

# Notes on singularly perturbed problems

Important concepts and results

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# 1 Useful notions

**Definition 1.1** ( $\epsilon$ -uniform methods). Miller, O’Riordan, and Shishkin [2] point out, that even simple examples of various one-dimensional problems involving singular perturbations cannot be solved numerically, in a completely satisfactory manner. This leads to the necessity for methods that behave uniformly well, whatever the value of the singular perturbation parameter may be. Such methods are called  $\epsilon$ -uniform methods,  $\epsilon$  being the singular perturbation parameter.

Valuable informations concerning the Experimental estimation of errors can be found in [1, Chapter 8, p. 157]. It is a known fact, that numerical methods for computing approximate solutions of partial differential equations are usually applied to problems with unknown exact solution. Now,  $u$  shall be the exact solution of any given problem,  $U^N$  shall be a numerical approximation on the mesh  $\Omega^N$ ,  $N$  being the number of mesh points in each coordinate direction of the discrete problem associated with the numerical method. The canonic criterion for judging the quality of a numerical method for solving problems in the case of non-singularly perturbed problems is an error estimate of the following form: there exist positive constants  $N_0, C = C(N_0)$  and  $p = p(N_0)$ , all independent of  $N$ , such that for all  $N > N_0$  one has the inequality

$$\|U^N - u\|_{\Omega^N} \leq CN^{-p} \quad (1)$$

The maximum pointwise error is in this case bounded by the error bound  $CN^{-p}$ .

**Definition 1.2** (order of local convergence). In the case that a numerical method is applied to a specific problem with a known exact solution  $u$ , the order of local convergence for the value  $N$  is defined by

$$p_{\text{exact}}^N = \log_2 \frac{\|U^N - u\|_{\Omega^N}}{\|U^{2N} - u\|_{\Omega^{2N}}} \quad (2)$$

and the asymptotic order of convergence by

$$p_{\text{exact}} = \lim_{N \rightarrow \infty} p_{\text{exact}}^N \quad (3)$$

# 2 Compatibility Conditions

[boydflyer1999] point out a severe problem that occurs in the case of diffusion or wave equations. One considers the domain  $[0, T] \times \Omega$ , where  $\Omega$  is the  $d$ -dimensional spatial domain with boundary  $\partial\Omega$ . One considers the elliptic operator

$$L = \sum_{i=1}^d \sum_{j=1}^d A_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{j=1}^d B_j(x) \frac{\partial}{\partial x_j} + \sum_{j=1}^d C_j(x) \quad (4)$$

Then the generalized linear diffusion problem reads

$$u_t = Lu, \quad u(x, t = 0) = u_0(x) \in \Omega \quad (5)$$

For simplicity, one considers first only homogeneous Dirichlet boundary conditions, i.e.

$$u = 0 \quad \text{if } u \in \partial\Omega \, \forall t.$$

One readily sees, that the boundary condition is independent of time. This entails, that the time derivatives of the solution must be zero on the boundary for any order  $j \geq 0, j \in \mathbb{N}$ :

$$\frac{\partial^j u(t=0)}{\partial t^j} = 0 \quad \text{on } [0, T] \times \partial\Omega \tag{6}$$

But it follows from this observation, that the left-hand side of the diffusion equation must be 0 on the boundary, so one obtains

$$Lu = 0 \quad u \in [0, T] \times \partial\Omega.$$

Now comes the critical point: if one would have

$$Lu_0 \neq 0 \quad u \in \partial\Omega$$

this would cause a contradiction.  $Lu$  has a jump discontinuity on the boundary at  $t = 0$  with magnitude  $Lu_0$  on  $\partial\Omega$ .

## References

- [1] P.A Farrell. *Robust computational techniques for boundary layers*. Vol. 16. Applied mathematics / [Chapman & Hall]. Boca Raton, Fla. [etc.]: Chapman & Hall, 2000.
- [2] J. J. H. Miller, E. O’Riordan, and G. I. Shishkin. *Fitted Numerical Methods For Singular Perturbation Problems*. WORLD SCIENTIFIC, 2012. DOI: [10.1142/8410](https://doi.org/10.1142/8410).

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