Sensitivity Analysis in Linear Programming

Optimization Techniques (ENGG*6140)

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Matrices in Linear Programming

Matrices in linear programming

Consider $\underline{n \text{ variables}}$ and $\underline{m \text{ constraints}}$ (excluding the constraints for $\underbrace{x_1, \ldots, x_n \geq 0}$).
After having slack variables, we can have:

$c_j x_j$

$$
\begin{array}{ll}
\underset{x_1,\ldots,x_n}{\text{maximize}} & c = c_1 x_1 + \cdots + c_n x_n \\
\text{subject to} & \left\{ \begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m, \end{array} \right. \\
& \underbrace{x_1, \ldots, x_n \geq 0.}
\end{array}
$$

a_{ij} → x_j in i-th constraint

$b_i \rightarrow$ RHS of ith constraint

Matrices in linear programming

Example:

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 \leq 48, \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

It is converted to:

$$\begin{aligned} \underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{aligned}$$

Matrices in linear programming

$$\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

Assume we solve it until the end and at the end, the basic variables are s_1, x_3, x_1 and the non-basic variables are x_2, s_2, s_3.

- basic and non-basic variables:

define

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top$$

- the coefficients of basic and non-basic variables in the objective function:

$$\boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top$$

- the coefficients of the variables in the constraints:

$$\begin{aligned}
&\boldsymbol{a}_{x_1} := [8, 4, 2]^\top, \boldsymbol{a}_{x_2} := [6, 2, 1.5]^\top, \boldsymbol{a}_{x_3} := [1, 1.5, 0.5]^\top, \\
&\boldsymbol{a}_{s_1} := [1, 0, 0]^\top, \boldsymbol{a}_{s_2} := [0, 1, 0]^\top, \boldsymbol{a}_{s_3} := [0, 0, 1]^\top, \\
&\boldsymbol{b} := [48, 20, 8]^\top
\end{aligned}$$

Matrices in linear programming

$C_b^T x_b + C_n^T x_n$ ≥ 0

$n \leq 6$

$$\begin{aligned} \underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{aligned}$$

$m = 3$

b

- basic and non-basic variables:

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top$$

$B x_b = b$

$x_b = B^{-1} b$

- the coefficients of the variables in the constraints:

$$\boldsymbol{a}_{x_1} := [8, 4, 2]^\top, \boldsymbol{a}_{x_2} := [6, 2, 1.5]^\top, \boldsymbol{a}_{x_3} := [1, 1.5, 0.5]^\top,$$
$$\boldsymbol{a}_{s_1} := [1, 0, 0]^\top, \boldsymbol{a}_{s_2} := [0, 1, 0]^\top, \boldsymbol{a}_{s_3} := [0, 0, 1]^\top$$

- the matrices of coefficients of the variables in the constraints, for basic and non-basic variables: $\boldsymbol{B} \in \mathbb{R}^{m \times m}$, $\boldsymbol{N} \in \mathbb{R}^{m \times (n-m)}$

$$\boldsymbol{B} := [\boldsymbol{a}_{s_1}, \boldsymbol{a}_{x_3}, \boldsymbol{a}_{x_1}] = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix} \qquad \boldsymbol{N} := [\boldsymbol{a}_{x_2}, \boldsymbol{a}_{s_2}, \boldsymbol{a}_{s_3}] = \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix}$$

s_1, x_3, x_1

$\in \mathbb{R}^m$

Matrices in linear programming

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^T B^{-1} a_j - c_j}_{c_b^T B^{-1} N - c_n^T}$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

	x_2	s_2	s_3	s_1	x_3	x_1	RHS
s_1	-2	2	-8	1	0	0	24
x_3	-2	2	-4	0	1	0	8
x_1	5/4	-1/2	3/2	0	0	1	2
c	5	10	10	0	0	0	280

(x_n: x_2, s_2, s_3; x_b: s_1, x_3, x_1)

$$[0 \;\; 20 \;\; 60]\begin{bmatrix}24\\8\\2\end{bmatrix} = (0\times 24) + (20\times 8) + (60\times 2)$$

$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top$

$\boldsymbol{a}_{x_1} := [8, 4, 2]^\top, \boldsymbol{a}_{x_2} := [6, 2, 1.5]^\top, \boldsymbol{a}_{x_3} := [1, 1.5, 0.5]^\top,$

