Time Series Econometrics

Moments of Distributions

Ben Gilbert[‡]

[‡] *Colorado School of Mines, Division of Economics & Business*

Overview

- Know the first four moments of a distribution
  - Mean, Variance/Covariance, Skewness, and Excess Kurtosis
- Distinction between population vs. sample moment
- Law of Large Numbers, Central Limit Theorem, and Sampling distribution of sample moments
- Chi-squared distribution, t-distribution
- t-tests and Jarque-Bera test of Normality
- Factors driving Non-normality

Population vs. Sample Moments

Moment	*Population*	*Sample*
Mean	$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$	$\hat{\mu}_X = \frac{1}{T}\sum_{t=1}^{T} x_t$
Variance	$\sigma_X^2 = E\left[(X-\mu_X)^2\right]$	$\hat{\sigma}_X^2 = \frac{1}{T-1}\sum_{t=1}^{T}(x_t-\hat{\mu}_X)^2$
Covariance	$\sigma_{XY} = E\left[(X-\mu_X)(Y-\mu_Y)\right]$	$\hat{\sigma}_{XY} = \frac{1}{T-1}\sum_{t=1}^{T}(x_t-\hat{\mu}_X)(y_t-\hat{\mu}_Y)$
Skewness	$S(X) = E\left[\frac{(X-\mu_X)^3}{\sigma_X^3}\right]$	$\hat{S}_X = \frac{1}{\sigma_X^3(T-1)}\sum_{t=1}^{T}(x_t-\hat{\mu}_X)^3$
Kurtosis	$K(X) = E\left[\frac{(X-\mu_X)^4}{\sigma_X^4}\right]$	$\hat{K}_X = \frac{1}{\sigma_X^4(T-1)}\sum_{t=1}^{T}(x_t-\hat{\mu}_X)^4$

- $K(X) = 3$ for a Normal distribution, so typically consider $K(X) - 3$ as “Excess Kurtosis”.

Sampling distributions of sample moments

- We never know the population moment, but estimate it from sample.
- Different samples produce different estimates.
- How can we describe the distribution of estimates across samples?

Sampling distributions of sample moments

- Law of Large Numbers (LLN):
  - If observations are independent and identically distributed (*i.i.d.*) then as the sample size grows, the sample estimate converges to the population value.

Sampling distributions of sample moments

- Central Limit Theorem (CLT):
  - The normalized sum of independent random variables converges to a Normal as the number of samples grows
  - (even if the underlying data are NOT Normal)
  - Notice each sample moment is a sum of transformed observations.

Sampling distributions of sample moments

- Conclusion: Sample moments are “asymptotically” Normally distributed around their population values.
- **Standard Error**: standard deviation (square root of variance) of sample estimate.

Sampling distributions of sample moments

- $\hat{S}_X \sim^A N(S(X), \frac{6}{T})$
- $\hat{K}_X \sim^A N(K(X), \frac{24}{T})$
- We can use this information to test hypotheses about the sample moments of data of interest.

Hypothesis Testing

- Student's t-statistic:

$$t = \frac{\textit{sample estimate} - \textit{hypothesized value}}{\textit{standard error of sample estimate}}$$

- If the sample estimate is farther from the hypothesized value than we are willing to accept, we reject the hypothesis.
- How far? Farther than would occur by accident in 95% of samples.
- Need to know the distribution of the t-statistic.

Hypothesis Testing

- Student's t-statistic:

$$t = \frac{sample\ estimate - hypothesized\ value}{standard\ error\ of\ sample\ estimate}$$

- Numerator: Normal minus a constant is Normal.

Hypothesis Testing

- Student's t-statistic:

$$t = \frac{\textit{sample estimate} - \textit{hypothesized value}}{\textit{standard error of sample estimate}}$$

- Denominator: the square root of a variance.
- A variance is a sum of squared Normals:

$$\frac{1}{T-1}\sum_{t=1}^{T}(x_t - \hat{\mu}_X)^2$$

- Chi-squared or χ^2 distribution: the sum of squared Normals.

Hypothesis Testing

- Student's t-statistic:

$$t = \frac{\textit{sample estimate} - \textit{hypothesized value}}{\textit{standard error of sample estimate}}$$

- The t-distribution is a special distribution that is a Normal divided by the square root of a χ^2.
- The t-distribution converges to a Normal as the sample grows.

Hypothesis Testing

- **Conclusion**: If the t-statistic for your hypothesis test falls in the tails of the t-distribution for a small sample or the tails of a Normal distribution for a large sample, then you reject the hypothesis. Otherwise, fail to reject. “Tail” is subjective, but typically less than 95% probability of observing that t-stat.

Hypothesis Testing

- t-test for the hypothesis of no skewness in your distribution:

$$\frac{\hat{S}(X)}{\sqrt{6/T}}$$

- t-test for the hypothesis of no excess kurtosis in your distribution:

$$\frac{\hat{K}(X)-3}{\sqrt{24/T}}$$

Joint Hypothesis Testing

- Jarque-Bera Normality Test

$$JB = \frac{\hat{S}(X)^2}{6/T} + \frac{(\hat{K}(X)-3)^2}{24/T} \sim \chi^2(2)$$

- t-stat is Normally distributed in large samples
- Sum of squared Normals has a χ^2 distribution

$$\hat{t}_S^2 + \hat{t}_K^2 \sim \chi^2(2)$$

- 2 degrees of freedom, one for each element in the sum.

Factors Driving Non-Normality of Returns

- Sample statistics are Normal, but raw returns data are often not. Why not?
  - returns of related equities/commodities, other variables
  - macro/market/industry shocks and market news
  - past shocks in own returns or other variables
  - unobservable error processes