ARCH/GARCH

(Generalized) Autoregressive Conditional Heteroskedasticity

(Generalized) Autoregressive Conditional Heteroskedasticity

- Volatility (risk) may be changing in a predictable way.
- Volatility is the conditional standard deviation of returns.
- Used a lot in financial time series analysis, but a lot of recent research in energy & commodities.
- Financial time series often ARMA(0,0) but strong autocorrelation in volatility (e.g., volatility clustering).
- Characterizing autocorrelation in volatility is helpful for forecasting, capturing kurtosis, option pricing, portfolio optimization, risk management
  - Can forecast volatility, and increase precision of forecasted returns.
  - Even without autocorrelation in the error or dependent variable, **variance of the errors may be autocorrelated.**
  - OLS would be unbiased, but not efficient

One application: Value at Risk

- Value at Risk: A measure of potential financial loss.
- For a given quantile of the returns distribution, what is the maximal loss from a given financial position during a given time period?
- Suppose I hold a \$100,000 long position in an equity. For the 95% quantile of the returns distribution, what is the greatest potential loss from market fluctuations between today and tomorrow?
- For what potential loss is there a 95% probability that actual losses will be smaller?

Value at Risk: Example

- You have \$10 million invested in IBM.
- You estimate daily returns as r_t with mean $\mu_r = -0.001$ and variance $\sigma^2 = 0.0002$ using ARMA(0,0)-GARCH(0,0).
- The 95% returns quantile is $-0.001 + 1.65 \cdot \sqrt{0.0002} = 0.0223$
- Your VaR for holding this position until tomorrow is $0.0233 \cdot 10,000,000 = \$233,000$.

Value at Risk: Example

- You estimate an ARMA(2,0)-GARCH(1,1) model of daily returns:

$$r_t = -0.00066 - 0.0247 r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.00000389 + 0.0799 a_{t-1}^2 + 0.9073 \sigma_{t-1}^2$$

- Based on this model, you forecast $\hat{r}(1) = -0.00071, \quad \hat{\sigma}^2(1) = 0.0003211$
- The 95% returns quantile is $-0.00071 + 1.65 \cdot \sqrt{0.0003211} = 0.0288$.
- Your VaR for holding the position until tomorrow is $0.0288 \cdot 10,000,000 = \$288,000$.
- By taking into account that volatility has been high recently, and forecasting tomorrow's volatility, you get a better picture of the relevant risk for today's decision.

ARCH

- ARCH model structure:
  - $r_t = \mu_t + a_t$
  - $\mu_t = E(r_t|F_{t-1})$, $\sigma_t^2 = Var(a_t|F_{t-1})$, where F_t is the information set.
  - $\mu_t = \mu$, or $\phi(L)r_t + \theta(L)a_t$, or $x_t'\beta$, or a combination
- A simple ARCH(1) model:
  - $r_t = x_t'\beta + a_t$
  - $a_t = \epsilon_t\sqrt{\alpha_0 + \alpha_1 a_{t-1}^2} = \epsilon_t\sigma_t$
  - $\epsilon_t \sim N(0, 1)$ (changing the variance of ϵ will just rescale the α's)
  - Could model ϵ_t using a different distribution (standardized Student-t, generalized error, and skewed versions of these) to deal with non-Normal volatility
  - By iterated expectations $E(a_t|x_t) = 0$ and $E(r_t|x_t) = x_t'\beta$, and this is a classical regression

ARCH

- $\sigma_t^2 = Var(a_t|a_{t-1}) = E(a_t^2|a_{t-1}) = E(\epsilon_t^2)(\alpha_0 + \alpha_1 a_{t-1}^2) = \alpha_0 + \alpha_1 a_{t-1}^2$
  - a_t is heteroskedastic conditional on past errors, not on x.
  - $Var(a_t) = \alpha_0 + \alpha_1 E(a_{t-1}^2) = \alpha_0 + \alpha_1 Var(a_{t-1})$
    - As long as a_t is covariance stationary,
    - $Var(a_t) = Var(a_{t-1}) = \alpha_0 + \alpha_1 Var(a_{t-1}) = \frac{\alpha_0}{1-\alpha_1}$
  - If we calculate the 4th moment, ARCH also implies fatter tails/more excess kurtosis than a Normal dist.