$\boldsymbol{a}_{s_1} := [1, 0, 0]^\top, \boldsymbol{a}_{s_2} := [0, 1, 0]^\top, \boldsymbol{a}_{s_3} := [0, 0, 1]^\top, \boldsymbol{b} := [48, 20, 8]^\top,$

$$\boldsymbol{B} := [a_{s_1}, a_{x_3}, a_{x_1}] = \begin{bmatrix}1 & 1 & 8\\0 & 1.5 & 4\\0 & 0.5 & 2\end{bmatrix}, \boldsymbol{N} := [a_{x_2}, a_{s_2}, a_{s_3}] = \begin{bmatrix}6 & 0 & 0\\2 & 1 & 0\\1.5 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{B}^{-1} = \begin{bmatrix}1 & 1 & 8\\0 & 1.5 & 4\\0 & 0.5 & 2\end{bmatrix}^{-1} = \begin{bmatrix}1 & 2 & -8\\0 & 2 & -4\\0 & -0.5 & 1.5\end{bmatrix}, \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix}1 & 2 & -8\\0 & 2 & -4\\0 & -0.5 & 1.5\end{bmatrix}\begin{bmatrix}48\\20\\8\end{bmatrix} = \begin{bmatrix}24\\8\\2\end{bmatrix}$$

Matrices in linear programming

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$c_b^T B^{-1}a_j - c_j^T$, $c_b^T B^{-1}N - c_n^T$	$\underbrace{0^T}_{c_b^T B^{-1}B - c_b^T}$	$c_b^T B^{-1}b$

	x_n: x_2	s_2	s_3	x_b: s_1	x_3	x_1	RHS
x_b: s_1	-2	2	-8	1	0	0	24
x_3	-2	2	-4	0	1	0	8
x_1	$5/4$	$-1/2$	$3/2$	0	0	1	2
c	5	10	10	0	0	0	280

$\boldsymbol{c}_b := [0, 20, 60]^\top$, $\boldsymbol{c}_n := [30, 0, 0]^\top$, $\boldsymbol{b} := [48, 20, 8]^\top$, $\boldsymbol{B}^{-1}\boldsymbol{b} = [24, 8, 2]^\top$.

$$\boldsymbol{B}^{-1}\boldsymbol{a}_j \implies \boldsymbol{B}^{-1}\boldsymbol{N} = \begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 5/4 & -1/2 & 3/2 \end{bmatrix}, \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I},$$

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{a}_j - c_j \implies$$

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_n^\top = [0, 20, 60] \begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 5/4 & -1/2 & 3/2 \end{bmatrix} - [30, 0, 0] = [5, 10, 10],$$

$$\boldsymbol{c}_n^\top \boldsymbol{B}^{-1}\boldsymbol{B} - \boldsymbol{c}_n^\top = \boldsymbol{0}^\top,$$

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{b} = [0, 20, 60] \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} = 280.$$

Cases for Sensitivity Analysis

Cases for sensitivity analysis

- **Sensitivity analysis** analyzes how much effect some change in something has on the optimization.
- We can have different cases of change in linear programming:
  1. change in **coefficient** of a variable (basic or nonbasic) in the **objective function**
     - ★ $\boxed{\text{1-1:}}$ change for **nonbasic** variable → c_j
     - ★ 1-2: change for **basic** variable
  2. change in **coefficient** of a variable (basic or nonbasic) in the $\boxed{\textbf{constraint(s)}}$ → a_{ij}
     - ★ 2-1: change for **nonbasic** variable
     - ★ 2-2: change for **basic** variable
  3. **adding** a new **variable** to optimization
  4. **adding** a new **constraint** to optimization

Note: we can have a **combination** of changes, too!

Case 1-1 of Change

Case 1-1 for sensitivity analysis

⋆ Change in **coefficient** of a **nonbasic** variable in the **objective function**.

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

32

36

$$\begin{cases} \underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{cases}$$

The company is able to increase the profit of the second product, $\boldsymbol{x}_2$, to (a) \$32 and (b) \$36. Do you recommend this change to the manager?

Case 1-1 for sensitivity analysis

⋆ Change in **coefficient** of a **nonbasic** variable in the **objective function**.

