12}^{(4)} &= -\tfrac{1}{4}\sin\gamma(E_{11}-E_{22}) + \tfrac{1}{2}(1+\cos\gamma)E_{12}. \end{aligned}\right\} \tag{5.30}$$

The fifth eigenvalue, $2c_{44}$, is of multiplicity 2 and is associated with the isochoric shearing deformation characterized by the strain eigentensor $\mathbf{E}^{(5,6)}$ given by $(5.25)_2$.

The results for tetragonal (6) symmetry are obtained from those of tetragonal (7) by allowing the elastic constant c_{16} and hence the angle γ defined by (5.28) to vanish. Again, there are five distinct eigenvalues. As before, two are associated with different dilatational modes and three correspond to various isochoric-shear–extensional modes. The two dilatational modes, being independent of the elastic constant c_{16}, are identical to those of hexagonal (7), hexagonal (6), transversely isotropic, and tetragonal (7) symmetries. Hence, they are characterized by equations (5.10) to (5.13). The third eigenvalue, $c_{11}-c_{12}$, is associated with the strain eigentensor

$$\mathbf{E}^{(3)} = \begin{bmatrix} \frac{1}{2}(E_{11}-E_{22}) & 0 & 0 \\ 0 & -\frac{1}{2}(E_{11}-E_{22}) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.31}$$

The fourth eigenvalue, $2c_{66}$, is associated with the simple shearing deformation characterized by

$$\mathbf{E}^{(4)} = \begin{bmatrix} 0 & E_{12} & 0 \\ E_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.32}$$

As in the case of tetragonal (7) symmetry, the fifth eigenvalue for tetragonal (6) symmetry, $2c_{44}$, is of multiplicity 2 and is associated with the isochoric shearing deformation characterized by the strain eigentensor $\mathbf{E}^{(4,5)}$ given by $(5.25)_2$.

5.4. *Orthotropic symmetry*

For orthotropic symmetry there are nine distinct elastic constants c_{11}, c_{12}, c_{13}, c_{22}, c_{23}, c_{33}, c_{44}, c_{55} and c_{66} in the Voigt notation. In this case (4.1) takes the form

$$\begin{bmatrix} c_{11}-\Lambda & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22}-\Lambda & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33}-\Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44}-\Lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{55}-\Lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_{66}-\Lambda \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \\ \hat{E}_5 \\ \hat{E}_6 \end{bmatrix} = \mathbf{0}. \tag{5.33}$$

The eigenvalues in this case are all distinct. The first three, which are denoted by Λ_1, Λ_2 and Λ_3, satisfy the relation

$$\begin{bmatrix} c_{11}-\Lambda & c_{12} & c_{13} \\ c_{12} & c_{22}-\Lambda & c_{23} \\ c_{13} & c_{23} & c_{33}-\Lambda \end{bmatrix}\begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{bmatrix} = \mathbf{0}, \tag{5.34}$$

and the last three are

$$\Lambda_4 = 2c_{44}, \qquad \Lambda_5 = 2c_{55}, \qquad \Lambda_6 = 2c_{66}. \tag{5.35}$$

To find the eigentensors associated with Λ_1, Λ_2, and Λ_3, we first note that

$$\hat{E}_4^{(K)} = \hat{E}_5^{(K)} = \hat{E}_6^{(K)} = 0 \quad \text{for } K = 1, 2, 3. \tag{5.36}$$

Next, form a system of nine independent algebraic equations in $\hat{E}_1^{(K)}$, $\hat{E}_2^{(K)}$ and $\hat{E}_3^{(K)}$ $(K = 1, 2, 3)$ using (5.34) and $(4.7)_1$. Solving this system of equations we obtain, after a lengthy calculation,

$$\mathbf{E}^{(K)} = \bar{K}(\Lambda_K, \Lambda_I, \Lambda_J) \times$$
$$\begin{bmatrix} c_{12}c_{23} - c_{13}(c_{22}-\Lambda_K) & 0 & 0 \\ 0 & c_{12}c_{13} - c_{23}(c_{11}-\Lambda_K) & 0 \\ 0 & 0 & (c_{11}-\Lambda_K)(c_{22}-\Lambda_K) - c_{12}^2 \end{bmatrix}, \tag{5.37}$$

