Eigenproblems Assignment: Due 2pm Thursday 26 September 2013

Introduction

The aim of this assignment is to implement methods for numerically calculating the eigenvalues and eigenvectors of a system. The methods will be applied to the problem of mass-spring vibrations.

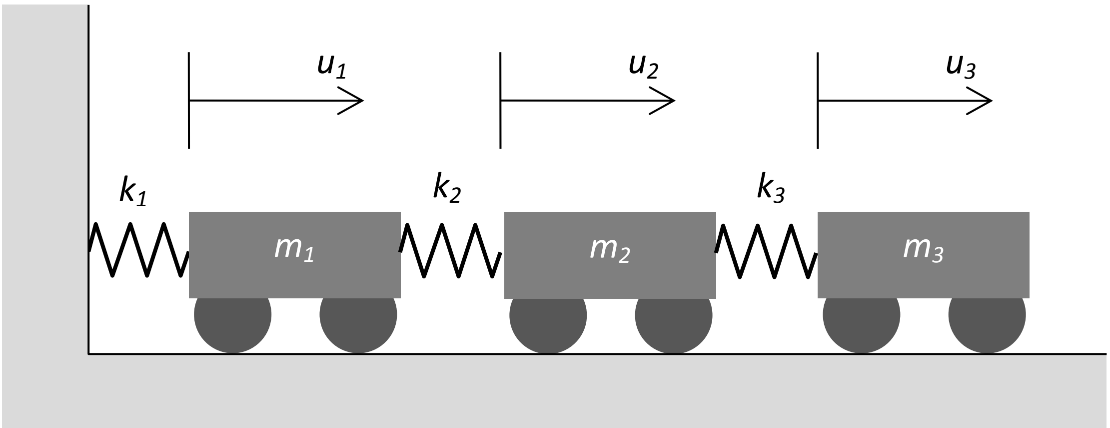

Figure 1: Spring-mass system with masses m_1 to m_3, and spring constants k_1 to k_3.

Consider the simple spring-mass system shown in Figure 1. This can be used to model spring surging, a resonant phenomenon known to occur in the valve springs of high speed engines. This behaviour can be avoided by ensuring that the surging frequencies lie well above the engine vibration frequencies. Applying Newton's second law to each of the masses gives the following system of equations:

$$m_1 \frac{\partial^2 u_1}{\partial t^2} = -k_1 u_1 - k_2(u_1 - u_2)$$
$$m_2 \frac{\partial^2 u_2}{\partial t^2} = -k_2(u_2 - u_1) - k_3(u_2 - u_3)$$
$$m_3 \frac{\partial^2 u_3}{\partial t^2} = -k_3(u_3 - u_2)$$

Assuming a harmonic solution form, $u_i = x_i \cos(\omega t)$ can be substituted for $i = 1, 2, 3$:

$$-\omega^2 x_1 \cos\omega t = \left(\frac{-k_1 - k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2\right) \cos\omega t$$
$$-\omega^2 x_2 \cos\omega t = \left(\frac{k_2}{m_2} x_1 + \frac{-k_2 - k_3}{m_2} x_2 + \frac{k_3}{m_2} x_3\right) \cos\omega t$$
$$-\omega^2 x_3 \cos\omega t = \left(\frac{k_3}{m_3} x_2 - \frac{k_3}{m_3} x_3\right) \cos\omega t$$

Eliminating $\cos(\omega t)$ from the equation yields the following system:

$$-\omega^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} & 0 \\ \frac{k_2}{m_2} & \frac{-k_2 - k_3}{m_2} & \frac{k_3}{m_2} \\ 0 & \frac{k_3}{m_3} & -\frac{k_3}{m_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Or $\mathbf{A}x = \lambda x$ where $\lambda = -\omega^2$. This eigenproblem can be solved to give the modes of vibration for the spring-mass system. Here the mode-shapes are given by the eigenvectors (x), and the mode frequencies are given by the square root of the negative eigenvalues.

