Linear Programming

Optimization Techniques (ENGG*6140)

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Linear Programming

Linear programming

A **linear programming** problem is of the form:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \text{linear function in } \boldsymbol{x} \\ \text{subject to} & \text{affine inequality constraints in } \boldsymbol{x}, \\ & \text{affine equality constraints in } \boldsymbol{x}. \end{array}$$

Standard linear programming

A **standard linear programming** problem is of the form:

Maximization:

$$
\begin{aligned}
\underset{\boldsymbol{x}=[x_1,\ldots,x_n]^\top}{\text{maximize}} \quad & \boldsymbol{\alpha}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{G}\boldsymbol{x} \preceq \boldsymbol{h}, \\
& \boldsymbol{x} \succeq \mathbf{0},
\end{aligned}
$$

Minimization:

$$
\begin{aligned}
\underset{\boldsymbol{x}=[x_1,\ldots,x_n]^\top}{\text{minimize}} \quad & \boldsymbol{\alpha}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{G}\boldsymbol{x} \succeq \boldsymbol{h}, \\
& \boldsymbol{x} \succeq \mathbf{0},
\end{aligned}
$$

where $\boldsymbol{G} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{h} \in \mathbb{R}^m$.

Standard linear programming

Equivalently:

$$\begin{aligned} \underset{x_1,\dots,x_n}{\text{minimize/maximize}} \quad & \alpha_1 x_1 + \cdots + \alpha_n x_n \\ \text{subject to} \quad & \text{linear inequality constraint } 1, \\ & \vdots \\ & \text{linear inequality constraint } m, \\ & x_1, \dots, x_n \geq 0, \end{aligned}$$

where $m \geq n$.

For **example**:

$$\begin{aligned} \underset{x_1,x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 40, \\ & x_1 + x_2 \geq 30, \\ & x_1, x_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \underset{x_1,x_2}{\text{maximize}} \quad & 40x_1 + 30x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 12, \\ & 2x_1 + x_2 \leq 16, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Practical Examples

Practical Example 1

- A company has two products. Let x_1 and x_2 denote the amount of the first and second products to be produced (with some scale), respectively. Therefore, $x_1, x_2 \geq 0$.
- The company has profits \$60 and \$30 for the first and second products. Therefore, the total profit of company:

$$c = \$(60x_1 + 30x_2).$$

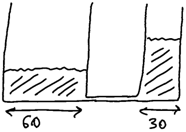
- The resources for these products are limited, so we have the following restrictions:
  - We do not want the first product, with proportion 8, and the second product, with proportion 3, to spend more than \$48, so: $8x_1 + 3x_2 \leq \$48$.
  - For four of the first product and three of the second product, we have the budget to spend at least \$25, so: $4x_1 + 2x_2 \geq \$25$.

The optimization becomes:

$$\begin{aligned} \underset{x_1, x_2}{\text{maximize}} \quad & c = 60x_1 + 30x_2 \\ \text{subject to} \quad & 8x_1 + 3x_2 \leq 48, \\ & 4x_1 + 2x_2 \geq 25, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Practical Example 2

- We have two 2D tanks of water which are connected from their bottom. Let x_1 and x_2 denote the height of water (with some scale) in the first and second tanks, respectively. Therefore, $x_1, x_2 \geq 0$.
- The widths of the two tanks are 60 and 30 (with some scale), respectively. Therefore, the total amount of water in these tanks is $c = 60x_1 + 30x_2$.



- There are some linear physical restrictions on the amount of water poured in these tanks (because of previous tanks which water has passed to reach these tanks): $8x_1 + 3x_2 \leq 48$ and $4x_1 + 2x_2 \geq 25$.

The optimization becomes:

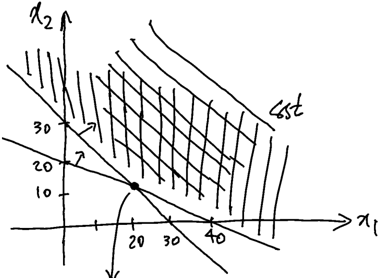
$$\begin{aligned} \underset{x_1, x_2}{\text{maximize}} \quad & c = 60x_1 + 30x_2 \\ \text{subject to} \quad & 8x_1 + 3x_2 \leq 48, \\ & 4x_1 + 2x_2 \geq 25, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solving linear programming by visualization

Visualization: example 1

Minimization example:

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 40, \\ & x_1 + x_2 \geq 30, \\ & x_1, x_2 \geq 0. \end{aligned}$$

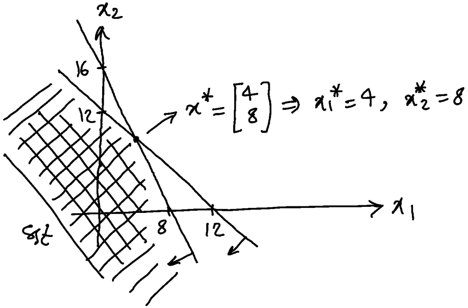


$x^* = \begin{bmatrix} 20 \\ 10 \end{bmatrix} \Rightarrow x_1^* = 20, \ x_2^* = 10$

Visualization: example 2

Maximization example:

