Synthetic Control

Monday, December 4, 2023 2:08 PM

$J = 1, \ldots, J+1$ $\quad t = 1, \ldots, T_0, \ldots, T$

$\uparrow$ event / policy

Y_{it}^N = not treated outcome $\quad Y_{it}^I$ = treated outcome

$$\alpha_{1t} = Y_{1t}^I - Y_{1t}^N \quad \text{for } t = T_0+1, \ldots, T$$

$\nwarrow$ estimate! don't observe

Find units w/ similar trends and characteristics!

we have Y_{it}, $t \le T_0$ and $\underset{r\times 1}{\underline{U}_i}$ $\to$ other covariates, summary statistics from pretreatment period.

$\updownarrow$

$\underline{K} = (k_1, \ldots, k_{T_0})'$

each k is a different linear combination of pretreatment outcomes

$$\bar{Y}_i^k = \sum_{s=1}^{T_0} k_s Y_{is}$$

$M \le T_0$ possible different linear combinations of pretreatment outcomes

looking for optimal weights $\underline{W} = (w_2, \ldots, w_{J+1})'$

for now $w_j \ge 0$ $\quad \sum w_j = 1$

select $\underline{W}^*$ such that

$$\bar{Y}_1^{k_1} \approx \sum_{j=2}^{J+1} w_j^* \bar{Y}_j^{k_1} \quad \text{for each linear combination } k$$

$$U_1 \approx \sum_{j=2}^{J+1} w_j^* U_j$$

Then
$$\hat{\alpha}_{1t} = Y_{1t} - \sum_{j=2}^{J+1} w_j^* Y_{jt}$$

Putting into practice:

let $\underset{(r+M)\times 1}{\underline{X}_1} = (\underline{U}_1', \bar{Y}_1^{k_1}, \ldots, \bar{Y}_1^{k_M})$

$$\underset{(r+M)\times J}{\underset{\sim}{X}_0} = \begin{bmatrix} \underline{u}_j', & \bar{Y}_j^{k_1}, & \cdots & \bar{Y}_j^{k_M} \end{bmatrix}$$

Need to write your own optimization routine

$$\min_{\underline{W}} \sqrt{(\underset{(r+M)\times 1}{\underline{X}_1} - \underset{(r+M)\times J}{\underset{\sim}{X}_0} \cdot \underset{J\times 1}{\underline{W}})' \underset{(r+M)\times(r+M)}{\underset{\sim}{V}} (\underline{X}_1 - \underset{\sim}{X}_0 \cdot \underline{W})} + \text{LASSO / Ridge / ElasticNet penalties}$$

↳ inverse of variance of the predictors (of the matching variables)

① First solve for $\underline{W}^*(\underset{\sim}{V})$

② $\min_{\underset{\sim}{V}} (\underset{T_0\times 1}{\underline{Y}_1} - \underset{T_0\times J}{\underset{\sim}{Y}_0} \cdot \underset{J\times 1}{\underline{W}^*(\underset{\sim}{V})})'(\underline{Y}_1 - \underset{\sim}{Y}_0 \cdot \underline{W}^*(\underset{\sim}{V}))$ ← MSPE get $\underset{\sim}{V}^*$

③ plug in $\underline{W}^*(\underset{\sim}{V}^*)$ now we have optimal weights

Package : SCUL Synthetic Control Using LASSO

$$\min_{\mu, \underline{W}} \sum_{t=1}^{T_0} \left(Y_{1t} - \mu - \sum_{j=2}^{J+1} w_j Y_{jt}\right)^2 + \underbrace{\lambda \| w_j \|}_{\text{LASSO Penalty}}$$

restricts $\underline{X}_1 = (Y_{11}, Y_{12}, \ldots, Y_{1T_0})$

$$\underset{\sim}{V} = \underset{\sim}{I}$$

$$\underset{\sim}{X}_0 = \begin{bmatrix} Y_{j1}, & \cdots & Y_{jT_0} \end{bmatrix}$$

allows $\mu \neq 0$, $w_j \gtrless 0$, $\sum_{j=2}^{J+1} w_j \neq 1$