Estimating Vector Autoregressions (VAR)s for forecasting, impulse response function, understanding market structure. (1)

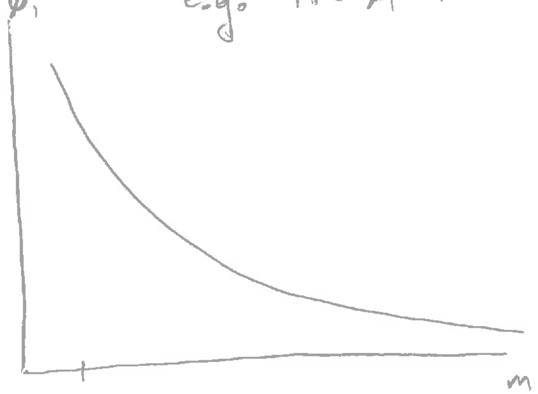
What is an impulse response function (IRF)?

univariate AR(1) : $y_t = \phi_1 y_{t-1} + a_t \Rightarrow y_{t+m} = \phi_1^{m+1} y_{t-1} + \sum_{s=0}^{m} \phi_1^s a_{t+m-s}$

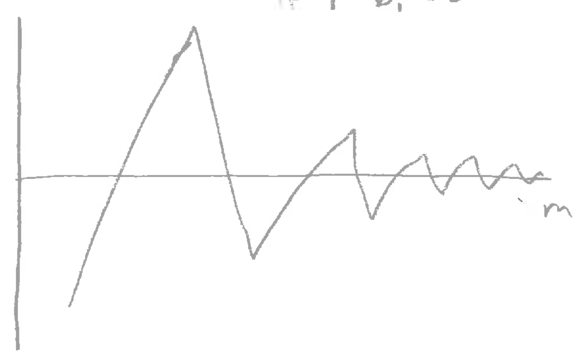
$$\frac{\partial y_{t+m}}{\partial a_t} = \phi_1^m$$

IRF = response of future y_{t+m} to past shock a_t as a function of the gap "m".

$\frac{\partial y_{t+m}}{\partial a_t} = \phi_1^m$ e.g. if $0<\phi_1<1$



ϕ_1^m if $-1<\phi_1<0$



VAR(1) case

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \underset{2\times1}{\underline{y}_t} = \underset{2\times2}{\underset{\sim}{F}}\ \underset{2\times1}{\underline{y}_{t-1}} + \underset{2\times1}{\underline{v}_t} = \underset{\sim}{F}^2 \underline{y}_{t-2} + \underset{\sim}{F}\underline{v}_{t-1} + \underline{v}_t \qquad \underset{\sim}{F}^2 = \underset{\sim}{F}\cdot\underset{\sim}{F}$$

$$\underline{y}_{t+m} = \underset{\sim}{F}^{m+1}\cdot \underline{y}_{t-1} + \sum_{s=0}^{m} \underset{\sim}{F}^s \underline{v}_{t+m-s}$$

$$\frac{\partial y_{i,t+m}}{\partial v_{jt}} = \text{row } i \text{, column } j \text{ element of } \underset{\sim}{F}^m = \frac{\partial \underline{y}_{t+m}}{\partial \underline{v}_t^T}$$

e.g. $$\underset{2\times2}{\underset{\sim}{F}^m} = \begin{bmatrix} \frac{\partial y_{1,t+m}}{\partial v_{1t}} & \frac{\partial y_{1,t+m}}{\partial v_{2t}} \\ \frac{\partial y_{2,t+m}}{\partial v_{1t}} & \frac{\partial y_{2,t+m}}{\partial v_{2t}} \end{bmatrix} \qquad \underline{v}_t = \begin{bmatrix} v_{1t} \\ v_{2t} \end{bmatrix}$$

Think about a univariate AR(2)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + a_t = \phi_1(\phi_1 y_{t-2} + \phi_2 y_{t-3} + a_{t-1}) + \phi_2(\phi_1 y_{t-3} + \phi_2 y_{t-4} + a_{t-2}) + a_t$$

$$= \phi_1^2 y_{t-2} + 2\phi_1\phi_2 y_{t-3} + \phi_2^2 y_{t-4} + \phi_1 a_{t-1} + \phi_2 a_{t-2} + a_t$$

it gets crazy to describe $\frac{\partial y_{t+m}}{\partial a_t} = f(\phi_1, \phi_2, m)$

