Autocorrelation, White Noise, and Linear Time Series

This Lecture

By now we know that autocorrelation in the variables and residuals is important. How do we test for it?

1. Autocorrelation Function (ACF)
2. Test individual autocorrelations at a given lag
3. Test joint autocorrelations (Portmanteau, Box-Pierce, Ljung-Box, Q-test)
4. Interpreting an ACF correlogram
5. Partial ACF and correlogram
6. White noise
7. Properties of linear time series

Recall: Stationarity

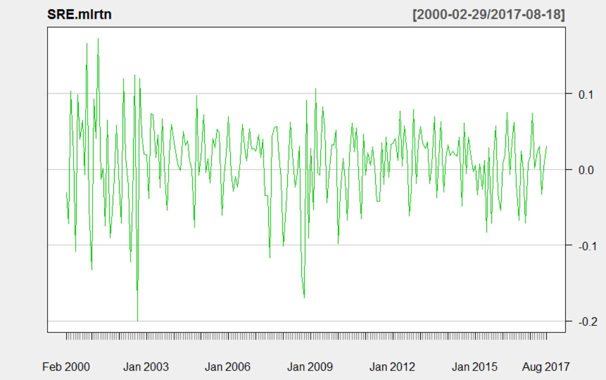
- *Weak/Covariance Stationarity*: The mean, variance, and autocovariance are constant through time, i.e., $E(r_t) = \mu$ (a constant) and $Cov(r_t, r_{t-s}) = \gamma_s$ only depends on the distance between the observations s and not time period t.
- Note also that in general $\gamma_0 = Var(r_t)$ and $\gamma_s = \gamma_{-s}$ (because $Cov(r_t, r_{t-s}) = Cov(r_{t+s}, r_t)$.

NONSTATIONARY

Sempra Energy equity price

STATIONARY

Sempra Energy monthly returns



Autocorrelation Function

- Correlation between two random variables X and Y:

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sqrt{E(X-\mu_X)^2 E(Y-\mu_Y)^2}}$$

- For stationary r_t, if $X = r_t$ and $Y = r_{t-s}$ we have the Autocorrelation Function (ACF):

$$\rho_s = \frac{Cov(r_t, r_{t-s})}{\sqrt{Var(r_t) \cdot Var(r_{t-s})}} = \frac{Cov(r_t, r_{t-s})}{Var(r_t)} = \frac{\gamma_s}{\gamma_0}$$

- This is a function of the gap, s.

Autocorrelation Function

- We can estimate this for our variable r_t if we want to build a model with lags of r_t
- We can also do this for our regression residuals if we are worried about time series errors.

$$r_{it} = \hat{\beta}_0 + \hat{\beta}_1 r_{jt} + \hat{e}_t$$

  - Calculate and test autocorrelations in $\hat{e}_t$

$$\frac{Cov(\hat{e}_t, \hat{e}_{t-s})}{Var(\hat{e}_t)}$$

Testing one at a time

- Use t-test for single autocorrelation at a specific lag

$$H_0 : \hat{\rho}_s = 0, \quad H_a : \hat{\rho}_s \neq 0$$

- By Central Limit Theorem, $\hat{\rho}_s$ converges to Normal.

$$t = \frac{\hat{\rho}_s - 0}{se(\hat{\rho}_s)}$$

- What is $se(\hat{\rho}_s)$?
  - If your null is $r_t \sim i.i.d.$, then $se(\hat{\rho}_s) = \sqrt{1/T}$.
  - If your null is $\gamma_k \neq 0$ for $k < s$, then

$$se(\hat{\rho}_s) = \sqrt{\left(1 + 2\sum_{i=1}^{s-1} \hat{\rho}_i^2\right) / T}$$

  - (inflate the variance due to autocorrelation at lower lags)

Jointly test ACF at multiple lags

- Use a version of a χ^2-test for

$$H_0 : \hat{\rho}_1 = \cdots = \hat{\rho}_m = 0, \quad H_a : \hat{\rho}_i \neq 0 \text{ for some } i \in \{1, \ldots, m\}$$

- Box-Pierce (Portmanteau):

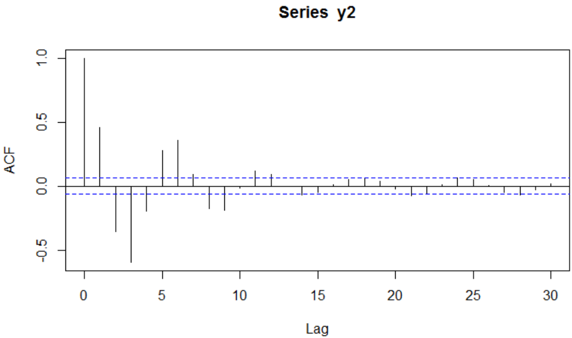
$$Q^\star(m) = T \sum_{s=1}^{m} \hat{\rho}_s^2$$

- Sum of squared Normals $\sim \chi^2(m)$
- Ljung-Box sample size adjustment (used in practice):

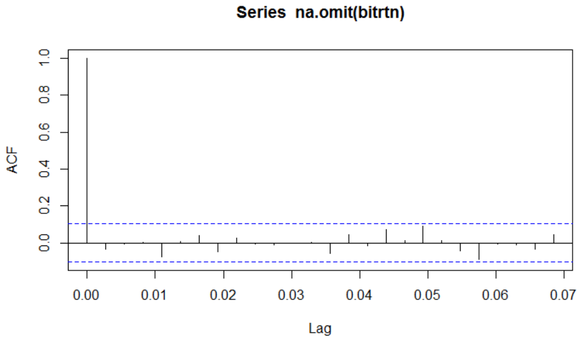
$$Q(m) = T(T+2) \sum_{s=1}^{m} \frac{\hat{\rho}_s^2}{T-s}$$

- m is whatever your hypothesis is, or $\approx ln(T)$ (unless you have seasonality).

