Pocklington's and Hallen's Equations
DRAFT – NEEDS CHECKING and Some Figures

Introduction:

The following discusses the formulation and implementation of Pocklington's and Hallen's equations for analysis of thick antennas with gaps. We start with Pocklington's equation followed by Hallen's equation. The first par discusses the theory, followed by the implementation using the method of moments.

Pocklington's Equation:

Pocklington's equation is a formulation of Maxwell's equations in an integral form suitable for numerical analysis of wire antennas of finite thickness and finite gap. There are three steps involved in the analysis:

1. Formulation of Maxwell's equations
2. Method of Moments formulation.
3. Definition of sources

The following details these steps for the two methods.

1. Formulation of Maxwell's equations

Starting with Maxwell's equations, we manipulate them in a form that is amenable to numerical solution. We do so by invoking the magnetic vector potential **A** through its definition:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \rightarrow \quad \mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{1}{\mu}\nabla \times \mathbf{A} \qquad (1)$$

With this, Maxwell's first equation is as follows:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} = -j\omega\left(\nabla \times \mathbf{A}\right) \quad \rightarrow \quad \nabla \times \mathbf{E} + j\omega\left(\nabla \times \mathbf{A}\right) = 0$$

Or:

$$\nabla \times \mathbf{E} + \nabla \times \left(j\omega\mathbf{A}\right) = \nabla \times \left(\mathbf{E} + j\omega\mathbf{A}\right) = 0$$

Now, since both terms in the brackets are electric field intensities and since their curl is zero, we have:

$$\mathbf{E} + j\omega\mathbf{A} = -\nabla V$$

where V is the electric potential (see Helmholtz's theorem in Chapter 2). Thus, the electric field intensity is:

$$\boxed{\mathbf{E} = -j\omega\mathbf{A} - \nabla V} \qquad (2)$$

Now we use Maxwell's 2nd equation:

$$\nabla \times \mathbf{H} = \mathbf{J}_s + j\omega\varepsilon\mathbf{E}$$

where $\mathbf{J}_s$ is a source current density (such as the current density on the surface of an antenna). Substituting for **H** from (1) and for **E** from (2):

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J}_s + j\omega\varepsilon\left(-j\omega\mathbf{A} - \nabla V \right)$$

Multiplying both sides by μ and expanding the terms:

$$\nabla \times \left(\nabla \times \mathbf{A} \right) = \mu\mathbf{J}_s + \omega^2\mu\varepsilon\mathbf{A} - j\omega\mu\varepsilon\nabla V$$

Since $\nabla \times \left(\nabla \times \mathbf{A} \right) = -\nabla^2\mathbf{A} + \nabla\left(\nabla \cdot \mathbf{A} \right)$, we have:

$$\nabla^2\mathbf{A} - \nabla\left(\nabla \cdot \mathbf{A} \right) = -\mu\mathbf{J}_s - \omega^2\mu\varepsilon\mathbf{A} + \nabla\left(j\omega\mu\varepsilon V \right)$$

Note that because $\omega\mu\varepsilon$ is a constant, we can move it into the gradient. Now, recall that the divergence of **A** can be selected in a convenient manner so that:

$$\nabla\left(\nabla \cdot \mathbf{A} \right) = -\nabla\left(j\omega\mu\varepsilon V \right) \qquad (3)$$

That is,

$$\boxed{\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon V} \qquad (4)$$

(We have called this the Lorenz gauge). With this, we have:

$$\boxed{\nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J}_s} \qquad (5)$$

We will come back to this, but for now we look into (3). Rewriting it:

$$\nabla V = -\frac{1}{j\omega\mu\varepsilon}\nabla\left(\nabla \cdot \mathbf{A} \right)$$

Substituting this in (2):

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla V = -j\omega\mathbf{A} + \frac{1}{j\omega\mu\varepsilon}\nabla \cdot \mathbf{A}$$

or:

$$\boxed{\mathbf{E} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\left(\nabla \cdot \mathbf{A} \right)} \qquad (6)$$

What we have done so far is to write the electric field intensity in terms of the magnetic vector potential alone. We will use this equation as the basis for the derivation of Pocklington's equation. It looks complicated (and it is) but it really is only a sequence of substitutions.

Connection to Antennas

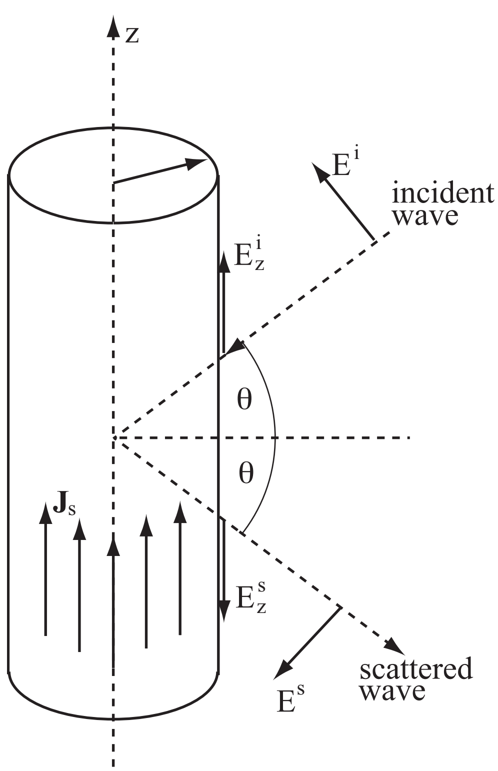
To connect all this to antennas, consider a conductor of radius *a* as in **Figure A**. An incident waves with electric field intensity as shown impinges on the conductor. It is reflected and the electric field intensity in the reflected wave is as shown. We will call this a scattered field because this name is more common in antenna work. You can think of the incident field as a field received by the antenna whereas the scattered field as that transmitted by the antenna in transmit mode. At the surface of the antenna, the total tangential component (*z*-directed, along the antenna) must be zero (assuming the antenna is made of a very good conductor). Since the electric field intensity must be in the direction of **A**, at the surface, **A** is in the *z*-direction, and any current in the antenna must also be in that direction. Also, because antennas operate at high frequencies, the current only exists at the surface as a surface current density [A/m].

