Unit Roots in Vector Time Series

Time Series Econometrics

Spurious Regression

- Suppose we have two independent random walks
  - $\Delta y_{1t} = \epsilon_{1t}, \; \Delta y_{2t} = \epsilon_{2t}$
  - $\epsilon_{\mathbf{t}} \sim iid(\mathbf{0}, \Omega)$ where $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\Omega = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$

Spurious Regression

- Suppose we regress one random walk on the other
  - $y_{1t} = \alpha + \gamma y_{2t} + u_t$
  - true $\alpha = \gamma = 0$
  - However, $\hat{\gamma}_{ols}$ and $\hat{\alpha}_{ols}$ have nonstandard limiting distributions and are not mean zero.
  - Also, the t-stats do not have a limiting distribution, so there are no correct critical values.
  - Also, $R^2 = 1 - \frac{SSE}{SST}$ which is a ratio of variances of random walks. This converges to a nonstandard distribution which doesn't tell us much about the true variation explained.
- spurious regression: estimated relationship looks awesome when there is really nothing.

Spurious Regression

- How to recognize and characterize a spurious regression:
  - The error in a spurious regression will **not** be I(0) stationary.
  - $y_t = x_t'\beta + u_t$
  - If I can find a β that makes $u_t \sim I(0)$, then this is **not** spurious.
  - If no such β exists, then it is spurious.
  - Spurious case:
  - $y_{1t} = \alpha + \gamma y_{2t} + u_t$
  - true $\alpha = \gamma = 0 \implies u_t = y_{1t} = \epsilon_{11} + \epsilon_{12} + ... + \epsilon_{1t}$
  - $u_t = y_{1t} \sim I(1)$.

Spurious Regression

- How to avoid a spurious regression
  1. Difference all variables before regression, or otherwise make sure all series are stationary.
     - $\Delta y_{1t} = \alpha + \gamma \Delta y_{2t} + \Delta u_t$
     - $\hat{\alpha}, \ \hat{\gamma}$ in this case converge normally.
  2. Include lags of both LHS and RHS variables as additional regressors (ARDL model).
     - $y_{1t} = \alpha + \phi y_{1,t-1} + \gamma y_{2t} + \delta y_{2,t-1} + u_t$
     - true $\alpha = \gamma = \delta = 0$, $\phi = 1$, $u_t = \epsilon_{1t} = \Delta y_{1t}$
     - t-tests of α, γ, δ are asymptotically valid, but not ϕ or F-tests

Unit Roots in Vector Autoregressions

- Suppose true model is: $\Delta \mathbf{y_t} = \zeta_1 \Delta y_{t-1} + ... + \zeta_{\mathbf{p-1}} \Delta \mathbf{y_{t-p+1}} + \epsilon_\mathbf{t}$
  - the data **should** have been differenced. (Case 2)
  - Stationarity assumption for this model in VAR looks like
  - $|\mathbf{I_n} - \zeta_1 z - \zeta_2 z^2 - ... - \zeta_{\mathbf{p-1}} z^{p-1}| = 0$ with $||z|| < 1 \implies \Delta \mathbf{y_t} \sim I(0)$
    - VAR(p-1) with no drift (no intercept).

Unit Roots in Vector Autoregressions

- Suppose we did not difference, estimated model VAR in levels with a constant.
- $\mathbf{y_t} = \alpha + \Phi_1 \mathbf{y_{t-1}} + \Phi_2 \mathbf{y_{t-2}} + ... + \Phi_p \mathbf{y_{t-p}} + \epsilon_\mathbf{t}$
- As with Dickey-Fuller test, this is a rotation of
- $\mathbf{y_t} = \alpha + \rho \mathbf{y_{t-1}} + \zeta_\mathbf{1} \Delta \mathbf{y_{t-1}} + ... + \zeta_\mathbf{p-1} \Delta \mathbf{y_{t-p+1}} + \epsilon_\mathbf{t}$
- True $\alpha = \mathbf{0}$ and $\rho = \mathbf{I_n}$
- As before, $\Phi_p = -\zeta_\mathbf{p-1}$,
- $\Phi_s = \zeta_\mathbf{s} - \zeta_\mathbf{s-1}$ for $s = 2, ..., p-1$
- $\Phi_1 = \rho + \zeta_\mathbf{1}$

Unit Roots in Vector Autoregressions

- As before, all the $\zeta_{\mathbf{s}}$'s are asymptotically normal, can use standard t-tests and F-tests.
- $\hat{\alpha}$ and $\hat{\rho}$ will be nonstandard but consistent (converge to something).
- Implications for estimating the levels regression:
  - t-tests and F-tests of elements of Φ_s are asymptotically valid
  - confidence intervals of impulse-response functions are asymptotically valid.
  - What's not okay: testing joint hypotheses that involve ρ or α.
  - For example, in testing for Granger causality, we might want to test that a particular element in $\mathbf{y_{t-s}}$ for $s = 1, ..., p$ has predictive power, or joint test of $\Phi_1^{(2,1)} = \Phi_2^{(2,1)} = ... = \Phi_p^{(2,1)} = 0$.
    - But this involves ρ and ζ, so we can't do it.
- On the other hand, if we should NOT difference, and we do, our model is misspecified - you will wash out important relationships, including cointegration.