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$c_b^T B^{-1} a_j - c_j^T$ $c_b^T B^{-1} N - c_n^T$	0^T $c_b^T B^{-1} B - c_b^T$	$c_b^T B^{-1} b$

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

$\boldsymbol{x}_2$ is a nonbasic variable. We have change in $c_{\boldsymbol{x}_2}$ in $\boldsymbol{c}_n$ so:

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{a}_{\boldsymbol{x}_2} - c_{\boldsymbol{x}_2} = [0, 20, 60]\begin{bmatrix}-2\\-2\\5/4\end{bmatrix} - \overbrace{(30+\delta)}^{30+\delta} = 35 - 30 - \delta = 5 - \delta$$

column of x_2 in $B^{-1}N$
(first column of $B^{-1}N$)

For not having change in optimization:

$$5 - \delta \geq 0 \Longrightarrow \delta \leq 5 \Longrightarrow c_{\boldsymbol{x}_2,\text{new}} = 30 + \delta \leq 35.$$

$30 \downarrow$ 35

Case 1-1 for sensitivity analysis

- For not having change in optimization: $5 - \delta \geq 0 \implies \delta \leq 5 \implies c_{x_2,\text{new}} = 30 + \delta \leq 35$.
- Therefore, if profit of x_2 is \$32 ≤ \$35, we do not recommend it as it does not change the previous optimal solution for production of the company.
- If profit of x_2 is \$36 > \$35, we should continue the optimization:

	x_2	s_2	s_3	s_1	x_3	x_1	RHS	min test:
s_1	-2	2	-8	1	0	0	24	—
x_3	-2	2	-4	0	1	0	8	—
x_1	$5/4$	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	1	2	$\frac{2}{5/4} = \frac{8}{5}$
Previous c row → c	5	10	10	0	0	0	280	
updated → c	-1	10	10	0	0	0	280	

		x_2	s_2	s_3	s_1	x_3	x_1	RHS	
$r_1 + \frac{8}{5} r_3$	s_1	0	1.2	-5.6	1	0	1.6	27.2	$\Rightarrow s_1^* = 27.2$
$r_2 + \frac{8}{5} r_3$	x_3	0	1.2	-1.6	0	1	1.6	11.2	$\Rightarrow x_3^* = 11.2$
$\frac{4}{5} r_3$	x_2	1	-0.4	1.2	0	0	0.8	1.6	$\Rightarrow x_2^* = 1.6$
$r_4 + \frac{4}{5} r_3$	c	0	9.6	11.2	0	0	0.8	281.6	$\Rightarrow c^* = \boxed{281.6}$

all nonnegative

compare!

Case 1-2 of Change

Case 1-2 for sensitivity analysis

⋆ Change in **coefficient** of a **basic** variable in the **objective function**.

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$
\begin{aligned}
\underset{x_1, x_2, x_3, s_1, s_2, s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

The company is decreasing the profit of the first product, $\boldsymbol{x}_1$, to (a) \$58 and (b) \$30. Do you recommend this change to the manager?

Case 1-2 for sensitivity analysis

	$x_n^\top$	$x_B^\top$	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^\top B^{-1}a_j - c_j^\top}_{c_b^\top B^{-1}N - c_n^\top}$	$\underbrace{0^\top}_{c_b^\top B^{-1}B - c_b^\top}$	$\boxed{c_b^\top B^{-1}b}$

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

$\boldsymbol{x}_1$ is a basic variable. We have change in c_{x_1} in $\boldsymbol{c}_b$ so:

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_n^\top = [0, 20, 60 + \delta] \begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 5/4 & -1/2 & 3/2 \end{bmatrix} - [30, 0, 0]$$
$$= [5 + 1.25\delta, 10 - 0.5\delta, 10 + 1.5\delta]$$

For not having change in optimization:

$$5 + 1.25\delta \geq 0 \implies \boxed{\delta \geq -4,} \quad 10 - 0.5\delta \geq 0 \implies \boxed{\delta \leq 20,} \quad 10 + 1.5\delta \geq 0 \implies \boxed{\delta \geq -6.6,}$$
$$\implies -4 \leq \delta \leq 20, c_{x_1} = 60 + \delta \implies 56 \leq c_{x_1} \leq 80.$$

Case 1-2 for sensitivity analysis

- For not having change in optimization: $56 \le c_{x_1} \le 80$.
- Therefore, if profit of $\boldsymbol{x}_1$ decreases to $\underbrace{\$58 \in [56, 80]}$, this decrease does not change the overall profit and it can be recommended.
- If profit of $\boldsymbol{x}_1$ is decreased to \$30 < \$56, we should continue the optimization:

$$\star \quad \underbrace{\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_n^\top} = [0, 20, 30] \begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 5/4 & -1/2 & 3/2 \end{bmatrix} - [30, 0, 0] = \underbrace{[-32.5, 25, -35]},$$

$$\star \quad \boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{b} = [0, 20, 30] \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} = 220.$$