We can write in general (not only on the conductor).

$$\mathbf{E}_s = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\mathbf{A}\right) = \frac{-j\omega^2\mu\varepsilon\mathbf{A} - j\nabla\left(\nabla\cdot\mathbf{A}\right)}{\omega\mu\varepsilon} = -\frac{j}{\omega\mu\varepsilon}\left(k^2\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A}\right)\right)$$

On the conductor, we can write for the tangential components of **E**:

$$\mathbf{E}_z^s(r=a) + \mathbf{E}_z^s(r=a) = 0 \quad \rightarrow \quad \mathbf{E}_z^s(r=a) = -\mathbf{E}_z^i(r=a) \tag{6a}$$



From the previous equation:

$$\boxed{\mathbf{E}_z^s = -\frac{j}{\omega\mu\varepsilon}\left(k^2\mathbf{A}_z + \nabla\left(\nabla\cdot\mathbf{A}_z\right)\right)} \tag{7}$$

Again, you may want to think about this as the transmitted electric field intensity given here in terms of the magnetic vector potential. Now, given a conductor carrying a current *I*, the magnetic vector potential (magnitude only) due to this current at a distance *R* (see calculation for the arbitrarily long antenna):

$$A = \frac{\mu}{4\pi} \int_{z'=-l/2}^{z'=l/2} I \frac{e^{-jkR}}{R} dz'$$

This is simply the Biot-Savart law. But, if the current is on the surface of the conductor as a surface current density, the total current in the conductor of radius *a* is:

$$I_z = \int_{\phi'=0}^{\phi'=2\pi} J_z (a\, d\phi') \qquad (8)$$

Thus, the magnetic vector potential due to a surface current density J_z (the current density can only have a *z*-component since it must flow along the conductor) on a conductor of radius *a* is:

$$A = \frac{\mu}{4\pi} \int_{z'=-l/2}^{z'=l/2} \left[\int_{\phi'=0}^{\phi'=2\pi} J_z \frac{e^{-jkR}}{R} a\, d\phi' \right] dz' \qquad (9)$$

If the conductor is not very thick (compared to the wavelength), and carries a total current $I_z(z')$, the current density may be assumed to be:

$$J_z(z') = \frac{I_z(z')}{2\pi a} \quad \left[\frac{\text{A}}{\text{m}} \right]$$

Note that the current and hence the current density vary with *z'* as we have seen in the section on arbitrarily long antennas where the current was sinusoidal. Here we don't know what the current is (that's what we are trying to calculate) and we cannot assume anything except that we know it varies with position along the conductor. Thus (9) becomes:

$$A_z = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \left[\frac{1}{2\pi a} \int_{\phi'=0}^{\phi'=2\pi} I_z(z') \frac{e^{-jkR}}{R} a\, d\phi' \right] dz' \qquad (10)$$

Before attempting to calculate *A*, it is useful to look at *R* in cylindrical coordinates. Taking a point (a, ϕ', z') on the surface of the conductor and a point (r, ϕ, z) at which we are calculating *A*, then *R* is:

$$R = \sqrt{r^2 + a^2 - 2ra\cos\left(\phi - \phi'\right) + \left(z - z'\right)} \qquad (10a)$$

Now, suppose we calculate *R* at a point on the surface of the conductor, that is, at $r = a$ at $\phi = 0$ (we take this value for ϕ since regardless of ϕ, the distance *R* will be the same, that is, scattering is symmetric):

$$R(r=a)=\sqrt{2a^2-2a^2\cos\left(-\phi'\right)+\left(z-z'\right)}$$
$$=\sqrt{4a^2\left(\frac{1-\cos\phi'}{2}\right)+\left(z-z'\right)}=\sqrt{4a^2\sin^2\frac{\phi'}{2}+\left(z-z'\right)} \qquad (11)$$

A pause for an explanation on why we did that:
The purpose of Pocklington's equation is to calculate the current distribution on the antenna given a known field distribution. We calculate the current based on a known electric field intensity at the surface of the antenna. We will see how we can know the field at the surface but, for now, as an example, if the antenna is a conductor (and it always is), then the electric field intensity in and on the antenna is approximately zero (it is zero for a perfect conductor). We will talk about this later but for now, it is sufficient to remember that the magnetic vector potential in (10) will be calculated at the surface of the antenna, from which **E** in (7) is then calculated and that will provide a means of calculating $I_z(z')$ in (10). Once we have this current, we will view the antenna as an arbitrarily long antenna with this current along it and that will allow us to calculate the field at any point in space. It is a long and convoluted method but in the end we will obtain a rather simple method that can be used for any antenna (with some restrictions on dimensions: radius and gap must be small compared to the wavelength).

At $r = 0$,

$$R(r=0)=\sqrt{a^2+\left(z-z'\right)}$$

that is, this is the distance between an arbitrary point z on the surface of the conductor and an arbitrary point z' on the axis of the conductor.

At $r = a$, (10) becomes:

$$A_z=\mu\int_{-l/2}^{l/2} I_z(z')\left[\frac{1}{2\pi}\int_{\phi'=0}^{\phi'=2\pi}\frac{e^{-jkR}}{4\pi R}d\phi'\right]dz' \qquad (11a)$$

The term in square brackets only depends on R and R depends on z and z' as can be seen in (11). This term is called the Green's function for the conductor. You may view it as the unit response of the antenna for current:

$$\boxed{G(z,z')=\frac{1}{2\pi}\int_{\phi'=0}^{\phi'=2\pi}\frac{e^{-jkR}}{4\pi R}d\phi'} \qquad (12)$$

With all this, the magnetic vector potential becomes:

$$\boxed{A_z = \mu \int_{-l/2}^{l/2} I_z(z')G(z,z')dz'} \quad (13)$$

Now we can substitute this in (7) but before we do that we note that because **A** only has a z-component, (7) can be simplified since $\nabla \cdot \mathbf{A} = \dfrac{\partial A_z}{\partial z} = \dfrac{dA_z}{dz}$ and therefore $\nabla\left(\nabla \cdot \mathbf{A}\right) = \dfrac{d^2 A_z}{dz^2}$. Thus,

$$E_z^s = -\frac{j}{\omega\mu\varepsilon}\left(k^2 + \frac{d^2}{dz^2}\right)A_z = -\frac{j}{\omega\varepsilon}\left(k^2 + \frac{d^2}{dz^2}\right)\int_{-l/2}^{l/2} I_z(z')G(z,z')dz'$$

We re-write this in the following way:

$$\boxed{\left(k^2 + \frac{d^2}{dz^2}\right)\int_{-l/2}^{l/2} I_z(z')G(z,z')dz' = -j\omega\varepsilon E_z^s} \quad \text{(at } r = a\text{)} \qquad (14)$$

This is Pocklington's Equation.

