Time Series Econometrics

Probability Distributions

Ben Gilbert[‡]

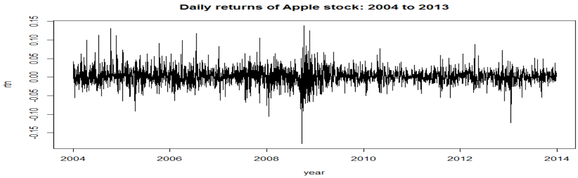
[‡] *Colorado School of Mines, Division of Economics & Business*

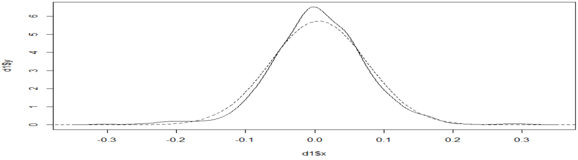
Overview

- Use distributions as a tool for describing risk & uncertainty, estimating parameters, and making predictions.
- Using conditional distributions to arrive at independence
- Some commonly used distributions in finance and economics
  - Normal, Lognormal, Cauchy, Scale Mixture of Normals, Finite Mixture of Normals
  - Next time: Chi-squared and t-distributions
- Log-likelihood functions

Using Distributions: Describing Risk, Making Predictions

- How often will my returns by very high, mediocre, very low?
- Can I make precise probability statements about that?





Using Distributions: Estimating Parameters, Making Predictions

- Recall one approach: pick parameters that maximize the likelihood of having observed your data.
- Recall $e_t = y_t - \alpha - \beta x_t = y_t - \hat{y}_t$.
- Suppose $f(e_1, ..., e_t, ..., e_T)$ is the joint probability distribution of the residuals, e.g., $e_t \sim N(0, \sigma^2)$.

$$max_{\alpha,\beta} f(e_1, ..., e_t, ..., e_T)$$

- If all the e_t are **independent** from each other:

$$max_{\alpha,\beta} f(e_1) \cdot f(e_2) \cdot ... \cdot f(e_T)$$

$$max_{\alpha,\beta} \sum_{t=1}^{T} ln\, f(e_t)$$

- Maximum Likelihood Estimation (MLE). Useful in more settings than OLS, shares many similar properties.

Conditional Distributions and Independence

- Observations may be independent once we **condition on** (or control for) past observations or related variables.
- Example: joint distribution of returns for one asset across T periods, possible dependence on the past.
- (Do I need to condition on *all* past values?)

$$
\begin{aligned}
f(r_1, ..., r_T) &= f(r_1) \cdot f(r_2 | r_1) \cdot ... \cdot f(r_T | r_{T-1}, r_{T-2}, ..., r_1) \\
&= f(r_1) \Pi_{t=2}^{T} f(r_t | r_{t-1}, ..., r_1)
\end{aligned}
$$

- (Do I need to condition on *all* past values?)

Normal Distribution

- Common bell curve
- Unconditional:

$$f(r_{it};\underline{\theta}) = f(r_{it};\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-0.5\left(\frac{r_{it}-\mu}{\sigma}\right)^2\right)$$

- Conditional: Suppose $r_{it} = \alpha + \beta r_{i,t-1} + \varepsilon_{it}$.
- My model for the mean is $\alpha + \beta r_{i,t-1}$

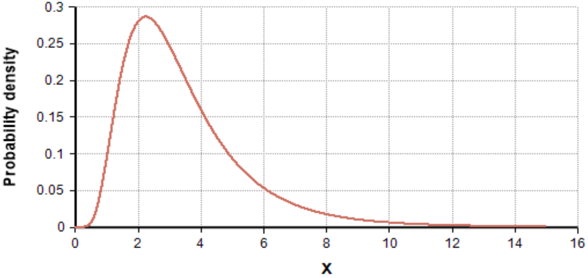
$$f(r_{it}|r_{i,t-1}) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-0.5\left(\frac{r_{it}-\alpha-\beta r_{i,t-1}}{\sigma}\right)^2\right)$$

- Sum of Normals is Normal, e.g., cumulative log returns over time:

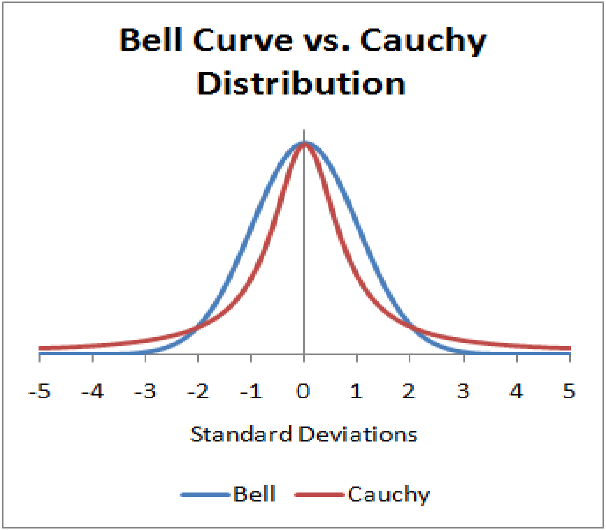
$$r_{i1} + r_{i2} + ... + r_{iT} \sim N(\mu T, \sigma^2 T)$$

Lognormal Distribution

- If log return $r_{it} \sim N(\mu, \sigma^2)$
- Then $log\left(\frac{P_{it}}{P_{i,t-1}}\right) = logP_{it} - logP_{i,t-1} \sim N(\mu, \sigma^2)$
- Which implies $P_{it},\ P_{i,t-1} \sim$ $lognormal\left(exp(\mu + 0.5\sigma^2), \left(exp(\sigma^2) - 1\right) \cdot exp(2\mu + \sigma^2\right)$
- Non-negative, positive skew and excess kurtosis (fat tail)



Fat tails in returns: Cauchy



Fat tails in returns: Mixture Distributions

- Scale Mixture of Normal Distributions:
  - $r_{it} \sim N(\mu, \sigma_t^2)$
  - $\sigma_t^2 \sim$ some other distribution, e.g., Gamma.
- Finite Mixture of Normal Distributions:
  - $r_{it} \sim (1-X) \cdot N(\mu, \sigma_1^2) + X \cdot N(\mu, \sigma_2^2)$
  - $Pr(X=1) = \alpha,\ Pr(X=0) = 1-\alpha$

Maximum Likelihood Estimation

- If conditional distribution is Normal:

$$f(r_t|r_{t-1},...) \sim N(\mu,\sigma^2)$$

- Then the likelihood of *having observed the data that I did* is

$$f(r_1,...,r_T) = f(r_1)\Pi_{t=1}^{T}\frac{1}{\sqrt{2\pi\sigma_t^2}}exp\left(-0.5\left(\frac{r_{it}-\mu_t}{\sigma_t}\right)^2\right)$$

- where μ_t contains my model for r_t, including past values of r_t.
- σ_t^2 might be a constant or might contain a model for time-varying volatility, including past values of σ_t^2.
- Choose the parameters of this model to maximize $lnf(\cdot)$