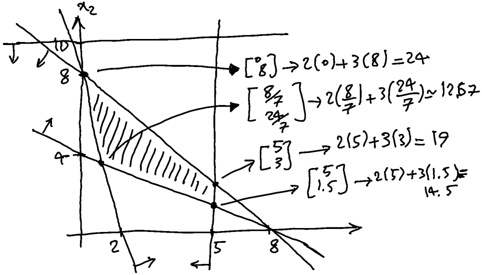
$$
\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & 40x_1 + 30x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 12, \\
& 2x_1 + x_2 \leq 16, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$



Visualization: example 3

Example with more number of constraints:

$$
\begin{aligned}
\underset{x_1, x_2}{\text{minimize}} \quad & 2x_1 + 3x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \geq 8, \\
& 2x_1 + 0.5x_2 \geq 4, \\
& x_1 + x_2 \leq 8, \\
& x_1 \leq 5, \\
& x_2 \leq 10, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$



$$x_1 + 2x_2 - 8 = 2x_1 + 0.5x_2 - 4$$

$$\hookrightarrow \quad 1.5x_2 = x_1 + 4 \to x_1 - 1.5x_2 + 4 = 0$$

$$x_1 + 2x_2 - 8 = 0$$

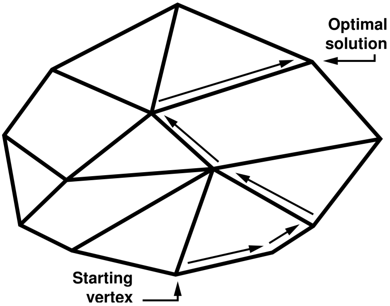
$$(1.5x_2 - 4) + 2x_2 - 8 = 0 \to 3.5x_2 = 12 \to x_2 = \frac{24}{7}$$

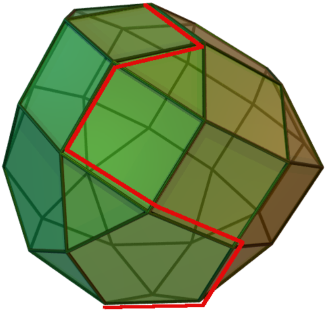
$$\hookrightarrow x_1 = 1.5x_2 - 4 = \frac{3}{\cancel{2}}\left(\frac{\overset{12}{\cancel{24}}}{7}\right) - 4 = \frac{36}{7} - 4 = \frac{8}{7}$$

Simplex Method Description

Simplex method description

- As you saw in the pictures, the feasible set (determined by the constraints) in the linear programming has **affine/linear boundaries**.
- It is because the constraints are affine/linear.
- Therefore, the feasible set is like a **simplex** with linear edges and some corners.
- The corners of the feasible set are named the **extreme points**.

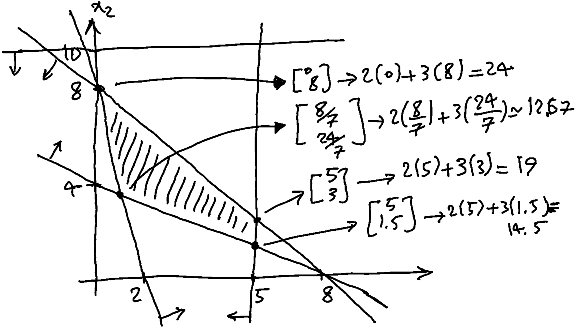




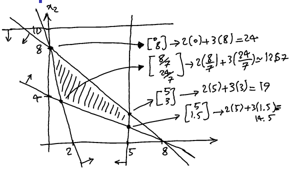
The images are taken from Wikipedia.

Simplex method description

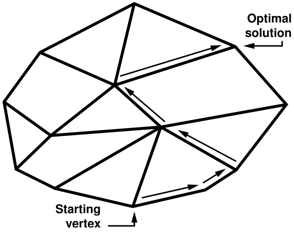
- The **simplex algorithm** was initially proposed in 1947 [1].
- It works on the **linear boundaries (edges)** and **extreme points** of the simplex feasible set.
- Obviously, the solution is at one of the **extreme points**.

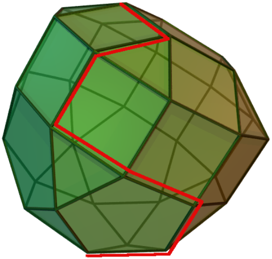


Simplex method description



- The simplex algorithm starts from an extreme point and it goes to one of its neighbor extreme points having the smallest/largest cost function at that point (only if the neighbor extreme point has smaller/larger cost value compared to the current extreme point).
- It continues this procedure until we reach an extreme point whose neighbor extreme points do not have smaller/larger cost value.





The images are taken from Wikipedia.

One of the methods for Simplex Algorithm: Tableau Method for Maximization

Slack variables

Consider this example:

$$
\begin{aligned}
\underset{x_1,x_2,x_3}{\text{maximize}} \quad & 6x_1 + 5x_2 + 4x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 \leq 240, \\
& x_1 + 3x_2 + 2x_3 \leq 360, \\
& 2x_1 + x_2 + 2x_3 \leq 300, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

- We convert each inequality $\leq$ constraint to an equality constraint by adding **slack variables**.
- Slack variables are **positive scalars** which are added to the left hand side of inequality $\leq$ constraint to make it equality.
- Example:

$$
\begin{aligned}
& 2x_1 + x_2 + x_3 3 \leq 240 \implies 2x_1 + x_2 + x_3 + s_1 = 240, \\
& x_1 + 3x_2 + 2x_3 \leq 360 \implies x_1 + 3x_2 + 2x_3 + s_2 = 360, \\
& 2x_1 + x_2 + 2x_3 \leq 300 \implies 2x_1 + x_2 + 2x_3 + s_3 = 300, \\
& s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