where $I, J, K = 1, 2, 3$ $(I \neq J \neq K)$, and where

$$\begin{aligned} \bar{K}(\Lambda_K, \Lambda_I, \Lambda_J) = {} & \frac{1}{(\Lambda_K - \Lambda_I)(\Lambda_K - \Lambda_J)[c_{12}(c_{13}^2 - c_{23}^2) - c_{13}c_{23}(c_{11} - c_{22})]} \\ & \times \{[c_{12}c_{13} - c_{23}(c_{11} - \Lambda_I)][(c_{11} - \Lambda_J)E_{11} + c_{12}E_{22} + c_{13}E_{33}] \\ & - [c_{12}c_{23} - c_{13}(c_{22} - \Lambda_I)][c_{12}E_{11} + (c_{22} - \Lambda_J)E_{22} + c_{23}E_{33}]\}. \end{aligned} \tag{5.38}$$

Note that

$$\bar{K}(\Lambda_K, \Lambda_I, \Lambda_J) = \bar{K}(\Lambda_K, \Lambda_J, \Lambda_I). \tag{5.39}$$

The eigentensors associated with Λ_4, Λ_5 and Λ_6 are easily found from (5.33) and (5.35). They are

$$\mathbf{E}^{(4)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & E_{23} \\ 0 & E_{23} & 0 \end{bmatrix}, \quad \mathbf{E}^{(5)} = \begin{bmatrix} 0 & 0 & E_{13} \\ 0 & 0 & 0 \\ E_{13} & 0 & 0 \end{bmatrix}, \quad \mathbf{E}^{(6)} = \begin{bmatrix} 0 & E_{12} & 0 \\ E_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.40}$$

As a numerical example consider the following.

The orthotropic elastic constants for human femoral cortical bone at a certain anatomical site are given by

$$\hat{\mathbf{c}} = \begin{bmatrix} 18 & 9\cdot98 & 10\cdot1 & 0 & 0 & 0 \\ 9\cdot98 & 20\cdot2 & 10\cdot7 & 0 & 0 & 0 \\ 10\cdot1 & 10\cdot7 & 27\cdot6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12\cdot46 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11\cdot22 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8\cdot04 \end{bmatrix}, \tag{5.41}$$

where the units are GPa. The eigenvalues for this matrix are

$$\left.\begin{matrix} \Lambda_1 = 43\cdot1, & \Lambda_2 = 13\cdot66, & \Lambda_3 = 9\cdot04, \\ \Lambda_4 = 12\cdot46, & \Lambda_5 = 11\cdot22, & \Lambda_6 = 8\cdot04, \end{matrix}\right\} \tag{5.42}$$

where the units are again GPa and the associated strain eigentensors, from (5.37), are given by

$$\mathbf{E}^{(1)} = \begin{bmatrix} 0\cdot24E_{11} + 0\cdot26E_{22} + 0\cdot34E_{33} & 0 & 0 \\ 0 & 0\cdot26E_{11} + 0\cdot28E_{22} + 0\cdot37E_{33} & 0 \\ 0 & 0 & 0\cdot34E_{11} + 0\cdot37E_{22} + 0\cdot48E_{33} \end{bmatrix}, \tag{5.43}$$

$$\mathbf{E}^{(2)} = \begin{bmatrix} 0\cdot17E_{11} + 0\cdot23E_{22} - 0\cdot30E_{33} & 0 & 0 \\ 0 & 0\cdot23E_{11} + 0\cdot31E_{22} - 0\cdot40E_{33} & 0 \\ 0 & 0 & -0\cdot30E_{11} - 0\cdot40E_{22} + 0\cdot52E_{33} \end{bmatrix}, \tag{5.44}$$

$$\mathbf{E}^{(3)} = \begin{bmatrix} 0\cdot59E_{11} - 0\cdot49E_{22} - 0\cdot04E_{33} & 0 & 0 \\ 0 & -0\cdot49E_{11} + 0\cdot41E_{22} + 0\cdot03E_{33} & 0 \\ 0 & 0 & -0\cdot04E_{11} + 0\cdot03E_{22} + 0\cdot0E_{33} \end{bmatrix}, \tag{5.45}$$

and $\mathbf{E}^{(4)}$, $\mathbf{E}^{(5)}$ and $\mathbf{E}^{(6)}$ are as in (5.40). It is easily verified that these six strain eigentensors form an orthogonal set. It is also easy to verify that the sum of the six strain eigentensors is the strain tensor $\mathbf{E}$.

