Introduction to the Theory of Computation
Homework #7

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$\boxed{1}$ Proof: By Mathematical Induction

- Let T be any binary tree.
- Let $height(x)$ be the height of any binary tree x.
- Let $leaves(x)$ be the number of leaves in any binary tree x.
- Suppose for any $n \in N$ if $height(T) < n$ then, $leaves(T) < 2^n$
- **Basis Step:**
- We have the binary tree T with one node, and no leaves.
- Since $height(T) = 0$ and $leaves(T) = 0$ then $0 < n$ is true and $0 < 2^n$ is also true.
- **Induction:**
- Assume an $n \in N$.
- Let T be any arbitrary tree of size s, and by definition of a binary tree T is the root with two sub-tree's.
- Let $height(T)$ of a binary tree be defined as: $height(T) = 1 + max(height(T_L), height(T_R))$
- Since $leaves(T)$ must be an integer it can be said that: $leaves(T) \leq \lceil \frac{s}{2} \rceil$
- Accordingly it can also be said that $height(T) \leq \lceil \frac{s}{2} \rceil$
- Thus if $\lceil \frac{s}{2} \rceil < n$ then, $\lceil \frac{s}{2} \rceil < 2^n$ must also hold.
- Therefore we have established for any $n \in N$ if $height(T) < n$ then, $leaves(T) < 2^n$

$\square$

$\boxed{2}$ Proof: By Contradiction

- Suppose $L = \{a^i | i \text{ is a square } \}$ is a CFL.
- Then the pumping lemma applies and there exists a number n so that any sufficiently long string $z \in L$ can be pumped.
- Pick $z = a^n$
- Since $|z| \geq n$, it follows that $z = uvwzy$, $|vwx| \leq n$, and $|vx| > 0$.
- Since $|vwx| \leq n$, vx can contain at most the two different symbols. Further, if there are two symbols they must be adjacent.
- Suppose xv contains one symbol. Then that symbol can be increased and so a^n is in L for $k > n$. But none of those are in L.
- Thus we arrive at a contradiction.
- So Suppose vx contains two symbols. Then two adjacent symbols can be increased by the same amount.
- If such an increase leads to alteration, we immediately get a contradiction.
- Otherwise, only adjacent symbols are increased, and nonadjacent symbols are left balanced again leading to a contradiction.
- For example $aa^n \notin L$ or $a^{11}a^n \notin L$.
- Thus in all cases, we obtain a contradiction so L is a CFL can not be the case.

$\square$

$\boxed{3}$ Are the following languages CFLs?

(a) Yes

(b) Yes

(c) No

(d) Yes

(e) No

$\boxed{4}$ Sipser, excercise 3.1

(c) 000.

$\vdash_m q_1000\sqcup$
$\vdash_m \sqcup q_200\sqcup$
$\vdash_m \sqcup xq_30\sqcup$
$\vdash_m \sqcup x0q_4\sqcup$
$\vdash_m q_{reject}$

(d) 000000.

$\vdash_m q_1000000$
$\vdash_m \sqcup q_200000$
$\vdash_m \sqcup xq_30000$
$\vdash_m \sqcup x0q_4000$
$\vdash_m \sqcup x0xq_300$
$\vdash_m \sqcup x0x0q_40$
$\vdash_m \sqcup x0x0xq_3$
$\vdash_m \sqcup x0x0xq_3$
$\vdash_m \sqcup x0x0q_5x$
$\vdash_m \sqcup x0xq_50x$
$\vdash_m \sqcup x0q_5x0x$
$\vdash_m \sqcup xq_50x0x$
$\vdash_m q_5 \sqcup x0x0x$
$\vdash_m \sqcup q_2x0x0x$
$\vdash_m \sqcup xq_20x0x$
$\vdash_m \sqcup xxq_3x0x$
$\vdash_m \sqcup xxxq_30x$
$\vdash_m \sqcup xxx0xq_4$
$\vdash_m q_{reject}$

$\boxed{5}$ Sipser, exercies 3.2

(c) 1##1

$\vdash_m q_11\#\#1$
$\vdash_m xq_3\#\#1$
$\vdash_m x\#q_5\#1$
$\vdash_m q_{reject}$

(d) 10##11

$\vdash_m q_110\#\#11$
$\vdash_m xq_30\#\#11$
$\vdash_m x0q_3\#\#11$
$\vdash_m x0\#q_5\#11$
$\vdash_m q_{reject}$

$\boxed{6}$ A machine which accepts a's then b's. The number of b's have to be greater than the number of a's but less than twice the number of a's