Slack variables

So, this problem:

$$\begin{aligned}
\underset{x_1, x_2, x_3}{\text{maximize}} \quad & 6x_1 + 5x_2 + 4x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 \leq 240, \\
& x_1 + 3x_2 + 2x_3 \leq 360, \\
& 2x_1 + x_2 + 2x_3 \leq 300, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

is converted to:

$$\begin{aligned}
\underset{x_1, x_2, x_3, s_1, s_2, s_3}{\text{maximize}} \quad & 6x_1 + 5x_2 + 4x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 + s_1 = 240, \\
& x_1 + 3x_2 + 2x_3 + s_2 = 360, \\
& 2x_1 + x_2 + 2x_3 + s_3 = 300, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

Forming equalities

$$\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & 6x_1 + 5x_2 + 4x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 + s_1 = 240, \\
& x_1 + 3x_2 + 2x_3 + s_2 = 360, \\
& 2x_1 + x_2 + 2x_3 + s_3 = 300, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

The cost function is: $c := 6x_1 + 5x_2 + 4x_3 \implies c - 6x_1 - 5x_2 - 4x_3 = 0$.
Therefore:

$$\begin{aligned}
2x_1 + x_2 + x_3 + s_1 &= 240, \\
x_1 + 3x_2 + 2x_3 + s_2 &= 360, \\
2x_1 + x_2 + 2x_3 + s_3 &= 300, \\
c - 6x_1 - 5x_2 - 4x_3 &= 0.
\end{aligned}$$

Forming the table in the tableau method

$$2x_1 + x_2 + x_3 + s_1 = 240,$$
$$x_1 + 3x_2 + 2x_3 + s_2 = 360,$$
$$2x_1 + x_2 + 2x_3 + s_3 = 300,$$
$$c - 6x_1 - 5x_2 - 4x_3 = 0.$$

	x_1	x_2	x_3	s_1	s_2	s_3	RHS
s_1	2	1	1	1	0	0	240
s_2	1	3	2	0	1	0	360
s_3	2	1	2	0	0	1	300
c	-6	-5	-4	0	0	0	0

Pivot and min test

1. In **maximization** problem, choose the **most negative** value for the **pivot column**.
2. Do the **min test**: divide RHS values (of rows except the c row) to the values of the pivot column. **Ignore** the negative or zero values in min test.
3. Get the minimum division value for the **pivot row**. The intersection of pivot row and pivot column gives the **pivot value**.

pivot row

pivot value

	x_1	x_2	x_3	s_1	s_2	s_3	RHS
s_1	2	1	1	1	0	0	240
s_2	1	3	2	0	1	0	360
s_3	2	1	2	0	0	1	300
c	-6	-5	-4	0	0	0	0

min test

$\frac{240}{2} = 120$

$\frac{360}{1} = 360$

$\frac{300}{2} = 150$

most negative value in c row

pivot column

Simplifying the pivot column

1. Make the pivot value **one** and other values **zero** in the pivot column.
2. For every row, use **the row itself** and the **pivot row** only.
3. **Replace the name** of the pivot row with the name of the pivot column.

	x_1	x_2	x_3	s_1	s_2	s_3	RHS
s_1	2	1	1	1	0	0	240
s_2	1	3	2	0	1	0	360
s_3	2	1	2	0	0	1	300
c	-6	-5	-4	0	0	0	0

		x_1	x_2	x_3	s_1	s_2	s_3	RHS
$r_1/2$	x_1	1	0.5	0.5	0.5	0	0	120
$r_2 - \frac{r_1}{2}$	s_2	0	2.5	1.5	-0.5	1	0	240
$r_3 - r_2$	s_3	0	0	1	-1	0	1	60
$r_4 + 3r_1$	C	0	-2	-1	3	0	0	720

Continuing the table

- In the **maximization** problem, we continue the table until all the values in the c row are non-negative (positive or zero).

	x_1	x_2	x_3	s_1	s_2	s_3	RHS
s_1	2	1	1	1	0	0	240
s_2	1	3	2	0	1	0	360
s_3	2	1	2	0	0	1	300
c	-6	-5	-4	0	0	0	0

		x_1	x_2	x_3	s_1	s_2	s_3	RHS
$r_1/2$	x_1	1	0.5	0.5	0.5	0	0	120
$r_2 - \frac{r_1}{2}$	s_2	0	2.5	1.5	-0.5	1	0	240
$r_3 - r_2$	s_3	0	0	1	-1	0	1	60
$r_4 + 3r_1$	C	0	-2	-1	3	0	0	720

Continuing the table

	x_1	x_2	x_3	s_1	s_2	s_3	RHS	min test
x_1	1	0.5	0.5	0.5	0	0	120	$\frac{120}{0.5} = 240$
s_2	0	2.5	1.5	-0.5	1	0	240	$\frac{240}{2.5} = 96$
s_3	0	0	1	-1	0	1	60	—
C	0	-2	-1	3	0	0	720	

		x_1	x_2	x_3	s_1	s_2	s_3	RHS
$r_1 - \frac{r_2}{5}$	x_1	1	0	$1/5$	$3/5$	$-1/5$	0	72
$r_2/2.5$	x_2	0	1	$3/5$	$-1/5$	$2/5$	0	96
r_3	s_3	0	0	1	-1	0	1	60
$r_4 + \frac{4}{5} r_2$	C	0	0	$\frac{1}{5}$	$\frac{13}{5}$	$\frac{4}{5}$	0	912

all non-negative

maximum cost function (c^*)