Each of the six strain eigentensors is associated with a distinct type of deformation. The first three eigentensors represent combined simple extension or contraction along the various symmetry axes while the last three eigentensors represent simple shearing in the symmetry planes.

5.5. *The monoclinic and triclinic symmetries*

The method of calculation of the eigentensors described in the previous four subsections can be formally extended to the monoclinic and triclinic symmetries. Although there are no algebraic difficulties in performing these extensions, the results do not illuminate the physics or the geometry of the situation and are not presented here. In the case of monoclinic symmetry there are six distinct eigentensors. Four of the eigentensors are different linear combinations of the three normal components and one shearing component. The other two distinct eigentensors are different linear combinations of the two remaining shearing components. In the case of triclinic symmetry there are also six distinct eigentensors. These six eigentensors are six different linear combinations of the six distinct components of the tensor.

6. Invariants of eigentensors

Irreducible sets of the invariants of any second-rank tensor for each of the anisotropic elastic symmetries are tabulated, for example, by Green and Adkins (**12**). An irreducible set of the isotropic invariants of $\mathbf{E}$ are $\operatorname{tr}\mathbf{E}$, $\operatorname{tr}\mathbf{E}^2$ and $\operatorname{tr}\mathbf{E}^3$. Of course, any function of these quantities is also an isotropic invariant of $\mathbf{E}$.

We seek here those functions of the invariants of a second-rank tensor for which the corresponding stress and strain invariants of the identical form are proportional. This objective was illustrated in the Introduction in the material preceding the statement of property D. For isotropic materials these special invariants are specified in (1.9). Noting that $\operatorname{tr}\tilde{\mathbf{E}}^2$ can be written in terms of $\operatorname{tr}\mathbf{E}$ and $\operatorname{tr}\mathbf{E}^2$,

$$\operatorname{tr}\tilde{\mathbf{E}}^2 = \operatorname{tr}\mathbf{E}^2 - \tfrac{1}{3}(\operatorname{tr}\mathbf{E})^2, \tag{6.1}$$

and that a similar relation holds for stress, it follows that the two relations (1.9) are expressible in terms of $\operatorname{tr}\mathbf{E}$ and $\operatorname{tr}\mathbf{E}^2$. Since we are dealing with linear elasticity, terms higher than quadratic are neglected; hence we do not construct an eigentensor invariant involving $\operatorname{tr}\mathbf{E}^3$.

We consider next cubic symmetry. The only cubic-symmetry invariants of $\mathbf{E}$ that are linear or quadratic in $\mathbf{E}$ are, from Green and Adkins (**11**), $\operatorname{tr}\mathbf{E}$, $E_{11}E_{22}+E_{22}E_{33}+E_{33}E_{11}$ and $E_{12}^2+E_{23}^2+E_{31}^2$. The associated invariants of the strain eigentensors for cubic symmetry are

$$\left.\begin{aligned}&\operatorname{tr}\mathbf{E}^{(1)} = \operatorname{tr}\mathbf{E}, \quad \operatorname{tr}[\mathbf{E}^{(2,3)}]^2 = \tfrac{2}{3}(\operatorname{tr}\mathbf{E})^2 - 2(E_{11}E_{22}+E_{22}E_{33}+E_{33}E_{11}),\\ &\operatorname{tr}[\mathbf{E}^{(4,5,6)}]^2 = E_{12}^2+E_{23}^2+E_{31}^2.\end{aligned}\right\} \tag{6.2}$$

