Notizen

Paper

Brice Girol

May 25, 2021

Table of Contents

1 Useful notions

Definition 1.1 (ϵ-uniform methods)**.** Miller, O'Riordan, and Shishkin [2] point out, that even simple examples of various one-dimensional problems involving singular perturbations cannot be solved numerically, in a completely satisfactory manner. This leads to the necessity for methods that behave uniformly well, whatever the value of the singular perturbation parameter may be. Such methods are called ϵ-uniform methods, ϵ being the singular perturbation parameter.

Valuable informations concerning the Experimental estimation of errors can be found in [1, Chapter 8, p. 157]. It is a known fact, that numerical methods for computing approximate solutions of partial differential equations are usually applied to problems with unknown exact solution. Now, u shall be the exact solution of any given problem, U^N shall be a numerical approximation on the mesh Ω^N, N being the number of mesh points in each coordinate direction of the discrete problem associated with the numerical method. The canonic criterion for judging the quality of a numerical method for solving problems in the case of non-singularly perturbed problems is an error estimate of the following form: there exist positive constants $N_0, C = C(N_0)$ and $p = p(N_0)$, all independent of N , such that for all $N > N_0$ one has the inequality

$$\left\|U^N - u\right\|_{\Omega^N} \leq CN^{-p} \tag{1}$$

The maximum pointwise error is in this case bounded by the error bound CN^{-p}.

Definition 1.2 (order of local convergence)**.** In the case that a numerical method is applied to a specific problem with a known exact solution u, the order of local convergence for the value N is defined by

$$p^N_{\text{exact}} = \log_2 \frac{\left\|U^N - u\right\|_{\Omega^N}}{\left\|U^{2N} - u\right\|_{\Omega^{2N}}} \tag{2}$$

and the asymptotic order of convergence by

$$p_{\text{ exact}} = \lim_{N\to\infty} p^N_{\text{exact}} \tag{3}$$

2 Compatibiity Conditions

[**boydflyer1999**] point out a severe problem that occurs in the case of diffusion or wave equations. One considers the domain $[0, T] \times \Omega$, where Ω is the d-dimensional spatial domain with boundary $\partial\Omega$. One considers the elliptic operator

$$L = \sum_{i=1}^{d}\sum_{j=1}^{d} A_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{j=1}^{d} B_j(x)\frac{\partial}{\partial x_j} + \sum_{j=1}^{d} C_j(x) \tag{4}$$

Then the generalized linear diffusion problem reads

$$u_t = Lu, \quad u(x, t = 0) = u_0(x) \in \Omega \tag{5}$$

For simplicity, one considers first only homogeneous Dirichlet boundary conditions, i.e.

$$u = 0 \quad \text{if } u \in \partial\Omega \, \forall t.$$

One readily sees, that the boundary condition is independent of time. This entails, that the time derivatives of the solution must be zero on the boundary for any order $j \geq 0, j \in \mathbb{N}$:

$$\frac{\partial^j u(t=0)}{\partial t^j} = 0 \quad \text{on } [0,T] \times \partial\Omega \tag{6}$$

But it follows from this observation, that the left-hand side of the diffusion equation must be 0 on the boundary, so one obtains

$$Lu = 0 \quad u \in [0,T] \times \partial\Omega.$$

Now comes the critical point: if one would have

$$Lu_0 \neq 0 \quad u \in \partial\Omega$$

this would cause a contradiction. Lu has a jump discontinuity on the boundary at $t = 0$ with magnitude Lu_0 on $\partial\Omega$.

References

[1] P.A Farrell. *Robust computational techniques for boundary layers*. Vol. 16. Applied mathematics / [Chapman & Hall]. Boca Raton, Fla. [etc.]: Chapman & Hall, 2000.

[2] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin. *Fitted Numerical Methods For Singular Perturbation Problems*. WORLD SCIENTIFIC, 2012. DOI: 10.1142/8410.

Alphabetical Index