On Conjecture 2 of T. Lamont Smith (2021)
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Background

Sloan (1973) introduced the concept of *persistence* of a number, the number of times you can multiply an integer's digits together before it reaches a fixed point. For example, consider 77. $7 * 7 = 49$, and $4 * 9 = 36$, and $3 * 6 = 18$, and $1 * 8 = 8$, which is a fixed point. Thus 77 has persistence 4. This can be generalized to any base b with the *sloan map* $S(n, b)$ which is the product of the digits of n when represented in base b. We will sometimes omit the parameter b when it is clear from context the base we're working in.

It is easy to show $S(n, b) < n$ and the fixed points in any base are the integers $0 \leq n < b$. Then the persistence of n in base b is the smallest k such that $S^{(k)}(n, b) < b$. The absolute persistence of n is the maximum persistence of n in any base (see Section 3 in Lamont (2021)).

It was conjectured in Lamont (2021) that the set of integers with absolute persistence less than s is bounded for all s. This appears to be correct.

Absolute persistence is bounded

Theorem 1 of Smith shows that for persistence s and base $b > s!$, the smallest integer n with persistence s in base b is

$$n = \left\lceil \frac{((s+1)! - 1)b}{s!} \right\rceil$$

The argument used in the proof of Theorem 1 of Lamont can be extended slightly in the following way, to prove the following. For convenience, let

$$f(s) = \frac{((s+1)! - 1)}{s!}$$

Theorem 1 In any base b, any integer $f(s)b < n < (s+1)b$ has persistence s.

When $s = 1$, the inequality is $b < n < 2b$. So n has the form $1 \circ d$ in base b. The sloan map of n is then d, a single digit with persistence 0. Thus n has persistence 1.

Assuming the theorem holds for $s - 1$, we consider the sloan map of n when $f(s)b < n < (s+1)b$. In base b, n has the form $s \circ d$, where $d > \frac{(s!-1)b}{s!}$ (see Table 1 in Lamont). Thus the sloan map of n satisfies

$$S(n) = sd > \frac{(s! - 1)b}{(s-1)!} = f(s-1)b$$

And since $d < b$ (since it's a digit in base b), $S(n) = sd < sb$. So by induction, the theorem holds for all persistence $s >= 1$. □

Note that we do not imply any restrictions on the base here, but only when $b > s!$ is there guaranteed to be an integer satisfying the inequalities.

Theorem 2 Given persistence s, there exists N such that for all $n > N$, n has absolute persistence at least s for all $n/(s+1) < b < n/f(s)$.

Proof: Rearranging the terms in the inequalities in Theorem 1, we see that for any n and b with $n/(s+1) < b < n/f(s)$, n has persistence s in base b. When

$$\frac{n}{f(s)} - \frac{n}{s+1} = n\left(\frac{1}{f(s)} - \frac{1}{s+1}\right) > 1$$

there must be at least one integer b between $n/f(s)$ and $n/(s+1)$. Thus, there is at least one base in which n has persistence s when

$$n > \frac{1}{\frac{1}{f(s)} - \frac{1}{s+1}} = (s+1)((s+1)! - 1)$$

By definition, n has absolute persistence at least s, proving the theorem. □

Therefore Conjecture 2 of Lamont (2021) is correct, because the absolute persistence level $s-1$ is bounded above by $(s+1)((s+1)!-1)+1$ (and below by 1, though of course there are stronger bounds).

Corollary When $n > (s+1)((s+1)!-1$, the largest base in which n achieves persistence s is $n/(s+1)-1$ when $s+1|n$ and $\lfloor n/(s+1) \rfloor$ otherwise.

In each case, the given base is the largest integer satisfying the inequalities in Theorem 2. For any larger base, $n < f(s)b$, so by Theorem 1 of Lamont (2021), it has persistence less than s.

There are a few interesting consequences of these results:

(a) For n above the cutoff for persistence s, it is computationally easy to find an explicit base in which it achieves persistence s. As far as I know, the best known algorithm to search for such a base prior to this was by brute force.

(b) By Theorem 1 of Lamont (2021), the smallest number n in base b with persistence s is greater than or equal to sb. When combined with Theorem 2, this means any n above the cutoff for s achieves its absolute persistence in a base less than $n/(s-1)$. Since the cutoff for s grows with $s(s!)$, this provides a slight improvement in the time complexity of finding the absolute persistence of n.

(c) Since the size of the set of bases for which n has persistence s is proportional to n, by the prime number theorem there exists an N such that any $n > N$ has persistence s in a prime base.

(d) The smallest base in which n has persistence s (and n is a 2-digit number) is $n/(s+1)+1$ when $s+1$ divides n, and $\lceil n/(s+1) \rceil$ otherwise.

References

N. J. A. Sloane, The persistence of a number, *J. Recr. Math.* **6** (1973), 97–98.

Tim Lamont-Smith, Multiplicative Persistence and Absolute Multiplicative Persistence, *J. Int. Seq.* **24** (2021), Article 21.6.7.