For hexagonal (7) symmetry, the invariants of $\mathbf{E}$ that are linear or quadratic in $\mathbf{E}$ are

$$\left.\begin{aligned}&E_{33}, \quad E_{11}+E_{22}, \quad E_{11}E_{22}-E_{12}^2, \quad E_{13}^2+E_{23}^2,\\ &(E_{22}-E_{11})E_{23}-2E_{12}E_{13}, \quad (E_{11}-E_{22})E_{13}-2E_{12}E_{13}.\end{aligned}\right\} \tag{6.3}$$

The associated invariants of the strain eigentensors for hexagonal (7) symmetry are

$$\left.\begin{aligned}
\operatorname{tr}\mathbf{E}^{(1)} &= \tfrac{1}{2}(1-\sin\alpha+2^{\frac{1}{2}}\cos\alpha)E_{33}+\tfrac{1}{2}(1+\sin\alpha+\tfrac{1}{2}2^{\frac{1}{2}}\cos\alpha)(E_{11}+E_{22}),\\
\operatorname{tr}\mathbf{E}^{(2)} &= \tfrac{1}{2}(1+\sin\alpha-2^{\frac{1}{2}}\cos\alpha)E_{33}+\tfrac{1}{2}(1-\sin\alpha-\tfrac{1}{2}2^{\frac{1}{2}}\cos\alpha)(E_{11}+E_{22}),\\
\operatorname{tr}[\mathbf{E}^{(3,4)}]^2 &= \tfrac{1}{4}(1+\cos\beta)(E_{11}+E_{22})^2-(1+\cos\beta)(E_{11}E_{22}-E_{12}^2)\\
&\quad+(1+\cos\beta)(E_{13}^2+E_{23}^2)-4r_{14}\cos\beta[E_{23}(E_{22}-E_{11})-2E_{12}E_{13}]\\
&\quad+4r_{15}\cos\beta[E_{13}(E_{11}-E_{22})-2E_{12}E_{23}],\\
\operatorname{tr}[\mathbf{E}^{(5,6)}]^2 &= \tfrac{1}{4}(1-\cos\beta)(E_{11}+E_{22})^2-(1-\cos\beta)(E_{11}E_{22}-E_{12}^2)\\
&\quad+(1+\cos\beta)(E_{13}^2+E_{23}^2)+4r_{14}\cos\beta[E_{23}(E_{22}-E_{11})-2E_{12}E_{13}]\\
&\quad-4r_{15}\cos\beta[E_{13}(E_{11}-E_{22})-2E_{12}E_{23}].
\end{aligned}\right\} \tag{6.4}$$

For hexagonal (6) symmetry, the only invariants of $\mathbf{E}$ that are linear or quadratic in $\mathbf{E}$ are E_{33}, $E_{11}+E_{22}$, $E_{11}E_{22}-E_{12}^2$, $E_{13}^2+E_{23}^2$ and $E_{23}(E_{22}-E_{11})-2E_{12}E_{13}$. By making use of (5.22), the invariants $\operatorname{tr}\mathbf{E}^{(1)}$ and $\operatorname{tr}\mathbf{E}^{(2)}$ remain the same but $(6.4)_{3,4}$ reduce to the following:

$$\left.\begin{aligned}
\operatorname{tr}[\mathbf{E}^{(3,4)}]^2 &= \tfrac{1}{4}(1+\cos\beta)(E_{11}+E_{22})^2-(1+\cos\beta)(E_{11}E_{22}-E_{12}^2)\\
&\quad+(1-\cos\beta)(E_{13}^2+E_{23}^2)-\sin\beta[E_{23}(E_{22}-E_{11})-2E_{12}E_{13}],\\
\operatorname{tr}[\mathbf{E}^{(5,6)}]^2 &= \tfrac{1}{4}(1-\cos\beta)(E_{11}+E_{22})^2-(1-\cos\beta)(E_{11}E_{22}-E_{12}^2)\\
&\quad+(1+\cos\beta)(E_{13}^2+E_{23}^2)+\sin\beta[E_{23}(E_{22}-E_{11})-2E_{12}E_{13}].
\end{aligned}\right\} \tag{6.5}$$