Before this can be solved, we make a couple of simple assumptions as follows:

1. The antenna is of finite thickness but is thin with respect to the wavelength. That is, $a << \lambda$, where a is the radius of the antenna..
2. The gap between the two monopoles forming the dipole is small, that is, $d << \lambda$, where d is the gap between the elements in the dipole or between one element and ground in a monopole.
3. Because the antenna is thin, we can assume that the current can be assumed to be on the center line of the antenna (at r' =0) and the field is calculated at the surface of the antenna at $r = a$. Substituting these into (10a), we have:

$$R(r=0) \approx \sqrt{a^2 + \left(z - z'\right)} \qquad (15)$$

The first of these assumptions means that Green's function is independent of the angle ϕ', which in turn means that the distance R is independent of the angle ϕ'. That means that the Green function is:

$$G(z,z') = \frac{1}{2\pi}\int_{\phi'=0}^{\phi'=2\pi} \frac{e^{-jkR}}{4\pi R} d\phi' \approx \frac{2\pi}{2\pi}\frac{e^{-jkR}}{4\pi R}$$

or:

$$\boxed{G(z,z') \approx \frac{e^{-jkR}}{4\pi R}} \qquad (16)$$

With the results in (15) and (16), performing the second order derivative in the brackets and expanding the terms, Pocklington's equation can be written as follows:

$$\boxed{\int_{z'=-l/2}^{z'l/2} I_z(z') \frac{e^{-jkR}}{4\pi R^5}\left[\left(1+jkR\right)\left(2R^2-3a^2\right)+\left(kaR\right)^2\right]dz' = -j\omega\varepsilon E_z^s} \qquad (17)$$

This is Poclington's Integral equation. It is a more explicit form than (14) and we will use it for the next steps.

Again, it looks complicated because of all the details. But looking at (14), we simply assume that E_z is known and attempt to calculate the current along the antenna that produces that field. This is an integro-differential equation with the unknown inside the integral so it is not solvable analytically. But the Method of Moments will take care of that in a rather simple way, after we make some reasonable assumptions.

Hallen's Integral Equation

We start with Eq. (6) above:

$$\boxed{\mathbf{E} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\mathbf{A}\right)} \qquad (6)$$

This equation calculates the current along the antenna given the gap voltage. Hallen's approach also assumes the antenna to be a perfect conductor whereas Pocklington's approach does not. But we start the process with the total electric field intensity and the magnetic vector potential.
Instead of calculating the scattered field as we have done in Eq. (7), we can calculate the total electric field intensity (that is, the sum of the incident and scattered fields), since this relation is totally general. However, on the surface of the perfect conductor, the total field is zero (see (6a)). That is, for the total field we have:

$$-j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\mathbf{A}\right) = 0$$

Multiplying both sides by j and then by $\omega\mu\varepsilon$, we have:

$$\omega^2\mu\varepsilon\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A}\right) = 0$$

Since the current in the antenna is in the z-direction, the magnetic vector potential is also in the z-direction and we have:

$$\boxed{\frac{d^2 A_z}{dz^2} + \omega^2 \mu\varepsilon A_z = 0} \qquad (18)$$

This is called the Helmholtz equation but the important thing is that it is a wave equation and has a standard solution as follows:

$$A_z = -j\sqrt{\mu\varepsilon}\left[B_1 \cos\left(kz'\right) + C_1 \sin\left(k\left|z'\right|\right)\right] \qquad (19)$$

Although we don't show how this solution is obtained, it can be verified by substitution back into (18). The constants B_1 and C_1 will have to be defined (shortly).

However, for a current carrying antenna, we have calculated the magnetic vector potential in Eq. (11a)

$$A_z = \mu \int_{-l/2}^{l/2} I_z(z') \left[\frac{1}{2\pi} \int_{\phi'=0}^{\phi'=2\pi} \frac{e^{-jkR}}{4\pi R} d\phi'\right] dz'$$

With the assumptions of a thin antenna (see assumptions and Eqs. (15) and (16)), the Green function is:

$$G(z,z') = \frac{1}{2\pi} \int_{\phi'=0}^{\phi'=2\pi} \frac{e^{-jkR}}{4\pi R} d\phi' \approx \frac{2\pi}{2\pi} \frac{e^{-jkR}}{4\pi R} = \frac{e^{-jkR}}{4\pi R}$$

where R is given in (15). Thus:

$$A_z = \mu \int_{-l/2}^{l/2} I_z(z') \frac{e^{-jkR}}{4\pi R} dz' \qquad (20)$$

Clearly (20) and (19) must equal and we have:

$$\mu \int_{-l/2}^{l/2} I_z(z') \frac{e^{-jkR}}{4\pi R} dz' = -j\sqrt{\mu\varepsilon}\left[B_1 \cos\left(kz'\right) + C_1 \sin\left(k\left|z'\right|\right)\right]$$

or, dividing both sides my μ:

$$\boxed{\int_{z'=-l/2}^{z'=l/2} I_z(z') \frac{e^{-jkR}}{4\pi R} dz' = -j\sqrt{\frac{\varepsilon}{\mu}}\left[B_1 \cos\left(kz'\right) + C_1 \sin\left(k\left|z'\right|\right)\right]} \qquad (21)$$

This is Hallen's Integral Equation

Some notes:

1. One can use either Poclington's equation in (17) or Hallen's equation in (21) to solve for the current along the antenna.
2. The solutions are not exactly the same and in some cases, one is better than the other. For example, as the gap decreases, Pocklington's equation becomes more accurate whereas for larger gaps Hallen's equation is better.
3. Both equations are approximate and are only exact in the limit (that is as the radius and the gap tend to zero).
4. These are not the only methods of analyzing antennas but we will limit ourselves to these two.
5. The sources for the Pocklington's equation are the electric field intensity at the surface of the antenna. For Hallen's equation, the source is the voltage across the gap (V_{gap}) shown in (21) as C_1. In fact $C_1 = V_{gap}/2$. The sources will be discussed below.

