Orthogonalized IRF's

$$\underline{y}_{t+m} = \underline{\varepsilon}_{t+m} + \underset{\sim}{\Psi}_1 \underline{\varepsilon}_{t+m-1} + \underset{\sim}{\Psi}_2 \underline{\varepsilon}_{t+m-2} + \dots + \underset{\sim}{\Psi}_{m-1} \underline{\varepsilon}_{t+1} + \underset{\sim}{\Psi}_m \underline{y}_t + \underset{\sim}{F}_{12}^{(m)} \underline{y}_{t-1}$$

$\underset{\sim}{\Psi}_m$: upper 1,1 block of $\underset{\sim}{F}^m$ $\quad$ $\underset{\sim}{F}_{12}^{(m)}$: upper 1,2 block of $\underset{\sim}{F}^m$

a shock of ε_{1t} may have contemporaneous info about $\varepsilon_{2t}, \varepsilon_{3t}, \dots \varepsilon_{nt}$

e.g. suppose we learn ε_{1t} first. Can we update our forecast based on how well ε_{1t} predicts $\varepsilon_{2t}, \dots \varepsilon_{nt}$?

$$E(\underline{y}_{t+m} \mid \underline{y}_t, \underline{y}_{t-1}, \dots) = \hat{\underline{y}}_{t+m|t} = \underset{\sim}{\Psi}_m \underline{y}_t + \underset{\sim}{F}_{12}^{(m)} \underline{y}_{t-1}$$

What if we only know ε_{1t}? This is correlated w/ $\varepsilon_{2t}, \dots \varepsilon_{nt}$ through

first column: $$\underset{\sim}{\Omega} = \begin{bmatrix} \sigma_1^2 & \sigma_{21} & \cdots & \\ \sigma_{12} & \sigma_2^2 & & \\ \sigma_{13} & & \ddots & \\ \vdots & & & \sigma_n^2 \\ \sigma_{1n} & & & \end{bmatrix}$$

$$E(\underline{y}_{t+m} \mid \varepsilon_{1t}, \underline{y}_{t-1}, \dots) = \underset{\sim}{\Psi}_m E(\underline{\varepsilon}_t \mid \varepsilon_{1t}, \underline{y}_{t-1}, \dots) + \underset{\sim}{F}_{12}^{(m)} \underline{y}_{t-1}$$

Let $$\underline{a}_1 \varepsilon_{1t} = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} \varepsilon_{1t} = E(\underline{\varepsilon}_t \mid \varepsilon_{1t})$$

$$\Rightarrow E(\underline{y}_{t+m} \mid \varepsilon_{1t}, \underline{y}_{t-1}) = \underset{n\times n}{\underset{\sim}{\Psi}_m} \underset{n\times 1}{\underline{a}_1} \varepsilon_{1t} + \underset{\sim}{F}_{12}^{(m)} \underline{y}_{t-1}$$

$$\frac{\partial E(\underline{y}_{t+m} \mid \varepsilon_{1t}, \underline{y}_{t-1})}{\partial y_{1t}} = \underset{\sim}{\Psi}_m \underline{a}_1 \qquad \underline{a}_1 = \frac{\partial E(\underline{\varepsilon}_t \mid \varepsilon_{1t})}{\partial \varepsilon_{1t}}$$

where do we get $\underline{a}_1$?

Orthogonalized IRF's

$$\hat{\underset{\sim}{\Omega}} = \frac{1}{T}\sum_{t=1}^{T} \hat{\underline{\varepsilon}}_t \hat{\underline{\varepsilon}}_t^T \quad \text{because} \quad \underset{\sim}{\Omega} = E(\underline{\varepsilon}\,\underline{\varepsilon}^T)$$

We can factor as $\hat{\underset{\sim}{\Omega}} = \hat{\underset{\sim}{A}}\,\hat{\underset{\sim}{D}}\,\hat{\underset{\sim}{A}}^T$

lower triangular matrix with ones on diagonal

diagonal matrix w/ variances on diagonal

$\hat{\underline{a}}_1$ is the first column of $\hat{\underset{\sim}{A}}$ → how ε_{1t} linearly predicts $\varepsilon_{2t}, \ldots \varepsilon_{nt}$

$\hat{\underline{a}}_2$ is the second column of $\hat{\underset{\sim}{A}}$ → how ε_{2t} linearly predicts $\varepsilon_{3t}, \ldots \varepsilon_{nt}$