The only transversely isotropic invariants of $\mathbf{E}$ that are linear or quadratic in $\mathbf{E}$ are E_{33}, $E_{11}+E_{22}$, $E_{11}E_{22}-E_{12}^2$ and $E_{13}^2+E_{23}^2$. The associated invariants of the strain eigentensors for transversely isotropic symmetry are $(6.4)_1$, $(6.4)_2$ and the following two:

$$\operatorname{tr}[\mathbf{E}^{(3,4)}]^2=\tfrac{1}{2}(E_{11}+E_{22})^2-2(E_{11}E_{22}-E_{12}^2),\qquad \operatorname{tr}[\mathbf{E}^{(5,6)}]^2=2(E_{13}^2+E_{23}^2), \tag{6.6}$$

which can be found from (6.5) by setting $\beta=0$.

For tetragonal (7) symmetry, the invariants of $\mathbf{E}$ that are linear or quadratic in $\mathbf{E}$ are E_{33}, $E_{11}+E_{22}$, $E_{13}^2+E_{23}^2$, E_{12}^2, $E_{11}E_{22}$ and $E_{12}(E_{11}-E_{22})$. The associated invariants of the strain eigentensors are $(6.4)_1$, $(6.4)_2$ and the following three which are obtained from (5.29), (5.30) and $(5.25)_2$:

$$\left.\begin{aligned}
\operatorname{tr}[\mathbf{E}^{(3)}]^2 &= \tfrac{1}{4}(1+\cos\gamma)(E_{11}+E_{22})^2-(1+\cos\gamma)E_{11}E_{22}\\
&\quad+(1-\cos\gamma)E_{12}^2+\sin\gamma E_{12}(E_{11}-E_{22}),\\
\operatorname{tr}[\mathbf{E}^{(4)}]^2 &= \tfrac{1}{4}(1-\cos\gamma)(E_{11}+E_{22})^2+(1-\cos\gamma)E_{11}E_{22}\\
&\quad+(1+\cos\gamma)E_{12}^2-\sin\gamma E_{12}(E_{11}-E_{22}),\\
\operatorname{tr}[\mathbf{E}^{(5,6)}]^2 &= 2(E_{13}^2+E_{23}^2).
\end{aligned}\right\} \tag{6.7}$$

For tetragonal (6) symmetry, the invariants of $\mathbf{E}$ that are linear or quadratic in $\mathbf{E}$ are E_{33}, $E_{11}+E_{22}$, $E_{12}^2+E_{23}^2$, E_{12}^2 and $E_{11}E_{22}$. The associated invariants of eigentensors in this case are $(6.4)_{1,2}$, (6.7) and the following two which are obtained from $(6.7)_1$, and $(6.7)_2$ by setting $\gamma=0$:

$$\operatorname{tr}[\mathbf{E}^{(3)}]^2=\tfrac{1}{2}(E_{11}+E_{22})^2-2E_{11}E_{22}, \qquad \operatorname{tr}[\mathbf{E}^{(4)}]^2=2E_{12}^2. \tag{6.8}$$

The only orthotropic invariants of $\mathbf{E}$ that are linear or quadratic in $\mathbf{E}$ are E_{11}, E_{22}, E_{33}, E_{13}^2, E_{23}^2 and E_{12}^2. The associated invariants of the strain eigentensors for orthotropic symmetry are, from (5.37) to (5.40),

$$\begin{aligned}\operatorname{tr}\mathbf{E}^{(K)}=\bar{K}(\Lambda_K,\Lambda_I,\Lambda_J)[&c_{12}c_{23}+c_{12}c_{13}-c_{12}^2-c_{23}(c_{11}-\Lambda_K)\\ &-c_{13}(c_{22}-\Lambda_K)+(c_{11}-\Lambda_K)(c_{22}-\Lambda_K)],\end{aligned} \tag{6.9}$$

with $K\neq I\neq J$ and $K, I, J=1, 2, 3$,

$$\operatorname{tr}(\mathbf{E}^{(4)})^2=E_{23}^2, \qquad \operatorname{tr}(\mathbf{E}^{(5)})^2=E_{13}^2, \qquad \operatorname{tr}(\mathbf{E}^{(6)})^2=E_{12}^2.$$

Although we have presented invariants of the eigentensors for only eight of the ten distinct linear anisotropic elastic symmetries, we have established the general pattern for constructing these invariants and demonstrated the features that they possess. It would not be difficult to accomplish the formal construction of the eigentensors for the monoclinic and triclinic symmetries as pointed out in section 5.5, but we have not done so for the reasons indicated there.