The Method of Moments

To solve Pocklington's or Hallen's equation we use the Method of Moments. The steps are described next, starting with Hallen's equation. The whole idea in the MoM is to replace the integral equations by an equivalent system of linear equations that can be solved for the current along the antenna.

MoM for Hallen's equation

We start with Eq. (21) but first note that in air, $\sqrt{\varepsilon/\mu} = 1/\eta_0$ and we will also substitute $C_1 = V_{gap}/2$ as discussed above. Thus, the equation to solve is:

$$\int_{z'=-l/2}^{z'=l/2} I_z(z')\frac{e^{-jkR}}{4\pi R}dz' = -\frac{j}{\eta_0}\left[B_1\cos(kz') + \frac{V_{gap}}{2}\sin(k|z'|)\right] \qquad (21a)$$

Recall as well that

$$R \approx \sqrt{a^2 + (z-z')}$$

To simplify notation in the following, we define:

$$G(z,z') = \frac{e^{-jkR}}{4\pi R} \quad \text{and} \quad D(z) = -\frac{j}{\eta_0}\left[B_1\cos(kz) + \frac{V_{gap}}{2}\sin(k|z|)\right] \qquad (21b)$$

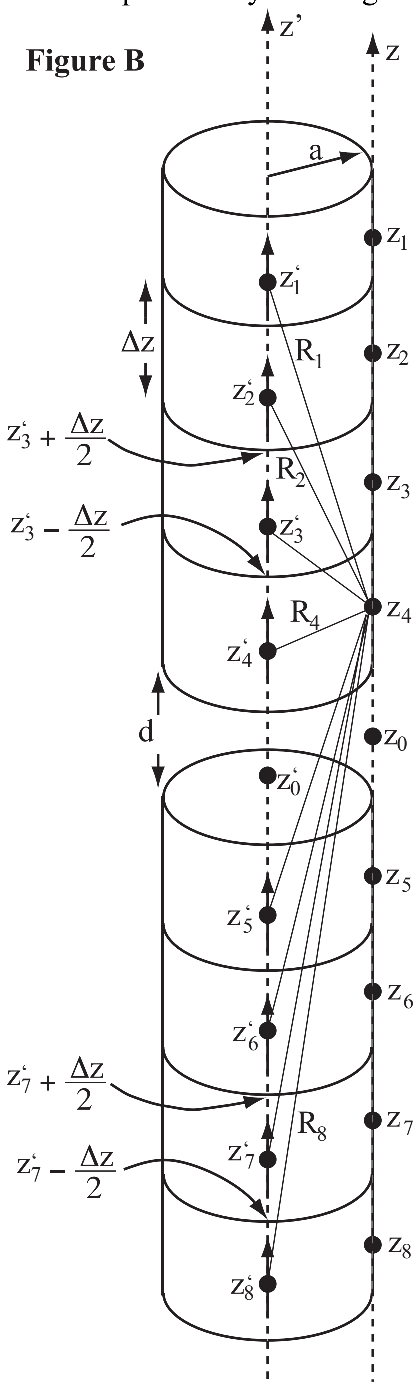
and rewrite (21a) as

$$\int_{z'=-l/2}^{z'=l/2} I_z(z')G(z,z')dz' = D(z) \quad (21c)$$

This will economize on writing.

Now we proceed by dividing the antenna into an arbitrary number of sections – kind of slicing the conductor into short segments, each of length *Δz'* as shown in **Figure B**. The axis of the antenna is along indicated as *z'* and the current is assumed to be along this axis. The surface of the antenna is on another axis we denote as *z*, a distance *a* from *z'*. In each section (we will call it an element) of the antenna, we assume an unknown current $I_n(z')$, which varies from element to element but it is constant within the element. This can be viewed as a stack of Hertzian dipoles of length *Δz'*. In **Figure B** the two halves of the dipole were divided into 4 elements each. The elements do not actually have to be of the same size but for simplicity and for implementation it is easier if they are of the same size (Δ*z* in this case). Note that the currents only exist in the conductor ,but we also marked the center of the gap as z'_0 on the *x'* axis and z_0 on the *z*-axis. More on that in the implementation section

Figure B



We write the current at any point of the antenna as:

$$I(z') = \sum_{n=1}^{N} I_n\left(U_n(z')\right) \quad (22)$$

where

$$U_n(z') = \begin{cases} 1 \quad \textit{for} \quad \left(z'_n - \Delta z'/2\right) < z' < \left(z'_n + \Delta z'/2\right) \\ \qquad\qquad 0 \quad \textit{elsewhere} \end{cases}$$

where $n = 1, 2, \ldots N$, and N is the total number of segments into which the antenna is divided.

These are called basis or expansion functions. It is important to note that what this function does, is to force I_n(z') in (22) to be zero everywhere except in element *n*. Thus, in element 1, we have a current I_1, in element 2, a current I_2, etc., and hence (22) is a kind of staircase function describing the current along the antenna. The idea is that as the number of elements increases, the staircase approximation becomes more accurate. Substituting (22) in (21a):

$$\int_{z'=-l/2}^{z'=l/2} \left[\sum_{n=1}^{N} I_n\left(U_n(z')\right)\right] G(z,z')dz' = D(z)$$

Now we multiply both sides of the antenna by a new function *w*(*z*) (it is called a weight or weighting function) and integrate both sides of this equation over the length of the antenna along the *z*-axis as follows:

$$w_m(z) = \delta(z - z_m), \quad m = 1,2,....N+1 \qquad (23)$$

$$\int_{z=-l/2}^{z=l/2} \left\{\int_{z'=-l/2}^{z'=l/2} \left[\sum_{n=1}^{N} I_n\left(U_n(z')\right)\right] \frac{e^{-jkR}}{4\pi R} dz'\right\} w_m(z)dz = \int_{z=-l/2}^{z=l/2} \left\{-\frac{j}{\eta_0}\left[B_1\cos\left(kz\right) + \frac{V_{gap}}{2}\sin\left(k\left|z\right|\right)\right]\right\} w_m(z)dz$$