ARCH

- Estimate by MLE:
  - For normal linear model: $lnL = -\frac{n}{2} ln2\pi - \frac{n}{2} ln\sigma^2 - \frac{1}{2\sigma^2} \sum a_i^2$
  - Here we have $\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2$, so
    - $lnL = \frac{-T}{2} ln2\pi - \frac{1}{2} \sum ln(\alpha_0 + \alpha_1 a_{t-1}^2) - \frac{1}{2} \sum \frac{a_t^2}{\alpha_0 + \alpha_1 a_{t-1}^2}$
  - Construct the residuals $\hat{a}_t$ from the estimated parameters of μ_t.
  - Construct the volatilities $\hat{\sigma}_t$ from the estimates of α and the squared $\hat{a}_t$.
  - If we pick a different distribution for ϵ in order to capture skewness and kurtosis in volatility, just maximize a different likelihood function.

ARCH

- In ARCH(1) we have $\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2$
- In ARCH(q) we could have $\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \alpha_2 a_{t-2}^2 + \ldots + \alpha_q a_{t-q}^2$
  - This is a q-th order moving average (MA(q)) in the squared residuals.
- We could estimate this using GMM.
  - We have one of the same old moment conditions and one new one:
  - $E(x_t \epsilon_t) = E\left[(y_t - x_t'\beta)x_t\right] = 0$
  - $E\left[(\sigma_t^2 - a_t^2)a_{t-i}^2\right] = 0$ for $i = 1, \ldots, q$

Weaknesses of ARCH

- Assumes positive and negative shocks have same effect on volatility - does not allow for leverage effects
- The ARCH α coefficients are constrained to a limited range in order to maintain finite variance and kurtosis. This restriction limits how well the ARCH model can fully describe kurtosis that exists.
- It provides little insight into the financial markets and what causes volatility to vary - just a description of observed statistical behavior of a variable.
- Respond slowly to large isolated shocks.

Model Building

Tsay's 4 steps

1. specify a model for the mean (e.g., ARMA, covariate regression)
2. use the residuals from (1) to test for ARCH
3. If ARCH present, specify a volatility model and estimate jointly with the model for the mean in (1). Use PACF of squared residuals as a guide.
4. Check the fit and refine if needed. Standardized model residuals should be iid.

Testing for ARCH

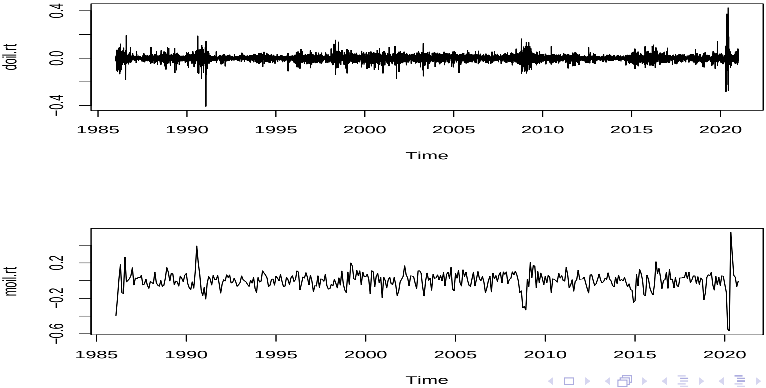
Two approaches:

1. Construct the $\hat{a}_t^2$ series from the model residuals. Apply Portmanteau/Ljung-Box tests for autocorrelation to this series.
2. Estimate an AR model for $\hat{a}_t^2$ and do an F-test for joint significance of the lags (ARCH-LM test)

Clues: look at ACF/PACF of the $\hat{a}_t^2$ series.