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^T B^{-1} a_j - c_j^T}_{c_b^T B^{-1} N - c_n^T}$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

	x_2	s_2	s_3	s_1	x_3	x_1	RHS	min test:
s_1	-2	2	-8	1	0	0	24	—
x_3	-2	2	-4	0	1	0	8	—
x_1	5/4	-1/2	3/2	0	0	1	2	$\frac{2}{3/2} = \frac{4}{3}$
c	5	10	10	0	0	0	$\boxed{280}$	
updated → c	-32.5	25	-35	0	0	0	$\boxed{220}$	

Case 1-2 for sensitivity analysis

s_1, x_3, x_1

	x_2	s_2	s_3	s_1	x_3	x_1	RHS
s_1	-2	2	-8	1	0	0	24
x_3	-2	2	-4	0	1	0	8
x_1	5/4	-1/2	3/2	0	0	1	2
c	5	10	10	0	0	0	280
c	-32.5	25	-35	0	0	0	220

min test:
—
—
$\frac{2}{3/2} = \frac{4}{3}$

	x_2	s_2	s_3	s_1	x_3	x_1	RHS
s_1	14/3	-2/3	0	1	0	16/3	104/3
x_3	4/3	2/3	0	0	1	8/3	40/3
s_3	5/6	-1/3	1	0	0	2/3	4/3
c	-10/3	40/3	0	0	0	70/3	800/3

min test:
$\frac{104/3}{14/3} = \frac{104}{14}$
$\frac{40}{4} = 20$
$\frac{4/3}{5/6} = \frac{24}{15}$

	x_2	s_2	s_3	s_1	x_3	x_1	RHS
s_1	0	6/5	-28/5	1	0	8/5	136/5
x_3	0	6/5	-8/5	0	1	8/5	56/5
x_2	1	-2/5	6/5	0	0	4/5	8/5
c	0	12	4	0	0	26	272

all nonnegative.

$\Rightarrow s_1^* = 136/5$
$\Rightarrow x_3^* = 56/5$
$\Rightarrow x_2^* = 8/5$
$\Rightarrow \boxed{c^* = 272}$

compare to 280

s_1, x_3, x_2

So, changing profit of $\boldsymbol{x}_1$ to \$30 decreases the total profit to \$272 from \$280.

Case 2-1 of Change

Case 2-1 for sensitivity analysis

a_{ij}

⋆ change in **coefficient** of a **nonbasic** variable in the **constraint(s)**.

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + \overset{\to 50}{30}x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \quad \to \text{before} \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

The company is changing the resources for $\boldsymbol{x_2}$ as $8x_1 + 5x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 4x_2 + 0.5x_3 \leq \$8$. Also, the company is changing the profit of that product to 50. What is your recommendation to the manager?

↳ combination of changes

after

Case 2-1 for sensitivity analysis

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c row → c	$\underbrace{\boxed{c_b^T B^{-1} a_j - c_j^T}}_{c_b^T B^{-1} N - c_n^T}$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

c_n, a_{ij} ↓ for N

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [\overset{50}{\cancel{30}}, 0, 0]^\top.$$

$\boldsymbol{x}_2$ is a nonbasic variable. We have change in $a_{\boldsymbol{x}_2}$ in $\boldsymbol{N}$, and a change in $c_{\boldsymbol{x}_2}$ so:

$$\underbrace{\boldsymbol{c}_b^\top \boldsymbol{B}^{-1} \boldsymbol{a}_{\boldsymbol{x}_2} - c_{\boldsymbol{x}_2}} = \underbrace{[0, 20, 60]}_{c_b^T} \underbrace{\begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}}_{B^{-1}} \underbrace{\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}}_{a_{x_2}} - 50 = 10 \geq 0.$$

It does not change the optimal solution so it does not change the total profit.
If that would become negative, we should have continued the table!

Case 2-2 of Change

Case 2-2 for sensitivity analysis

⋆ change in **coefficient** of a **basic** variable in the constraint(s). $\to a_{ij}$ for basic (in B)

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$
\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

The company is changing the resources for $\boldsymbol{x}_1$ as $5x_1 + 6x_2 + x_3 \leq \$48$, $5x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $x_1 + 1.5x_2 + 0.5x_3 \leq \$8$. What is your recommendation to the manager?