Basic and non-basic variables

Once the **table is over**:

- A row with having only one 1 and the rest 0 is a **basic variable**.
- The other columns are **non-basic variables**.

non-basic variables → x_3, s_1, s_2

basic variables → x_1, x_2, s_3

		x_1	x_2	x_3	s_1	s_2	s_3	RHS
$r_1 - \frac{r_2}{5}$	x_1	1	0	$1/5$	$3/5$	$-1/5$	0	72
$r_2/2.5$	x_2	0	1	$3/5$	$-1/5$	$2/5$	0	96
r_3	s_3	0	0	1	-1	0	1	60
$r_4 + \frac{4}{5} r_2$	c	0	0	$1/5$	$13/5$	$4/5$	0	912

all non-negative (table over)

Checking the optimal values

- Once the table is over, the RHS of the c row is the **optimal cost function**. Here it is $c^* = 912$.
- The optimal values for the variables are the RHS of the rows. In other words, the optimum basic variables are the RHS of rows. Here they are $x_1^* = 72, x_2^* = 96, s_3^* = 60$.
- The optimum value for the rest of the variables (the **non-basic variables**) is **zero**. Here they are $x_3^* = 0, s_1^* = 0, s_2^* = 0$.
- We can check if the optimal cost is correct:

$$c := 6x_1 + 5x_2 + 4x_3 \implies c^* = 6x_1^* + 5x_2^* + 4x_3^* = 6(72) + 5(96) + 4(0) = 912 \quad \checkmark$$

		x_1	x_2	x_3	s_1	s_2	s_3	RHS
$r_1 - \frac{r_2}{5}$	x_1	1	0	1/5	3/5	-1/5	0	72
$r_2/2.5$	x_2	0	1	3/5	-1/5	2/5	0	96
r_3	s_3	0	0	1	-1	0	1	60
$r_4 + \frac{4}{5}r_2$	c	0	0	1/5	13/5	4/5	0	912

all non-negative

maximum cost function (c^*)

Big M method

When to use the big M method

We should use the big M method when there are one or some $\geq$ constraints and/or $=$ constraints. In other words, whenever we have **mixed constraints**.

Consider this example with $\leq$ and $\geq$ constraints:

$$
\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & c = 3x_1 + 4x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 600, \\
& x_1 + x_2 \leq 225, \\
& 5x_1 + 4x_2 \leq 1000, \\
& x_1 + 2x_2 \geq 150, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

- For $\leq$ constraints, we use slack variables as before:

$$
\begin{aligned}
& 2x_1 + x_2 \leq 600 \implies 2x_1 + x_2 + s_1 = 600, \\
& x_1 + x_2 \leq 225 \implies x_1 + x_2 + s_2 = 225, \\
& 5x_1 + 4x_2 \leq 1000 \implies 5x_1 + 4x_2 + s_3 = 1000, \\
& s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

Big M method: $\geq$ constraints

$$\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & c = 3x_1 + 4x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 600, \\
& x_1 + x_2 \leq 225, \\
& 5x_1 + 4x_2 \leq 1000, \\
& x_1 + 2x_2 \geq 150, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

- For $\geq$ constraints, we can't use slack variables because the slack variable will not be non-negative anymore:

$$x_1 + 2x_2 \geq 150 \implies x_1 + 2x_2 + s_4 = 150 \implies s_4 \leq 0.$$

- For $\geq$ constraints, we use excess variables e and artificial variables a:

$$\begin{aligned}
& x_1 + 2x_2 \geq 150 \implies x_1 + 2x_2 + a_4 - e_4 = 150, \\
& a_4, e_4 \geq 0.
\end{aligned}$$

- We want the additional variable to be very small ($a_4 = \epsilon$) so we add it to the cost function with a very big multiplication factor $M \gg 1$:

$$\underset{x_1, x_2, x_3}{\text{maximize}} \quad c = 3x_1 + 4x_2 - Ma_4,$$

because if $M \gg 1$, then $a_4 \to 0$ to cancel its effect in the cost function.

Tableau method with the big M method

$$
\begin{aligned}
\underset{x_1,x_2,s_1,s_2,s_3,a_4,e_4}{\text{maximize}} \quad & c = 3x_1 + 4x_2 - Ma_4 \\
\text{subject to} \quad & 2x_1 + x_2 + s_1 = 600, \\
& x_1 + x_2 + s_2 = 225, \\
& 5x_1 + 4x_2 + s_3 = 1000, \\
& x_1 + 2x_2 + a_4 - e_4 = 150, \\
& x_1, x_2, s_1, s_2, s_3, a_4, e_4 \geq 0.
\end{aligned}
$$

- We make zero the column value of additional variable in the c row, because the value of a_4 should be about zero rather than M.

		x_1	x_2	s_1	s_2	s_3	a_4	e_4	RHS
	s_1	2	1	1	0	0	0	0	600
	s_2	1	1	0	1	0	0	0	225
	s_3	5	4	0	0	1	0	0	1000
	a_4	1	2	0	0	0	1	-1	150
	c	-3	-4	0	0	0	M	0	
$r_5 - Mr_4$	c	$-3-M$	$-2M-4$	0	0	0	0	M	$-150M$