Notice we can write univariate AR(2) is VAR(1) in companion form:

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} a_t \\ 0 \end{bmatrix}$$

$$\underset{\sim}{y}_t = \underset{\sim}{F}\, \underset{\sim}{y}_{t-1} + \underline{v}_t \quad \text{and we know} \quad \underset{\sim}{F}^m = \begin{bmatrix} \frac{\partial y_{t+m}}{\partial a_t} & 0 \\ \frac{\partial y_{t+m-1}}{\partial a_t} & 0 \end{bmatrix} \quad \underline{v}_t = \begin{bmatrix} a_t \\ 0 \end{bmatrix}$$

here's

here's our impulse response function $F^{(m)}_{1,1}$

notice for $n=1$, we want the 1,1 upper left block of $\underset{\sim}{F}^m$

or univariate AR(p)

$y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + a_t$ equivalent to

$$\begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & & \phi_p \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \vdots \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} a_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \underset{p\times 1}{\underset{\sim}{y}_t} = \underset{p\times p}{\underset{\sim}{F}}\, \underset{p\times 1}{\underset{\sim}{y}_{t-1}} + \underset{p\times 1}{\underline{v}_t}$$

$$\frac{\partial y_{t+m}}{\partial a_t} = F^{(m)}_{11} \quad \text{or} \quad \underset{\sim}{F}^m = \begin{bmatrix} \frac{\partial y_{t+m}}{\partial a_t} & 0 & \cdots & 0 \\ \frac{\partial y_{t+m-1}}{\partial a_t} & & & \\ \vdots & & & \\ & & & \end{bmatrix}$$

We also saw we can write VAR(p) as a VAR(1) in companion form:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \underset{2\times1}{\underset{\sim}{y}_t} = \underset{2\times2}{\underset{\sim}{\Phi}_1} \underset{2\times1}{\underset{\sim}{y}_{t-1}} + \underset{2\times2}{\underset{\sim}{\Phi}_2} \underset{2\times1}{\underset{\sim}{y}_{t-2}} + \underset{2\times1}{\underline{\varepsilon}_t} \quad \text{equivalent to}$$

$$\underset{\sim}{\xi}_t = \begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{1t-1} \\ y_{2t-1} \end{bmatrix} = \begin{bmatrix} \underset{2\times1}{\underset{\sim}{y}_t} \\ \underset{2\times1}{\underset{\sim}{y}_{t-1}} \end{bmatrix} = \begin{bmatrix} \underset{2\times2}{\underset{\sim}{\Phi}_1} & \underset{2\times2}{\underset{\sim}{\Phi}_2} \\ \underset{2\times2}{\underset{\sim}{I}} & \underset{2\times2}{\underset{\sim}{0}} \end{bmatrix} \begin{bmatrix} \underset{\sim}{y}_{t-1} \\ \underset{\sim}{y}_{t-2} \end{bmatrix} + \begin{bmatrix} \underset{2\times1}{\underline{\varepsilon}_t} \\ \underset{2\times1}{\underline{0}} \end{bmatrix}$$

Recall

$$\underset{\sim}{\Omega} = E\left[\underline{\varepsilon}_t \cdot \underline{\varepsilon}_t^T\right] = E\left(\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} \end{bmatrix}\right) = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

$$\underset{\sim}{\xi}_t = \underset{\sim}{F}\, \underset{\sim}{\xi}_{t-1} + \underline{v}_t$$

$$\underset{\sim}{\xi}_{t+m} = \underset{\sim}{F}^m \underset{\sim}{\xi}_t + \underline{v}_{t+m} + \underset{\sim}{F}\, \underline{v}_{t+m-1} + \dots + \underset{\sim}{F}^{m-1} \underset{\sim}{v}_{t+1}$$

$$\frac{\partial \underset{\sim}{\xi}_{t+m}}{\partial \underset{\sim}{\xi}_t^T} = \underset{\sim}{F}^m$$

we are really only interested in the upper left $n\times n$ or 2×2 block

$$\underset{\substack{4\times4 \\ np\times np}}{\underset{\sim}{F}^m} = \begin{bmatrix} \frac{\partial y_{1t+m}}{\partial y_{1t}} & \frac{\partial y_{1,t+m}}{\partial y_{2t}} & \text{other junk} \\ \frac{\partial y_{2t+m}}{\partial y_{1t}} & \frac{\partial y_{2t+m}}{\partial y_{2t}} & \\ \text{other junk} & & \text{other junk} \end{bmatrix}$$