Note that there are two variables here: *z* and *z'*. The current only depends on *z'* but the magnetic vector potential depends on *z* and *z'* through *R*. Also, in (23), $\delta(z - z_m)$ is the Kronecker delta function (see **Figure C**)

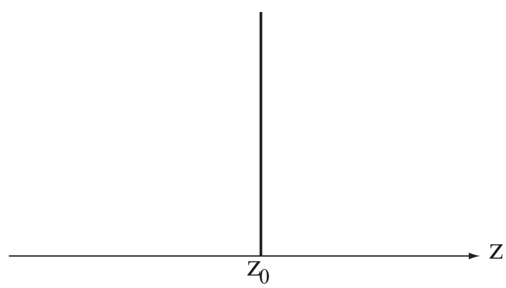


Figure C. Delta function $\delta(z - z_0)$

$$\int_{z=-l/2}^{z=l/2} \left\{\int_{z'=-l/2}^{z'=l/2} \left[\sum_{n=1}^{N} I_n\left(U_n(z')\right)\right] G(z,z')dz'\right\} w_m(z)dz = \int_{z=-l/2}^{z=l/2} D(z) w_m(z)dz$$

Now, since the integral of a function multiplied by the delta function $\delta(z - z_m)$ is equal to the value of the function at $z = z_m$:

$$\int_{z'=-l/2}^{z'=l/2} \left[\sum_{n=1}^{N} I_n\left(U_n(z')\right)\right] G(z_m,z')dz' = D(z_m)$$

Now we interchange between the sum and the integration:

$$\sum_{n=1}^{N} I_n \int_{z'=-l/2}^{z'=l/2} U_n(z')G(z_m,z')dz' = D(z_m)$$

But, since the function $U_n(z')$ is zero outside element n and 1 within element n, we only need to integrate on each element independently:

$$\sum_{n=1}^{N} I_n \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} (z')G(z_m,z')dz' = D(z_m) \qquad (23b)$$

Re-introducing the notation in (21b), the equation becomes:

$$\sum_{n=1}^{N} I_n \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} \frac{e^{-jkR_m}}{4\pi R_m} dz' = -\frac{j}{\eta_0}\left[B_1 \cos\left(kz_m\right) + \frac{V_{gap}}{2}\sin\left(k\left|z_m\right|\right)\right]$$

Now we are almost done except we note the following:

1. On the left hand side we have N unknown values of I that need to be evaluated.
2. On the right hand side we have an additional unknown value of B_1.
3. There are $N+1$ values of z_m. The first N values are the centers of the N elements of the antenna whereas the $N+1$th is the center of the gap.

Note however that for each value of z_m, we must integrate the left hand side. That is, there are in fact $N+1$ equations in $N+1$ unknowns. Perhaps the best way to understand this is to look at a few equations. Since z_m is the center of element m, we can write, for example, assuming there are a total of 8 elements, 4 in each half of the dipole and a gap **(See Figure B):** At the center of element 1 (this point is on the surface of the conductor):

$$\sum_{n=1}^{10} I_n \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} \frac{e^{-jkR_1}}{4\pi R_1} dz' + \frac{j}{\eta_0} B_1 \cos\left(kz_1\right) = -j\frac{V_{gap}}{2\eta_0}\sin\left(k\left|z_1\right|\right)$$

where:

$$R_1 = \sqrt{a^2 + \left(z_1 - z'\right)^2}$$

At the center of element 6 (for example):

$$\sum_{n=1}^{10} I_n \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} \frac{e^{-jkR_6}}{4\pi R_6} dz' + \frac{j}{\eta_0} B_1 \cos\left(kz_6\right) = -j\frac{V_{gap}}{2\eta_0}\sin\left(k\left|z_6\right|\right)$$

where

$$R_6 = \sqrt{a^2 + \left(z_6 - z'\right)^2}$$

At the center of element 8 (for example):

$$\sum_{n=1}^{8} I_n \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} \frac{e^{-jkR_8}}{4\pi R_8} dz' + \frac{j}{\eta_0} B_1 \cos(kz_8) = -j\frac{V_{gap}}{2\eta_0}\sin(k|z_8|)$$

where

$$R_8 = \sqrt{a^2 + (z_8 - z')^2}$$

Finally, at the center of the gap

$$\sum_{n=1}^{8} I_n \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} \frac{e^{-jkR_9}}{4\pi R_9} dz' + \frac{j}{\eta_0} B_1 \cos(kz_9) = -j\frac{V_{gap}}{2\eta_0}\sin(k|z_9|)$$

where:

$$R_9 = \sqrt{a^2 + (z_9 - z')^2}$$

In most cases z_{11} (the center of the gap) will be at $z = 0$ but of course it does not have to be.

Note also that the term containing the unknown B_1 was moved to the left hand side. This in effect is an $N + 1$ system of equations in $N + 1$ unknowns as follows:

$$\begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} & \frac{j}{\eta_0}\cos(kz_1) \\ K_{21} & K_{22} & \cdots & K_{2N} & \frac{j}{\eta_0}\cos(kz_2) \\ \vdots & \vdots & & \vdots & \vdots \\ K_{N+1,1} & K_{N+1,2} & \cdots & K_{N+1,N} & \frac{j}{\eta_0}\cos(kz_{N+1}) \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \\ B_1 \end{Bmatrix} = \begin{Bmatrix} -j\frac{V_{gap}}{2\eta_0}\sin(k|z_1|) \\ -j\frac{V_{gap}}{2\eta_0}\sin(k|z_2|) \\ \vdots \\ -j\frac{V_{gap}}{2\eta_0}\sin(k|z_{N+1}|) \end{Bmatrix} \quad (24)$$

where:

$$K_{m,n} = \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} \frac{e^{-jkR_m}}{4\pi R_m} dz', \qquad R_m = \sqrt{a^2 + (z_m - z')^2}$$

Once the coefficients are evaluated, the system of equations is solved. The unknown B_1 is calculated but it is discarded since we are only interested in the currents along the antenna. More explanations will be given in the section on implementation.