Tableau method with the big M method

	x_1	x_2	s_1	s_2	s_3	a_4	e_4	RHS	min test
s_1	2	1	1	0	0	0	0	600	$\frac{600}{1} = 600$
s_2	1	1	0	1	0	0	0	225	$\frac{225}{1} = 225$
s_3	5	4	0	0	1	0	0	1000	$\frac{1000}{4} = 250$
a_4	1	2	0	0	0	1	-1	150	$\frac{150}{2} = 75$
C	-3	-4	0	0	0	M	0		
C	$-3-M$	$-2M-4$	0	0	0	0	M	$-150M$	

		x_1	x_2	s_1	s_2	s_3	a_4	e_4	RHS
$r_1 - \frac{r_4}{2}$	s_1	$3/2$	0	1	0	0	$-1/2$	$1/2$	575
$r_2 - r_4/2$	s_2	$\frac{1}{2}$	0	0	1	0	$-1/2$	$1/2$	150
$r_3 - 2r_4$	s_3	3	0	0	0	1	-2	2	700
$r_4/2$	x_2	$\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	75
	C	-1	0	0	0	0	$M+2$	-2	300

$r_5 + \left(\frac{2M+4}{2}\right) r_4 = r_5 + (M+2) r_4$

Tableau method with the big M method

	x_1	x_2	s_1	s_2	s_3	a_4	e_4	RHS	min test
s_1	$3/2$	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	575	$\frac{575}{0.5}=1150$
s_2	$\frac{1}{2}$	0	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	150	$\frac{150}{0.5}=300$
s_3	3	0	0	0	1	-2	2	700	$\frac{700}{2}=350$
x_2	$\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	75	—
c	-1	0	0	0	0	$M+2$	-2	300	

		x_1	x_2	s_1	s_2	s_3	a_4	e_4	RHS
r_1-r_2	s_1	1	0	1	0	0	0	0	375
$2r_2$	e_4	1	0	0	2	0	-1	1	300
r_3-4r_2	s_3	1	0	0	-4	1	0	0	100
r_4+r_2	x_2	1	0	0	1	0	0	0	225
r_5+4r_2	c	1	0	0	4	0	M	0	900

all positive

Therefore: $s_1^* = 375, e_4^* = 300, s_3^* = 100, x_2^* = 225, x_1^* = 0, s_2^* = 0, s_3^* = 0, a_4^* = 0, c^* = 900.$
Check: $c^* = 3x_1^* + 4x_2^* = 3(0) + 4(225) = 900$ $\surd$

Example 2 for mixed constraints

Consider another example with mixed constraints:

$$
\begin{aligned}
\underset{x_1, x_2, x_3}{\text{maximize}} \quad & c = x_1 - x_2 + 3x_3 + 4 \\
\text{subject to} \quad & x_1 + x_2 \leq 20, \\
& x_1 + x_3 = 5, \\
& x_2 + x_3 \geq 10, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

- We drop the DC value from the cost for now:

$$c = x_1 - x_2 + 3x_3.$$

- We have:

$$
\begin{aligned}
& x_1 + x_2 \leq 20 \implies x_1 + x_2 + s_1 = 20, \\
& x_1 + x_3 = 5 \implies x_2 + x_3 + a_1 = 5, \\
& x_2 + x_3 \geq 10 \implies x_2 + x_3 + a_2 + e_2 = 10, \\
& s_1, a_1, a_2, e_2 \geq 0.
\end{aligned}
$$

Example 2 for mixed constraints

The problem is converted to:

$$\begin{aligned}
\underset{x_1, x_2, x_3}{\text{maximize}} \quad & c = x_1 - x_2 + 3x_3 - Ma_1 - Ma_2 \\
\text{subject to} \quad & x_1 + x_2 + s_1 = 20, \\
& x_2 + x_3 + a_1 = 5, \\
& x_2 + x_3 + a_2 + e_2 = 10, \\
& s_1, a_1, a_2, e_2 \geq 0.
\end{aligned}$$

		x_1	x_2	x_3	s_1	a_1	a_2	e_2	RHS	min test
	s_1	1	1	0	1	0	0	0	20	—
	a_1	1	0	1	0	1	0	0	5	$5/1 = 5$
	a_2	0	1	1	0	0	1	-1	10	$10/1 = 10$
	c	-1	$+1$	-3	0	$+M$	$+M$	0	0	
$r_4 - Mr_2$	c	$-M-1$	1	$-M-3$	0	0	M	0	$-5M$	
$r_5 - Mr_3$	c	$-M-1$	$-M+1$	$-2M-3$	0	0	0	M	$-15M$	

Example 2 for mixed constraints

		x_1	x_2	x_3	s_1	a_1	a_2	e_2	RHS	min test:
	s_1	1	1	0	1	0	0	0	20	—
	a_1	1	0	1	0	1	0	0	5	$5/1=5$
	a_2	0	1	1	0	0	1	-1	10	$10/1=10$
	c	-1	$+1$	-3	0	$+M$	$+M$	0	0	
r_4-Mr_2	c	$-M-1$	1	$-M-3$	0	0	M	0	$-5M$	
r_5-Mr_3	c	$-M-1$	$-M+1$	$-2M-3$	0	0	0	M	$-15M$	

		x_1	x_2	x_3	s_1	a_1	a_2	e_2	RHS	min test:
r_1	s_1	1	1	0	1	0	0	0	20	$20/1=20$
r_2	x_3	1	0	1	0	1	0	0	5	—
r_3-r_2	a_2	-1	1	0	0	-1	1	-1	5	$\frac{5}{1}=5$
$r_6+(2M+3)r_2$	c	$M+2$	$-M+1$	0	0	$2M+3$	0	M	$-5M+15$	

