Introduction to the Theory of Computation
Homework #8
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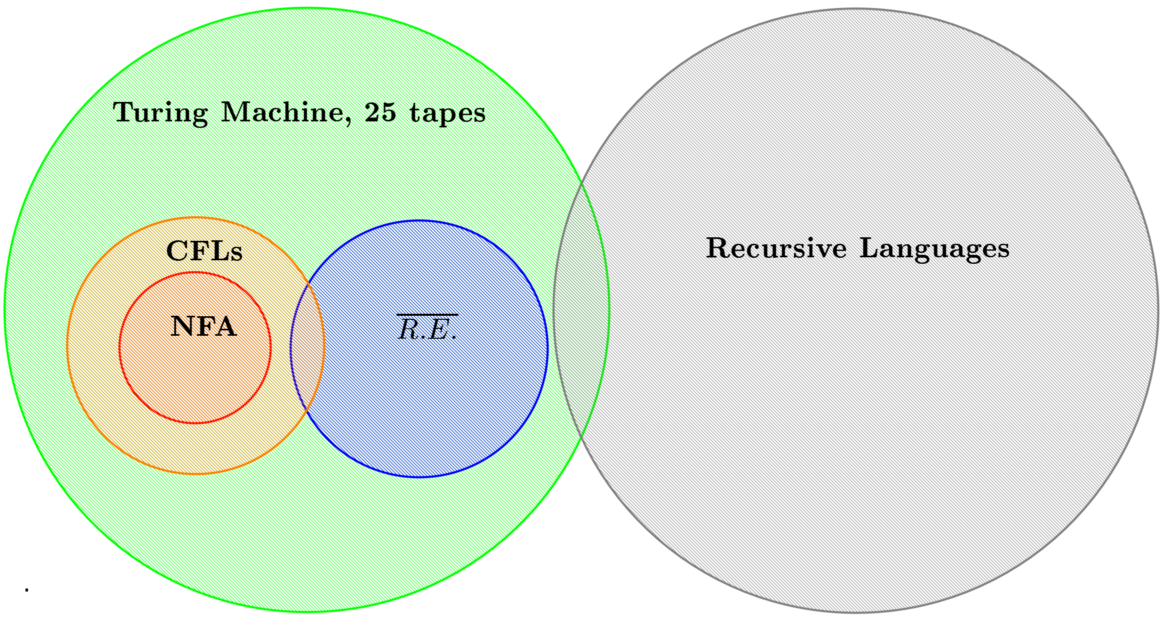
$\boxed{1}$ Venn Diagram:



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$\boxed{2}$ **Proof:**

- M = "On input x check if $x = (G)$ where G is an CFG .
  1. Convert G into Chomsky normal form.
  2. Mark all non-terminals NT which have some rule $NT \to n$
  3. Repeat until no new non-terminals are marked:
     (a) Mark the non-terminal NT if there is a rule
     (b) $NT \to XZ$ such that Z and X are already marked.
  4. If W is marked ($L(G) \neq \emptyset$) then M rejects, else accepts."
- Since decider M can be constructed for the problem: "Given a CFG G, is $L(G)$ not empty?" then it must be decidable

$\square$

$\boxed{3}$ **Proof:**

- Assume B is countable and that $f\colon \mathbb{N} \rightarrow \mathbb{B}$ is a functional correspondence.
- Let $f_i(j)$ denote the i^{th} symbol in $f(j)$.
- Consider the string s whose i^{th} element differs from $f_i(i)$, then the i^{th} symbol in s is 1 and vice versa.
- Then s cannot be in the image of f since it differs from every string in the image of f by at least one symbol.
- Furthermore, s is in B since it is an infinite sequence over $\{0, 1\}$.
- Therefore, B is uncountable.

□

$\boxed{4}$ **Proof:**

- $M =$ "On input $< R, S >$ where R and S are regular expressions
  1. Convert R to DFA A and S to DFA B
  2. Construct DFA C such that $L(C) = L(B)?L(A)$
  3. Submit $< A, C >$ to the decider for EQDFA
  4. If it accepts, *accept*.
  5. If it rejects, *reject*."
- M is a decidable since steps 1, 2, 4, and 5 will not create and infinite loops and step 2 calls a decider.
- Also, M accepts $< R, S >$ if and only if $L(R) = L(R)\ ?\ L(S)$.
- Therefore, M is a decider for A so A is decidable.

□

$\boxed{5}$ **Problem:** Given a CFG G, is $L(G) = \Sigma^*$

(a) CFGs are strictly more powerful than REs, this problem is in both RE and $\overline{RE}$.

(b) Proof Sketch:

- Given that ALL_{CFG} is is undecidable, $L(M_1) = L(M_2)$ is a similar problem.
- As it asks if each FA produces the same set of all possible strings.
- As the set of strings is countably infinite, the question is answerable but it would in essence never return.
- For example any case similar to a^*.
- In summary, the problem is atleast as hard as ALL_{CFG}, and would create a endless loop.
- Thus the problem is undecidable.

□

$\boxed{6}$ **Proof:**

- We show that the language T is undecidable by showing that $H_{TM} \geq T$.
- The reduction function f takes as input (M, w) where M is a Turing Machine and w is a string.
- f outputs $f(M, w) = (N_{M,w})$, where $N_{M,w}$ is the Turing Machine that does the following:
  - On input x:
    1. If $x = 01$, then *accept*.
    2. Else if $x \neq 10$, then *reject*.
    3. Else then simulate M on w. If M halts on w, then *accept*.
- Clearly, the function f is computable. If $(M, w) \in H_{TM}$ or M halts on w then $L(N_{M,w}) = \{01, 10\}$, thus $(N_{M,w}) \notin T$.
- We conclude that $(M, w) \in H_{TM}$ if and only if $f(M, w) \in T$.
- Therefore f is a reduction from H_{TM} to T. It follows that T is undecidable as H_{TM} is undecidable.

□

$\boxed{7}$ **Proof:**

- We show that $USELESS_{TM}$ is undecidable by showing that $\overline{H_{TM}} \geq USELESS_{TM}$.
- The reduction function f takes as input (M, w) where M is a Turing Machine and w is a string.
- f outputs $f(M, w) = (N_{M,w})$, where $N_{M,w}$ is the Turing Machine that does the following:
  - On input symbol 0, $N_{M,w}$ simply enters all it's states except for a special state $\overline{q}$.
  - On input symbol 1, $N_{M,w}$ simulates M on w, if M halts on w, then enters state $\overline{q}$.
- Clearly, the function f is computable.
- From the behavior of $N_{M,w}$ on input symbol 0, it is clear that only the special state $\overline{q}$ can be useless.
- From the behavior of $N_{M,w}$ on input symbol 1, it is clear that $\overline{q}$ is useless if and only if M does not halt on w.
- Therefore $(M, w) \in \overline{H_{TM}}$ if and only if $f(M, w) \in USELESS_{TM}$, and hence f is a reduction from $\overline{H_{TM}}$ to $USELESS_{TM}$.
- It follows that $USELESS_{TM}$ is undecidable as H_{TM} is undecidable.

□