Case 2-2 for sensitivity analysis

	$x_n^\top$	$x_B^\top$	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^\top B^{-1} a_j - c_j^\top}_{c_b^\top B^{-1} N - c_n^\top}$	$\underbrace{0^\top}_{c_b^\top B^{-1} B - c_b^\top \to 0}$	$c_b^\top B^{-1} b$

B

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

Previous $\boldsymbol{B}$ was: $\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}$.

$\boldsymbol{x}_1$ is a basic variable. We have change in $a_{\boldsymbol{x}_1}$ in $\boldsymbol{B}$, so:

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c}_n^\top = [0, 20, 60] \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1.5 & 5 \\ 0 & 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}.$$

We compute it. If any of the values becomes negative, we should continue the table; otherwise, the total profit does not change.

Case 3 of Change

Case 3 for sensitivity analysis

$\star$ **adding** a new **variable** to optimization.

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$
\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

The company is adding a new product $\boldsymbol{x}_4$ with profit (a) \$15 or (b) \$25, and the constraint coefficients $\boldsymbol{a} = [1, 1, 1]^\top$. What is your recommendation to the manager?

Case 3 for sensitivity analysis

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^T B^{-1} a_j - c_j^T}_{c_b^T B^{-1} N - c_n^T}$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

$\boldsymbol{x}_4$ is a nonbasic variable. We calculate its value in the last row of the table (if $c_{\boldsymbol{x}_4} = 15$):

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1} \boldsymbol{a}_{\boldsymbol{x}_4} - c_{\boldsymbol{x}_4} = [0, 20, 60] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 15 = 5 \geq 0.$$

It does not change the optimal solution so it does not change the total profit.

Case 3 for sensitivity analysis

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^T B^{-1} a_j - c_j^T}_{c_b^T B^{-1} N - c_n^T}$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

$\boldsymbol{x}_4$ is a nonbasic variable. We calculate its value in the last row of the table (if $c_{\boldsymbol{x}_4} = 25$):

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1} \boldsymbol{a}_{\boldsymbol{x}_4} - c_{\boldsymbol{x}_4} = [0, 20, 60] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 25 = -5 < 0.$$

We should continue the table.

Case 3 for sensitivity analysis

$$B^{-1}a_{x_4} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

	x_2	s_2	s_3	s_1	x_3	x_1	x_4	RHS	min test:
s_1	-2	2	-8	1	0	0	-5	24	—
x_3	-2	2	-4	0	1	0	-2	8	—
x_1	$5/4$	$-1/2$	$3/2$	0	0	1	1	2	$2/1=2$
c	5	10	10	0	0	0	-5	280	

	x_2	s_2	s_3	s_1	x_3	x_1	x_4	RHS	
s_1	$\frac{17}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	5	0	34	$\Rightarrow s_1^* = 34$
x_3	$\frac{1}{2}$	1	-1	0	1	2	0	12	$\Rightarrow x_3^* = 12$
x_4	$5/4$	$-1/2$	$3/2$	0	0	1	1	2	$\Rightarrow x_4^* = 2$
c	$\frac{45}{4}$	$\frac{15}{2}$	$\frac{35}{2}$	0	0	5	0	290	$\Rightarrow c^* = 290$

all nonnegative

So, the optimum objective function has increased and this addition of variable in beneficial.

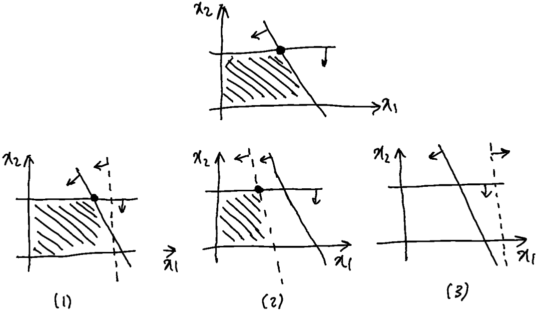
Case 4 of Change

Case 4 for sensitivity analysis

⋆ **adding** a new **constraint** to optimization.

This can result in three sub-cases:

- 4-1: The current optimal solution **satisfies** the new constraint.
- 4-2: The current optimal solution **doesn't satisfy** the new constraint but linear programming still **has a feasible solution**.
- 4-3: The current optimal solution **doesn't satisfy** the new constraint and linear programming **doesn't have a feasible solution**.



Question: Can adding a constraint improve the optimum value of objective function?