Example: Daily & Monthly WTI Returns

```
## [1] "DCOILWTICO"

## Warning in log(doil):  NaNs produced

## [1] "MCOILWTICO"
```



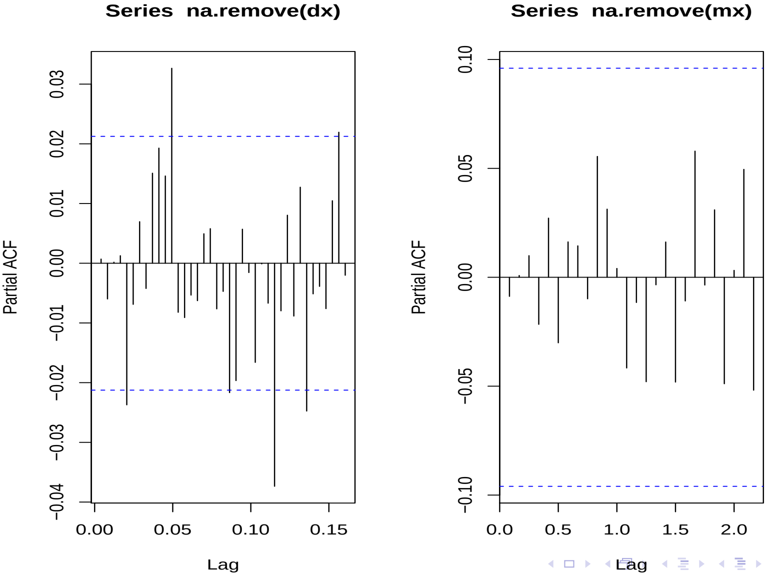
Example: Daily & Monthly WTI Returns

Find reasonable ARIMA models for the mean and capture residuals

```
# auto.arima(doil.rt) ARMA(0,0)
# ar(doil.rt) AR(24)
# auto.arima(moil.rt) SARMA(2,2)(1,2)
m = Arima(moil.rt,order=c(2,0,2),seasonal=list(order=c(1,0,2)
                              ,period=12),include.constant=TRUE)
mx = residuals(m)
mx2 = mx^2
# a little slow
dx = residuals(Arima(doil.rt,order=c(24,0,0),include.constant=FALSE))
dx2 = dx^2
```

Example: Daily & Monthly WTI Returns

No autocorrelation in the residuals



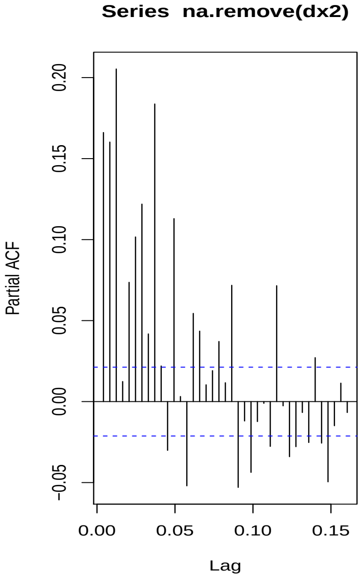
Example: Daily & Monthly WTI Returns

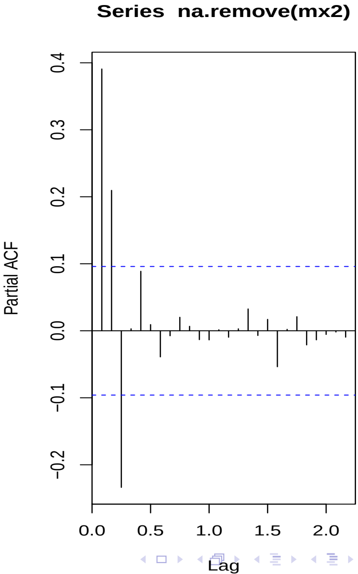
No autocorrelation in the residuals

```
Box.test(dx,lag=24,type='Ljung')

##
##  Box-Ljung test
##
## data:  dx
## X-squared = 0.69835, df = 24, p-value = 1

Box.test(mx,lag=15,type='Ljung')

##
##  Box-Ljung test
##
## data:  mx
## X-squared = 4.6918, df = 15, p-value = 0.9944
```

