Multivariate Volatility Models

Var in gold returns, changes in TIPS

$$\underset{\sim}{r}_t = \begin{bmatrix} gold_t \\ tips_t \end{bmatrix} = \underset{2\times 2}{\underset{\sim}{\Phi}_1} \begin{bmatrix} gold_{t-1} \\ tips_{t-1} \end{bmatrix} + \begin{bmatrix} a_{gt} \\ a_{it} \end{bmatrix} \qquad VAR(1)$$

a_{gt} has time varying volatility

$$\underbrace{\sigma^2_{gt} = \alpha_0 + \delta_1 \sigma^2_{gt-1} + \alpha_1 a^2_{gt-1}} + \underbrace{\beta_1 \sigma^2_{it} + \beta_2 \sigma^2_{it-1} + \beta_3 a^2_{it-1}}$$

$$\underset{2\times 1}{\underset{-}{r}_t} = \underbrace{\underset{2\times 2}{\underset{\sim}{\Phi}_1}\ \underset{2\times 1}{\underset{-}{r}_{t-1}}}_{\underset{2\times 1}{\underset{-}{\mu}_t}} + \underbrace{\underset{2\times 1}{\underset{-}{a}_t}}$$

univariate case

$$a_t = \varepsilon_t \sqrt{\alpha_0 + \alpha_1 a^2_{t-1}}$$

multivariate case

$$\underset{-}{a}_t = \underbrace{\underset{2\times 2}{\underset{\sim}{\Sigma}_t^{1/2}}}\ \underbrace{\underset{2\times 1}{\varepsilon_t}}$$

standard normal random vector

$$\varepsilon_t \sim N(\underset{2\times 1}{\underset{-}{0}}, \underset{2\times 2}{\underset{\sim}{I}})$$

$$Cov(\underset{-}{a}_t \mid F_{t-1}) = \underset{2\times 2}{\underset{\sim}{\Sigma}_t} = \begin{bmatrix} \sigma^2_{gt} & \sigma_{gi,t} \\ \sigma_{gi,t} & \sigma^2_{it} \end{bmatrix}$$

$\sigma_{gi} = \sigma_{ig}$

symmetric matrix

$\underset{\sim}{\Sigma}_t$ a conditional variance

$\rightarrow$ positive definite

$\Rightarrow$ 3 parameters to estimate at each t

k equations, $\frac{k(k+1)}{2}$ parameters to estimate each t

Recall correlation: $\rho_{gi,t} = \dfrac{\sigma_{gi,t}}{\sqrt{\sigma^2_{gt} \cdot \sigma^2_{it}}}$

Options:

1) BEKK model is "algebraically correct"

1) BEKK model is "algebraically correct"
computationally difficult

2) Exponentially-weighted moving average (EWMA)

3) Dynamic Conditional Correlation (DCC)

BEKK

estimate $\underset{\sim}{\Sigma}_1 = \begin{bmatrix} \sigma^2_{g1} & \sigma_{gi,1} \\ \sigma_{gi,1} & \sigma^2_{i1} \end{bmatrix}$

$$\underset{\sim}{\Sigma}_t = \underset{2\times2}{\underset{\sim}{A}_0} \underset{2\times2}{\underset{\sim}{A}_0^T} + \underset{2\times2}{\underset{\sim}{A}_1} \big(\underbrace{\underset{2\times1}{\underline{a}_{t-1}} \cdot \underset{1\times2}{\underline{a}_{t-1}^T}} \big) \underset{2\times2}{\underset{\sim}{A}_1^T} + \underset{2\times2}{\underset{\sim}{B}_1} \underset{\sim}{\Sigma}_{t-1} \underset{2\times2}{\underset{\sim}{B}_1^T}$$

EWMA

$$\underset{\sim}{\Sigma}_t = (1-\lambda)\, \underline{a}_{t-1} \cdot \underline{a}_{t-1}^T + \lambda \underset{\sim}{\Sigma}_{t-1}$$

$$= (1-\lambda) \begin{bmatrix} a_{gt-1} \\ a_{it-1} \end{bmatrix} \begin{bmatrix} a_{gt-1} & a_{it-1} \end{bmatrix} + \lambda \underset{2\times2}{\underset{\sim}{\Sigma}_{t-1}}$$

$$= (1-\lambda) \begin{bmatrix} a^2_{gt-1} & a_{gt-1} a_{it-1} \\ a_{gt-1} a_{it-1} & a^2_{it-1} \end{bmatrix} + \lambda \underset{\sim}{\Sigma}_{t-1} \qquad 0 < \lambda < 1$$

DCC

two steps

1) marginal model: estimate VAR, calculate residuals, estimate univariate GARCH on each residual separately

2) standardized VAR residuals $\underset{\sim}{\underline{a_{gt}}}$, $\underset{\sim}{a_{it}}$

2) standardized VAR residuals $\tilde{a}_{gt} = \dfrac{a_{gt}}{\sigma_{gt}}$, $\dfrac{a_{it}}{\sigma_{it}} = \tilde{a}_{it}$

calculate time-varying correlations between $\tilde{a}_{gt}$, $\tilde{a}_{it}$

$$\underset{\sim}{\Sigma}_t = \begin{bmatrix} \sigma_{gt} & 0 \\ 0 & \sigma_{it} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \rho_{git} \\ \rho_{git} & 1 \end{bmatrix}}_{\underset{\sim}{R}_t} \begin{bmatrix} \sigma_{gt} & 0 \\ 0 & \sigma_{it} \end{bmatrix} \qquad \rho_{gi,t} = \frac{\sigma_{gi,t}}{\sqrt{\sigma_{gt}^2 \cdot \sigma_{it}^2}}$$

$$\underset{\sim}{R}_t = \begin{bmatrix} \frac{1}{\sigma_{gt}} & 0 \\ 0 & \frac{1}{\sigma_{it}} \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_{gt}^2 & \sigma_{git} \\ \sigma_{git} & \sigma_{it}^2 \end{bmatrix}}_{\underset{\sim}{Q}_t} \begin{bmatrix} \frac{1}{\sigma_{gt}} & 0 \\ 0 & \frac{1}{\sigma_{it}} \end{bmatrix}$$

$$\underset{\sim}{Q}_t = (1-\theta_1-\theta_2)\underset{\sim}{R}_0 + \theta_1 \underset{\sim}{Q}_{t-1} + \theta_2 \underline{a}_{t-1} \cdot \underline{a}_{t-1}^T$$