ACF Correlograms



ACF Correlograms



Correlograms

- Vertical lines $= \hat{\rho}_s$ at each lag s.
  - At $s = 0$, always perfectly correlated - spike where $\hat{\rho}_1 = 1$.
- Dashed horizontal lines $=$ 95% confidence intervals for individual t-test at lag s.
- If a vertical line crosses dashed horizontal line, reject the null of zero autocorrelation at that lag.
- High frequency data: often many are just barely significant. Model the important ones.
- Joint significance?

Partial Autocorrelation Function (PACF)

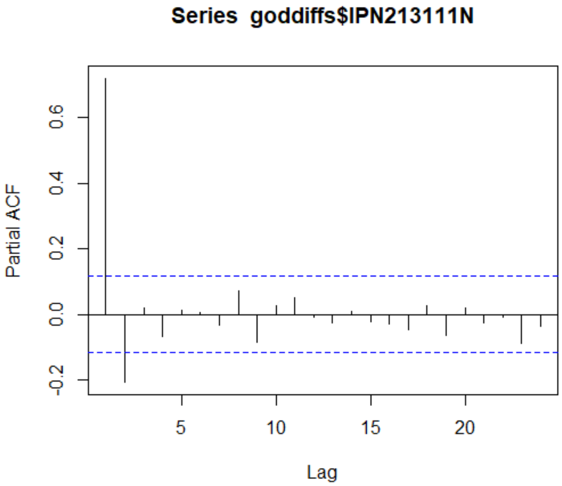
- ACF measures pairwise correlation at a given time gap, e.g., $corr(r_t, r_{t-3})$ but does not control for the lags in between, e.g., r_{t-1}, r_{t-2}.
- PACF can be helpful in choosing number of lags in autoregressive model.
- Sequentially estimate

$$
\begin{aligned}
r_t &= \phi_{0,1} + \phi_{1,1} r_{t-1} + e_{1,t} \\
r_t &= \phi_{0,2} + \phi_{1,2} r_{t-1} + \phi_{2,2} r_{t-2} + e_{2,t} \\
r_t &= \phi_{0,3} + \phi_{1,3} r_{t-1} + \phi_{2,3} r_{t-2} + \phi_{3,3} r_{t-3} + e_{3,t} \\
&\text{etc.}
\end{aligned}
$$

- The PACF is $(\phi_{1,1}, \phi_{2,2}, \phi_{3,3}, \ldots)$.
- Each successive estimate controls for the lags in between, measures the *additional* explanatory power of the next lag.

PACF Correlograms

PACF for US oil & gas drilling index. Similar format to ACF, but no 0 coefficient.



White noise

- *White Noise*: Sequence of i.i.d. (independent and identically distributed) random variables with finite mean and variance.
- *Gaussian White Noise*: Normally distributed white noise with mean 0 and variance σ^2.
- No significant ACFs.

Linear Time Series

- A linear time series can be written

$$r_t = \mu + a_t + \psi_1 a_{t-1} + \cdots = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i}$$

- μ is the mean, $\psi_0 = 1$
- a_t is mean-zero white noise.
- a_t is the shock/innovation/new information about r_t that arrives at time t.
- r_t can be summarized by the cumulative effect of all past shocks.
- μ can contain a model with other explanatory variables.

Linear Time Series

$$r_t = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i}$$

- Weights ψ_i on past shocks govern behavior.
- If all $\psi_i = 0$ except $\psi_0 = 1$, then r_t is white noise (no autocorrelation).
- For r_t to be stationary, ψ-weights on past shocks must die out.

$$E(r_t) = \mu,$$

$$E[(r_t - \mu)^2] = E[(\sum_{i=0}^{\infty} \psi_i a_{t-i})^2)] = \sigma_a^2 \sum_{i=0}^{\infty} \psi_i^2$$

- For a nonstationary series, past shocks are **permanent**.

Linear Time Series

$$r_t = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i}$$

- ψ-weights and autocorrelation:

$$\begin{aligned}
\gamma_s = Cov(r_t, r_{t-s}) &= E[(\textstyle\sum_{i=0}^{\infty} \psi_i a_{t-i})(\sum_{j=0}^{\infty} \psi_j a_{t-s-j})] \\
&= E[\textstyle\sum_{i,j=0}^{\infty} \psi_i \psi_j a_{t-i} a_{t-s-j}] \\
&= \textstyle\sum_{j=0}^{\infty} \psi_{j+s} \psi_j E[a_{t-s-j}^2] \\
&= \sigma_a^2 \textstyle\sum_{j=0}^{\infty} \psi_{j+s} \psi_j
\end{aligned}$$

$$\rho_s = \frac{\gamma_s}{\gamma_0} = \frac{\sigma_a^2 \sum_{j=0}^{\infty} \psi_{j+s} \psi_j}{\sigma_a^2 \sum_{j=0}^{\infty} \psi_j^2}$$

- ψ_j^2 has to die out enough as j grows (far enough back in history) to sum to a finite constant
- $\psi_{j+s}\psi_j$ has to die out as s grows (far enough between observations) for autocorrelation to die out.

Linear Time Series Takeaways

- Any linear process can be represented this way:
  - A cumulative weighted sum of all past shocks/innovations/information.
- Stationary:
  - Effect of (or weight on) past shocks dies out.
  - Autocorrelations die out as distance between observations grows
- Nonstationary:
  - Shocks are permanent
- White noise:
  - Past shocks get zero weight, autocorrelations are all not significantly differently from zero.