Example 2 for mixed constraints

		x_1	x_2	x_3	s_1	a_1	a_2	e_2	RHS	min test:
r_1	s_1	1	1	0	1	0	0	0	20	$20/1 = 20$
r_2	x_3	1	0	1	0	1	0	0	5	—
$r_3 - r_2$	a_2	-1	1	0	0	-1	1	-1	5	$\frac{5}{1} = 5$
$r_6 + (2M+3)r_2$	c	$M+2$	$-M+1$	0	0	$2M+3$	0	M	$-5M+15$	

		x_1	x_2	x_3	s_1	a_1	a_2	e_2	RHS	
$r_1 - r_3$	s_1	2	0	0	1	1	-1	1	15	$\Rightarrow s_1^* = 15$
r_2	x_3	1	0	1	0	1	0	0	5	$\Rightarrow x_3^* = 5$
r_3	x_2	-1	1	0	0	-1	1	-1	5	$\Rightarrow x_2^* = 5$
$r_4 + (M-1)r_3$	c	3	0	0	0	$M+4$	$M-1$	1	10	$\Rightarrow c^* = 10$

all non-negative!

Therefore: $s_1^* = 15, x_3^* = 5, x_2^* = 5, c^* = 10, x_1^* = a_1^* = a_2^* = e_2^* = 0$.
Check: $c^* = x_1^* - x_2^* + 3x_3^* = 0 - 5 + 3(5) = 10 \quad \surd$
The final answer for maximum actual cost is (we add back the DC value): $c^* = 10 + 4 = 14$.

The Reason for the Tableau Method

The reason for the tableau method

$$\begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{maximize}} \quad & c = 4x_1 + 6x_2 - 5x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 50, \\ & 2x_1 + 3x_2 + x_4 \leq 42, \\ & 3x_3 - x_4 \leq 250, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

is converted to:

$$\begin{aligned} \underset{x_1,x_2,x_3,x_4,s_1,s_2,s_3}{\text{maximize}} \quad & c = 4x_1 + 6x_2 - 5x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + s_1 = 50, \\ & 2x_1 + 3x_2 + x_4 + s_2 = 42, \\ & 3x_3 - x_4 + s_3 = 250, \\ & x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0. \end{aligned}$$

- \# variables: 7, # equations: 3
- We can set $7 - 3 = 4$ variables to zero (non-basic variables) and find the other 3 variables (basic variables).
- How many ways can we choose the three variables out of the 7 variables? $\binom{7}{3} = 35$.

Example variables to choose

One of the ways:

$$\text{non-basic variables: } x_1 = x_2 = x_3 = x_4 = 0,$$
$$\text{basic variables: } s_1, s_2, s_3.$$

$$\begin{aligned}
\underset{s_1, s_2, s_3}{\text{maximize}} \quad & c = 0 \\
\text{subject to} \quad & s_1 = 50, \\
& s_2 = 42, \\
& s_3 = 250, \\
& s_1, s_2, s_3 \geq 0.
\end{aligned}$$

Therefore, $s_1 = 50, s_2 = 42, s_3 = 250$.
The cost function becomes: $c = 0$.

Example variables to choose

One of the ways:

non-basic variables: $x_1 = x_4 = s_1 = s_2 = 0$,
basic variables: x_2, x_3, s_3.

$$
\begin{aligned}
\underset{x_2, x_3, s_3}{\text{maximize}} \quad & c = 6x_2 \\
\text{subject to} \quad & x_2 + x_3 = 50, \\
& 3x_2 = 42, \\
& 3x_3 + s_3 = 250, \\
& x_2, x_3, s_3 \geq 0.
\end{aligned}
$$

Therefore, $x_2 = 14, x_3 = 36, s_3 = 142$.
The cost function becomes: $c = 6(14) = 84$.

The reason for the pivot column

Which variable should we increase which maximizes the cost function the most?

$$c = 4x_1 + 6x_2 - 5x_4.$$

Increasing the variable x_2 has the most effect because it has the biggest multiplication factor, i.e., 6.
Recall that we had:

$$c - 4x_1 - 6x_2 + 5x_4 = 0.$$

That is why, in the tableau method, we find the most negative value in the c row. This is the reason for the **pivot column**.

The reason for the min test

$$
\begin{aligned}
\underset{x_1,x_2,x_3,x_4,s_1,s_2,s_3}{\text{maximize}} \quad & c = 4x_1 + 6x_2 - 5x_4 \\
\text{subject to} \quad & x_1 + x_2 + x_3 + s_1 = 50, \\
& 2x_1 + 3x_2 + x_4 + s_2 = 42, \\
& 3x_3 - x_4 + s_3 = 250, \\
& x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	min test
s_1	1	1	1	0	1	0	0	50	$\frac{50}{1}=50$
s_2	2	3	0	1	0	1	0	42	$\frac{42}{3}=14$
s_3	0	0	3	-1	0	0	1	250	—
c	-4	-6	0	5	0	0	0	0	

How much can we increase the x_2 variable?