Example: Daily & Monthly WTI Returns

Clear autocorrelation in the **squared** residuals





Example: Daily & Monthly WTI Returns

Clear autocorrelation in the **squared** residuals

```
Box.test(dx2,lag=24,type='Ljung')

##
##  Box-Ljung test
##
## data:  dx2
## X-squared = 4988.9, df = 24, p-value < 2.2e-16

Box.test(mx2,lag=6,type='Ljung')

##
##  Box-Ljung test
##
## data:  mx2
## X-squared = 110.55, df = 6, p-value < 2.2e-16
```

GARCH

- $r_t = x_t'\beta + a_t$
- F_t is an information set at time t.
- $a_t|F_t \sim N(0, \sigma_t^2)$, σ_t^2 is the conditional variance.
- $\sigma_t^2 = \alpha_0 + \delta_1\sigma_{t-1}^2 + \delta_2\sigma_{t-2}^2 + ... + \delta_p\sigma_{t-p}^2 + \alpha_1 a_{t-1}^2 + \alpha_2 a_{t-2}^2 + ... + \alpha_q a_{t-q}^2$
- $\sigma_t^2 = \gamma'\mathbf{z_t}$ where $\mathbf{z_t}$ $=(1, \sigma_{t-1}^2, ..., \sigma_{t-p}^2, a_{t-1}^2, ..., a_{t-q}^2)'$ and
  - $\gamma = (\alpha_0, \delta', \alpha')'$

GARCH

- The conditional variance is an ARMA(p,q) process in squared disturbances.
  - $\sigma_t^2 = \alpha_0 + D(L)\sigma_t^2 + A(L)a_t^2$
- This is GARCH(p,q)
- GARCH with small p and q performs better than ARCH with large q.
- Notice before we required stationarity of a_t. We need the moments to be finite.
- The stationarity condition is that the roots of $1 - D(z)$ lie outside the unit circle.
- There are also IGARCH models where the volatility process has a unit root.

GARCH

- To get the intuition, consider a GARCH(1,1) and a stronger assumption: $D(1) + A(1) < 1$, or that the coefficents on the AR and MA terms are small enough to limit dependence.
  - Now $Var(a_t) = \frac{1}{1-D(1)-A(1)}$
  - We still have $E(a_t) = 0$ and $Cov(a_t, a_s) = 0$ for $t \neq s$, so still a classical regression model.

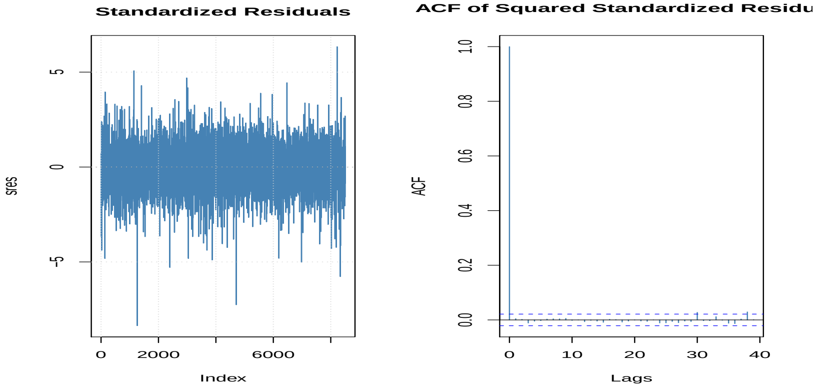
ARCH/GARCH Example: Daily WTI Returns

PACF of **daily** squared residuals suggests a long ARCH - maybe ARCH(29) or more. Using fGarch package:

```
# notice the ARCH order comes first, GARCH order second
# (opposite of arima syntax)
arch.d29 = garchFit(~garch(29,0),data=na.remove(dx),trace=F)

## Warning: Using formula(x) is deprecated when x is a character
vector of length > 1.
## Consider formula(paste(x, collapse = " ")) instead.

## summary(arch.d29) # summary output is very long. See in R.
```