MoM for Pocklington's equation

We turn now to the Pocklington's equation in (21):

$$\int_{z'=-l/2}^{z'l/2} I_z(z')\frac{e^{-jkR}}{4\pi R^5}\left[\left(1+jkR\right)\left(2R^2-3a^2\right)+\left(kaR\right)^2\right]dz' = -j\omega\varepsilon E_z^s$$

Now suppose make the following notation:

$$G(z,z') = \frac{e^{-jkR}}{4\pi R^5}\left[\left(1+jkR\right)\left(2R^2-3a^2\right)+\left(kaR\right)^2\right]$$

With:

$$R = \sqrt{a^2+\left(z-z'\right)^2}$$

$$D(z) = -j\omega\varepsilon E_z^s$$

With these, Pocklington's equation becomes:

$$\int_{z'=-l/2}^{z'l/2} I_z(z')G(z,z')dz' = D(z)$$

This is identical in form to (21c) which was Hallen's equation. We can therefore proceed in the same fashion by segmenting the antenna into $N+1$ elements. N should be an even number, with $N/2$ segments on each half of the dipole and one containing the gap. Figure D shows the division of an antenna into 4 elements for each half-dipole and one element in the gap. Once we do that we can proceed with the same steps as above and get:

$$\sum_{n=1}^{N} I_n \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} G(z_m,z')dz' = D(z_m)$$

This is the same as (23c) except, of course, that the terms are different. Re-substituting the notation, we have:

$$\sum_{n=1}^{N} I_n \int_{z'=z_n'-\Delta z'/2}^{z'=z_n'+\Delta z'/2} \left\{\frac{e^{-jkR_m}}{4\pi R_m^5}\left[\left(1+jkR_m\right)\left(2R_m^2-3a^2\right)+\left(kaR_m\right)^2\right]\right\}dz' = -j\omega\varepsilon E_{z_m}^s$$

where:

$$R_m = \sqrt{a^2+\left(z_m-z'\right)^2}$$

Note also that the function $w_m(z)$ in (23) only has $N+1$ terms.
Unlike the Hallen equation, there are only $N+1$ unknowns here (the currents) hence we get an $N+1$ by $N+1$ system of equations. The system is:

(25)

$$
\begin{bmatrix}
K_{1,1} & K_{1,2} & \cdots & \cdots & K_{1,N} & K_{1,N+1} \\
K_{2,1} & K_{2,2} & \cdots & \cdots & K_{2,N} & K_{2,N+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
K_{N/2,1} & K_{N/2,2} & \cdots & \cdots & K_{N/2,N} & K_{N/2,N+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
K_{N+1,1} & K_{N+1,2} & \cdots & \cdots & K_{N+1,N} & K_{N+1,N+1}
\end{bmatrix}
\begin{Bmatrix} I_1 \\ I_2 \\ \\ I_{N/2+1} \\ \\ I_{N+1} \end{Bmatrix}
=
\begin{Bmatrix} -j\omega\varepsilon E^s_{z_1} \\ -j\omega\varepsilon E^s_{z_2} \\ \\ -j\omega\varepsilon E^s_{z_{N/2+1}} \\ \\ -j\omega\varepsilon E^s_{z_{N+1}} \end{Bmatrix} \quad (25)
$$

where:

$$
K_{m,n} = \int_{z'=z'_n-\Delta z'/2}^{z'=z'_n+\Delta z'/2} \left\{ \frac{e^{-jkR_m}}{4\pi R_m^5} \left[\left(1+jkR_m\right)\left(2R_m^2-3a^2\right) + \left(kaR_m\right)^2 \right] \right\} dz'
$$

(Note: in **Figure D**, $N = 8$ and therefore, the gap element number is $N/2 + 1 = 5$).

The value $E^s_{z_m}$ is the known electric field intensity at the center of element m.

The calculation of the electric field intensity follows below.

Once the electric field intensity is calculated, the coefficients of the matrix are calculated in a manner similar to that for the Hallen equation. It should be noted that in Pocklington's equation we actually calculate the current in the gap. That is the feed current to the antenna.

Sources for the MoM Calculations

Hallen's equation: In this case the source is trivially simple: The only possible source is the voltage across the gap as in (21a). That is, once the constant C_1 is replaced with $V_{gap}/2$, the sources are set and calculation can proceed.

Pocklington's equation: Here the situation is very different. There are two basic methods to define E^s_z in (25).

1. Gap generator: This is the simplest method. It consists of setting E anywhere in the gap to equal V_{gap}/d and anywhere on the surface of the conductor to zero (See **Figure E(b).** However, this requires modification of the matrix in (25) as follows:

    a. When dividing the antenna into elements divide the gap itself as an element (hence the need for the total number of elements to be odd).
    b. Use the gap generator as described in (1). That is, on the gap, specify the electric field intensity as in (a), whereas on all other elements, the electric field intensity is zero.
    c. Solve for the current everywhere including the gap.

d. The voltage across the gap divided by the current in the gap is the antenna impedance (the real part is the radiation resistance).
e. The current distribution along the antenna is then used to calculate other parameters. Note however than when calculating the electric and magnetic field intensity in the far field, the gap current is not used – this current is really the current in the transmission line feeding the antenna, not a current in the antenna.

Note that we must use this method because if we do not model the gap, the field would be zero everywhere and the problem cannot be solved.