Alternatively factor $\hat{\underset{\sim}{\Omega}} = \hat{\underset{\sim}{P}}\,\hat{\underset{\sim}{P}}^T = (\hat{\underset{\sim}{A}}\,\hat{\underset{\sim}{D}}^{\frac{1}{2}})(\hat{\underset{\sim}{D}}^{\frac{1}{2}}\,\underset{\sim}{A}^T)$

$\hat{\underset{\sim}{P}}$ lower triangular w/ standard deviations along diagonal

Notice $\hat{\underset{\sim}{A}} = \hat{\underset{\sim}{P}}\,(\hat{\underset{\sim}{D}}^{\frac{1}{2}})^{-1}$

$\underset{\sim}{\Psi}_m$ is non-orthogonalized IRF.

$\underset{\sim}{\Psi}_m \hat{\underline{a}}_1$ is orthogonalized IRF: effect of a one-unit increase in y_{1t} on y_{t+m}

$\underset{\sim}{\Psi}_m \hat{\underline{P}}_1$ is orthogonalized IRF, scaled by standard deviations: effect of a one-standard deviation increase in y_{1t} on y_{t+m}

$\underset{\sim}{\Psi}_m \hat{\underline{a}}_2$ or $\underset{\sim}{\Psi}_m \hat{\underline{P}}_2$ is effect of y_{2t} on y_{t+m}, etc.

($\hat{\underline{a}}_2$ is 2nd column of $\hat{\underset{\sim}{A}}$, $\hat{\underline{P}}_2$ is 2nd column of $\hat{\underset{\sim}{P}}$).

Overall: $\hat{\underset{\sim}{\Psi}}_m \hat{\underset{\sim}{A}}$ or $\hat{\underset{\sim}{\Psi}}_m \hat{\underset{\sim}{P}}$.

Order matters. Put variables in order of exogeneity, timing of news.

Structural VAR

Suppose we think oil price return affects gas price return and drilling rate in current period, gas price affects drilling rate in current period

y_{1t} = oil $\quad y_{2t}$ = gas $\quad y_{3t}$ = drill

$$y_{1t} = \underline{b}_{11}\, y_{t-1} + \dots + \underline{b}_{1p}\, y_{t-p} + u_{1t}$$

not ε_{1t}, but related "structural residual" uncorrelated across equations

$$y_{2t} = a_{21} y_{1t} + \underline{b}_{21}\, y_{t-1} + \dots + \underline{b}_{2p}\, y_{t-p} + u_{2t}$$

$$y_{3t} = a_{31} y_{1t} + a_{32} y_{2t} + \underline{b}_{31}\, y_{t-1} + \dots + \underline{b}_{3p}\, y_{t-p} + u_{3t}$$

$$\Rightarrow \quad y_{1t} =$$

$$-a_{21} y_{1t} + y_{2t} =$$

$$-a_{31} y_{1t} - a_{32} y_{2t} + y_{3t} =$$

$$\underset{\sim}{A}^{-1} y_t = \underset{\sim}{B}_1 y_{t-1} + \dots + \underset{\sim}{B}_p y_{t-p} + \underline{u}_t$$

$$\Rightarrow \quad y_t = \underbrace{\underset{\sim}{A}\underset{\sim}{B}_1}_{\underset{\sim}{\Phi}_1} y_{t-1} + \dots + \underbrace{\underset{\sim}{A}\underset{\sim}{B}_p}_{\underset{\sim}{\Phi}_p} y_{t-p} + \underbrace{\underset{\sim}{A}\, \underline{u}_t}_{\underline{\varepsilon}_t}$$

Another way to think about orthogonalized IRF

$$\underset{\sim}{\Omega} = E\left(\underline{\varepsilon}_t \underline{\varepsilon}_t^T\right) = E\left(\underset{\sim}{A} \underbrace{\underline{u}_t \underline{u}_t^T}_{\substack{n \times n \\ \text{diagonal}}} \underset{\sim}{A}^T\right) = E\left(\underset{\sim}{A}\, \underset{\sim}{D}\, \underset{\sim}{A}\right)$$

$$\frac{\partial y_{t+m}}{\partial \underline{u}_t^T} = \frac{\partial y_{t+m}}{\partial \underline{\varepsilon}_t^T} \frac{\partial \underline{\varepsilon}_t}{\partial \underline{u}_t^T} = \underset{\sim}{\Psi}_m \underset{\sim}{A}$$