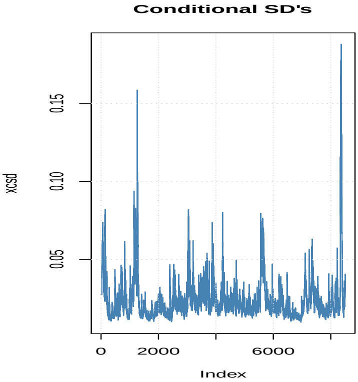
ARCH/GARCH Example: Daily WTI Returns

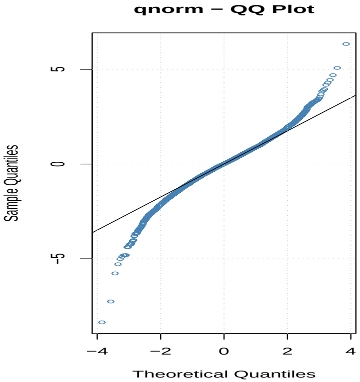
Residuals look
like white noise, autocorrelation mostly gone from squared residuals.



ARCH/GARCH Example: Daily WTI Returns

Also heavy tails from QQ plot





ARCH/GARCH Example: Daily WTI Returns

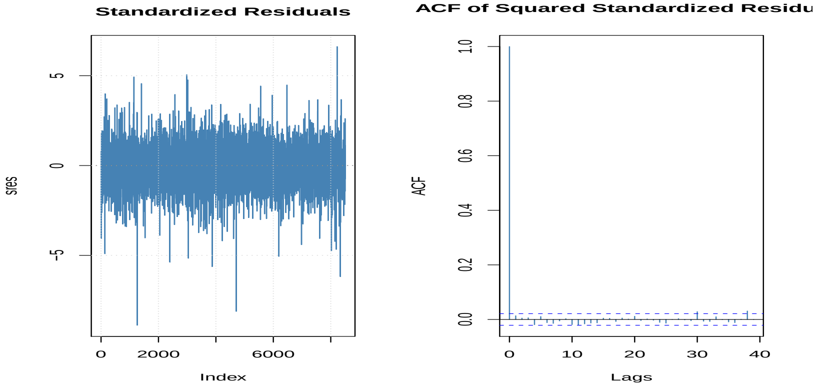
Try a GARCH(1,1) - simpler, smaller model

```
# notice the ARCH order comes first, GARCH order second
# (opposite of arima syntax)
arch.d11 = garchFit(~garch(1,1),data=na.remove(dx),trace=F)

## Warning: Using formula(x) is deprecated when x is a character
vector of length > 1.
## Consider formula(paste(x, collapse = " ")) instead.

## summary(arch.d11) # summary output is very long. See in R.
```