- In the first constraint, the worst case scenario is $x_1 = x_3 = s_1 = 0$ and the most we can increase x_2: $x_2 = 50$
- In the second constraint, the worst case scenario is $x_1 = x_4 = s_2 = 0$ and the most we can increase x_2: $3x_2 = 42 \implies x_2 = 42/3 = 14$
- In the third constraint, the worst case scenario is $x_3 = x_4 = s_3 = 0$ and the most we can increase x_2: $30x_2 = 250 \implies x_2 = \infty$
- Therefore, the minimum increase we can have for x_2 is: $\min(50, 42, \infty) = 42$.

Solving the Dual Problem for Minimization

Dual problem for minimization

An example **minimization** linear problem is:

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 40, \\ & x_1 + x_2 \geq 30, \\ & x_1, x_2 \geq 0. \end{aligned}$$

- When we have a minimization linear programming, we can **convert** the minimization problem to a maximization problem.
- We should find the **dual problem** for the minimization problem. The dual for the minimization is a maximization problem. We will learn the dual problem of linear programming soon.

Dual problem for minimization

An example **minimization** linear problem is:

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 40, \\ & x_1 + x_2 \geq 30, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Consider the constraints:

$$\begin{aligned} x_1 + 2x_2 \geq 40 \ &\overset{\times y_1}{\Longrightarrow} \ y_1 x_1 + 2y_1 x_2 \geq 40 y_1, \\ x_1 + x_2 \geq 30 \ &\overset{\times y_2}{\Longrightarrow} \ y_2 x_1 + y_2 x_2 \geq 30 y_2, \end{aligned}$$

where $y_1, y_2 \geq 0$. Summing the sides together gives:

$$(y_1 + y_2)x_1 + (2y_1 + y_2)x_2 \geq 40y_1 + 30y_2.$$

On the other hand, the cost of dual problem is a lower bound on the cost of the primal problem:

$$12x_1 + 16x_2 \geq (y_1 + y_2)x_1 + (2y_1 + y_2)x_2.$$

Dual problem for minimization

Summing the sides together gives:

$$(y_1 + y_2)x_1 + (2y_1 + y_2)x_2 \geq 40y_1 + 30y_2.$$

On the other hand, the cost of dual problem is a lower bound on the cost of the primal problem:

$$12x_1 + 16x_2 \geq (y_1 + y_2)x_1 + (2y_1 + y_2)x_2.$$

Therefore:

$$12x_1 + 16x_2 \geq (y_1 + y_2)x_1 + (2y_1 + y_2)x_2 \geq 40y_1 + 30y_2.$$

Hence:

$$\begin{aligned} y_1 + y_2 &\leq 12, \\ 2y_1 + y_2 &\leq 16. \end{aligned}$$

We want to find the best (maximum) lower bound, so:

$$\underset{y_1, y_2}{\text{maximize}} \quad 40y_1 + 30y_2.$$

Dual problem for minimization

Therefore:

$$\begin{aligned} \underset{y_1, y_2}{\text{maximize}} \quad & 40y_1 + 30y_2 \\ \text{subject to} \quad & y_1 + y_2 \leq 12, \\ & 2y_1 + y_2 \leq 16, \\ & y_1, y_2 \geq 0. \end{aligned}$$

is the dual problem for the following problem:

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 40, \\ & x_1 + x_2 \geq 30, \\ & x_1, x_2 \geq 0. \end{aligned}$$

This maximization problem can be solved as explained before.

Solving the problem by tableau method

$$\begin{aligned} \underset{y_1, y_2}{\text{maximize}} \quad & c = 40y_1 + 30y_2 \\ \text{subject to} \quad & y_1 + y_2 + s_1 = 12, \\ & 2y_1 + y_2 + s_2 = 16, \\ & y_1, y_2 \geq 0. \end{aligned}$$

	y_1	y_2	s_1	s_2	RHS
s_1	1	1	1	0	12
s_2	2	1	0	1	16
c	-40	-30	0	0	0

$\underbrace{\text{min test}}$

$12/1 = 12$

$16/2 = 8$

		y_1	y_2	s_1	s_2	RHS
$r_1 - \frac{r_2}{2}$	s_1	0	0.5	1	-0.5	4
$\frac{r_2}{2}$	y_1	1	0.5	0	0.5	8
$r_3 + 20r_2$	c	0	-10	0	20	320

Solving the problem by tableau method

	y_1	y_2	s_1	s_2	RHS
s_1	0	0.5	1	−0.5	4
y_1	1	0.5	0	0.5	8
c	0	−10	0	20	320

min test

$4/0.5 = 8$

$8/0.5 = 16$

		y_1	y_2	s_1	s_2	RHS
$2r_1$	y_2	0	1	2	−1	8
$r_2 - r_1$	y_1	1	0	−1	1	4
$r_3 + 20r_1$	c	0	0	20	10	400

all positive

Therefore: $y_2^* = 8, y_1^* = 4, s_1^* = 0, s_2^* = 0, c^* = 400$.
Check: $c^* = 40y_1^* + 30y_2^* = 40(4) + 30(8) = 400 \quad \surd$

The strong duality holds for linear programming, so:

$$c^* = 400 \text{ for the primal problem, too.}$$

Dual Simplex Method

Why we need the dual simplex method?