The matrix for the gap generator source looks as follows:

$$\begin{bmatrix} K_{1,1} & K_{1,2} & \cdots & \cdots & K_{1,N} & K_{1,N+1} \\ K_{2,1} & K_{2,2} & \cdots & \cdots & K_{2,N} & K_{2,N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{N/2+1,1} & K_{N/2+1,2} & \cdots & \cdots & K_{N/2=1,N} & K_{N/2+1,N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{N+1,1} & K_{N+1,2} & \cdots & \cdots & K_{N+1,N} & K_{N+1,N+1} \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N/2+1} \\ \vdots \\ I_{N+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ -j\omega\varepsilon V_{gap}/d \\ \vdots \\ 0 \end{Bmatrix}$$

2. The second method is the Frill generator discussed in class. This is much more accurate but also more complex. We will not go through the details because they require rather complex arguments and integration. But the method is as follows:

In this method, the electric field along the antenna is calculated as follows:

$$\begin{aligned} E_z^s = -V_{gap}\Bigg[\frac{k\left(b^2-a^2\right)e^{-jkR_0}}{8\ln(b/a)R_0^2}\Bigg\{2\left[\frac{1}{kR_0}+j\left(1-\frac{b^2-a^2}{2R_0^2}\right)\right] \\ +\frac{a^2}{R_0}\left(\left[\frac{1}{kR_0}+j\left(1-\frac{b^2+a^2}{2R_0^2}\right)\right]\left(-jk-\frac{2}{R_0}\right)+\left(\frac{1}{kR_0^2}+j\frac{b^2+a^2}{R_0^3}\right)\right]\Bigg\}\Bigg] \end{aligned} \tag{26}$$

where:

$$R_0 = \sqrt{a^2+z^2} \tag{27}$$

This field distribution along the antenna and gap is shown schematically in **Figure E(c).**

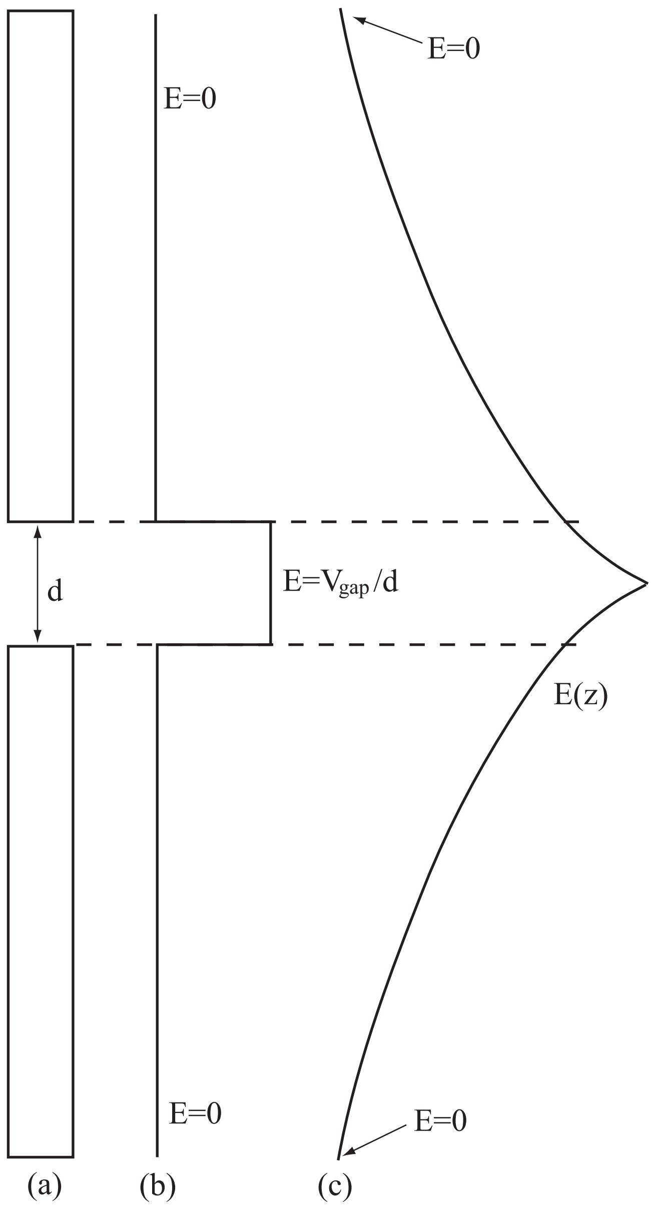


Figure E

You will note in (26) that b has not been defined (a is the radius of the antenna and is known). The value of b is defined as follows:

a. Start with the radiation resistance of the antenna R_{rad}. Assume for a moment that it is known.

b. Use the following formula for the impedance of the antenna (this assumes the antenna is fed with a coaxial cable, which is almost always true but even if it is not, the formula is a good approximation for our needs):

$$R_{rad} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right)$$

$$= \frac{120\pi}{2\pi}\ln\left(\frac{b}{a}\right) = 60\ln\left(\frac{b}{a}\right)$$

or:

$$\frac{R_{rad}}{60} = \ln\left(\frac{b}{a}\right) \quad \rightarrow \quad e^{R_{rad}/60} = \frac{b}{a}$$

That is:

$$\boxed{b = ae^{R_{rad}/60}} \qquad (28)$$

In general, R_{rad} for the antenna is not known so instead we use the radiation resistance of an ideal (infinitesimally thin antenna, no gap) of the same overall length as the antenna we want to analyze as a first approximation. For example, suppose we want to analyze a thick half-wavelength antenna with a gap. We do not know its radiation resistance so we use $R_{rad} = 73\ \Omega$ as an approximation. For any antenna length, use **Eq. (18.93)** in the E&M book or **Table 18.2** if the antenna length you are analyzing is listed. Once you have the radiation resistance, substitute it in (28) to find b and then substitute b in (26) to find the electric field intensity at any point of the antenna. Use the values calculated in (26) in (25) but please note that you only need the values at z_m (the center of the various elements).

Eq. (26) provides more accurate values for the electric field than the gap generator since the field on the antenna is not zero. But it is much more complex. An approximation to Eq. (26) can be used if the antenna is very thin (but still finite in thickness). This provides a simpler calculation for the electric field intensity on the antenna but is less accurate than (26). Use with caution.

$$E_z^s = -\frac{V_{gap}}{2\ln(b/a)}\left[\frac{e^{-jkR_1}}{R_1} - \frac{e^{-jkR_2}}{R_2}\right] \tag{29}$$

where:

$$R_1 = \sqrt{a^2+z^2}\ , \qquad R_2 = \sqrt{b^2+z^2}$$

This field is also demonstrated in **Figure E(c)** although it is slightly different than that produced by (26).

Implementation: Proceed with the same steps as for the gap generator and calculate the electric field intensity at the center of all elements in of the antenna including the gap (that is, the gap is one of the segments, whereas each half-dipole is segmented into *N*/2 elements). All other consideration given for the gap generator apply except of course that now the electric field intensity is not zero on the conducting antenna.