$$\hat{\sigma}_t^2 = 0.000006 + 0.9\sigma_{t-1}^2 + 0.098a_{t-1}^2$$

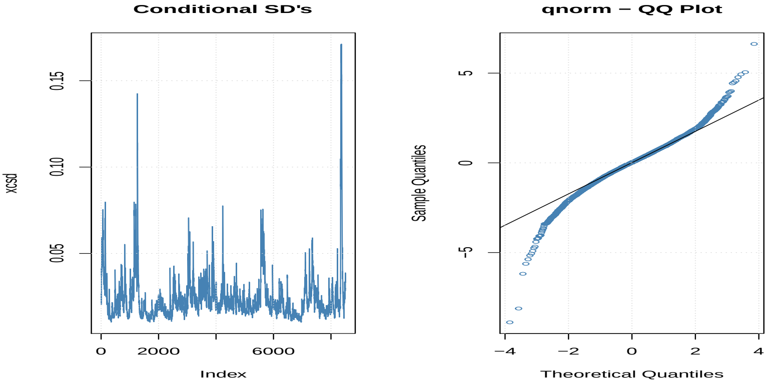
ARCH/GARCH Example: Daily WTI Returns

Similar result, much simpler



ARCH/GARCH Example: Daily WTI Returns

Still some heavy tails - should try a different distribution



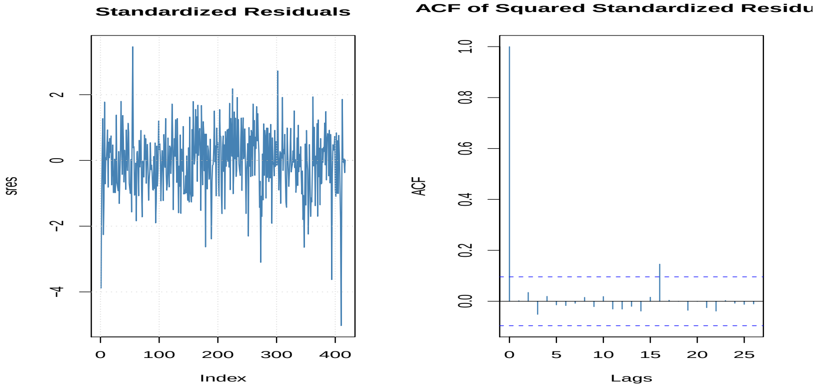
ARCH/GARCH Example: Monthly WTI Returns

PACF of **monthly** squared residuals suggests about ARCH(6):

```
# notice the ARCH order comes first, GARCH order second
# (opposite of arima syntax)
arch.m6 = garchFit(~garch(6,0),data=na.remove(mx),trace=F)

## Warning:  Using formula(x) is deprecated when x is a character
vector of length > 1.
##   Consider formula(paste(x, collapse = " ")) instead.

## summary(arch.m6) # summary output is very long. See in R.
```

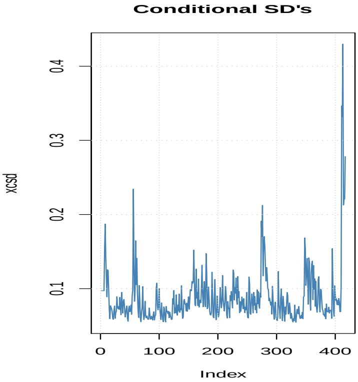
ARCH/GARCH Example: Monthly WTI Returns

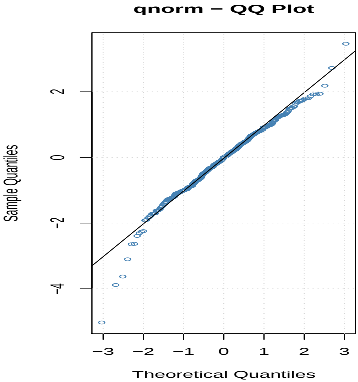
Residuals look
like white noise, autocorrelation mostly gone from squared residuals.