call this $\underset{\substack{2\times2 \\ n\times n}}{\underset{\sim}{\Psi}_m} = \underset{\sim}{F}_{1,1}^{(m)}$ a sub block of F

$$= \frac{\partial \underset{\sim}{y}_{t+m}}{\partial \underset{\sim}{y}_t^T} = \frac{\partial \underset{\sim}{y}_{t+m}}{\partial \underline{\varepsilon}_t^T}$$

$$\underset{\sim}{\xi}_{t+m} = \begin{bmatrix} y_{1t+m} \\ y_{2t+m} \\ y_{1,t+m-1} \\ y_{2,t+m-1} \end{bmatrix} \longrightarrow \underset{\sim}{y}_{t+m} = \underline{\varepsilon}_{t+m} + \underset{\sim}{\Psi}_1 \underline{\varepsilon}_{t+m-1} + \dots + \underset{\sim}{\Psi}_{m-1} \underline{\varepsilon}_{t+1} + \underset{\sim}{\Psi}_m \underset{\sim}{y}_t + \underset{\sim}{F}_{12}^{(m)} \underset{\sim}{y}_{t-1}$$

How do we know how $\frac{\partial y_{i,t+m}}{\partial v_{jt}}$ will behave as a function of m more generally?

In AR(1) $|\phi_1| < 1$. In AR(p) $\phi(L)$ lag polynomial roots.

What about $\underset{\sim}{\Psi}_m$ or $\underset{\sim}{F}^m$ in a VAR(p)?

We can usually factor $\underset{np \times np}{\underset{\sim}{F}} = \underset{np \times np}{\underset{\sim}{T}} \underset{\sim}{\Lambda} \underset{\sim}{T}^{-1}$ where $\underset{\sim}{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ & & & 0 \\ 0 & & 0 & \lambda_{np} \end{bmatrix}$

$$\underset{\sim}{F}^2 = (\underset{\sim}{T}\underset{\sim}{\Lambda}\underset{\sim}{T}^{-1})(\underset{\sim}{T}\underset{\sim}{\Lambda}\underset{\sim}{T}^{-1}) = \underset{\sim}{T}\underset{\sim}{\Lambda}^2\underset{\sim}{T}^{-1}$$

$$\underset{\sim}{F}^m = \underset{\sim}{T}\underset{\sim}{\Lambda}^m\underset{\sim}{T}^{-1}, \quad \underset{\sim}{\Lambda}^m = \begin{bmatrix} \lambda_1^m & 0 & 0 \\ 0 & \lambda_2^m & \vdots \\ & & 0 \\ 0 & 0 & \lambda_{np}^m \end{bmatrix}$$

$$\frac{\partial y_{i,t+m}}{\partial \varepsilon_{jt}} = a_{ij}^{(1)}\lambda_1^m + a_{ij}^{(2)}\lambda_2^m + \dots + a_{ij}^{(np)}\lambda_{np}^m$$

coefficients from $\underset{\sim}{T}, \underset{\sim}{T}^{-1}$

$= i,j$ element of $\underset{\sim}{T}\underset{\sim}{\Lambda}\underset{\sim}{T}^{-1}$.

The λ's are the <u>eigenvalues</u> of $\underset{\sim}{F}$, or values of λ that make

$$|\underset{\sim}{F} - \lambda \underset{\sim}{I}| = 0$$

$$\begin{vmatrix} f_{11}-\lambda & f_{12} & & f_{1,np} \\ f_{21} & f_{22}-\lambda & & \\ \vdots & & \ddots & \\ f_{np,1} & & & f_{np,np}-\lambda \end{vmatrix} = 0 = a_1\lambda + a_2\lambda^2 + \dots + a_{np}\lambda^{np}$$

has np roots

If roots are real and all are $|\lambda_i| < 1$, IRF dies out.

If any $|\lambda_i| = 1$, there's a <u>unit</u> <u>root</u>, IRF is permanent.

If roots are complex, stochastic cycles show up in IRF.

upper $n \times n$ block of $\underset{\sim}{F}^m$ is $\underset{\sim}{\Psi}_m$ as a function of m is called the "non-orthogonalized IRF" →

$\frac{\partial y_{i,t+m}}{\partial \varepsilon_{jt}}$ does not account for correlation of ε_i and ε_j