Finally, note that the number of eigentensor invariants for a particular elastic symmetry is equal to the number of linear and quadratic strain invariants associated with that particular elastic symmetry. The maximum number of eigentensor invariants is six.

7. Discussion

The eigentensors of particular elastic symmetries are significant because they identify preferred modes of deformation associated with particular anisotropic elastic symmetries. We speculate that the eigentensors will be useful in the design of composite materials and in the deeper appreciation of nature achieved by understanding the functioning of natural composites such as bone and wood.

The invariants of the eigentensors should be useful in formulating phenomenological theories of fracture for brittle anisotropic materials. There are two developments of the von Mises (**12**) criterion for isotropic materials presented in contemporary elementary texts. One suggests that the von Mises criterion is a consequence of the fact that there is a maximum distortional strain energy that a material can sustain and the other suggests that the criterion is independent of the constitutive equation of the material and depends only on a phenomenologically determined function of the stress invariants. If the interpretation in terms of strain energy is accepted, the construction of fracture criteria for anisotropic elastic materials is a straightforward development of the results presented in this paper.

While the tensorial representation of $\hat{\mathbf{c}}$ has been effective in this contribution, it is unlikely to be helpful in the solution of particular problems. This is because, while the stress–strain relations are simple in terms of $\hat{\mathbf{T}}$ and $\hat{\mathbf{E}}$, the equations of motion and the strain–displacement relations involve awkward factors of the square root of 2. The value we see in the representation $\hat{\mathbf{c}}$ stems from the eigenvalue problem considered here and the ease of doing coordinate transformations of $\hat{\mathbf{c}}$ compared with the coordinate transformation of $\mathbf{c}$.

Acknowledgements

The work of the first author was supported by the National Science Foundation (Solid Mechanics Division), grant MEA 8318967 to Tulane University. The work of the second author was supported by USPHS, research grant DE 06859 from the National Institute of Dental Research, National Institutes of Health, Bethesda, Maryland 20205.

REFERENCES

1. Lord Kelvin (W. Thomson), *Phil. Trans. R. Soc.* **166** (1856) 481.
2. ——, In *Encyclopaedia Britannica* **7** (1878) 796–825 (9th edition, London and Edinburgh).
3. I. Todhunter and K. Pearson, *A History of the Theory of Elasticity*, Vol. II, part II (1893) (reprinted by Dover, New York.)
4. W. Voigt, *Lehrbuch der Kristallphysik* (Leipzig 1910) 560.
5. R. F. S. Hearmon, *Applied Anisotropic Elasticity* (University Press, Oxford 1961).
6. L. J. Walpole, *Proc. R. Soc.* A**391** (1984) 149.
7. J. F. Nye, *Physical Properties of Crystals* (Clarendon Press, Oxford 1957).
8. W. Bond, *Bell Sys. tech. J.* **42** (1943) 1.
9. A. C. Pipkin, *J. Elast.* **6** (1976) 179.
10. L. J. Walpole, *Adv. appl. Mech.* **21** (1981) 169.
11. A. E. Green and J. E. Adkins, *Large Elastic Deformations and Nonlinear Continuum Mechanics* (Clarendon Press, Oxford 1960).
12. R. von Mises, *Z. angew. Math. Mech.* **8** (1928) 161.