Case 4-1 for sensitivity analysis

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$\begin{aligned} \underset{x_1, x_2, x_3, s_1, s_2, s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{aligned}$$

We saw in the table (see slide 8) that the solution is: $x_1^* = 2, x_2^* = 0, x_3^* = 8$.
The company is adding a new resource constraint:

$$x_1 + x_2 + x_3 \leq 11.$$

It satisfies the current solution:

$$2 + 0 + 8 = 10 \leq 11 \quad \surd$$

Case 4-2 for sensitivity analysis

We saw in the table (see slide 8) that the solution is: $x_1^* = 2, x_2^* = 0, x_3^* = 8$.
The company is adding a new resource constraint: $x_2 \geq 1$. It doesn't satisfy the current solution: $0 \ngeq 1$.
The new constraint:

$$x_2 \geq 1 \implies -x_2 \leq -1 \implies -x_2 + s_4 = -1.$$

	x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
s_1	-2	2	-8	1	0	0	0	24
x_3	-2	2	-4	0	1	0	0	8
x_1	$\frac{5}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	1	0	2
s_4	-1	0	0	0	0	0	0	-1
C	5	10	10	0	0	0	0	280

	x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
s_1	0	2	-8	1	0	0	-2	26
x_3	0	2	-4	0	1	0	-2	10
x_1	0	-0.5	1.5	0	0	1	1.25	0.75
x_2	1	0	0	0	0	0	-1	1
C	10	10	0	0	0	0	5	275

all nonnegative

$s_1^* = 26, x_3^* = 10, x_1^* = 0.75, x_2^* = 1, C^* = 275$

Note: we have used the dual simplex method above.

Case 4-3 for sensitivity analysis

We saw in the table (see slide 8) that the solution is: $x_1^* = 2, x_2^* = 0, x_3^* = 8$.
The company is adding a new resource constraint: $x_1 + x_2 \geq 12$. It doesn't satisfy the current solution: $2 \ngeq 12$.
The new constraint:

$$x_1 + x_2 \geq 12 \implies -x_1 - x_2 \leq -12 \implies -x_1 - x_2 + s_4 = -12.$$

	x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
s_1	-2	2	-8	1	0	0	0	24
x_3	-2	2	-4	0	1	0	0	8
x_1	5/4	-1/2	3/2	0	0	1	0	2
s_4	-1	0	0	0	0	-1	1	-12
c	5	10	10	0	0	0	0	280

min test:
$\left|\frac{5}{-1}\right| = 5$
$\left|\frac{0}{-1}\right| = 0$

		x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
	s_1	-2	2	-8	1	0	0	0	24
	x_3	-2	2	-4	0	1	0	0	8
$r_3 + r_4$	x_1	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	0	0	-10
$-r_4$	x_1	1	0	0	0	0	1	-1	12
	c	5	10	10	0	0	0	0	280

min test:
$\left|\frac{10}{-\frac{1}{2}}\right| = 20$

Note: we have used the dual simplex method above.

Case 4-3 for sensitivity analysis

		x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
	s_1	-2	2	-8	1	0	0	0	24
	x_3	-2	2	-4	0	1	0	0	8
r_3+r_4	x_1	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	0	0	-10
$-r_4$	x_1	1	0	0	0	0	1	-1	12
	c	5	10	10	0	0	0	0	280

min test: $\left|\frac{10}{-\frac{1}{2}}\right| = 20$

		x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
r_1+4r_3	s_1	-1	0	-2	1	0	0	4	-16
r_2+4r_3	x_3	-1	0	2	0	1	0	4	-32
$-2r_3$	s_2	$-\frac{1}{2}$	1	-3	0	0	0	-2	20
r_4	x_1	1	0	0	0	0	1	-1	12
r_5+20r_3	c	10	0	40	0	0	0	20	80

min test: $\left|\frac{10}{-1}\right| = 10$

		x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
r_1-r_2	s_1	0	0	-4	1	-1	0	0	16
$-r_2$	x_2	1	0	-2	0	-1	0	-4	32
$r_3-\frac{1}{2}r_2$	s_2	0	1	-4	0	-0.5	0	-4	36
r_4+r_2	x_1	0	0	2	0	1	1	3	-20
r_5+10r_2	c	0	0	60	0	10	0	60	-240

no negative entry in row!

Therefore, it does not have a feasible solution!

Acknowledgment

This lecture is inspired by the lectures of Prof. Shokoufeh Mirzaei on sensitivity analysis in linear programming: [Link]