- We converted the minimization linear problem to its **dual problem** which is the maximization linear problem. Then, we solved it using the simplex method for maximization.
- However, it only gave us the **optimal cost function** c^* and not the **optimum primal variables** $\{x_1^*, \ldots, x_n^*\}$.
- For finding these optimum primal variables in the minimization linear programming, we can use the **dual simplex method**.
- The dual simplex method only works for the **minimization** linear problem if:
  - **all** its multiplication factors in the **cost** function are **non-negative**.
  - **at least one** of the **inequality constraints** is $\geq$.

Dual simplex method: example

$$\begin{aligned} \underset{x_1,x_2}{\text{minimize}} \quad & c = 3x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 600, \\ & x_1 + x_2 \leq 225, \\ & 5x_1 + 4x_2 \leq 1000, \\ & x_1 + 2x_2 \geq 150, \\ & x_1, x_2 \geq 0. \end{aligned}$$

For inequality $\geq$, we have:

$$x_1 + 2x_2 \geq 150 \implies -x_1 - 2x_2 \leq -150$$

Using slack variables:

$$\begin{aligned} \underset{x_1,x_2}{\text{minimize}} \quad & c - 3x_1 + 4x_2 = 0 \\ \text{subject to} \quad & 2x_1 + x_2 + s_1 = 600, \\ & x_1 + x_2 + s_2 = 225, \\ & 5x_1 + 4x_2 + s_3 = 1000, \\ & -x_1 - 2x_2 + s_4 = -150, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Dual simplex method: example

$$
\begin{aligned}
\underset{x_1, x_2}{\text{minimize}} \quad & c - 3x_1 + 4x_2 = 0 \\
\text{subject to} \quad & 2x_1 + x_2 + s_1 = 600, \\
& x_1 + x_2 + s_2 = 225, \\
& 5x_1 + 4x_2 + s_3 = 1000, \\
& -x_1 - 2x_2 + s_4 = -150, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

1. Pivot row: Pick the most negative value in RHS
2. min test: Divide the non-zero values of c row by the negative values of the pivot row. Take absolute value in division.

	x_1	x_2	s_1	s_2	s_3	s_4	RHS
s_1	2	1	1	0	0	0	600
s_2	1	1	0	1	0	0	225
s_3	5	4	0	0	1	0	1000
s_4	-1	-2	0	0	0	1	-150
c	-3	-4	0	0	0	0	0

min test: $\begin{cases} \left|\frac{-3}{-1}\right| = 3 \\ \left|\frac{-4}{-2}\right| = 2 \end{cases}$

Dual simplex method: example

	x_1	x_2	s_1	s_2	s_3	s_4	RHS
s_1	2	1	1	0	0	0	600
s_2	1	1	0	1	0	0	225
s_3	5	4	0	0	1	0	1000
s_4	−1	−2	0	0	0	1	−150
c	−3	−4	0	0	0	0	0

min test: $\left\{ \begin{array}{l} \left|\frac{-3}{-1}\right| = 3 \\ \left|\frac{-4}{-2}\right| = 2 \end{array} \right.$

		x_1	x_2	s_1	s_2	s_3	s_4	RHS
	s_1	1.5	0	1	0	0	0.5	525
$r_2 + \frac{r_4}{2}$	s_2	0.5	0	0	1	0	0.5	150
$r_3 + 2r_4$	s_3	3	0	0	0	1	2	700
$\frac{-r_4}{2}$	x_2	0.5	1	0	0	0	−0.5	75
	c	−1	0	0	0	0	−2	300

all positive

Therefore: $s_1^* = 525, s_2^* = 150, s_3^* = 700, x_2^* = 75, c^* = 300, x_1^* = 0, s_4^* = 0$.
Check: $c^* = 3x_1^* + 4x_2^* = 3(0) + 4(75) = 300$ $\surd$

Dual simplex method for $\geq$ constraints in maximization

We can also use the **dual simplex method** for handling $\geq$ **constraints** in **maximization**. Example:

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 \leq 48, \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8, \\ & x_2 \geq 1, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

We can convert the $\geq$ constraints to $\leq$ constraints by **multiplying the sides of inequality by** -1:

$$x_2 \geq 1 \implies -x_2 \leq -1 \implies -x_2 + s_4 = -1.$$

So, the problem is converted to:

$$\begin{aligned} \underset{x_1,x_2,x_3,s_1,s_2,s_3,s_4}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\ & -x_2 + s_4 = -1, \\ & x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0. \end{aligned}$$

Dual simplex method for $\geq$ constraints in maximization

$$\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3,s_4}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& -x_2 + s_4 = -1, \\
& x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0.
\end{aligned}$$

	x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
s_1	-2	2	-8	1	0	0	0	24
x_3	-2	2	-4	0	1	0	0	8
x_1	5/4	-1/2	3/2	0	0	1	0	2
s_4	-1	0	0	0	0	0	0	-1
c	5	10	10	0	0	0	0	280

	x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
s_1	0	2	-8	1	0	0	-2	26
x_3	0	2	-4	0	1	0	-2	10
x_1	0	-0.5	1.5	0	0	1	1.25	0.75
x_2	1	0	0	0	0	0	-1	1
c	10	10	0	0	0	0	5	275

all nonnegative

$s_1^* = 26, x_3^* = 10, x_1^* = 0.75, x_2^* = 1, c^* = 275$

Acknowledgment

This lecture is inspired by the lectures of Prof. Shokoufeh Mirzaei on linear programming: [Link]

References

[1] G. B. Dantzig, "Reminiscences about the origins of linear programming," in *Mathematical Programming The State of the Art*, pp. 78–86, Springer, 1983.