Some Comments and hints:

Hallen's equation:

1. Divide each half of the dipole into equal number of elements. Do not include the gap.
2. The number of divisions should be reasonable: say you start with 50.
3. For practical purposes start numbering at one end of the dipole. This will simplify plotting.
4. Solve for the current along the antenna
5. Now increase the number of divisions. You can increase the number anyway you want but my recommendation is to double them. This simplifies data.
6. Calculate the currents again and compare with the previous calculation to see if there is any change. This is a difficult task. You can simply plot the two solutions and see visually if there is any change. A more accurate way of doing this is to calculate an average error per element and see if it is below a certain, predefined value. This is done as follows:

$$error = \frac{\sum_{n=1}^{N_{current}} I_n - \sum_{n=1}^{N_{previous}} I_n}{N_{current}} \tag{30}$$

where $N_{current}$ is the number of elements in the current calculation and $N_{previous}$ is the number of elements in the previous calculation. If the error is smaller than a certain percentage the current solution is assumed to be correct. The percentage is up to you and can be something like 1% (error $\leq$ 0.01) or something like that. Don't overdo it!. If you make this too small, the solution may take an inordinate amount of time or may not by possible. 0.01 or 0.001 should do just fine.

7. In some cases, the solution may look really bad. Assuming you are doing it correctly, this means that the basic assumptions of thin antenna and small gap are not satisfied. The conclusion is that the method id not suitable for these dimensions.
8. In the solution, you may observe some oscillations at the end of the antenna and sometimes around the gap. This is due to numerical errors and in such cases you should take the average value as the correct curve.
9. Calculation of parameters of the antenna after the evaluation of the currents follows the relations we saw for arbitrarily long antennas but you must start by viewing each element as a Hertzian dipole, calculating its E and H fields in the far field and then using superposition of the fields of the N elements. This can be done as in Section 18.7.1 by changing the integrals to sums or by viewing the N elements as an array and calculating the fields of the array. Once you have the fields you can calculate any parameter.
10. The feed current in the antenna is the current found in either of the elements next to the gap (elements 4 or 6 in **Figure B**). That is the current used to calculate the input impedance.
11. When plotting the current distribution along the antenna, the gap is of course omitted.

Pocklington's equation:
All comments for Hallen's equation apply here as well except:
1. The gap is viewed as an element and hence the electric field intensity in for all elements must be supplied using one of the methods for sources discussed above.
2. Unlike the Hallen's equation, the gap current is calculated and used to calculate the antenna impedance.

Integration:
In either of the methods (Poclington or Hallen, with any of the sources) there is a need to integrate over the elements. There are many methods of doing that. In some cases, analytical integration may be possible. In others, one may resort to symbolic integration but in most cases a numerical integration technique will be necessary. Here again there are many methods that one can use, some more accurate and some simpler than others. The following is the simplest method and is called Sympson's trapezoidal rule. It follows the basic method of integration and is described next.

Consider a continuous function $y = F(x)$, shown in **Figure F** as the black curve. The integral of this function over the limit X_1 and X_2 is:

$$W = \int_{x=X_1}^{x=X_2} F(x)\,dx$$

Suppose now that the x-coordinate is divided into any number of segments N, as shown in **Figure F**. The Integral can now be approximated by assuming that the continuous function can be assumed to be linear between each to values of x as shown by the red curve. The trapezoidal rule calculates the areas of each section by evaluating the function ate the center of each segment and multiplying by the width of the segment. The area of segment k is:

$$w_k = F\left(\frac{x_{k+1}+x_k}{2}\right)\times\left(x_{k+1}-x_k\right)$$

where $x_{k+1}-x_k$ is the width of the segment and $\left(x_{k+1}+x_k\right)/2$ is the center of the segment where the function is evaluated. This formula gives the average value of the area.
With this, the integral is approximated as:

$$W \approx \sum_{k=1}^{N} w_k = \sum_{k=1}^{N} F\left(\frac{x_{k+1}+x_k}{2}\right)\times\left(x_{k+1}-x_k\right)$$

To simplify calculation one may resort to making all segments equal in width: $x_{k+1}-x_k = \Delta x$. The integral now is:

$$W \approx \sum_{k=1}^{N} F\left(\frac{x_{k+1}+x_k}{2}\right)\times \Delta x$$

Since the exact function is approximated with a linear segment, the narrower the segments (the larger *N*), the better the approximation. Ne should however be reasonable and use a reasonable number of segments. In general, it is difficult to know what is reasonable and the number depends on how the function varies with *x*. One method that is often used is to start with a small number of segments (say 10), calculate the integral, then double the number of segments and calculate it again. An error similar to that in (3) can be calculated and when this error is smaller than a desired value, the last evaluated integral becomes the result.

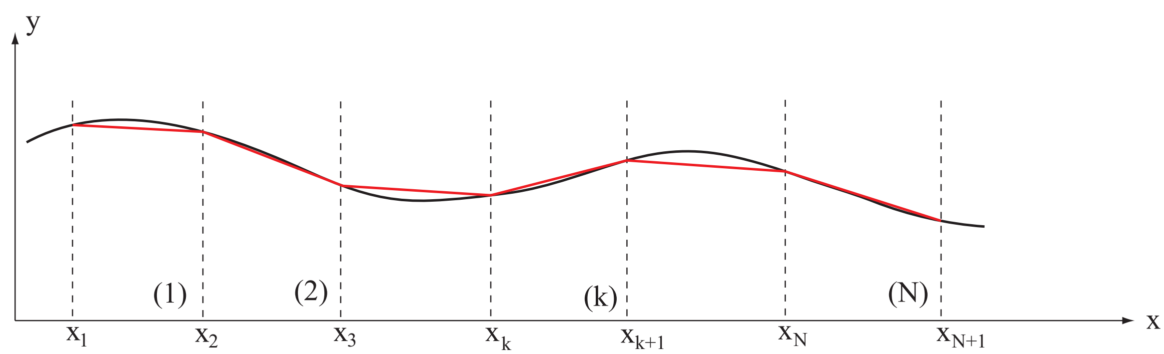


Figure F. Simpson's rule method of integration