ARCH/GARCH Example: Monthly WTI Returns

Still some heavy tails but not as severe as daily returns





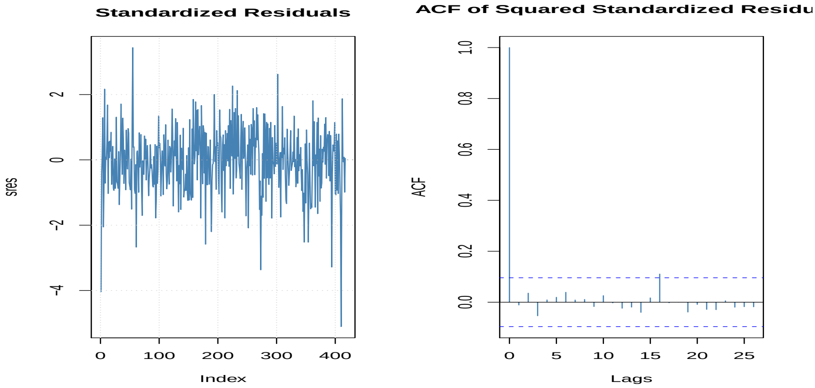
ARCH/GARCH Example: Monthly WTI Returns

Can still get a better fit with GARCH(1,1)

```
# notice the ARCH order comes first, GARCH order second
# (opposite of arima syntax)
arch.m11 = garchFit(~garch(1,1),data=na.remove(mx),trace=F)

## Warning:  Using formula(x) is deprecated when x is a character
vector of length > 1.
##  Consider formula(paste(x, collapse = " ")) instead.

## summary(arch.m11) # summary output is very long. See in R.
```

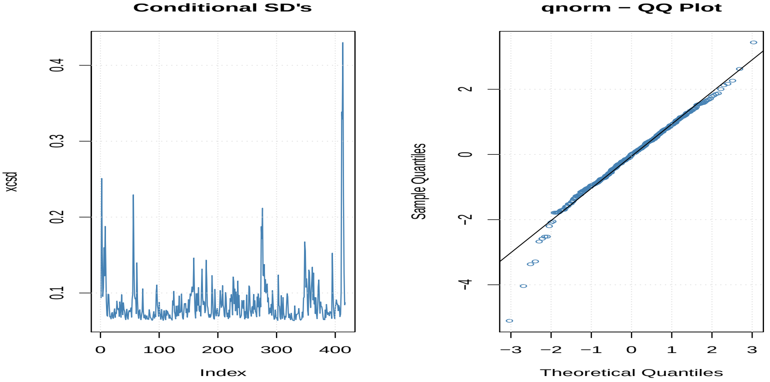
ARCH/GARCH Example: Monthly WTI Returns

Pretty similar to ARCH(6), but GARCH(1,1) has lower BIC/HQ



ARCH/GARCH Example: Monthly WTI Returns

Pretty similar to ARCH(6), but GARCH(1,1) has lower BIC/HQ



ARCH/GARCH Example: WTI Returns

We can estimate the models for the mean and volatility jointly

```
# Estimated daily mean and volatility models jointly
arch.d2 = garchFit(~arma(24,0)+garch(1,1),data=na.remove(doil.rt)
                                  ,trace=F,cond.dist='sstd')
summary(arch.d2)
plot(arch.d2)

# Estimate monthly jointly, but remove seasonal component first
sm = residuals(Arima(moil.rt,order=c(0,0,0),seasonal=
                        list(order=c(1,0,2),period=12)
                         ,include.constant=TRUE))
arch.m2 = garchFit(~arma(2,2)+garch(1,1),data=na.remove(sm)
                                      ,trace=F,cond.dist='std')
summary(arch.m2)
plot(arch.m2)
# forecast the mean return and volatility
predict(arch.m2,5)
```

GARCH Testing

- Testing for GARCH is easy.
  - Remember that a long ARCH(q) is probably better captured by a short GARCH(p,q).
  - Greene suggests a Chi-square LM test for a long ARCH(q) that is thus also evidence of GARCH.
  - Use a consistent estimate of β (like OLS) to calculate the a_t and square them.
  - Run a regression of a_t^2 on q lagged values.
  - The number of observations in this regression multiplied by r-squared $T \cdot R^2$ is Chi-square(q).
  - The null hypothesis is no ARCH effects.
  - Various similar LM tests can be constructed for testing ARCH(p) against ARCH(p+q) or ARCH(q) against GARCH(p,q) or GARCH(p,0) against GARCH(p,q). Just compare $T \cdot R^2$ from the various regressions.