Programming Tasks

You are given a main() function and some macros to get you started. We will discuss the shape of the program in class before you start coding it.

Routines to solve Eigenproblems:

1. Write a routine `power_method` implementing the power method to find the largest eigenvalue and its associated eigenvector of a given square matrix. This should be coded as a subroutine that takes an initial estimate of the eigenvector and returns a converged estimate for λ and the normalised eigenvector. The termination criterion should use a user-specified tolerance, Δ, on the eigenvalue. For example: $(|\lambda_{n+1} - \lambda_n|)/(|\lambda_{n+1}|) \leq \Delta$

2. Write a routine `deflate` implementing the deflation procedure to remove a specified eigenvalue/eigenvector pair from a given matrix.

3. Write a routine `eigen_shift` that computes the largest and smallest eigenvalues of a matrix, along with the corresponding eigenvectors, by making use of your `power_method` routine and the shifting algorithm.

4. Write a routine `eigen_all` to find all the eigenvalues and eigenvectors of a symmetric matrix, by making use of your `power_method` and deflate routines.

5. Check that your eigensolver routines are working correctly by comparing your output on a small test system with an independent solver (eg: matlab, excel, by hand). Do not continue until you are satisfied that your routines are working correctly.

Routines to explore the spring surging problem:

6. Write a routine to construct the $\mathbf{A}$ matrix for a spring-mass system like that shown in Figure 1 with N masses. The number of masses, N, the spring constants, k_i, and mass values, m_i, are user inputs.

7. Write a routine to display the natural frequency in Hz and the normalised eigenvector (mode-shape) for each mode. **Note:** the natural frequency in Hz, f, is related to the eigenvalues by:

$$f = \frac{\omega}{2\pi}$$

Modelling Task

A steel spring with 10 active coils can be modelled as a spring-mass system with 10 masses, according to the equations below. Each mass is given by m; the stiffness element joining the first mass to the wall is k_1; the stiffness elements joining one mass to the next are k_i, and the analytical solution to the natural frequencies of a spring are given by f_n, where the spring parameters are also given below.

$$m_i = \frac{\pi^2 D^2 \rho R}{2} \qquad k_1 = \frac{2GD^4}{64R^3} \qquad k_i = \frac{GD^4}{64R^3}, i = 2, 3 \ldots 10 \qquad f_n = \frac{nD\sqrt{G/2\rho}}{16\pi R^2 Na} \quad n = 1,3,5 \ldots$$

Parameter	Description	Value
G	shear modulus of the wire	steel = 7.929e10 N/m^2
ρ	density	steel = 7751 kg/m^3
D	wire diameter	5 mm
R	mean coil radius	53.2 mm
Na	number of active coils	10

Use your code to model a 10 coil spring system, and submit the results and discussion of you model's behaviour in a report.

Report

Write a report (maximum 2 pages) including the following:

1. Program output:
   a. Display the natural frequencies and eigenvectors for the lowest and highest calculated spring modes (power method and shifting)
   b. Display the natural frequencies and eigenvectors for all 10 modes (power method and deflation)
   c. Calculate the natural frequencies of the fixed / free spring using Equation 4.
2. Interpretation of the spring model:
   a. Use a graph to compare the natural frequencies from Equation 4 with the natural frequencies you obtained using the power method and deflation. Where do you think discrepancies and errors can come from here? Consider the order in which you computed the modes and frequencies. Does this make a difference? Where do you think errors might come from in general? Where have they come from in this case?
   b. Draw or sketch the mode-shapes for the first (highest frequency) and the last **two** (lower end frequency) modes.
3. Discussion: To support your representations, write a paragraph including the following:
   a. Define the term "mode shape".
   b. Describe what you are plotting and why this is representative of a mode shape.
   c. How are the mode shapes related to the computed eigenvectors?
   d. What is the significance of the sign of the eigenvector components?
4. Submit all relevant code as an Appendix to your report (not included in the two page limit)