APPENDIX

Employing the representation (3.1) of $\hat{\mathbf{Q}}$ in terms of $\mathbf{Q}$ and requiring both of these transformations to be orthogonal, we can find all the restrictions on the 36 components of $\hat{\mathbf{Q}}$. These restrictions are listed below. Note that the italic indices have a range of 3 and no summation is implied by the repeated indices.

$$\sum_{i=1}^{3} \hat{Q}_{\alpha i} = \begin{cases} 1 & \text{for } \alpha = 1, 2, 3, \\ 0 & \text{for } \alpha = 4, 5, 6, \end{cases} \tag{A1}$$

$$\hat{Q}_{i4}^2 = 2\hat{Q}_{i2}\hat{Q}_{i3}, \tag{A2}$$

$$\hat{Q}_{4i}^2 = 2\hat{Q}_{2i}\hat{Q}_{3i}, \tag{A3}$$

$$\hat{Q}_{i5}^2 = 2\hat{Q}_{i1}\hat{Q}_{i3}, \tag{A4}$$

$$\hat{Q}_{5i}^2 = 2\hat{Q}_{1i}\hat{Q}_{3i}, \tag{A5}$$

$$\hat{Q}_{i6}^2 = 2\hat{Q}_{i1}\hat{Q}_{i2}, \tag{A6}$$

$$\hat{Q}_{6i}^2 = 2\hat{Q}_{1i}\hat{Q}_{2i}, \tag{A7}$$

$$\hat{Q}_{44}^2 = \hat{Q}_{22}\hat{Q}_{33} + \hat{Q}_{23}\hat{Q}_{32} + \hat{Q}_{24}\hat{Q}_{34}, \tag{A8}$$

$$\hat{Q}_{45}^2 = \hat{Q}_{21}\hat{Q}_{33} + \hat{Q}_{23}\hat{Q}_{31} + \hat{Q}_{25}\hat{Q}_{35}, \tag{A9}$$

$$\hat{Q}_{46}^2 = \hat{Q}_{21}\hat{Q}_{32} + \hat{Q}_{22}\hat{Q}_{31} + \hat{Q}_{26}\hat{Q}_{36}, \tag{A10}$$

$$\hat{Q}_{54}^2 = \hat{Q}_{12}\hat{Q}_{33} + \hat{Q}_{13}\hat{Q}_{32} + \hat{Q}_{14}\hat{Q}_{34}, \tag{A11}$$

$$\hat{Q}_{55}^2 = \hat{Q}_{11}\hat{Q}_{33} + \hat{Q}_{13}\hat{Q}_{31} + \hat{Q}_{15}\hat{Q}_{35}, \tag{A12}$$

$$\hat{Q}_{56}^2 = \hat{Q}_{11}\hat{Q}_{32} + \hat{Q}_{12}\hat{Q}_{31} + \hat{Q}_{16}\hat{Q}_{36}, \tag{A13}$$

$$\hat{Q}_{64}^2 = \hat{Q}_{12}\hat{Q}_{23} + \hat{Q}_{13}\hat{Q}_{22} + \hat{Q}_{14}\hat{Q}_{24}, \tag{A14}$$

$$\hat{Q}_{65}^2 = \hat{Q}_{21}\hat{Q}_{13} + \hat{Q}_{23}\hat{Q}_{11} + \hat{Q}_{15}\hat{Q}_{25}, \tag{A15}$$

$$\hat{Q}_{66}^2 = \hat{Q}_{11}\hat{Q}_{22} + \hat{Q}_{21}\hat{Q}_{12} + \hat{Q}_{16}\hat{Q}_{26}. \tag{A16}$$

There are six equations in (A1), three each in (A2) to (A7) and nine equations in (A8) to (A16). Thus, there are 33 relations among the 36 components of $\hat{\mathbf{Q}}$ and, therefore, $\hat{\mathbf{Q}}$ has only three independent components.

Given a $\hat{\mathbf{Q}}$ with components that satisfy the above restrictions and $\hat{Q}_{ij} > 0$ $(i, j = 1, 2, 3)$, the corresponding 3-dimensional orthogonal transformation, that is, $\mathbf{Q}$, can be found from

$$Q_{ij}^2 = \hat{Q}_{ij}. \tag{A17}$$